Universal Protocol for Quantum Simulation of Gauge Theories in Arbitrary Dimensions and Gauge Groups

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Based on 2011.06576 [hep-th], 2401.12045 [hep-th], 2411.13161 [quant-ph], 2505.02553 [quant-ph] + a few more

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2 Sept. 2025 @Trento, Italy

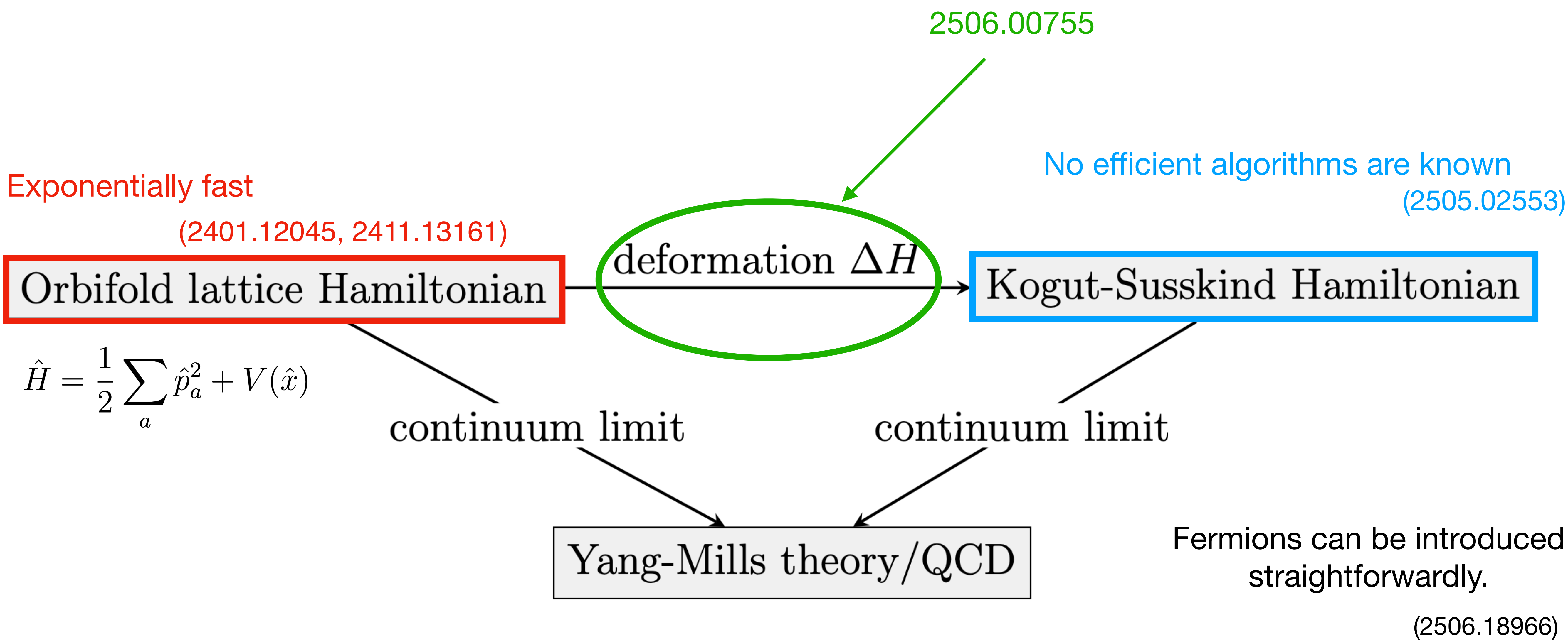
Main message

- Quantum computer can do only simple things at large scale.
- Efficient circuit and compilation strategy are crucial.
- Non-compact variables lead to exponential improvement.
- Compact variables are, most likely, dead end.

(Objections are welcome!)

