Matching Lagrangian and Hamiltonian Simulations in $(2+1)$-dimensional U(1) gauge theory

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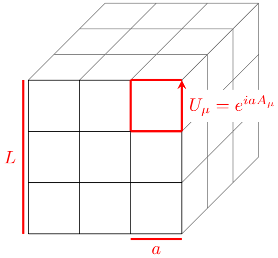
Based on arXiv:2503.11480

2nd September 2025
ECT* Workshop on Hamiltonian Lattice Gauge Theories

Comparison Lagrangian and Hamiltonian Formulation

Lagrangian
Discrete time
large lattice sizes
imaginary time
Metropolis-Algorithm
successful for simulating QCD



Hamiltonian
Continuous time
no sign problem
Exact Diagonalisation
suited for Quantum Computers
and Tensor Networks

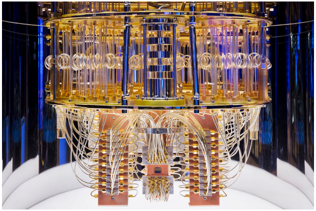


figure by Engagdet[1]

[1] *IBM Quantum computer* n.d.

Introduction

Both theories have a lot of advantages - combine them!

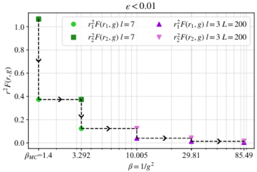
Comparison: Match theories: same a_s, $a_t \to 0$

Transfer Matrix[1]: temporal continuum limit on Lagrangian side

Application: determine the step scaling function[2]

Other approaches[3]

Anisotropic lattices introduced for glueball studies[4]



Crippa et al. 2024

[1] Creutz 1977

[2] Crippa et al. 2024

[3] Carena, Lamm et al. 2021; Carena, Gustafson et al. 2022

[4] Morningstar and Peardon 1997; Morningstar and Peardon 1999

Hamiltonian

Standard Kogut-Susskind Hamiltonian[1]

$$\hat{H}_{\text{tot}} = \frac{g^2}{2} \sum_{\vec{r}} \left(\hat{E}^2_{\vec{r},1} + \hat{E}^2_{\vec{r},2} \right) - \frac{1}{2a^2 g^2} \sum_{\vec{r}} \left(\hat{P}_{\vec{r}} + \hat{P}^\dagger_{\vec{r}} \right)$$

Reduce degrees of freedom with Gauss' law

$$\Big[\sum_{\mu=1,2} \left(\hat{E}_{\vec{r},\mu} - \hat{E}_{\vec{r}-\mu,\mu} \right) - Q_{\vec{r}} \Big] |\Phi\rangle = 0,$$

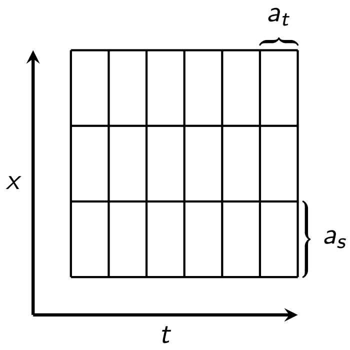
Determine ground state $\left|\Psi_0\right\rangle$ with exact diagonalization

$$\langle \mathcal{P} \rangle \equiv \left\langle \Psi_0 \right| \frac{1}{2V} \sum_{\vec{r}} \left(\hat{P}_{\vec{r}} + \hat{P}^\dagger_{\vec{r}} \right) \left| \Psi_0 \right\rangle,$$

U(1) elements are truncated to elements of $\mathbb{Z}_{2l+1}$
3×3 lattice with periodic boundary conditions

[1]Kogut and Susskind 1975

Lagrangian - Anisotropic Lattice



adapted from [1] [2]

$$S = \sum_{r,i} \frac{\beta}{\xi_{\text{input}}} \operatorname{Re}\operatorname{Tr}\left(1 - P_{0i}(r)\right) + \sum_{r,i>j>0} \beta \xi_{\text{input}} \operatorname{Re}\operatorname{Tr}\left(1 - P_{ij}(r)\right)$$

