Critical behavior of the Schwinger model via gauge-invariant variational uniform matrix product states

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Plan of the talk

- Motivations and background
- Continuum and lattice Schwinger models
- Gauge-invariant VUMPS
- MPS Transfer matrices
- Critical behaviors
- Conclusion

Hamiltonian lattice gauge theories

- Hamiltonian lattice gauge theories are free of the sign problem.
- They will be an important application of quantum computers in the future.
- Even today, Hamiltonian lattice gauge theories can yield new physical results, by applying tensor network techniques.

Continuum Schwinger model

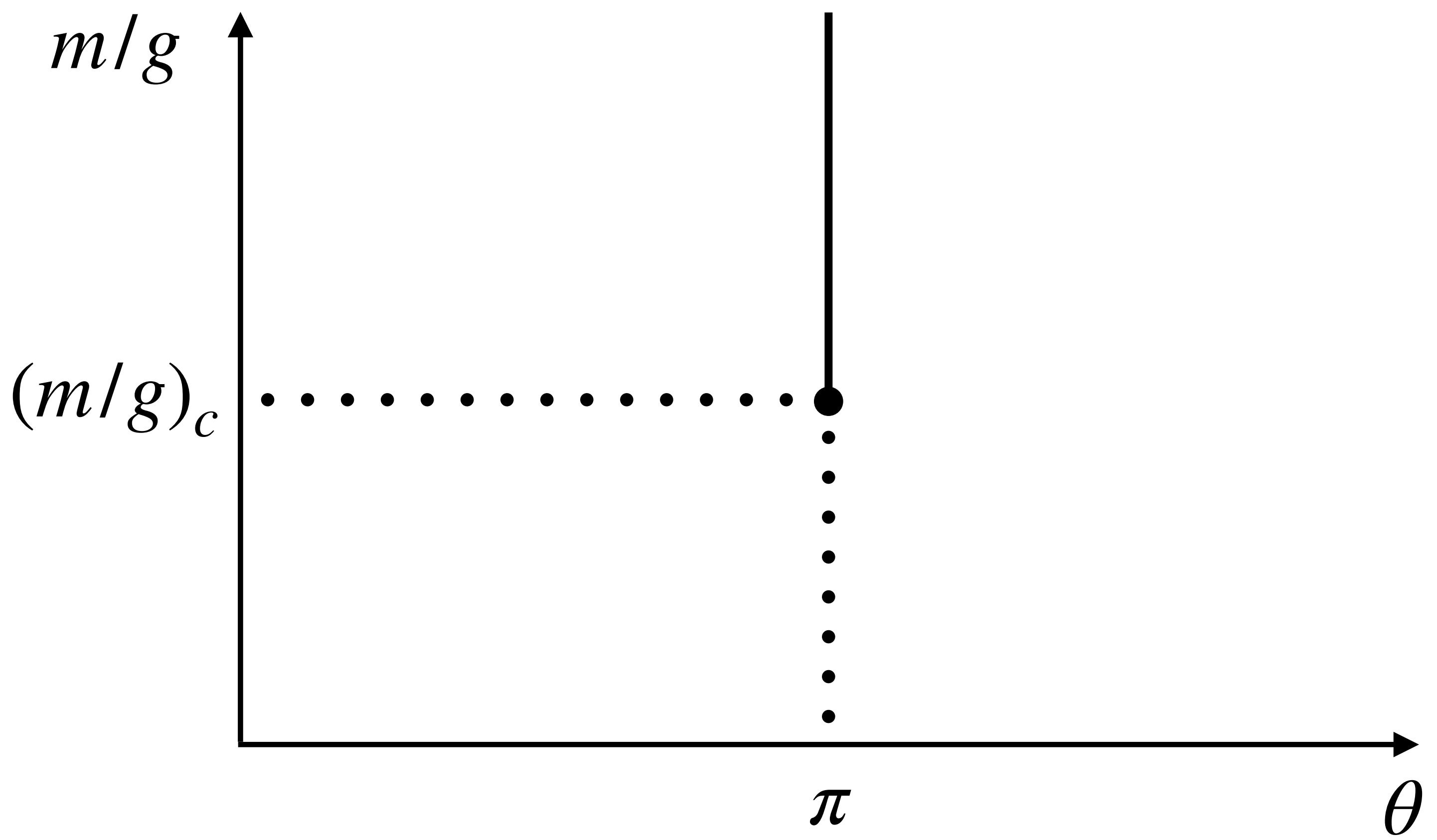
- The Schwinger model (QED in 1+1D) is a tractable toy model of QCD in 3+1D.
- The massive Schwinger model with a topological term is defined by the action

$$S = \int \mathrm{d}^2 x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{g\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu} + i\bar{\psi}(\gamma^\mu \partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi \right]$$

- It is a useful testbed for numerical techniques.
- Solvable for the vanishing fermion mass $m = 0$ by bosonization.

