Center-vortex semiclassics on $R^2 \times T^2$ and large-N adiabatic continuity

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Based on 2505.07467 (also 2201.06166, 2405.12402)

[cf. Gonzalez-Arroyo's talk on Monday]

Goal For 4d confining gauge theories,

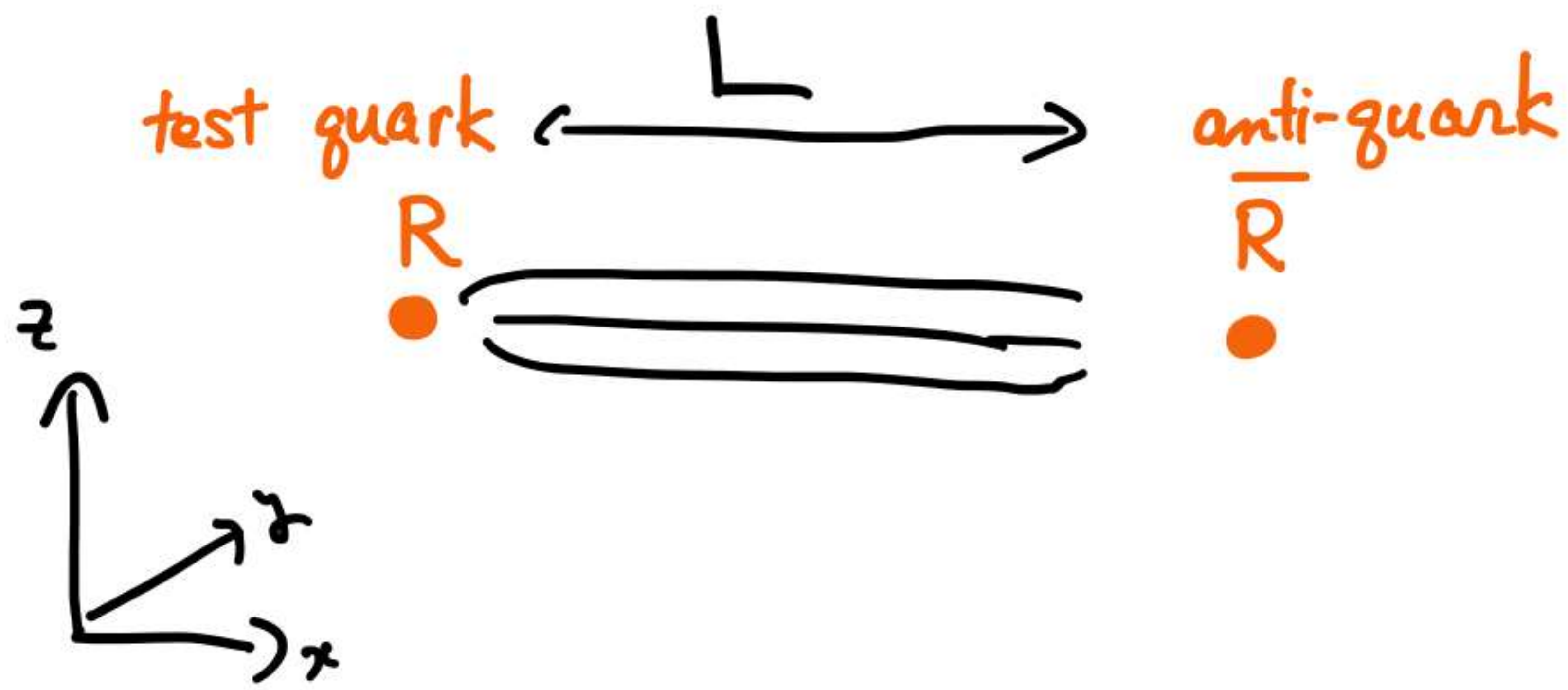
Weak-coupling (semiclassical) confinement on $\mathbb{R}^2 \times \underbrace{T^2}_{\text{'t Hooft flux}}$

adiabatic continuity

$\Longleftrightarrow$ Strong-coupling confinement on $\mathbb{R}^4$

- Introduction
  - ▷ Confinement in the $\mathbb{R}^2 \times T^2$ semiclassics
  - ▷ Review : Difficulty for large-N adiabatic continuity
    & Gonzalez-Arroyo, Okawa proposal : N-dependent twist.
- $\mathbb{R}^2 \times T^2$ semiclassics w/ N-dependent twist
  & large-N adiabatic continuity