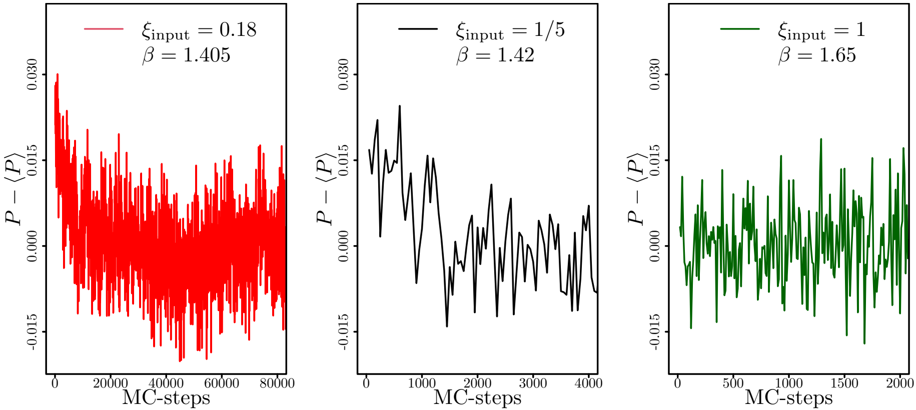
$$\xi_{\text{ren}} = \frac{a_t}{a_s}$$

$$\xi_{\text{input}} \text{ set as } \frac{L}{T}$$

[1] Loan, Brunner et al. 2003

[2] Loan, Byrnes and Hamer 2004

Lagrangian - Simulations



Thermalization and autocorrelation time grow for small ξ_{input}

Simulated: $\xi_{\text{input}} = 1, 4/5, 2/3, 1/2, 2/5, 1/3, 1/4, 1/5, 0.18$

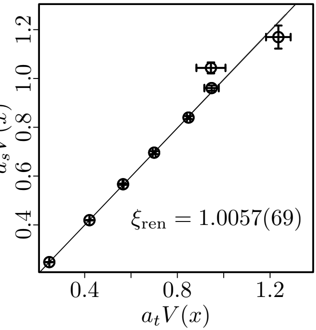
Normal potentials

$$\lim_{y\to\infty} \frac{W_{ss}(x/a_s,(y+1)/a_s)}{W_{ss}(x/a_s,y/a_s y)} = \exp\big(-a_s V_s(x/a_s)\big)$$

$$\lim_{t\to\infty} \frac{W_{st}(x/a_s,(t+1)/a_t)}{W_{st}(x/a_s,t/a_t)} = \exp\big(-a_t V_t(x/a_s)\big)$$

$$a_s V_s(x/a_s) = \frac{1}{\xi_{\text{ren}}} a_t V_t(x/a_s) + c$$

adapted from [1]



$L = 16, \; T = 16, \; \beta = 1.7$

[1] Loan, Byrnes and Hamer 2004

Normal potentials

$$\lim_{y\to\infty} \frac{W_{ss}(x/a_s,(y+1)/a_s)}{W_{ss}(x/a_s,y/a_s y)} = \exp\big(-a_s V_s(x/a_s)\big)$$

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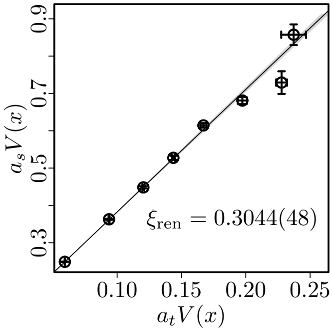


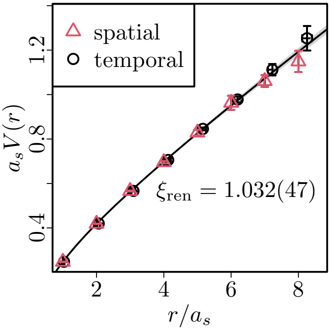
$L = 16,\ T = 48,\ \beta = 1.7$

[1] Loan, Byrnes and Hamer 2004

Sideways potentials

$$\lim_{x\to\infty} \frac{W_{ss}((x+1)/a_s, y/a_s)}{W_{ss}(x/a_s, y/a_s)} = \exp\big(-a_s V_s(y/a_s)\big)$$

$$\lim_{x\to\infty} \frac{W_{st}((x+1)/a_s, t/a_t)}{W_{st}(x/a_s, t/a_t)} = \exp\big(-a_s V_t(t/a_t)\big)$$



