Strong-Coupling QCD on a Quantum Annealer

Thomas Luu
t.luu@fz-juelich.de
IAS-4, FZJ & HISKP, Uni. Bonn

Partners in crime

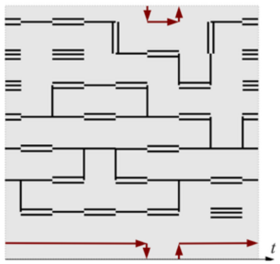
J.Kim, T.L., W.Unger, arXiv:2305.18179, arXiv:2311.07209, arXiv:2412.11677

Strong coupling limit of QCD (SC-LQCD)

- Strong coupling limit: $\beta = \frac{2N_c}{g^2} \to 0$
  - $\implies \quad S = S_g + S_f \to S_f$
  - gauge action vanishes
- Change order of integration (Rossi & Wolff '84)
  - Integrate out gauge fields first
  - Then integrate over Grassmann fields

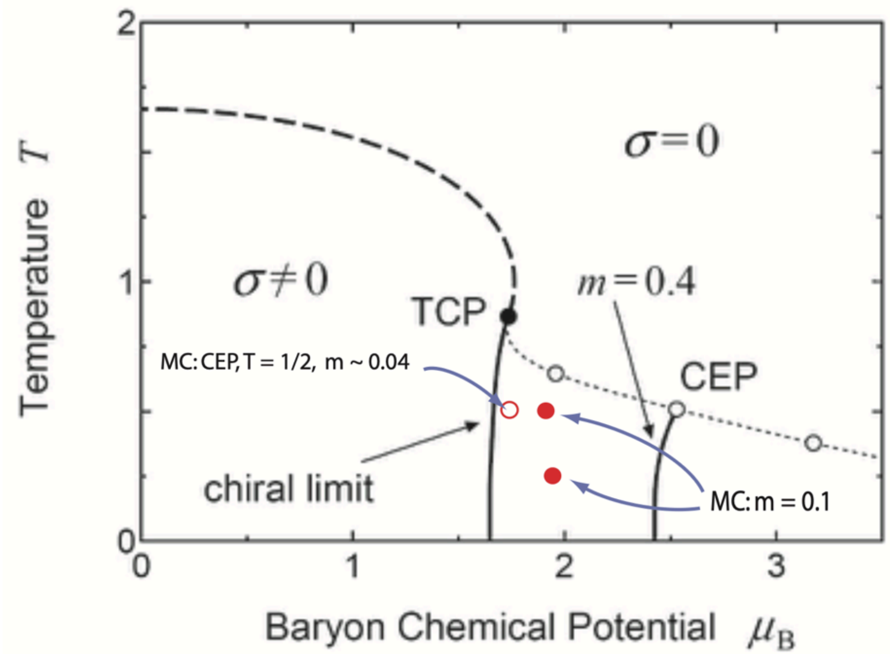
$$Z = \int d\bar{\chi} d\chi dU e^{-S_f[\bar{\chi},\chi,U]} = \sum_{\{k,n,\ell\}} \underbrace{\prod_{b=(x,\hat{\mu})} \frac{(N_c - k_b)!}{N_c! k_b!} \gamma^{2k_b \delta_{\hat{0},\hat{\mu}}}}_{\text{meson hoppings}} \underbrace{\prod_x \frac{N_c!}{n_x!} (2am_q)^{n_x}}_{\text{chiral condensate}} \underbrace{\prod_\ell w(\ell, \mu)}_{\text{baryon hoppings}}$$

- Confinement is manifest, giving new degrees of freedom
  - monomers, dimers (mesons), polymers (baryons)
  - all color singlets



Why is SC-LQCD interesting?

Phase diagram[1]



- Expected to exhibit chiral phase transition
- Sign-problem with dual variables is ***milder***
  - Easier to simulate at $\mu_B \neq 0$
- Past investigations have included
  - Mean-field (Drouffe & Zuber ’83)
  - Monte-Carlo (Karsch & Mütter ’89)
    - Ergodic? (Aloisio et al. ’00)
  - Worm algorithm (Fromm, Unger, de Forcrand)
- Understanding exact nature and location of phase transition requires low-T simulations
  - Worm algorithm, updating single link per step, is highly inefficient in this realm

Goal

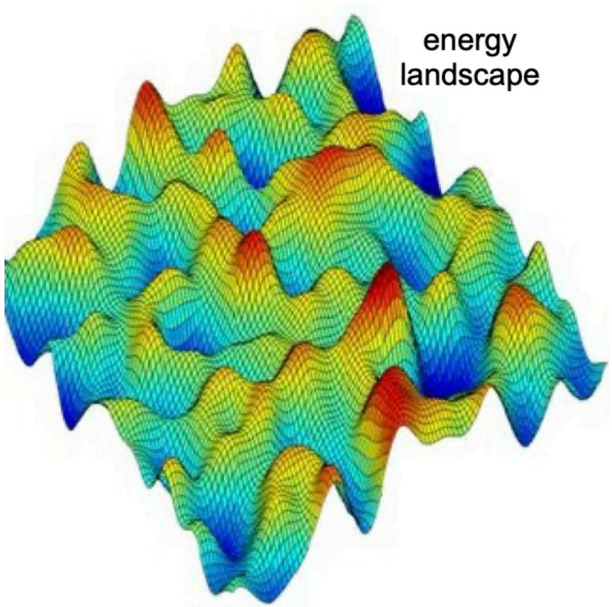
Simulate SC-LQCD on a Quantum Annealer where temperature plays no role

1 Fromm & de Forcrand PoS LATTICE2008 (2008) 191, Nishida Phys Rev D 69 (2004) 094501

Simulated (thermal) annealing

$$H_{Ising} = \sum_{i<j} K_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^z \quad ; \quad \sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

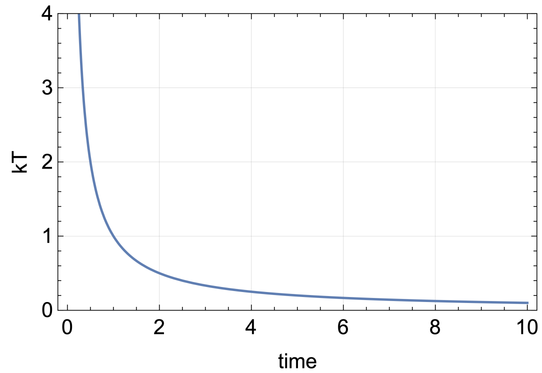
What is the absolute ground state of this system?