Key fact

Cartesian coordinate is easier than polar coordinate



2506.00755
Exponentially fast
(2401.12045, 2411.13161)
Orbifold lattice Hamiltonian
$\hat{H} = \frac{1}{2}\sum_a \hat{p}_a^2 + V(\hat{x})$
deformation ΔH
No efficient algorithms are known
(2505.02553)
Kogut-Susskind Hamiltonian
continuum limit
continuum limit
Yang-Mills theory/QCD
Fermions can be introduced straightforwardly.
(2506.18966)

Susskind

Easy BFSS matrix model

↓ Orbifold projection

(Kaplan, Katz, Unsal)

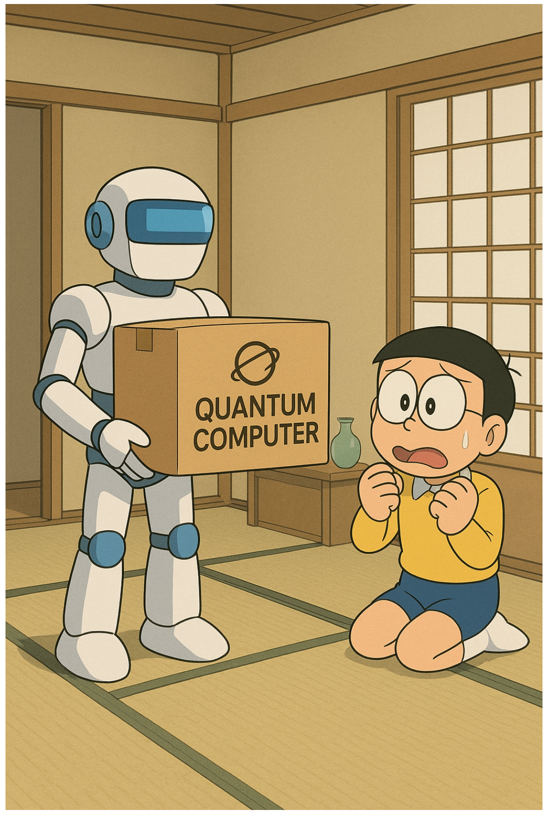
Easy Orbifold lattice

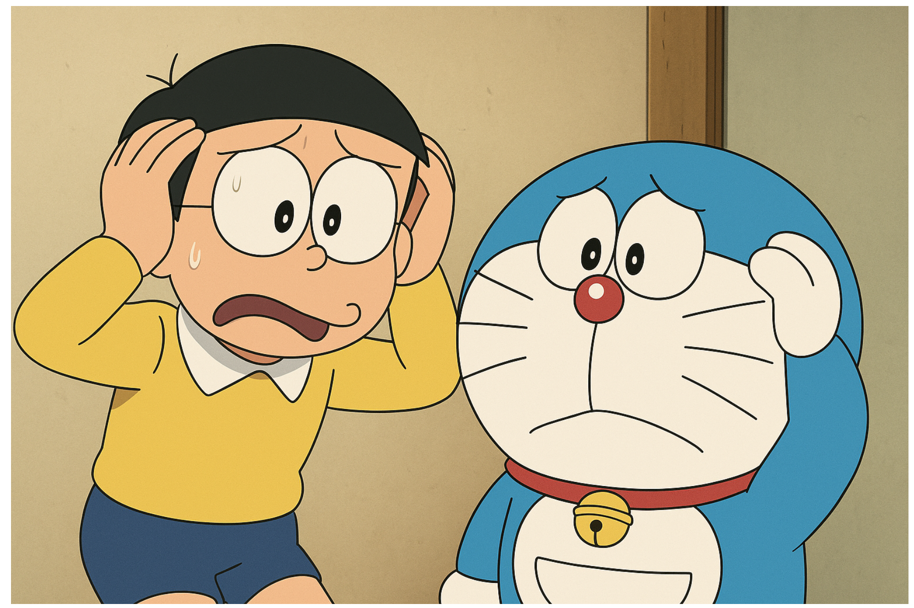
INSTITUTE for NUCLEAR THEORY

↓ Deformation & large mass

KS Hamiltonian

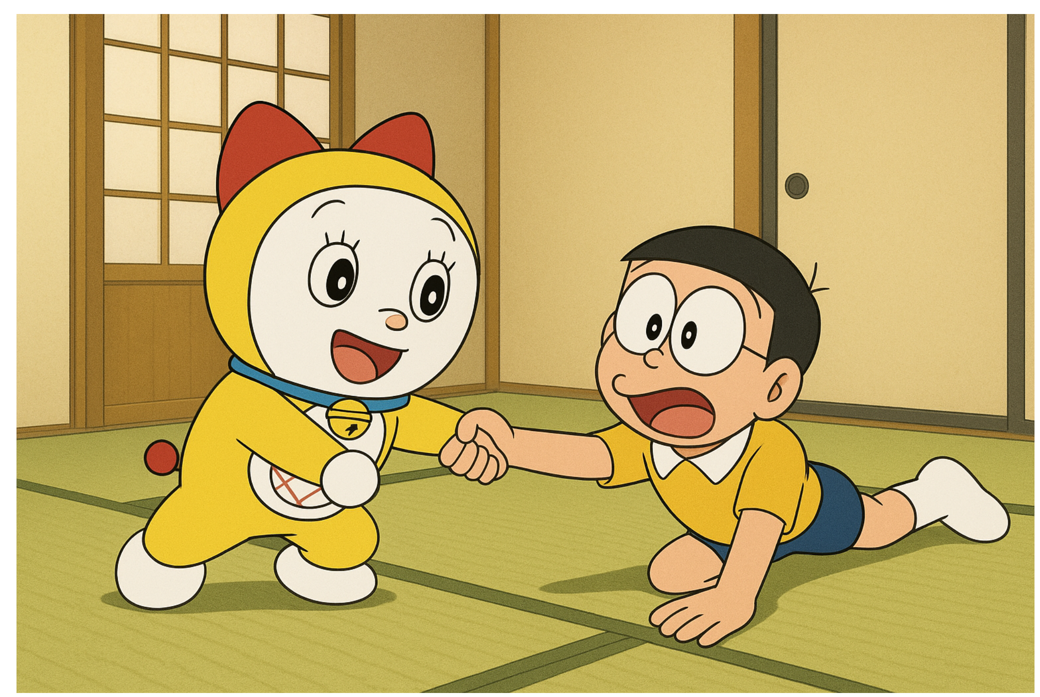
Implications





A fault-tolerant device has been delivered,
but we cannot write a program…

No problem!
We can use non-compact variables!