$L = 16,\ T = 16,\ \beta = 1.7$

$$V_s(y/a_s) = V_t\left(t/a_t\right) \quad \Rightarrow \quad y = t \quad \Rightarrow \quad \xi_{\text{ren}} = \frac{a_t y}{a_s t} = \frac{a_t}{a_s}$$

adapted from [1]

[1] Alford et al. 2001

Sideways potentials

$$\lim_{x\to\infty} \frac{W_{ss}((x+1)/a_s, y/a_s)}{W_{ss}(x/a_s, y/a_s)} = \exp\big(-a_s V_s(y/a_s)\big)$$

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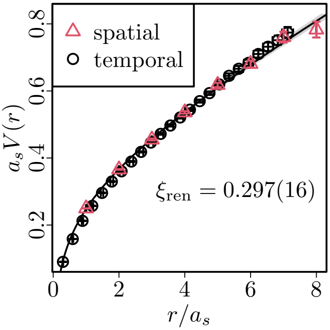

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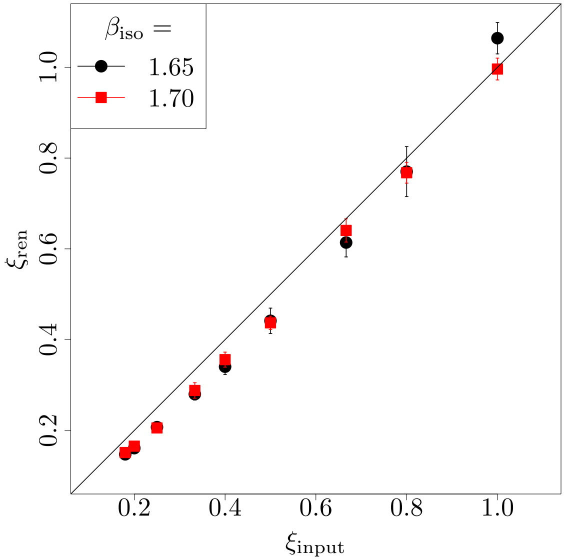
$$V_s(y/a_s) = V_t\left(t/a_t\right) \Rightarrow y = t \Rightarrow \xi_{\text{ren}} = \frac{a_t y}{a_s t} = \frac{a_t}{a_s}$$

adapted from [1]

[1] Alford et al. 2001

Renormalized Anisotropy



$\xi_{\text{ren}} > \xi_{\text{input}}$
But: effect small
ξ_{ren} does not depend on β

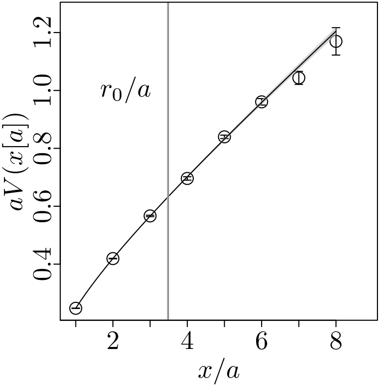
Setting the scale

$$V(r) = d + \sigma \cdot r + b \cdot \ln(r)$$

$$r^2 \frac{\mathrm{d}}{\mathrm{d}r} V(r)\Big|_{r=r_0} = c = 1.65$$

given in[1], [2]

Keep a_s constant along trajectory $a_t \to 0$.