Quantum phase transitions

- The model exhibits 1st- and 2nd-order phase transitions as the parameters $(\theta, m/g)$ are varied.



- The most precise estimate was $(m/g)_c \simeq 0.3335(2)$ by Byrnes et al. in '02. Our work (and another concurrent one) update this value.

Lattice Hamiltonian formulation

- The Hamiltonian on a 1D lattice is [Banks-Susskind-Kogut]

$$H_\theta = \frac{g^2 a}{2} \sum_n \left(L(n) + \frac{\theta}{2\pi} \right)^2 - \frac{i}{2a} \sum_n \left(\phi(n)^\dagger U_n \phi(n+1) - \mathrm{h.c.} \right)$$
$$+ m_{\mathrm{lat}} (-1)^n \phi^\dagger(n) \phi(n)$$

$$\{\phi(m), \phi^\dagger(n)\} = \delta_{mn}, \quad [L(m), U(n)] = \delta_{mn} U(m)\,.$$

- We relate the continuum mass m to m_{lat} by an $\mathcal{O}(a)$ improvement: $m \equiv m_{\mathrm{lat}} + g^2 a/8$. [Dempsey et al.] At $m = 0$, H_θ and $H_{\theta+\pi}$ are unitarily related by a discrete chiral symmetry transformation.

Gauss law constraint

- On physical states, we impose the Gauss law constraint $G(n)\,|\,\mathrm{phys}\rangle = 0$, where

$$G(n) \equiv L(n) - L(n-1) - \phi(n)^{\dagger}\phi(n) + \frac{1+(-1)^n}{2}.$$

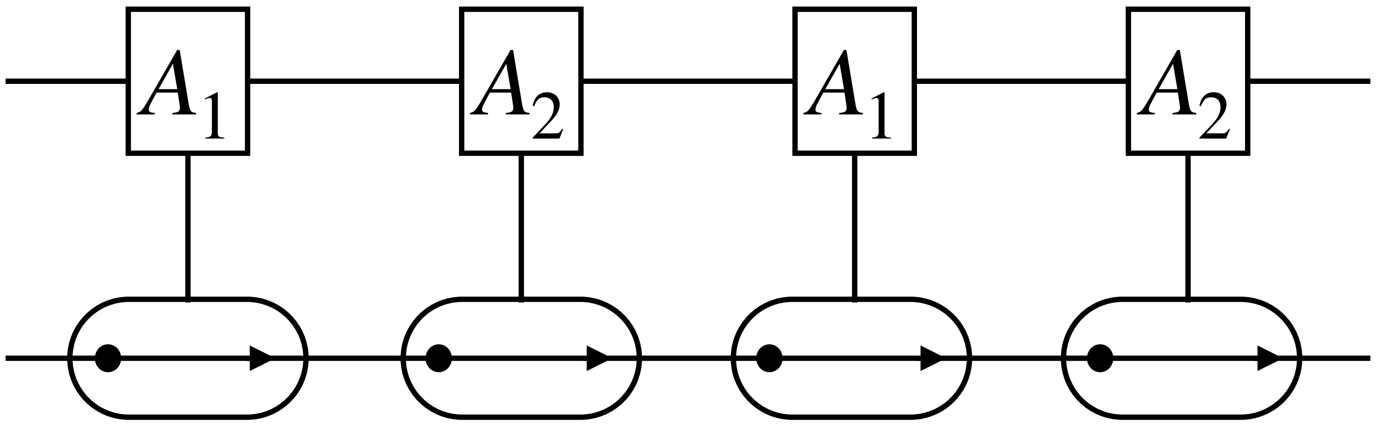
- The constraint may be used to eliminate $L(n)$.
- Instead, we will use a tensor network ansatz that automatically solves the constraint.

Uniform matrix product state

- The Hilbert space $H_{\text{ferm}} \otimes H_{\text{gauge}}$ is spanned by $|\{s_n, p_n\}\rangle = \bigotimes_{n\in\mathbb{Z}} |s_n, p_n\rangle$ labeled by $s_n = 2\phi(n)^\dagger\phi(n) - 1 \in \{+1, -1\}$ and the quantized electric fields $p_n = L(n) \in \mathbb{Z}$.