⚠ **Warning**

Determining the absolute dground state is a non-deterministic polynomially (NP) hard problem

- Initialize system at some temperature $kT > E_{max}$
- Slowly reduce the temperature



- Escaping local minima via ***thermal fluctuations***
- System settles to lowest energy configuration (hopefully)
- Has physical interpretation
  - annealing of metals, glasses, etc.
- Monte-Carlo via Metropolis-Hastings

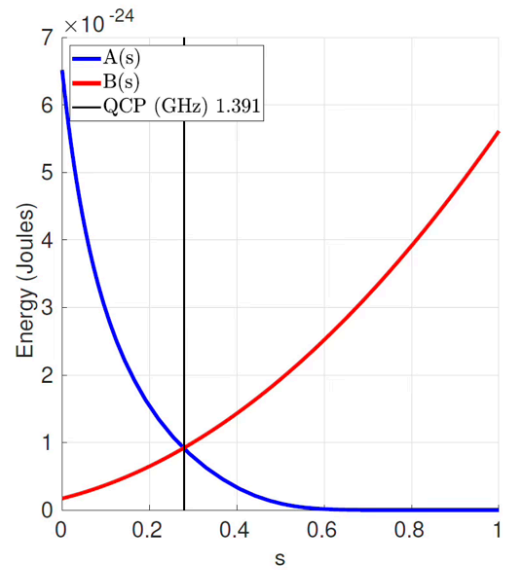
Simulated Quantum Annealing

- Introduce a non-commuting term to "target" Hamiltonian:

$$H(s) = -A(s)\sum_i \sigma_i^x + B(s)H_{Ising} \quad ; \quad \sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

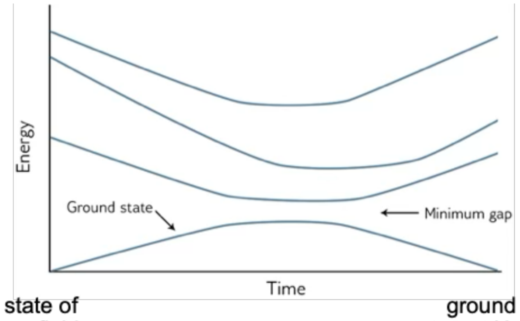
$$[\sigma^x, \sigma^z] \neq 0$$

- Keep temperature fixed, but vary $A(s)$ and $B(s)$
- System evolves from the ground state of transverse field to ground state of target Hamiltonian



Quantum fluctuations

Transverse field introduces ***quantum fluctuations*** that enables "tunnelling" between barriers, ensuring (hopefully) that the ground state of the instantaneous Hamiltonian $H(s)$ is reachable during evolution of s



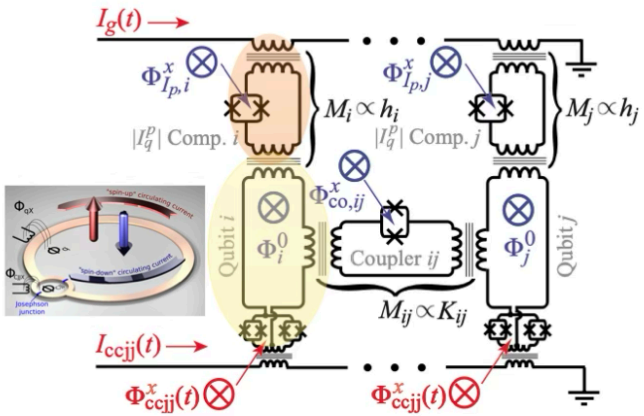
- Simulated quantum annealing (via quantum Monte Carlo) has been shown to be more efficient than simulated (thermal) annealing for various spin models[1]

Figure sources

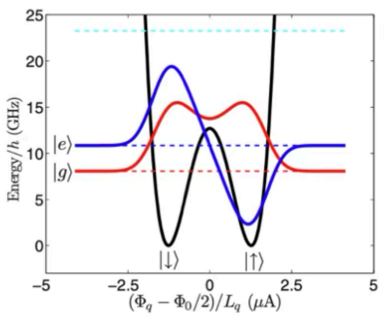
1 Santoro et al. Science v295 5564 (2247-2430)

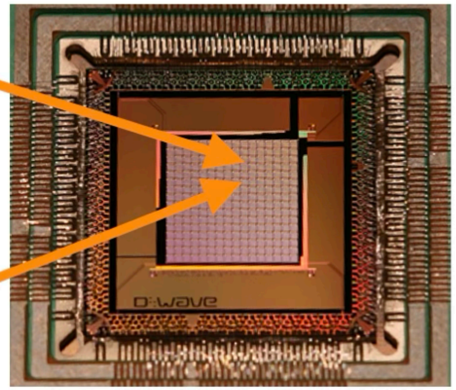
Physical Quantum Annealer

The D-Wave Quantum Annealer



Qubit composed of superconducting loop with Josephson junctions





$$H(s) = -\frac{1}{2}\sum_i \left[2\left|I^p_q(s)\right| M_i(s)\sigma_z^{(i)} + \Delta_q(s)\sigma_x^{(i)}\right] + \sum_{j>i} M_{ij}(s)\left|I^p_q(s)\right|^2 \sigma_z^{(i)}\sigma_z^{(j)}$$