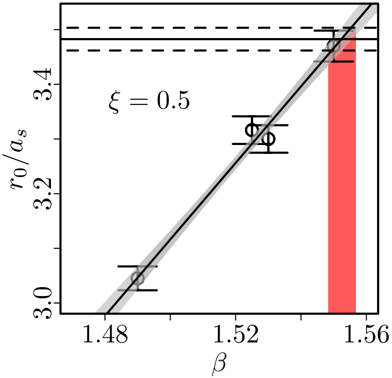
$L = 16,\ \beta = 1.7$

[1] Sommer 1994

[2] Loan, Brunner et al. 2003

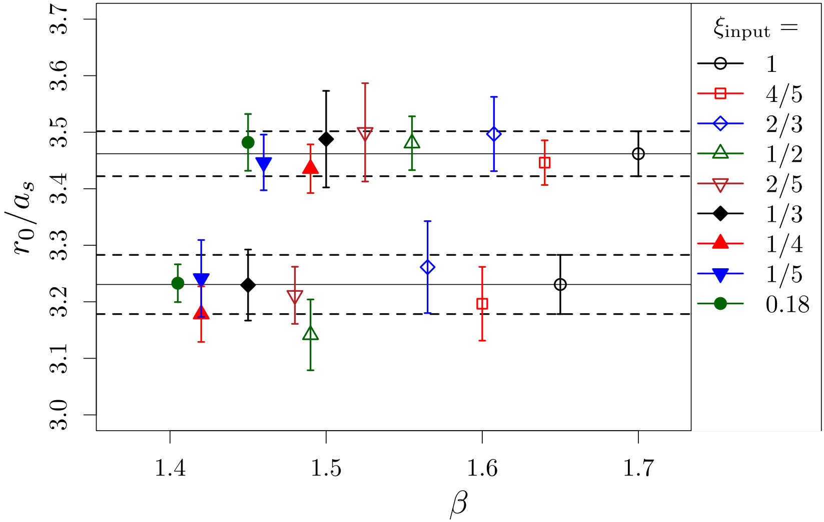
Renormalization of β

- r_0/a_s for constant lattice spacing
- Linear interpolations of $r_0/a_s(\beta)$, $\langle P\rangle(\beta)$
- $\xi_{\text{ren}}(\beta)$ = const.



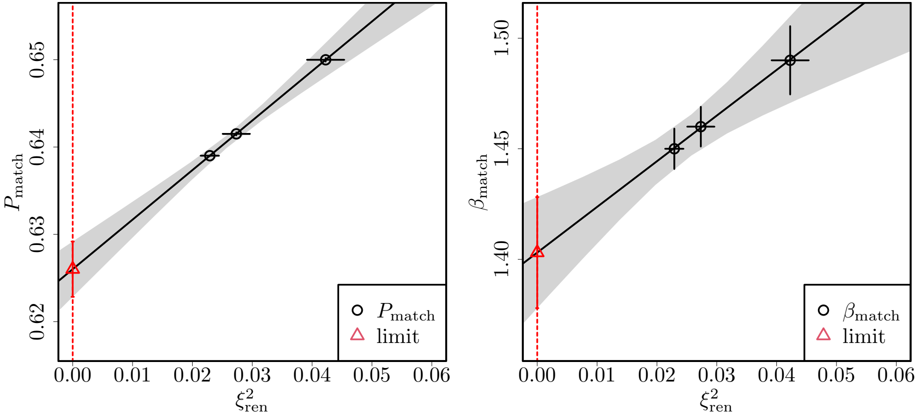
$L = 16, \; T = 32$

Renormalization of β



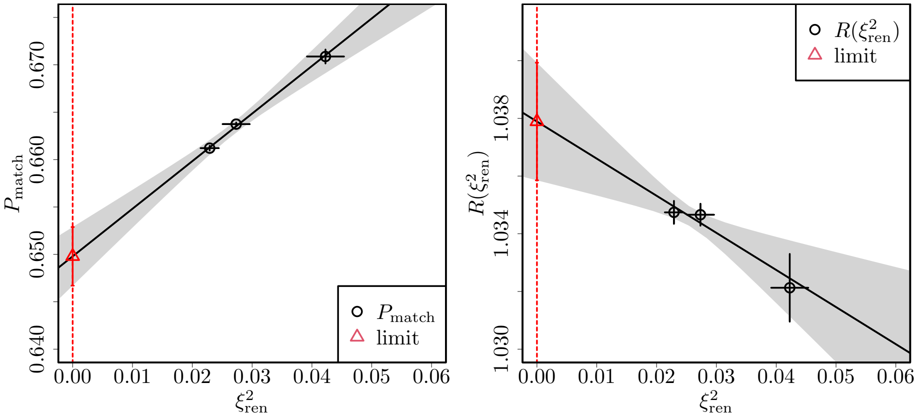
Select a matching β for each ξ_{input}.