- We use a 2-site translation invariant uniform matrix product state (uMPS) ansatz

$$|\Psi\rangle = \sum_{\{s_n,p_n\}} \left(\prod_{j\in\mathbb{Z}} A_1^{s_{2j-1},p_{2j-1}} A_2^{s_{2j},p_{2j}}\right) |\{s_n, p_n\}\rangle$$



Gauge-invariant MPS ansatz

- The Gauss law constraint is solved by the gauge-invariant ansatz [Buyens et al. '14]

$$(A_n^{s,p})_{(q\alpha_q;r\beta_r)} = (a_n^{s,q})_{\alpha_q,\beta_r}\delta_{p,q+(s+(-1)^n)/2}\delta_{p,r}$$

- Each bond $(q\alpha_q)$ carries the U(1) gauge charge q.

- Any MPS that satisfies the constraint can be transformed into this form.

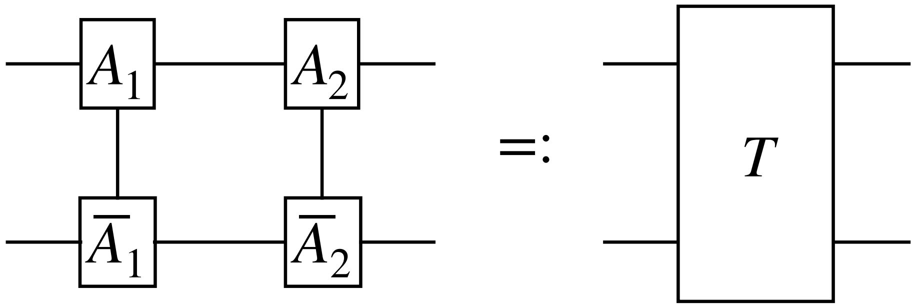
- For each q, $\alpha_q = 1,2,\ldots,D_q$. The total bond dimension is $D = \sum_q D_q$.

VUMPS algorithm

- We apply the VUMPS (variational uniform matrix product state) algorithm [Zauner-Stauber et al. '17] to the gauge-invariant MPS ansatz. The first such attempt.
- VUMPS seeks the ground state of a translationally invariant Hamiltonian recursively. As in DMRG, the extremum condition can be recast into eigenvalue problems that involve the effective environment Hamiltonian determined from a previous recursive step.
- MPS tensors are updated by singular value decomposition (SVD) and a linear eigensolver.

MPS transfer matrix

- Define the MPS transfer matrix T. [Zauner et al.'14]



- Let us view the spatial direction as a Euclidean time. In the limit $ga \to 0$ and $D \to \infty$, the transfer matrix T should describe an imaginary time evolution of a relativistic continuum QFT on $\mathbb{R}^2$, $T \sim e^{-aH}$.

- For finite D, the eigenvalues of T are discrete: $\lambda_0 = 1,\ \lambda_1 = e^{-\epsilon_1 + i\phi_1},\ \lambda_2 = e^{-\epsilon_2 + i\phi_2}, \ldots$ with $0 < \epsilon_1 \leq \epsilon_2 \leq \ldots$. The bond dimension D is interpreted as an IR cut-off.

Transfer matrix eigenvalues

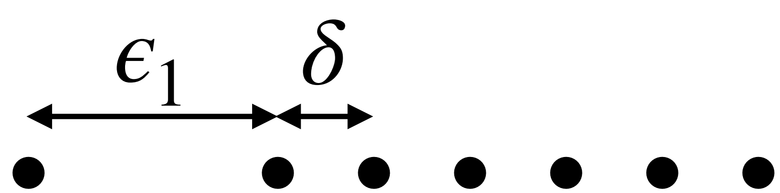
- $T = \sum_{i=0}^{\infty} \lambda_i \,|\, i)(i \,|$ with $(i \,|\, j) = \delta_{ij} \Rightarrow$
  $\langle O_1(0) O_2(2n+1) \rangle_{\text{conn}} \sim \sum_{j>0} Z_{12}^j e^{(-\epsilon_j + i\phi_j)n}$ for come constants (form factors) Z_{12}^j.
- $1/\epsilon_1$ is the (leading) correlation length.
- In practice, $\phi_j = 0$, $0 < \epsilon_1 < \epsilon_2 < \ldots$.