Inductances M_{ij} and M_i can be tuned *in situ*

$$-\sum_i \left(A(s)\sigma_i^x + B(s)h_i\sigma_i^z\right) + B(s)\sum_{i<j} K_{ij}\sigma_i^z\sigma_j^z$$

D-Wave is a "single-use, dedicated device"[1]

- Typical anneal times $\sim \mathcal{O}(100\ \mu s)$
- Each run consists of $\sim \mathcal{O}(500 - 1000)$ anneals
- Requires total of $\sim \mathcal{O}(500\ ms - 1\ s)$ per run
- When submitting run, get access to ***entire*** 'machine' (= QPU)
- After each run, perform post-processing of data

Total allocation per year: $\sim \mathcal{O}(\text{hrs})$

1 More about D-Wave here

Expected computational scaling

Simulated (thermal) annealing

$$\mathcal{O}\left(e^{aN^c}\right)$$

Simulated quantum annealing

$$\mathcal{O}\left(e^{aN^c}\right)$$

Physical quantum annealing

$$\mathcal{O}\left(e^{aN^c}\right)$$

It’s still an NP-Complete problem

D-Wave hopefully has much smaller coefficients a and c

Some examples showing potential scaling advantage

Phys. Rev. X 8, 031016 (2018)

Nature v617, 61–66 (2023)

Phys. Rev. Lett. 134, 160601 (2025)

Solving Optimization Problems with Quantum Annealing

- Naturally D-Wave can be used to investigate Ising spin glass systems
- But we can do more! Assume you want to “optimize” a specific problem with “objective” function $f(\alpha, \vec{x})$
  - Goal: map your problem into H_{Ising}
    - find mapping M that maps parameters of your problem $\{\alpha, \vec{x}\} \to \{K, h, \vec{s}\}$ parameters of H_{Ising}
    - use D-Wave to find solution $\vec{s}$ that minimizes $H_{Ising}(K, h, \vec{s})$
    - $\min H_{Ising}(K, h, \vec{s}) \implies \min f(\alpha, \vec{x})$
- What types of problems can be done this way?
  - In H_{Ising}, $\vec{s}$ represents discrete degrees of freedom $\implies \vec{x} \in \mathbb{Z}^n$ (discrete values)

Combinatorial Optimization (CO) problem

Integer (Linear) Programming (ILP)

$$\begin{aligned} \text{find extremum of } & c^T \cdot x \\ \text{subject to } & A \cdot x \leq b \\ & x \geq 0 \\ \text{and } & x \in \mathbb{Z}^n \end{aligned}$$

where c and b are vectors of coefficients and A is a matrix of coefficients

Example

$$\begin{aligned} \text{find } \min y &= 3x_1 + 2x_2 + x_3 + x_4 + 3x_5 + 2x_6 \\ \text{subject to } & x_1 + x_3 + x + 6 = 1 \\ & x_2 + x_3 + x_4 + x_6 = 1 \\ & x_3 + x_4 + x_5 = 1 \\ & x_1 + x_2 + x_4 + x_6 = 1 \\ \text{and } & x_i = 0 \text{ or } 1 \end{aligned}$$

⚠ **Warning**

Solving CO is a non-deterministic polynomially (NP) hard problem!

Examples of classical (heuristic) ILP solvers

CPLEX

Gurobi

Quadratic unconstrained binary optimization (QUBO)

- ILP is not suitable for D-Wave
- Need to cast problem ***w/ constraints*** into quadratic form $x^T Q x$ w/o constraints
- Include constraints via “penalty” factors in the optimization

$$\begin{aligned} \min y = & 3x_1 + 2x_2 + x_3 + x_4 + 3x_5 + 2x_6 \\ & + p_1(x_1 + x_3 + x_6 - 1)^2 + p_2(x_2 + x_3 + x_5 + x_6 - 1)^2 \\ & + p_3(x_3 + x_4 + x_5 - 1)^2 + p_4(x_1 + x_2 + x_4 + x_6 - 1)^2 \end{aligned}$$

- Penalty parameters $p_i \geq 0$ are free parameters, similar to Lagrange parameters
  - Satisfying the constraints means the penalty terms vanish under minimization
- Set $p_i = p = 10$ and use $x_i^2 = x_i$

$$\min y = \min x^T Q x + 40 \quad ; \quad Q = \begin{pmatrix} -17 & 10 & 10 & 10 & 0 & 20 \\ 10 & -18 & 10 & 10 & 10 & 20 \\ 10 & 10 & -29 & 10 & 20 & 20 \\ 10 & 10 & 10 & -19 & 10 & 10 \\ 0 & 10 & 20 & 10 & -17 & 10 \\ 20 & 20 & 20 & 10 & 10 & -28 \end{pmatrix} \quad ; \quad x_{sol} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Mapping this onto the D-Wave

- Q can be symmetric or upper (lower) triagular
- Previous problem required binary variables with $x_i^2 = x_i$
- We can relate to our Ising problem that uses $s_i^2 = 1$ via $x_i = (s_i + 1)/2$

$$x^T Q x = s^T K s + h^T s + C \quad ; \quad K = \frac{1}{4} Q \quad ; \quad h_i^T = \frac{1}{2} \sum_j Q_{ji} \quad ; \quad C = \frac{1}{4} \sum_{ij} Q_{ij}$$

- Given QUBO Q, or Ising K, h, the D-Wave API makes an embedding (determines connections of physical qubits to make logical qubits) to model the problem and subsequently performs QA

global minimum of QUBO cost function = global minimum of CO cost function

> ⚠ **Warning**
>
> Mapping a general CO problem into QUBO form is non-trivial, and the mapping is not unique.