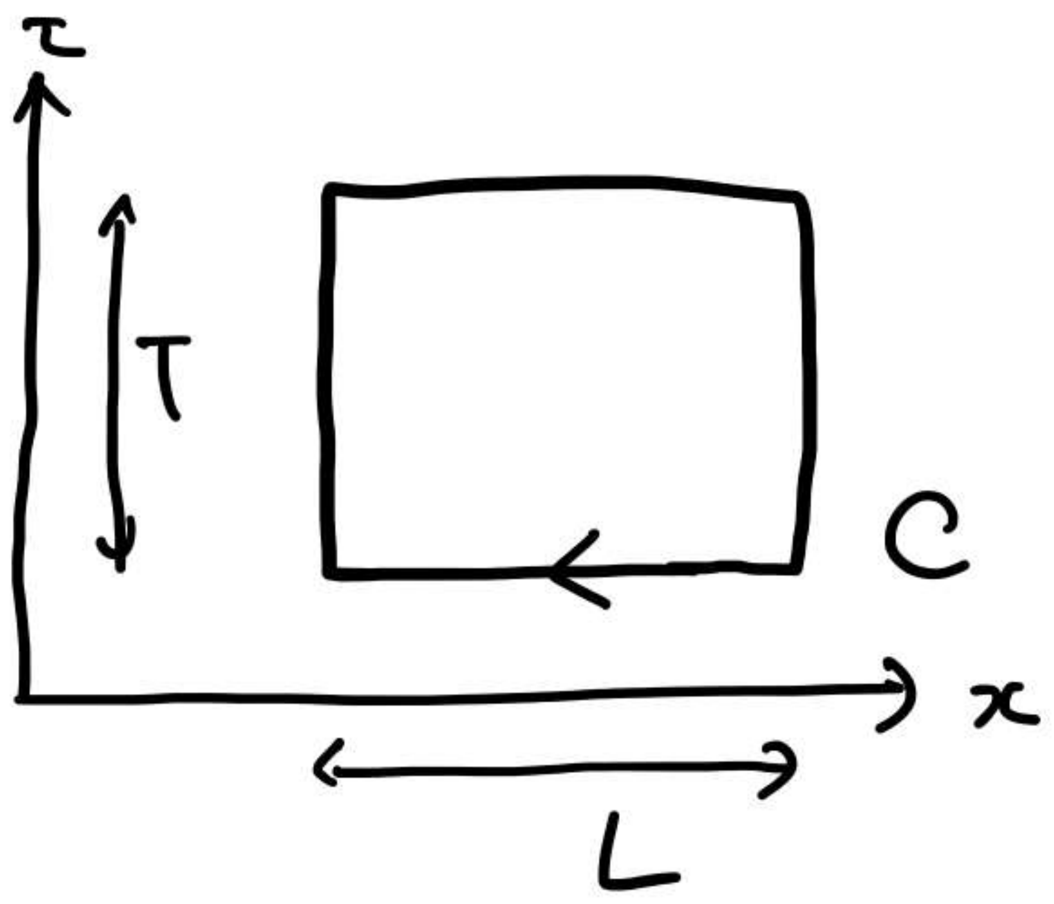
Confinement



Interquark potential :

$$V_{R\bar{R}}(L) \underset{L\to\infty}{\simeq} \begin{cases} \sigma_R L & (\text{confinement}) \\ \text{const.} & (\text{Higgs}) \end{cases}$$

Wilson loop : $W_R(C) = \mathrm{tr}_R \left[\mathcal{P} \exp\left(\oint_C a \right) \right]$.



$$\langle W_R(C) \rangle \simeq \exp\left(- T \cdot V_{R\bar{R}}(L) \right)$$

$$\simeq \begin{cases} \exp\left(- \sigma_R T L \right) & (\text{Area law}) \\ \exp\left(- \text{const.} \, (T+L) \right) & (\text{Perimeter law}). \end{cases}$$

This is an order parameter for the $\mathbb{Z}_N$ 1-form symmetry, or center symmetry.