$\delta(D)$ as an IR cut-off

- The effect of the finite bond dimension D can be quantified more clearly by $\delta(D) \equiv \epsilon_2 - \epsilon_1$ than by D itself. [Rams et al. '18, Vanhecke et al. '19]

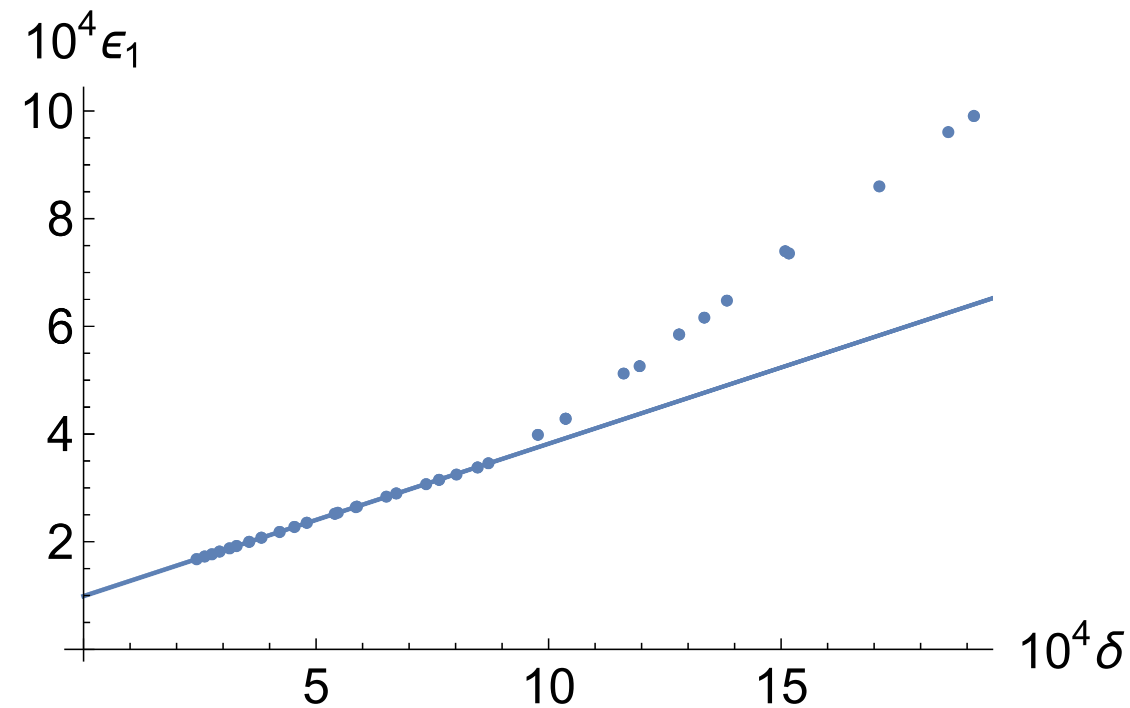
- $1/\delta(D)$ is interpreted as the effective size (IR cut-off) of a fictitious "space". $\Rightarrow$ Finite size scaling.

- ϵ_j's form a continuum as $D \to \infty, \delta \to 0$.



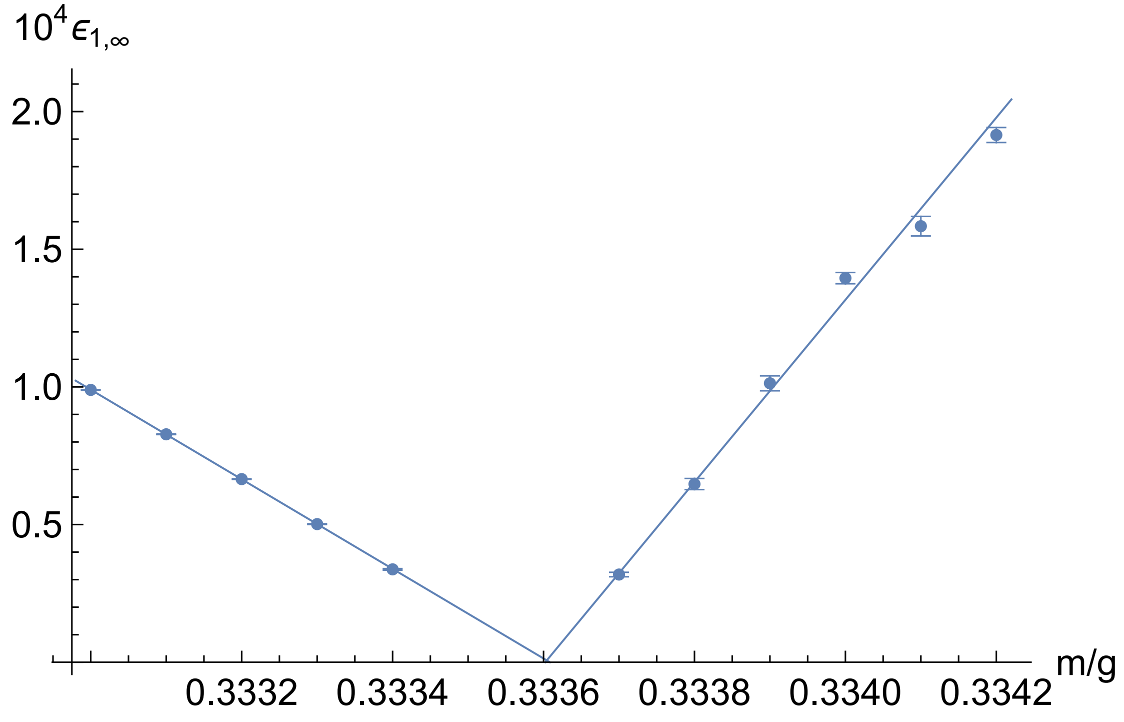
- $\epsilon_1(D) = \epsilon_{1,\infty} + c_1\delta(D)$