SC-LQCD as a QUBO Problem

- Strong coupling limit: $\beta = \frac{2N_c}{g^2} \to 0 \implies S = S_g + S_f \to S_f$
- Gauge links and fermion fields can be integrated formally

$$Z = \int d\bar{\chi} d\chi dU e^{-S_f[\bar{\chi},\chi,U]} = \sum_{\{k,n,\ell\}} \underbrace{\prod_{b=(x,\hat{\mu})} \frac{(N_c - k_b)!}{N_c! k_b!} \gamma^{2k_b \delta_{\hat{0},\hat{\mu}}}}_{\text{meson hoppings}} \underbrace{\prod_x \frac{N_c!}{n_x!} (2am_q)^{n_x}}_{\text{chiral condensate}} \underbrace{\prod_\ell w(\ell, \mu)}_{\text{baryon hoppings}}$$

- "Grassmann constraint"

$$n_x + \sum_{\mu=\pm 0,\cdots,\pm d} \left(k_\mu(x) + \frac{N_c}{2} |\ell_\mu(x)| \right) = N_c$$

Parameters of this "dual" theory take discrete (integer) values!

> ⓘ **Goal: Use D-Wave to generate a distribution of solutions**

An explicit example: $U(1)$ on a 2x2 lattice

$U(1) \implies$ no baryons

$$Z = \int d\bar{\chi} d\chi dU e^{-S[\bar{\chi},\chi,U]} = \sum_{\{k_b,n_x\}} e^{-S[k_b,n_x]}$$

$$S[k_b, n_x] = \sum_{b=(x,\hat{\mu})} D(k_b) + \sum_x M(n_x)$$

$$\text{dimer} \quad D(k_b) = -\log\left(\frac{(N_c - k_b)!}{N_c! k_b!}\right) - 2k_b \delta_{\hat{0},\hat{\mu}} \log(\gamma)$$

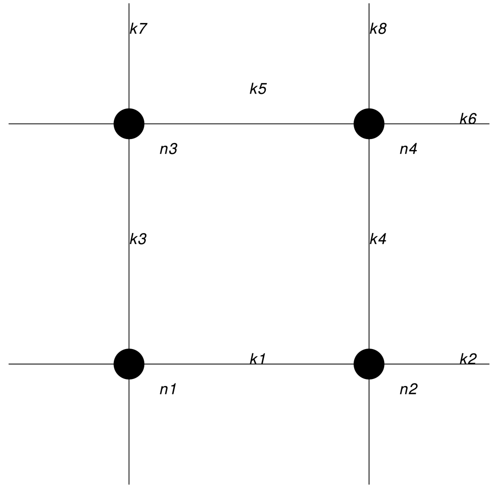
$$\text{monomer} \; M(n_x) = -\log\left(\frac{N_c!}{n_x!}\right) - n_x \log(2am_q) + c$$

Grassmann constraint

$$n_x + \sum_{\mu=\pm 0,\cdots,\pm d} k_\mu(x) = N_c$$

Discrete variables

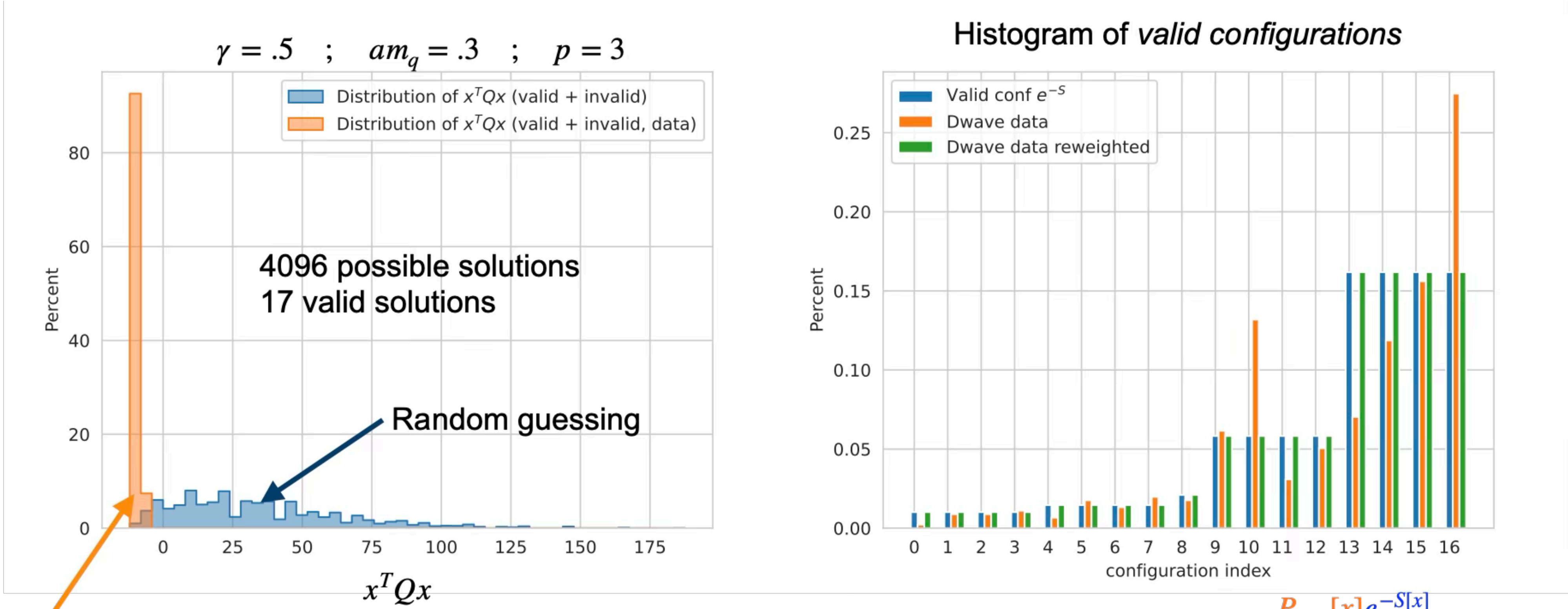
$$k_b \in \{0, 1\} \quad n_x \in \{0, 1\}$$



$$\implies Q = \begin{pmatrix} -2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -2 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 & 0 & -2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma = 1 \quad ; \quad 2am_q = 1$$