$\mathbb{Z}_N$ 1-form center symmetry & θ-periodicity anomaly

SU(N) link variable

$\mathbb{Z}_N$ plaquette variable (Background : $dB = 0$)

$$S_{\text{Wilson}}[U_\ell, B_p] = \beta \sum_p \mathrm{Re}\left\{ \mathrm{tr}\left(1 - e^{-\frac{2\pi i}{N} B_p} U_{\ell_4}^\dagger U_{\ell_3}^\dagger U_{\ell_2} U_{\ell_1}\right)\right\}$$

B : $\mathbb{Z}_N$-valued 2-form gauge field for center symmetry = 't Hooft flux/twist

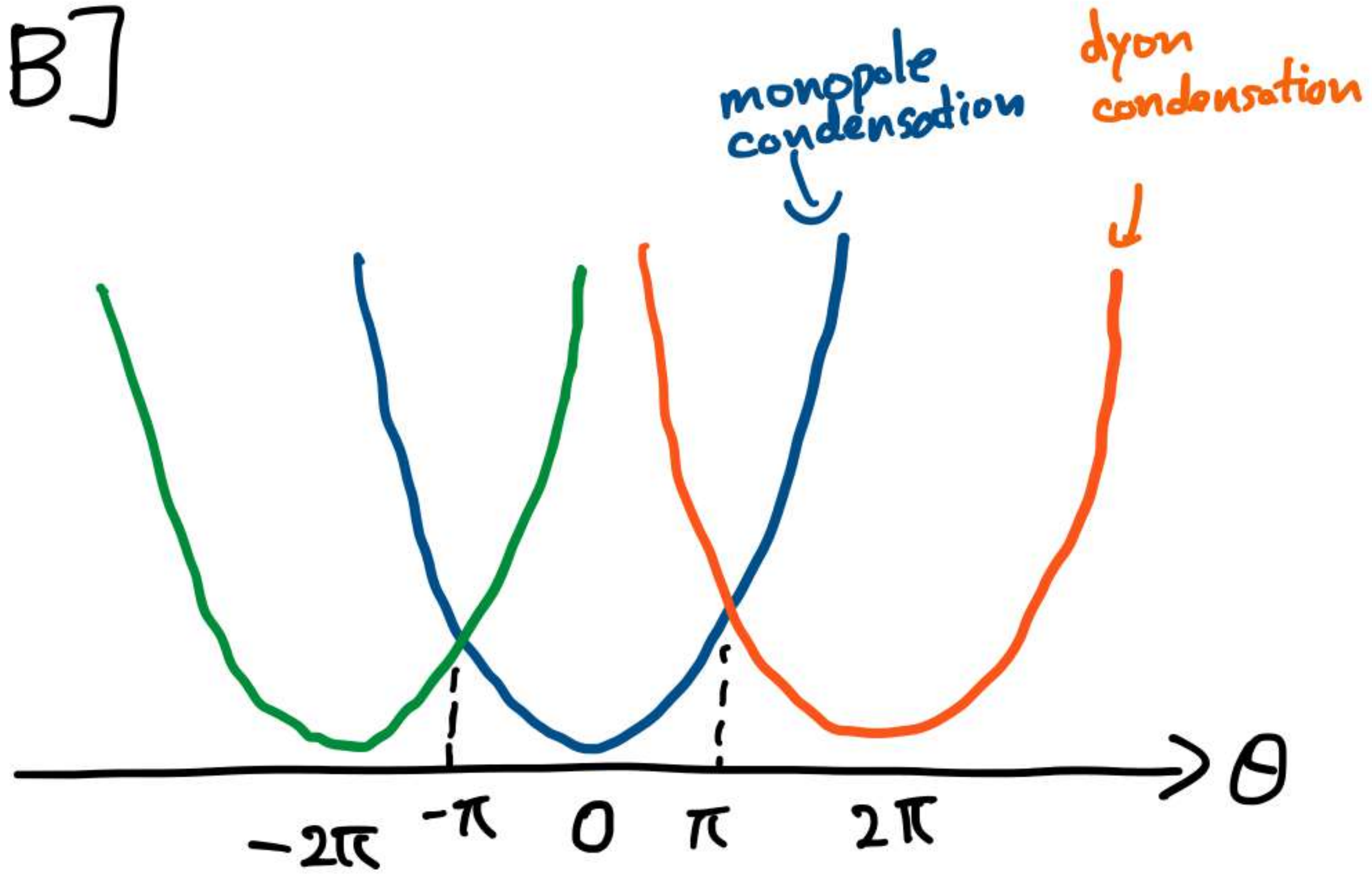
$$\begin{cases} B_p \mapsto B_p + (d\lambda)_p \\ U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell \end{cases}$$