1/(correlation length)
for $D = \infty$

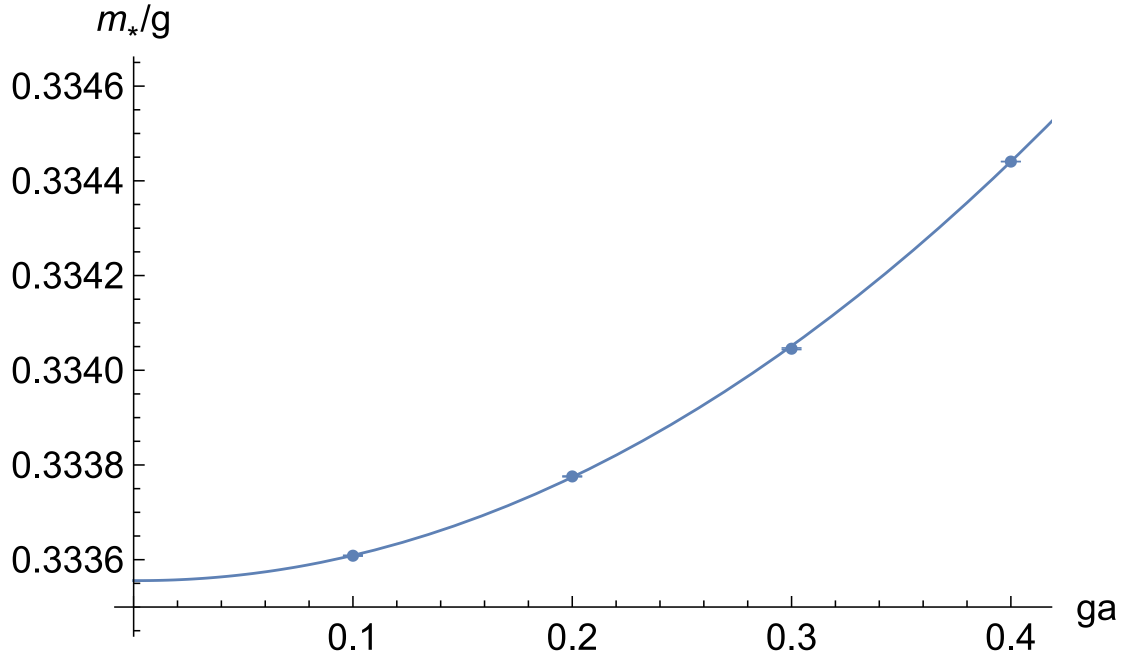


Estimation of the critical mass

- The ga-dependent critical mass $(m/g)_*$ is where the intercept $\epsilon_{1,\infty} = \epsilon_{1,\infty}(m/g)$ vanishes.



- As $ga \to 0$, $(m/g)_*$ approaches the critical mass $(m/g)_c$ in the continuum limit.



Result for the critical mass

- We obtained the value $(m/g)_c = 0.333556(5)$.
- An independent DMRG work by Arguello Cruz et al. published around the same time (last December): $(m/g)_c = 0.333561(4)$, a consistent result.
- We have an improvement by two orders of magnitude from the previous best estimate $(m/g)_c = 0.3335(2)$ by Byrnes et al. '02.

Scaling behaviors

If the critical behavior is described by the Ising universality class in the IR and a $c = 1$ CFT in the UV, the combinations