Distribution of D-Wave solutions

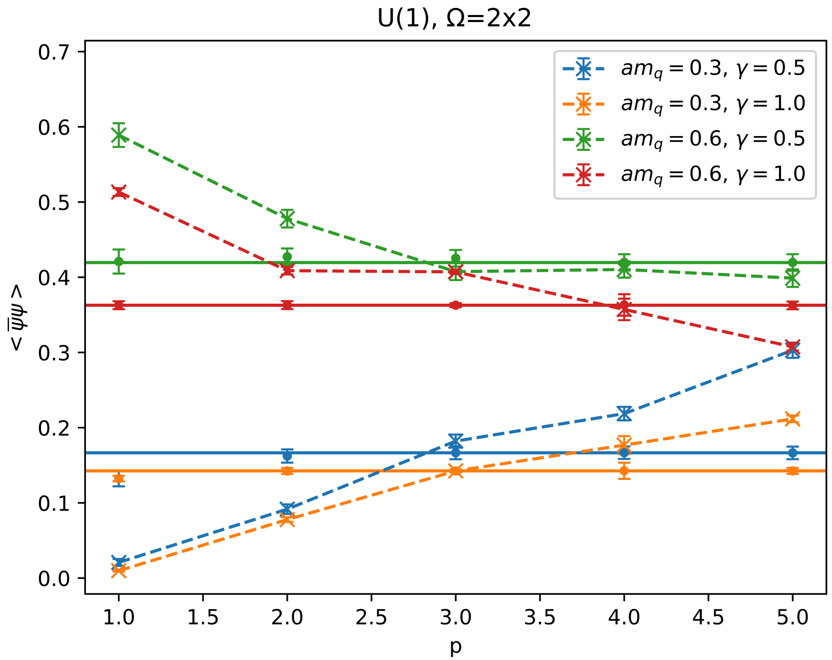


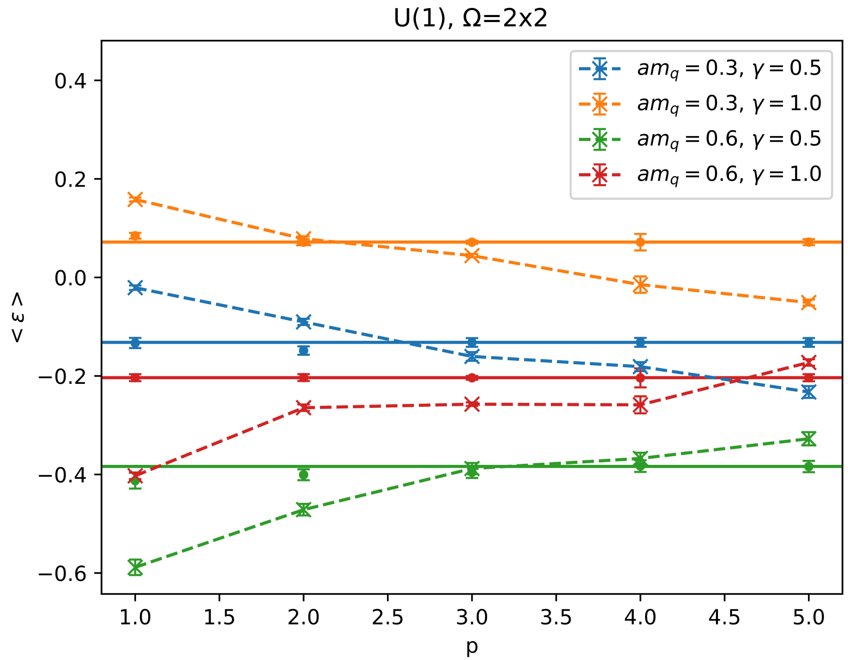
$P[x] \propto e^{-x^TQx} \neq e^{-S[x]}$

Histogram reweighting $P_S[x] = \frac{P_{DW}[x]e^{-S[x]}}{\sum_x P_{DW}[x]e^{-S[x]}}$

Ferrenberg & Swendsen, Phys.Rev.Lett. 61 (1988) 2635-2638

$U(1)$ Observables





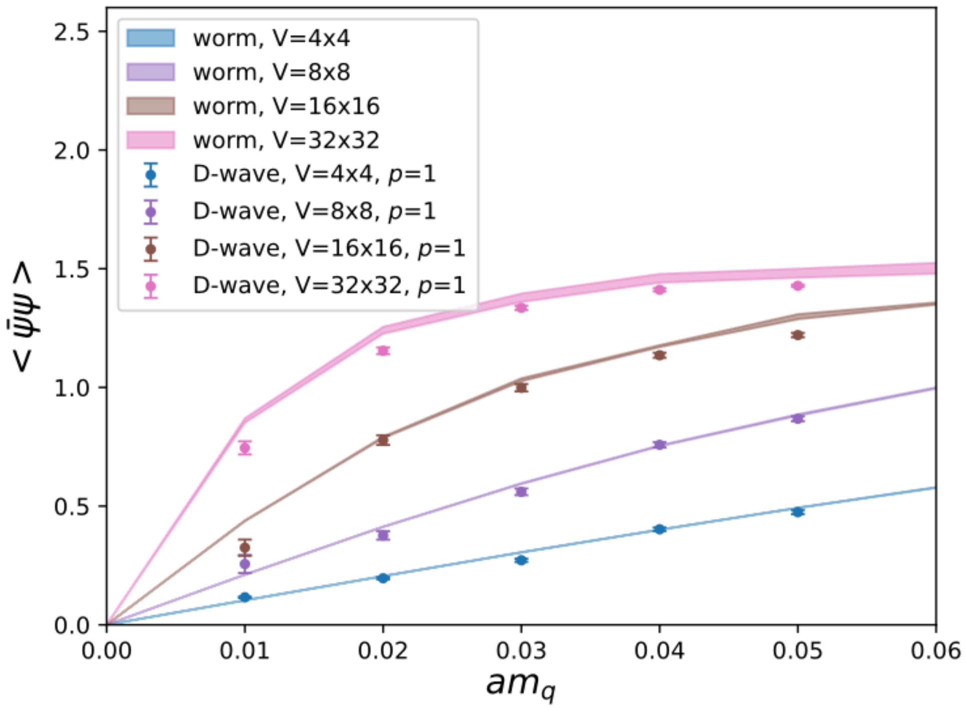
$$\langle \bar{\psi}\psi \rangle = \frac{1}{V}\langle M \rangle \quad ; \quad M = \sum_{x \in V} n_x$$