Images generated by ChatGPT

$$\hat{U}\,|U\rangle = U\,|U\rangle$$

$$U \in \mathrm{SU}(N)$$

Lattice regularization is nightmare

Electric field ~ momentum $\hat{E}$

$$\langle U|R, ij\rangle = \rho^{(R)}_{ij}(U)$$

'plane wave' = irreducible representation

Fourier transform is nightmare

$$\hat{x}\,|x\rangle = x\,|x\rangle$$

$$x \in \mathbb{R}$$

Lattice regularization is trivial

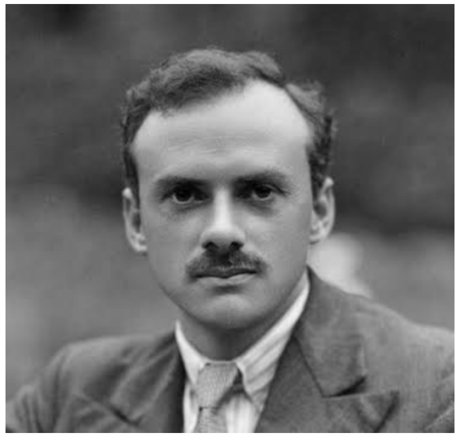
momentum $\hat{p}$

$$\langle x|p\rangle = e^{ipx}$$

$$[\hat{x}, \hat{p}] = i$$

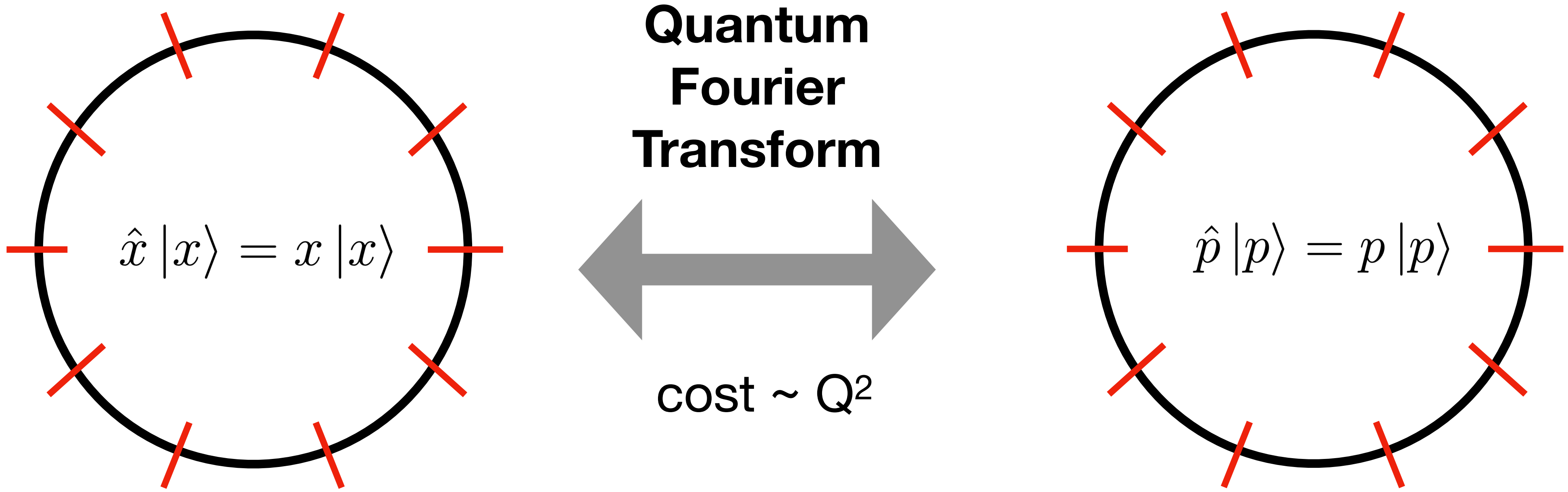
Fourier transform is trivial

$\hat{x}$ and $\hat{p}$ are simpler



$\Lambda = 2^Q$ points for each boson

Killer application on quantum computer



$$\hat{x}_a = -\delta_x \cdot \left(\frac{\hat{\sigma}_{z;a,1}}{2} + 2 \cdot \frac{\hat{\sigma}_{z;a,2}}{2} + \cdots + 2^{Q-1} \cdot \frac{\hat{\sigma}_{z;a,Q}}{2} \right)$$

$$\hat{p}_a = -\delta_p \cdot \left(\frac{\hat{\sigma}_{z;a,1}}{2} + 2 \cdot \frac{\hat{\sigma}_{z;a,2}}{2} + \cdots + 2^{Q-1} \cdot \frac{\hat{\sigma}_{z;a,Q}}{2} \right)$$

$$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$$

$\hat{x}$ and $\hat{p}$ are simpler

→ Scalar QFT is simpler

$$\hat{x}, \hat{p} \leftrightarrow \hat{\phi}_{\vec{n}}, \hat{\pi}_{\vec{n}}$$

(e.g., Jordan, Lee, Preskill, 2011)

$$[\hat{\phi}_{\vec{n}}, \hat{\pi}_{\vec{n}'}] = i\delta_{\vec{n}\vec{n}'}$$

→ Matrix model is simpler

$$\hat{x}, \hat{p} \leftrightarrow \hat{X}_{M,ij}, \hat{P}_{M,ij}$$

(e.g., Gharibyan, MH, Honda, Liu, 2021
Maldacena, 2023)

Don’t use (vanilla) Kogut-Susskind

- N^2-1 bosons cannot be treated separately.
- $Q(N^2-1)$ qubits must be treated together.
- 4^{Q(N²−1)} Pauli strings would be needed for each link.
- 4^{4Q(N²−1)} Pauli strings would be needed for each plaquette.

Key question: How large Q do we need?

- Fluctuation of $A_\mu \sim 1/a$ in (3+1)-d
- $\Lambda = 2^Q \sim 1/a$
- 4^{Q(N²−1)} ∼ (1/a)^{2(N²−1)} ∼ (1/a)^{16} for SU(3), for each link.
- 4^{4Q(N²−1)} ∼ (1/a)^{8(N²−1)} ∼ (1/a)^{64} for SU(3), for each plaquette.