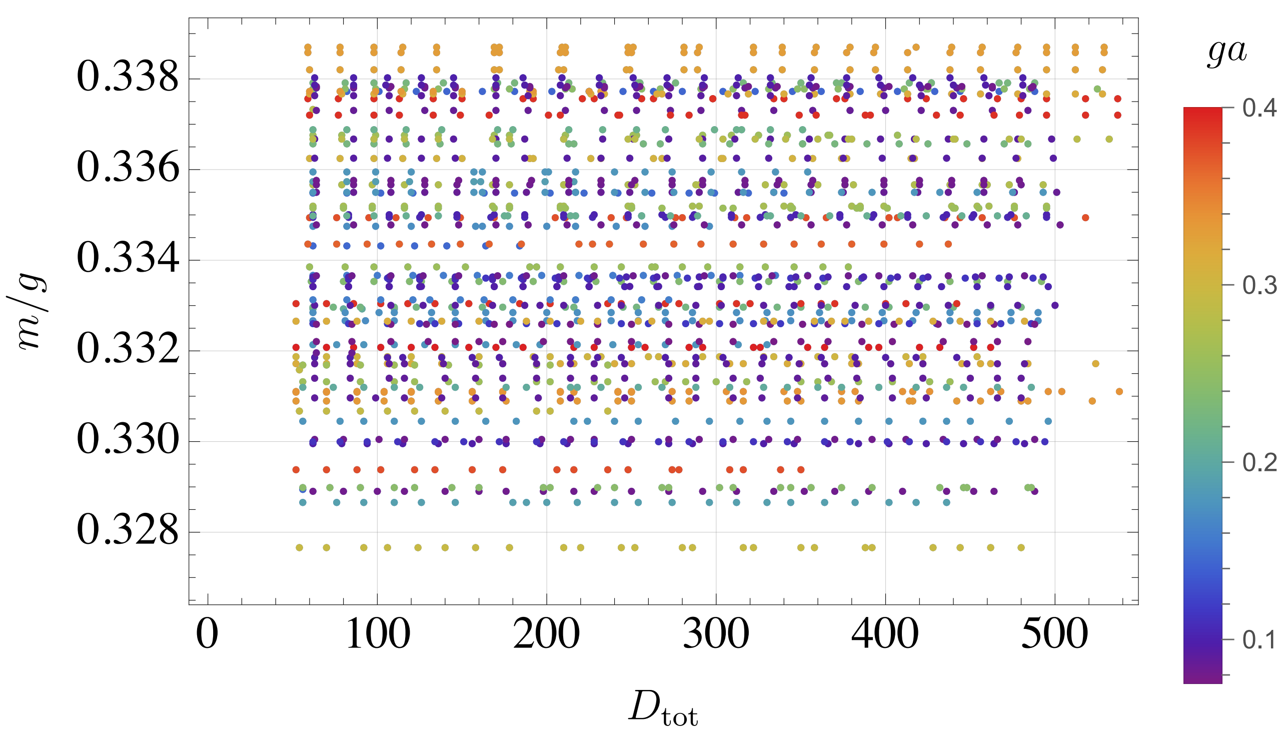
$$\tilde{\xi} = \xi/(\Lambda L)\,,\ \tilde{\phi} = L^{1/8}\phi\,,\ \tilde{S} = S - \frac{1/2}{6}\log L - \frac{1}{6}\log\Lambda$$

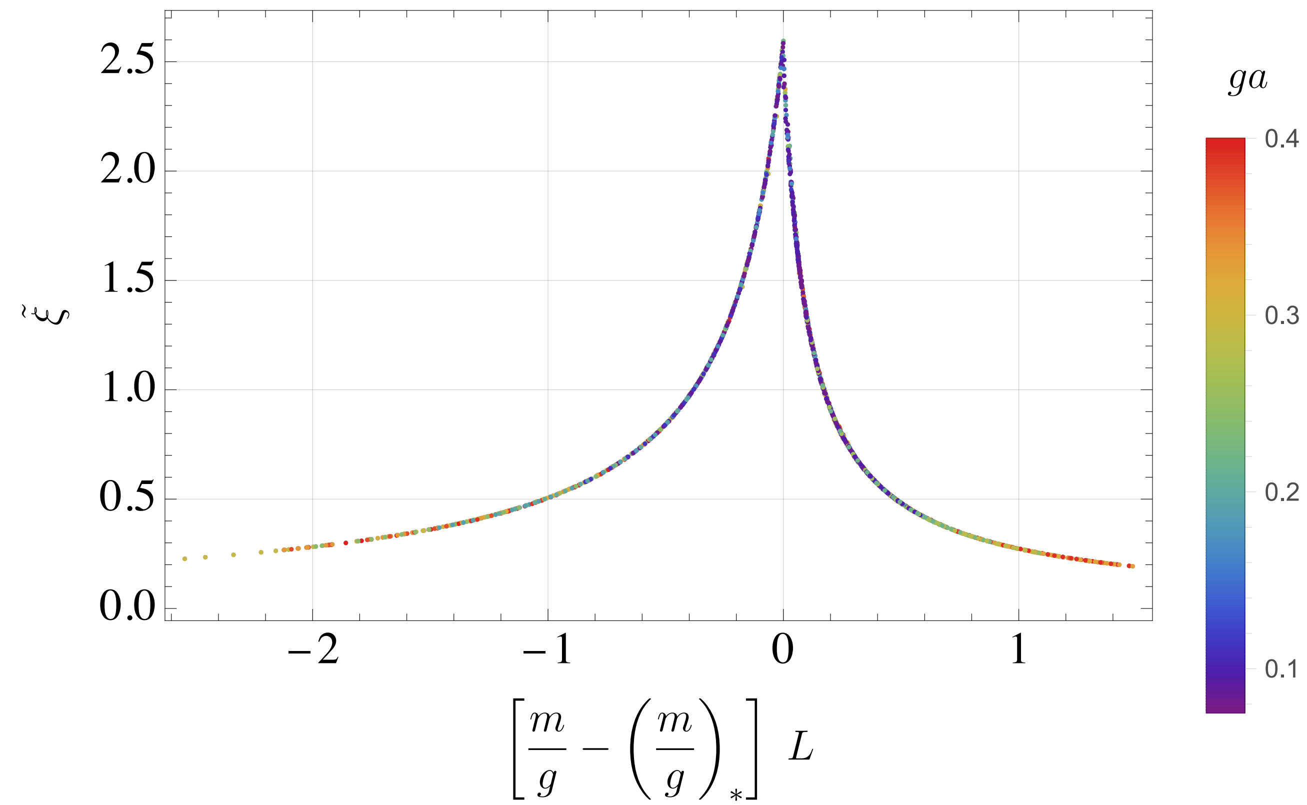
with $\phi = \left\langle \frac{L(n) + L(n+1) + 1}{2} \right\rangle$, $\xi = \frac{1}{\epsilon_1}$, $S =$ bipartite entanglement entropy, should exhibit a double data collapse as functions of