$$\langle \epsilon \rangle = \frac{1}{V}(\langle D_t \rangle - \langle M \rangle) \quad ; \quad D_t = \sum_{x \in V} k_{x,\mu=t}$$

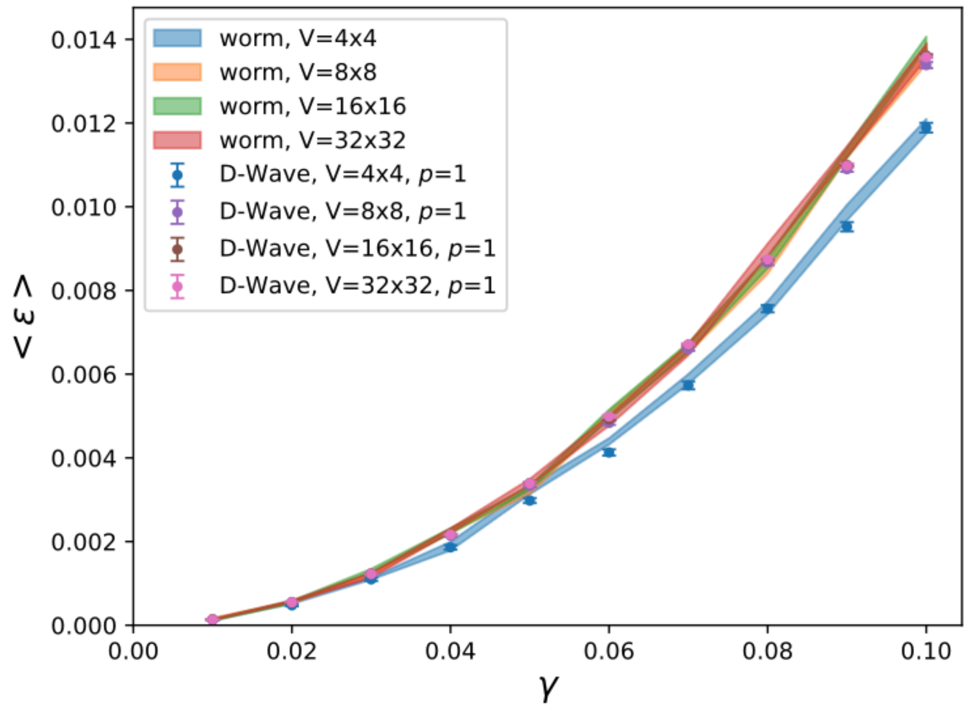
dashed line: raw D-Wave data

solid line: reweighted D-Wave data

$U(3)$ Observables



$\gamma = .1$

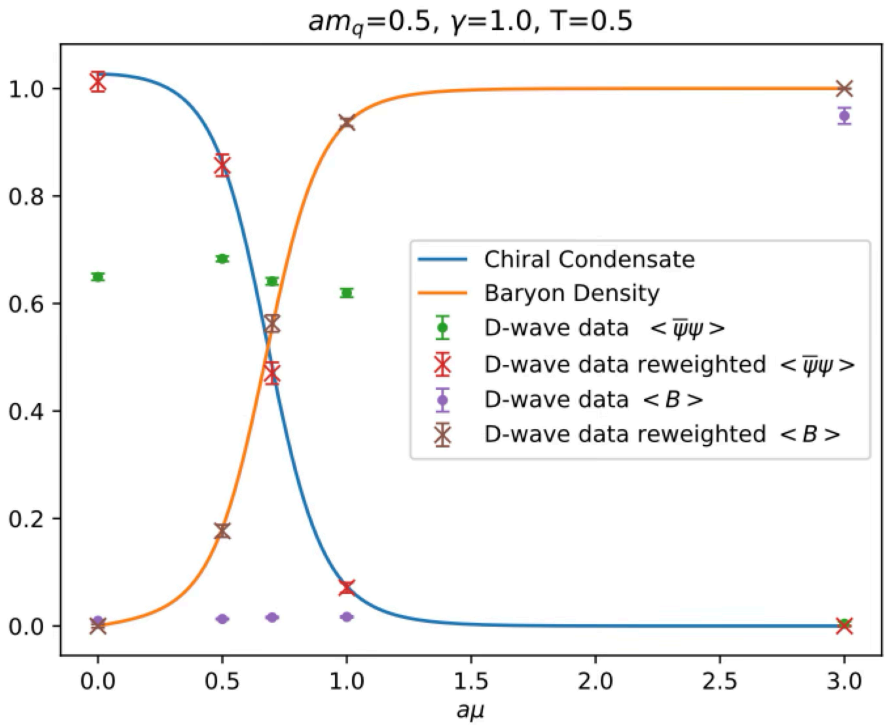


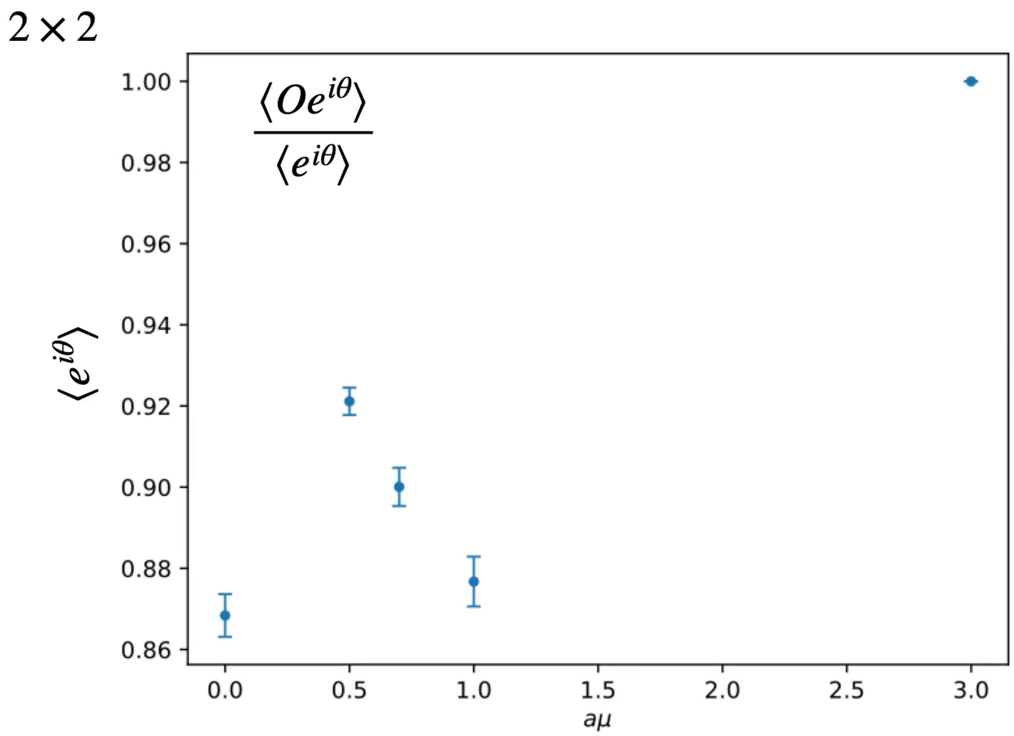
$am_q = .3$

Hybrid Classical/QA algorithm

Large volumes obtained via parallel D-Wave sampling of multiple sub-lattice volumes

$SU(3)$ Observables





$$\langle \bar{\psi}\psi \rangle = \frac{1}{2am_q V} \left\langle \sum_{x \in V} n_x \right\rangle$$