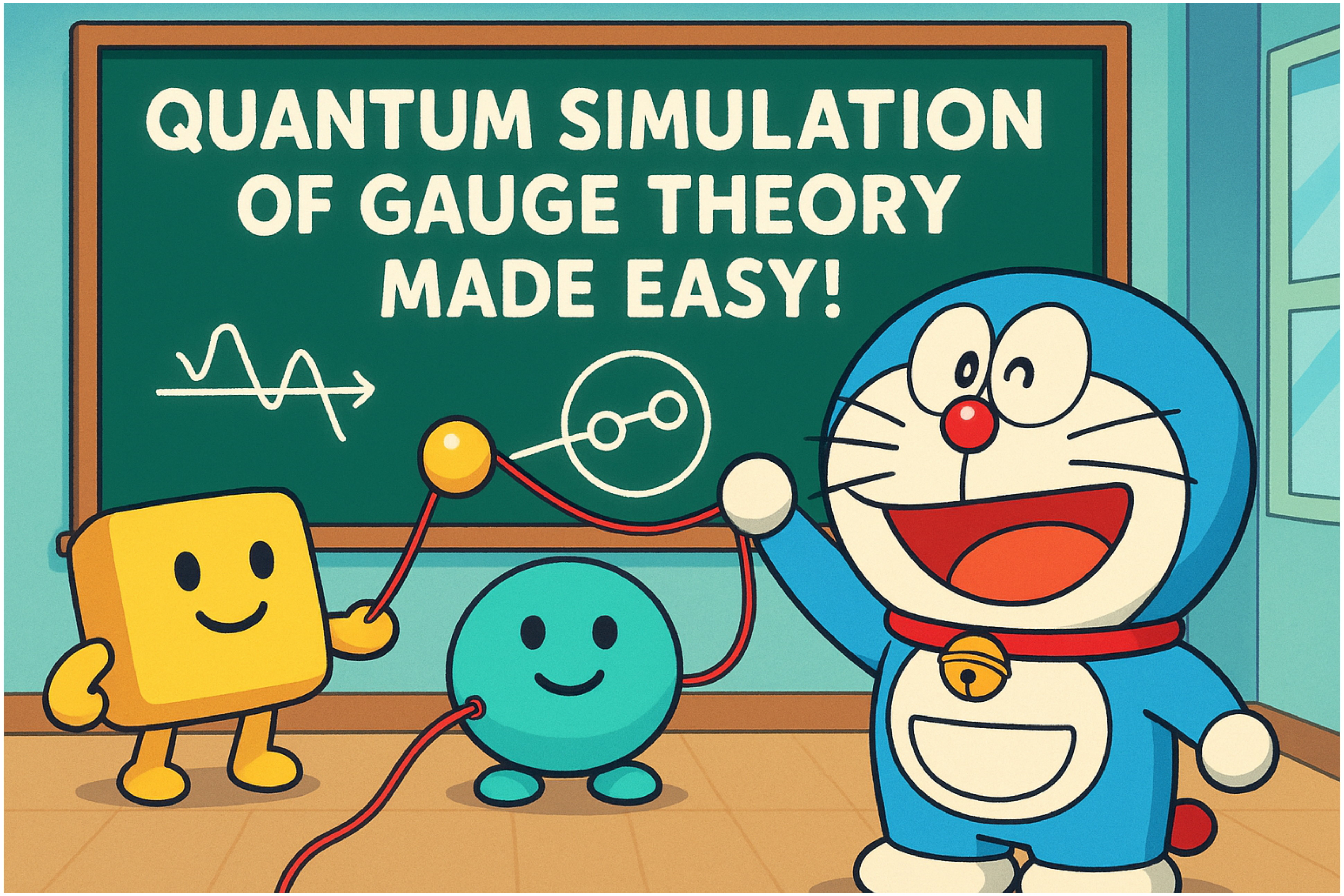


Image generated by ChatGPT

$$\mathrm{SU}(2) = \mathrm{S}^3 \subset \mathbb{R}^4$$

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1$$

- It's hard to write efficient circuit keeping this constraint.
- It's easy if this constraint is realized dynamically, adding

$$m^2 \times (|\alpha|^2 + |\beta|^2 - 1)^2$$

Scalar mass term

$$\mathbb{R}^4 = \mathrm{S}^3 \times \mathbb{R}_{>0}$$

Gauge field **Scalar field**

Orbifold lattice construction

(Kaplan, Katz, Unsal, 2002)

- Original motivation: supersymmetric lattice (not relevant now)
- Non-compact variables → Ideal for quantum simulations

(Buser, Gharibyan, MH, Honda, Liu, 2020; Bergner, MH, Rinaldi, Schafer, 2024, …)

$$\mathrm{SU}(N) \subset \mathbb{C}^{N^2} = \mathbb{R}^{2N^2}$$

$$\mathrm{SU}(2) = \mathrm{S}^3 \subset \mathbb{R}^4 \subset \mathbb{R}^8$$

scalar field

$$Z = c \cdot W \cdot U = c \cdot e^{as} \cdot e^{iaA}$$

gauge field

positive definite Hermitian (W)

unitary (U)

$$L = \int d^3x \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu s_I)^2 + \frac{g_{4\mathrm{d}}^2}{4}[s_I, s_J]^2 \right)$$