$\leftarrow$ gauged center transformation

With the presence of B, θ-periodicity is mildly violated by a contact term of B:

$$Z_{\theta+2\pi}[B] = e^{\frac{2\pi i}{N}\int \frac{1}{2} B \cup B} Z_\theta[B]$$

[Gaiotto, Kapustin, Komargodski, Seiberg 2017
(cf. van Baal '82)
(purely lattice derivation :
Abe, Morikawa, Onoda, Suzuki, YT '23)]



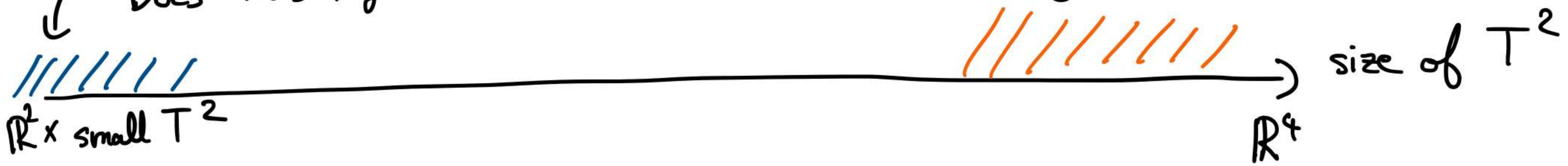
Semiclassical approach for confinement via T^2-compactification

Weakly coupled thanks to small T^2

Does this regime show confinement?

confinement of our interest

But, strongly coupled & difficult.



2d Symmetry

$\begin{cases} \mathbb{Z}_N^{(1)} & : \text{center sym. for 2d Wilson loop} \\ \mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)} & : \text{"conventional" center symmetry for Polyakov loops } P_3 \,\&\, P_4. \end{cases}$

$\xLeftarrow{T^2\text{-compact.}}$

4d Symmetry

$\mathbb{Z}_N^{(1)}$ in 4d

& $p = \int_{T^2} B \in \mathbb{Z}_N$: 't Hooft flux.

2d Anomaly

$$Z_{\theta+2\pi}[B_{2d}] = e^{i\frac{2\pi}{N} p \int B_{2d}} Z_\theta[B_{2d}]$$

$\Longleftarrow$

4d Anomaly

$$Z_{\theta+2\pi}[B] = e^{\frac{2\pi i}{N}\int \frac{1}{2} B \cup B} Z_\theta[B]$$

$\boxed{\text{Activation of 't Hooft flux } p \text{ w/ } \gcd(N,p)=1 \text{ preserves 4d anomaly maximally.}}$

Yang-Mills theory on $\mathbb{R}^2 \times T^2$ & 't Hooft flux

4d $SU(N)$ YM : $\mathbb{Z}_N^{(1)}$ center symmetry

$$\mathbb{R}^2 \times T^2 \Longrightarrow \begin{cases} \mathbb{Z}_N^{(1)} : & \text{Area vs Perimeter for 2d Wilson loops} \\ \mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)} : & \text{Conventional center symmetry for Polyakov loops } P_3, P_4 \end{cases}$$