Renormalized Continuum Limit



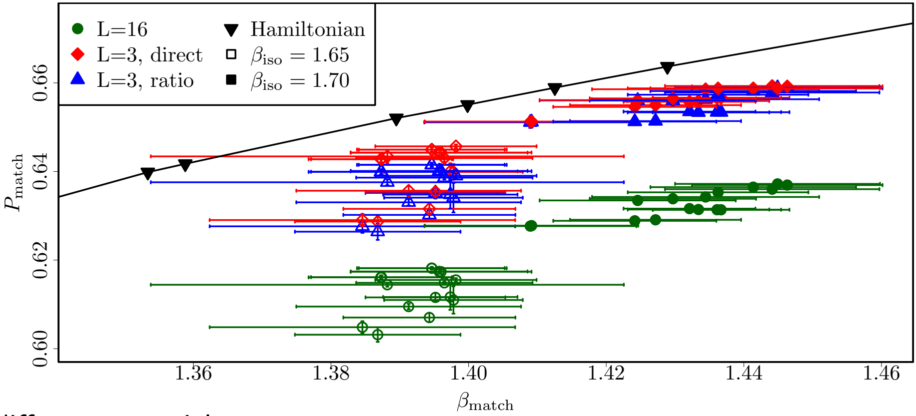
Linear fit with $\xi_{\mathrm{input}} = (0.18, 1/5, 1/4)$, $L = 16$, $T = (88, 80, 64)$, $\beta_{\mathrm{iso}} = 1.7$
fit in orders of $\xi_{\mathrm{ren}}^2 \propto a_t^2$

Small Volume limit



Linear fit with $\xi_{\mathrm{input}} = (0.18, 1/5, 1/4)$, $L = 3$, $T = (88, 80, 64)$, $\beta_{\mathrm{iso}} = 1.7$