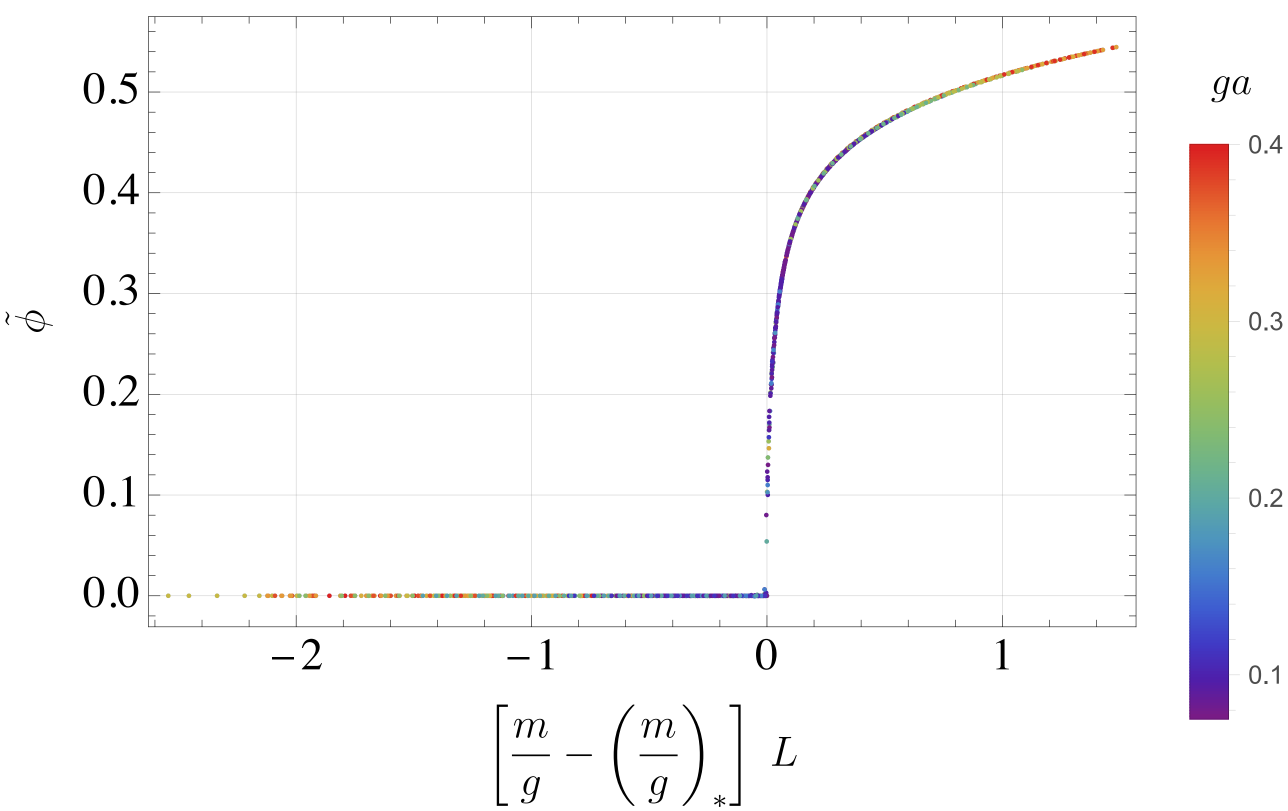
$$L = \frac{ga}{\delta}\,,\ \Lambda = \frac{1}{ga}\,,\ t = \left(\frac{m}{g}\right) - \left(\frac{m}{g}\right)_*$$