Orbifold Lattice Hamiltonian for Yang-Mills

$$\hat{H} = \frac{1}{2}\sum_a \hat{p}_a^2 + V(\hat{x})$$

$$\hat{H} = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{d} \hat{P}_{j,\vec{n}}\,\hat{\bar{P}}_{j,\vec{n}} + \frac{g_{\mathrm{d}}^2}{2a^d}\left| \sum_{j=1}^{d} \left(\hat{Z}_{j,\vec{n}}\,\hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}}\hat{Z}_{j,\vec{n}-\hat{j}} \right) \right|^2 \right.$$
$$\left. + \frac{2g_{\mathrm{d}}^2}{a^d} \sum \left| \hat{Z}_{j,\vec{n}}\,\hat{Z}_{k,\vec{n}+\hat{j}} - \hat{Z}_{k,\vec{n}}\,\hat{Z}_{j,\vec{n}+\hat{k}} \right|^2 \right) + \Delta\hat{H}\,,$$

$$\Delta\hat{H} \equiv \frac{m^2 g_d^2}{2a^{d-2}} \sum_{\vec{n}} \sum_{j=1}^{d} \mathrm{Tr}\left| \hat{Z}_{j,\vec{n}}\hat{\bar{Z}}_{j,\vec{n}} - \frac{a^{d-2}}{2g_d^2} \right|^2$$
$$+ \frac{m_{\mathrm{U}(1)}^2 a^{d-2}}{2g_d^2} \sum_{\vec{n}} \sum_{j=1}^{d} \left| \left(\frac{a^{d-2}}{2g_d^2} \right)^{-N/2} \det\left(\hat{Z}_{j,\vec{n}} \right) - 1 \right|^2 .$$

Large mass limit = Kogut-Susskind

Orbifold Lattice Hamiltonian for quark sector

$$\hat{H}_{\rm naive} = ia^3 \sum_{\vec{n}} \left\{ -\frac{1}{2a}\sqrt{\frac{2g_{\rm 4d}^2}{a}} \sum_{j=1}^{3} \left(\hat{\bar{\psi}}_{\vec{n}} \gamma^j \hat{Z}_{j,\vec{n}} \hat{\psi}_{\vec{n}+\hat{j}} - \hat{\bar{\psi}}_{\vec{n}+\hat{j}} \gamma^j \hat{\bar{Z}}_{j,\vec{n}} \hat{\psi}_{\vec{n}} \right) + m \hat{\bar{\psi}}_{\vec{n}} \hat{\psi}_{\vec{n}} \right\}$$

$$\hat{H}_{\rm Wilson} = a^3 \cdot \frac{i}{2a} \sum_{j=1}^{3} \sum_{\vec{n}} \left(\hat{\bar{\psi}}_{\vec{n}} - \sqrt{\frac{2g_{\rm 4d}^2}{a}} \hat{\bar{\psi}}_{\vec{n}+\hat{j}} \hat{\bar{Z}}_{j,\vec{n}} \right) \left(\hat{\psi}_{\vec{n}} - \sqrt{\frac{2g_{\rm 4d}^2}{a}} \hat{Z}_{j,\vec{n}} \hat{\psi}_{\vec{n}+\hat{j}} \right)$$