Systematic Errors

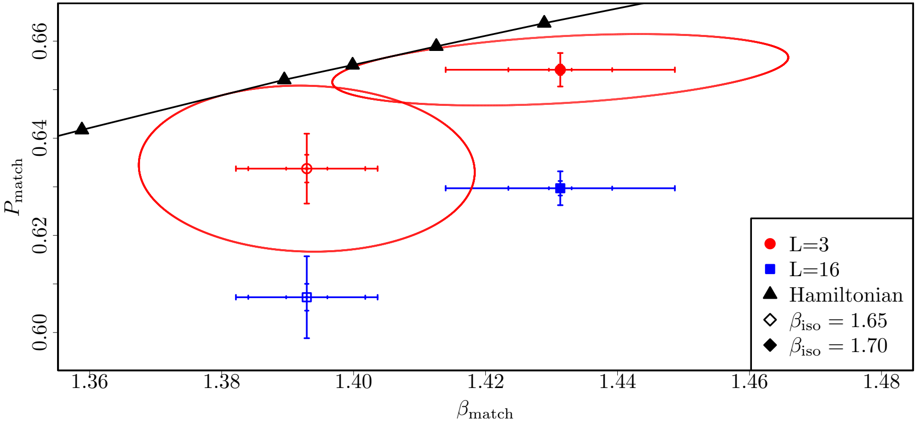


different potential types
different extrapolation polynomials

$\Rightarrow$ systematic error

Systematic error from choice of boundaries in potential determination

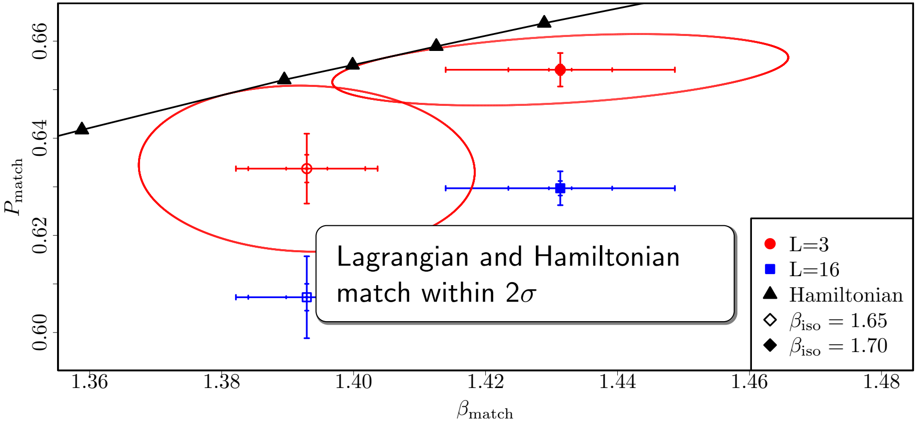
Comparing Lagrangian and Hamiltonian



Take correlation into account

One number to quantify agreement between the two sets

Comparing Lagrangian and Hamiltonian

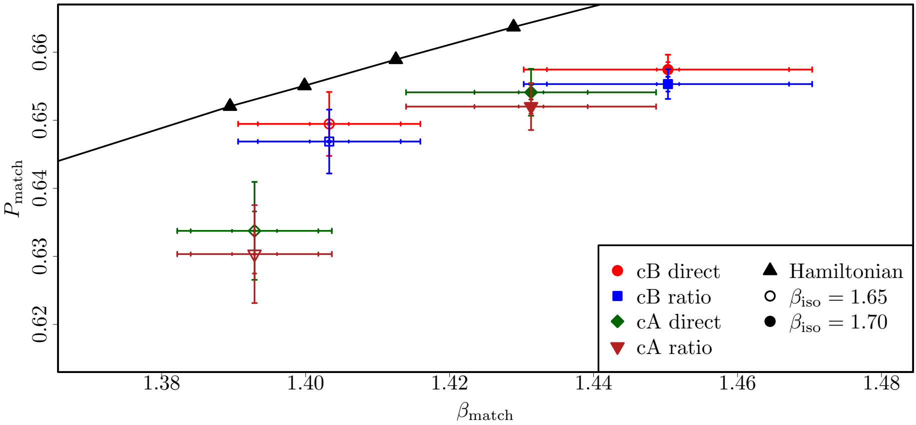


Lagrangian and Hamiltonian match within 2σ

Take correlation into account

One number to quantify agreement between the two sets

Comparing Lagrangian and Hamiltonian

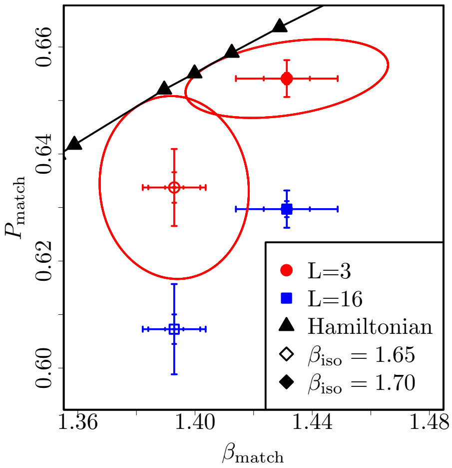


Check different finite volume extrapolations

cA includes smallest anisotropy ($\xi_{\text{input}} = 0.18$), cB does not

Summary

- Determine anisotropy and take temporal continuum limit
- Lagrangian and Hamiltonian simulations match within 2σ
- Slight mismatch remaining
- Extension to other groups conceptually straightforward



Summary

- Determine anisotropy and take temporal continuum limit
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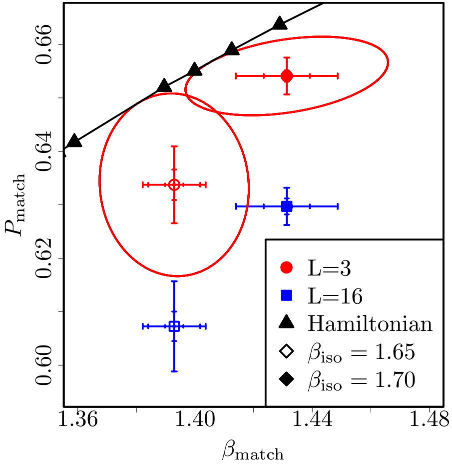
Thank you for your attention!