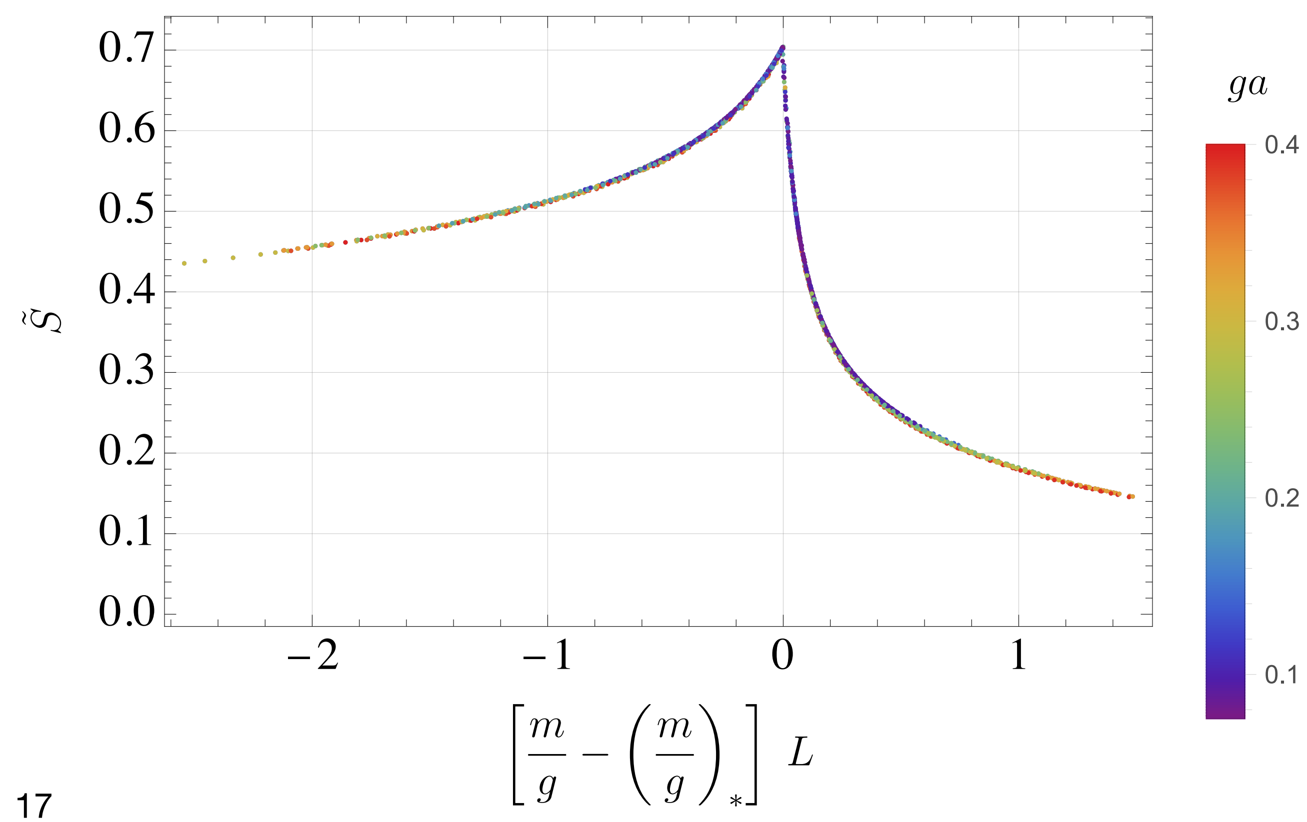
where $(m/g)_* = (m/g)_c + b_1\Lambda^{-1} + b_2\Lambda^{-2} + l_1/L$.

Double collapse of data









Conclusion

- Exhibited the precise critical behavior of the lattice Schwinger model.
- The ground state was obtained by the VUMPS algorithm applied to the special uMPS ansatz where all the variational degrees of freedom are restricted to the gauge-invariant subspace.
- Obtained the precise critical mass in the continuum limit $(m/g)_c = 0.333556(5)$.
- Demonstrated the double collapse of the randomly generated numerical values for several physical quantities.