$$\langle B \rangle = \frac{1}{N_s} \left\langle \sum_{l} w_l \right\rangle$$

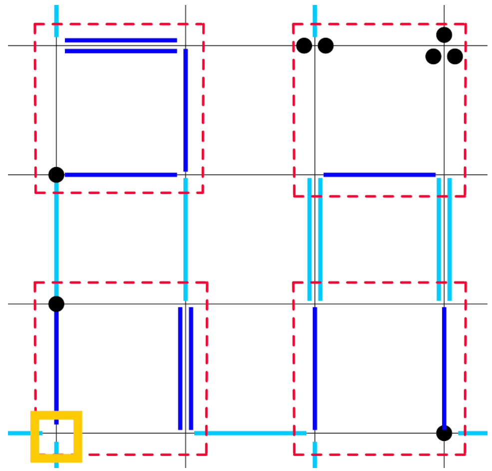
Recapitulation

- Phase diagram of SC-QCD is non-trivial
  - Tri-critical end point ($am = 0$) vs critical end point ($am \neq 0$)
  - Worm algorithm suffers when $T \to 0$
- Physically performing Quantum Annealing insensitive to T (of theory)
  - Annealing works on systems with discrete degrees of freedom
  - Minimisation of objective function (e.g. Hamiltonian)
  - Combinatorial Optimisation
- Recast into QUBO form
  - Optimisation w/o constraints
- Applied to Strong-coupling limit of select guage theories
  - Showed first results for U(1), U(3), and SU(3)