Further Background

For further background, look at C. F. Groß et al. 2025 and the references therein. For code and scripts, see Christiane Franziska Groß et al. 2025.

Final Results I

β_{iso}	set	β_{match}
1.65	cB	1.403(06)(10)(06)[13]
1.7	cB	1.450(06)(17)(08)[20]
1.65	cA	1.393(06)(08)(04)[11]
1.7	cA	1.431(05)(10)(13)[17]

Table: The coupling constant β at the matching point. Errors aare statistical, systematic from potential, systematic from continuum limit extrapolation and combined.

Final Results II

β_{iso}	1.65	1.7
set	cA	cA
$P(L=16)$	0.6073(06)(48)(69)[84]	0.6297(14)(18)(26)[35]
$P(L=3)$ direct	0.6337(07)(45)(56)[72]	0.6541(10)(16)(29)[34]
deviation dir.	1.49	1.88
$P(L=3)$ ratio	0.6303(07)(50)(72)[88]	0.6520(15)(18)(28)[36]
deviation rat.	1.87	1.87

Table: The different plaquette observables at the matching point for the extrapolation set cA. For explanation of the errors given see in the text. deviation measures the difference between the Lagrangian and Hamiltonian simulations.

Final Results III

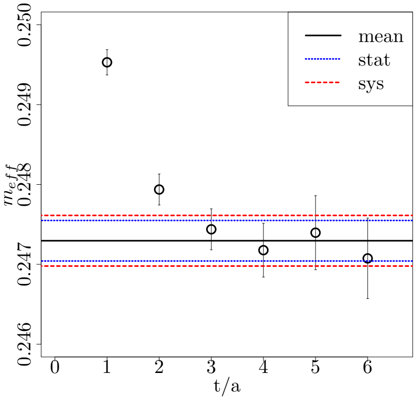
β_{iso}	1.65	1.7
set	cB	cB
$P(L=16)$	0.6233(04)(11)(67)[67]	0.6334(07)(18)(27)[34]
$P(L=3)$ direct	0.6495(03)(07)(46)[47]	0.6574(07)(14)(15)[22]
deviation dir.	1.57	0.61
$P(L=3)$ ratio	0.6469(04)(12)(69)[70]	0.6553(07)(19)(28)[34]
deviation rat.	0.66	1.77

Table: Same as table 2, but for extrapolation set cB

Selecting boundaries

Akaike Information Criterion

- $w = \exp\left(-\frac{1}{2}\cdot\left(\chi^2 + 2 - (t_2 - t_1)\right)\right)$
- mean=median
- statistical: bootstrap samples of median
- systematical: 16% and 84% quantiles



Systematics potential

label	Potential type	$I_{\xi_{\text{ren}}}$	I_{r_0}	included error
N1ES	Normal	$[2,7]$	$[1,7]$	statistical only
S1ES	Sideways	$[2,7]$	$[1,7]$	statistical only
N0ES	Normal	$[2,8]$	$[1,8]$	statistical only
S0ES	Sideways	$[2,8]$	$[1,8]$	statistical only
N1ET	Normal	$[2,7]$	$[1,7]$	statistical and systematic
S1ET	Sideways	$[2,7]$	$[1,7]$	statistical and systematic
N0ET	Normal	$[2,8]$	$[1,8]$	statistical and systematic
S0ET	Sideways	$[2,8]$	$[1,8]$	statistical and systematic