A universal framework

$$\hat{H} = \frac{1}{2}\sum_a \hat{p}_a^2 + V(\hat{x})$$

Orbifold lattice,
matrix model,
scalar QFT, …

Simple in momentum basis ⟷ (Fourier Transform) ⟷ Simple in coordinate basis

arXiv > quant-ph > arXiv:2411.13161

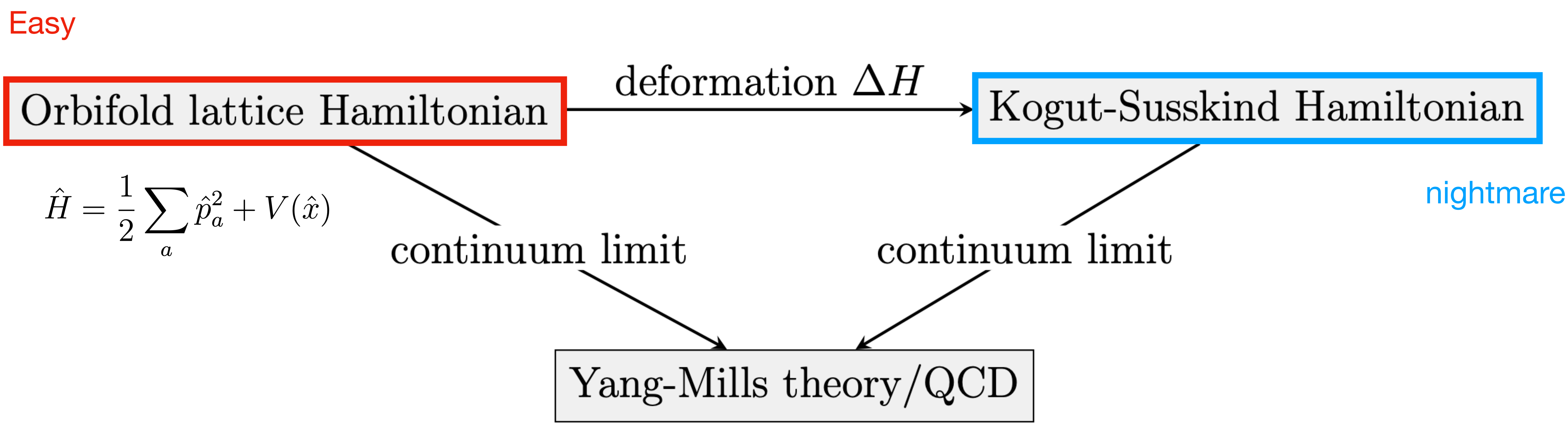
Quantum Physics

[Submitted on 20 Nov 2024]

A universal framework for the quantum simulation of Yang-Mills theory

Jad C. Halimeh, Masanori Hanada, Shunji Matsuura, Franco Nori, Enrico Rinaldi, Andreas Schäfer

→ any SU(N), any dimensions, any matter content



How easy?

Example: Trotter decomposition

$$\hat{H} = \frac{1}{2}\sum_a \hat{p}_a^2 + \underline{V(\hat{x})}$$

quartic for orbifold lattice

(Roughly speaking, $\hat{U}\hat{U}\hat{U}^\dagger\hat{U}^\dagger \to \hat{Z}\hat{Z}\hat{Z}^\dagger\hat{Z}^\dagger$)

$$\hat{x}_a = -\delta_x \cdot \left(\frac{\hat{\sigma}_{z;a,1}}{2} + 2 \cdot \frac{\hat{\sigma}_{z;a,2}}{2} + \cdots + 2^{Q-1} \cdot \frac{\hat{\sigma}_{z;a,Q}}{2} \right)$$

$$\boxed{\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4 \sim \sum \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z}$$

$$\hat{p}^2 \sim \sum \hat{\sigma}_z \otimes \hat{\sigma}_z$$

$$\hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 \sim \sum \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z$$

$$\rightarrow \boxed{\exp(-\mathrm{i}\hat{p}^2 \Delta t)} \rightarrow \text{Q.F.T.} \rightarrow \boxed{\exp(-\mathrm{i}V(\hat{x})\Delta t)} \rightarrow \text{Q.F.T.} \rightarrow \cdots$$

$$\begin{aligned}&\exp\left(-\mathrm{i}\theta C'_{pqrs}\hat{\sigma}_{z,p}\hat{\sigma}_{z,q}\hat{\sigma}_{z,r}\hat{\sigma}_{z,s}\right)\\&\quad=\mathrm{CX}_{p,q}\mathrm{CX}_{q,r}\mathrm{CX}_{r,s}\exp\left(-\mathrm{i}\theta C'_{pqrs}\hat{\sigma}_{z,s}\right)\mathrm{CX}_{r,s}\mathrm{CX}_{q,r}\mathrm{CX}_{p,q}\end{aligned}$$

$$\mathrm{CX}_{p,q}\left|b_p\right\rangle_p\left|b_q\right\rangle_q=\left|b_p\right\rangle_p\left|b_p\oplus b_q\right\rangle_q$$

- Go to Momentum basis via Quantum Fourier Transform.
- In the momentum basis, $\hat{p}$ is diagonal.