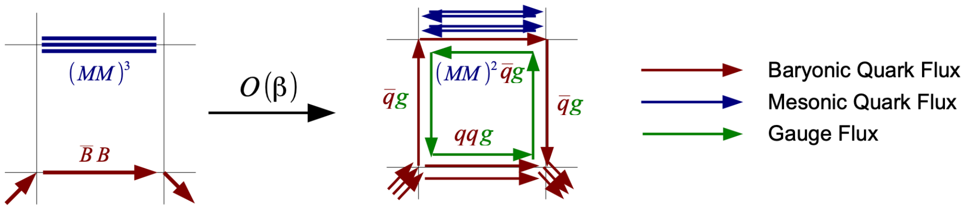
Backup slides

Challenges of using a Quantum Annealer

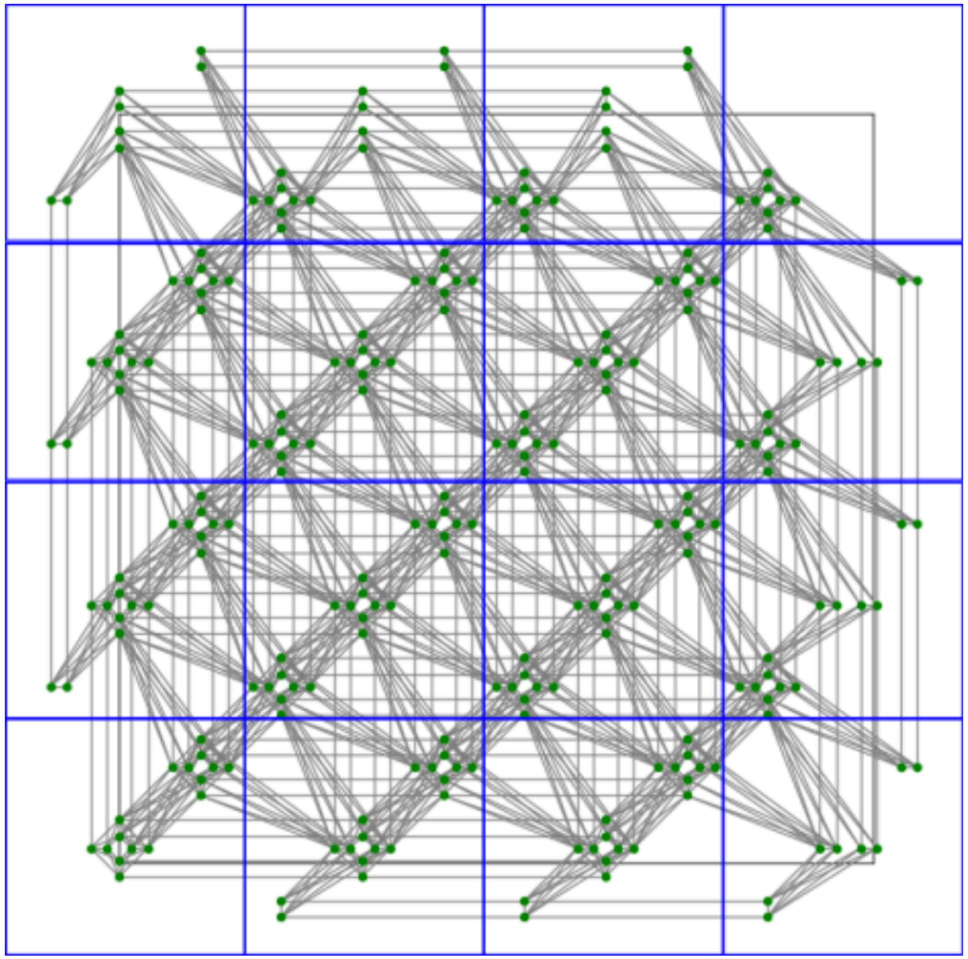
- Number of valid solutions falls precipitiously as the volume increases
  - perform parallel sublattice volumes updates with hybrid classical/QA calculation arXiv:2412.11677
  - looking to extend to large sublattice volumes (3x3, 2x2x2,4x4,…)



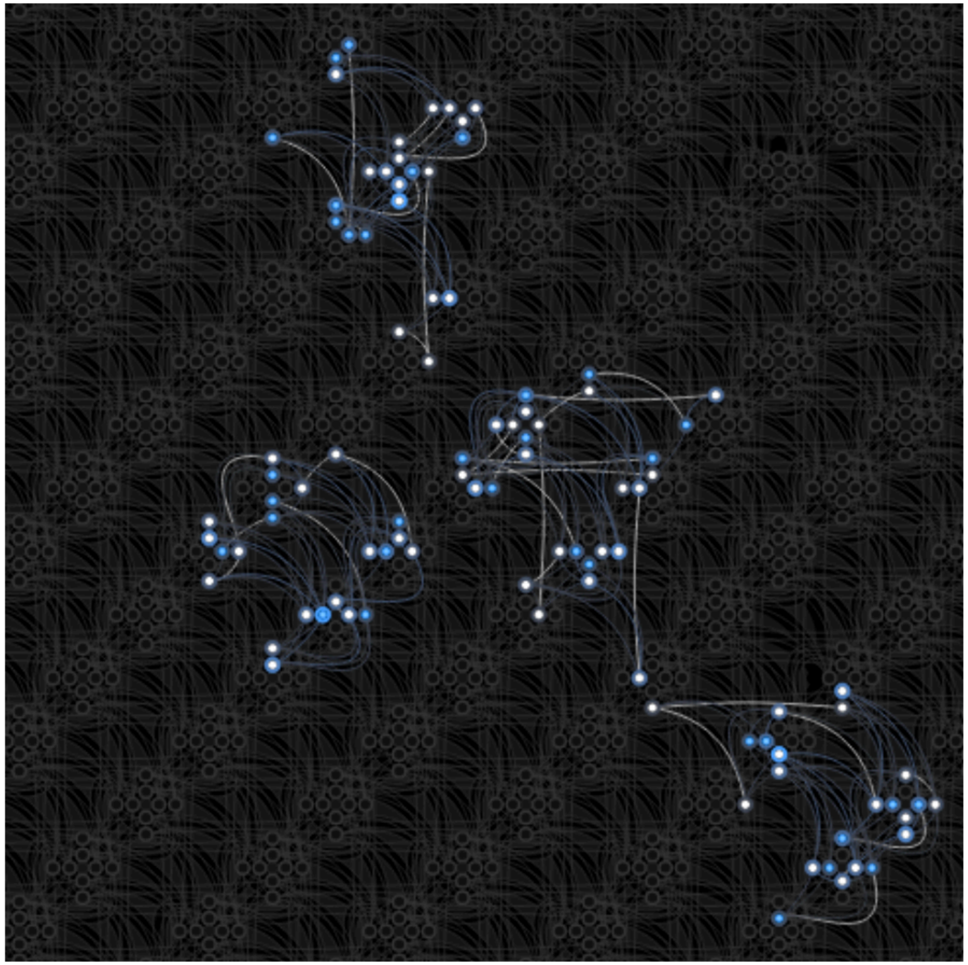
- incorporate $\mathcal{O}(\beta^n)$ gauge corrections to our formalism



Topology of D-Wave Advantage system



(a) Pegasus Topology



(b) Embedding of 4 sub-lattices

Figure source