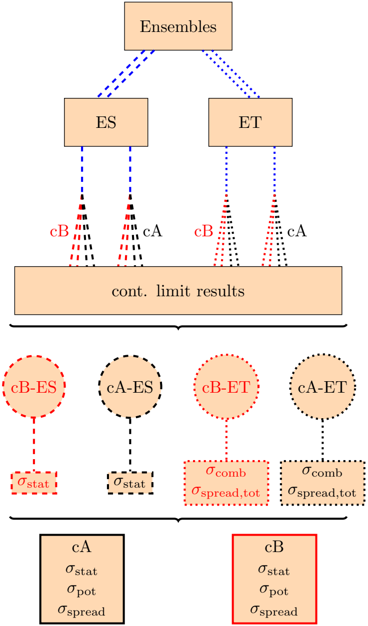
Table: Summary of the different analysis chains. The intervals $I_{\xi_{\text{ren}}}$ and I_{r_0} indicate the range of distances in units of the spatial lattice spacing a_s used to determine ξ_{ren} and r_0/a_s, respectively. The systematic error arises from uncertainty in choosing the fit range for the effective masses. In the following, we refer to the different analysis chains with the labels in the first column (see text).

Systematics Extrapolation

cA		cB	
ξ_{input}-values	n_p	ξ_{input}-values	n_p
$0.18, 1/5, 1/4$	1	$1/5, 1/4, 1/3$	1
$0.18, 1/5, 1/4, 1/3$	1	$1/5, 1/4, 1/3, 2/5$	1
$0.18, 1/5, 1/4, 1/3, 2/5$	2	$1/5, 1/4, 1/3, 2/5, 1/2$	2
$0.18, 1/5, 1/4, 1/3, 2/5, 1/2$	2	$1/5, 1/4, 1/3, 2/5, 1/2, 2/3$	2

Table: Two sets of extrapolations cA and cB used to calculate a combined continuum limit of the fits. The same fit ranges are used for all trajectories and for all fits to the continuum limit – for β, the plaquettes and for R. We list the anisotropies that are included in the polynomial fits and the degrees of the polynomials.

Flowchart Analysis Procedure



determine $\xi_{\text{ren}}, r_0/a_s$ in analysis chains

determine trajectory of constant a_s

extrapolate to cont. limit

collect in sets

extract errors from average and spread

calculate individual errors

List of parameters for continuum limit

- total or statistical error
- normal or sideways potential
- upper bound of potential
- lower bound of potential, for r_0/a_s and for ξ_{ren}
- degree of extrapolation
- points used for extrapolation
- interpolate to constant lattice spacing or choose a fixed point
- extrapolate at L=3 or extrapolate ratio

Error analysis

- do bootstrapping with blocklength 1
- determine autocorrelation time
- rescale error, bootstrapsamples such that error is correct
- can then use covariance matrix in effective mass fits, use χ^2 for AIC

Confidence Ellipse

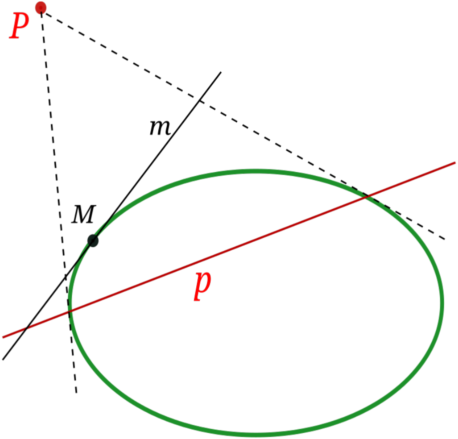


figure from wikipedia

- Ellipse angle from correlation P, β
- Ellipse aspect ratio from errors
- take Hamiltonian interpolation as polar to ellipse
- rescale ellipse radius until pole is on ellipse
- polar is tangent
- convert area of ellipse into matching probability

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Bibliography IV

Groß, Christiane Franziska et al. (2025). *Code and scripts for "Matching Lagrangian and Hamiltonian Simulations in (2+1)-dimensional U(1) Gauge Theory"*. Version V1. DOI: 10.60507/FK2/AUJJ08. URL: https://doi.org/10.60507/FK2/AUJJ08.