Fourier transform is easy, unlike Kogut-Susskind

Trotter decomposition

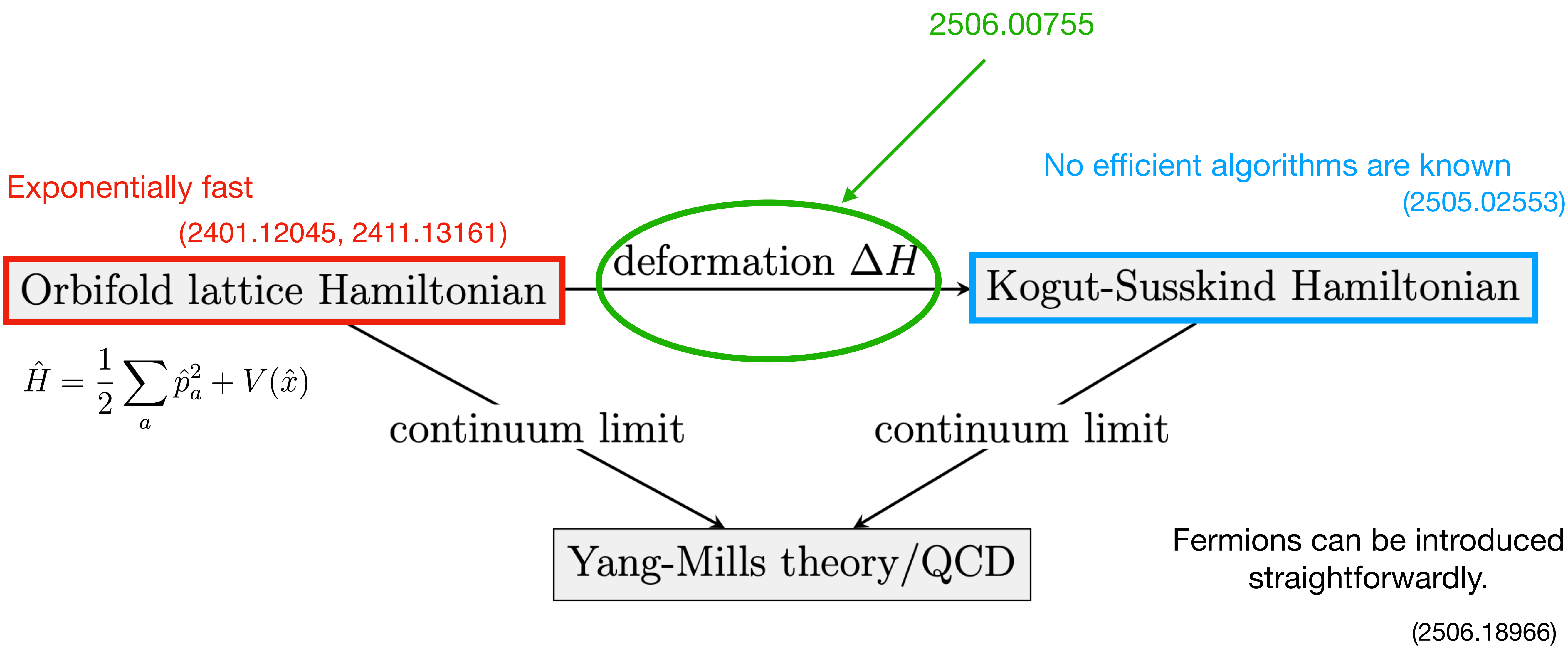
$$\hat{p}^2 \sim \sum \hat{\sigma}_z \otimes \hat{\sigma}_z$$

$$\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4 \sim \sum \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \hat{\sigma}_z$$

$$\to \exp(-\mathrm{i}\hat{p}^2\Delta t) \to \mathrm{Q.F.T.} \to \exp(-\mathrm{i}V(\hat{x})\Delta t) \to \mathrm{Q.F.T.} \to \cdots$$

- Polynomial in Q = logΛ, i.e., *exponential speedup*
- Exponentially fast *because nothing special is used*.

2401.12045 [hep-th], 2411.13161 [quant-ph]



- Orbifold lattice → exponential speed up for Yang-Mills/QCD
- Kogut-Susskind is a special case of orbifold lattice. (Scalar mass →∞)
- By using orbifold lattice, Kogut-Susskind can be simulated exponentially fast, too.

Conclusion

Don’t use polar coordinates. **(Unitary links)**

Use Cartesian coordinates. **(Complex links)**

Exponential speedup follows.

We need only elementary math.

🙅‍♂️

Exponential speedup follows although we need only elementary math.

☑️

Exponential speedup follows because we need only elementary math.