Role of 't Hooft flux p

① 4d anomaly is maximally preserved in 2d effective theory

② Classical vacuum is unique & $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)}$ symmetric :

$$P_3 P_4 = e^{\frac{2\pi i}{N} p} P_4 P_3$$

$\leftarrow$ $SU(N)$ clock & shift algebra

③ Classical vacuum "violates" $\mathbb{Z}_N^{(1)}$ but semiclassically restored as

4d instanton $\xrightarrow{\mathbb{R}^2 \times T^2_{p\text{-twist}}}$ N center-vortex constituents.

[Gonzalez-Arroyo, Montero '98]

Fun w/ center-vortex / fractional instantons & fermions

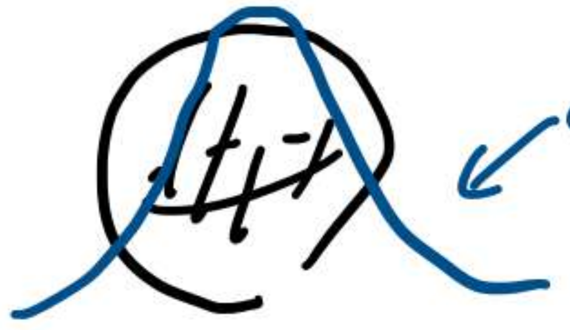
$\mathbb{R}^2\times T^2$ – semiclassics applies not only pure YM but QCD!!

$$\left(U(1)_{\text{baryon}} = \frac{U(1)_{\text{quark}}}{\mathbb{Z}_{N_c}} \ :\ \begin{array}{c} 1 \text{ baryon magnetic flux} \\ \int_{T^2} dA_B = 2\pi \end{array} \Rightarrow \begin{array}{c} 1 \text{ 't Hooft flux} \\ \int_{T^2} B_p = 1 \end{array} \right)$$

η' – mass via $U(1)_A$ anomaly $= -\cos\left(\frac{\eta' - \theta}{N}\right)$.

"fractionalized" Kobayashi – Maskawa – 't Hooft vertex

4d instanton

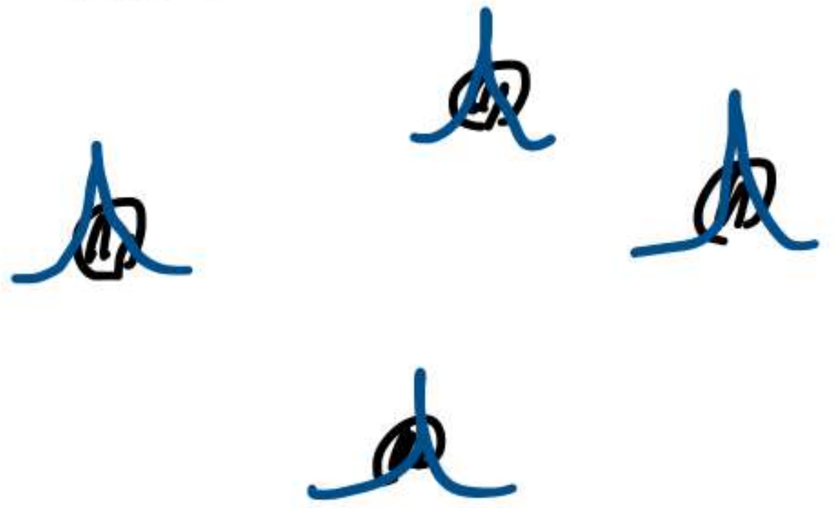


chiral zero mode

$\bar{\psi}_L \psi_R \sim e^{i\eta'}$

$\Longrightarrow$

N center-vortex fractional constituents



(YT, Ünsal 2201.06166
Hayashi, YT 2402.04320)

$\mathrm{Ind}(\not{D}) = 1$,

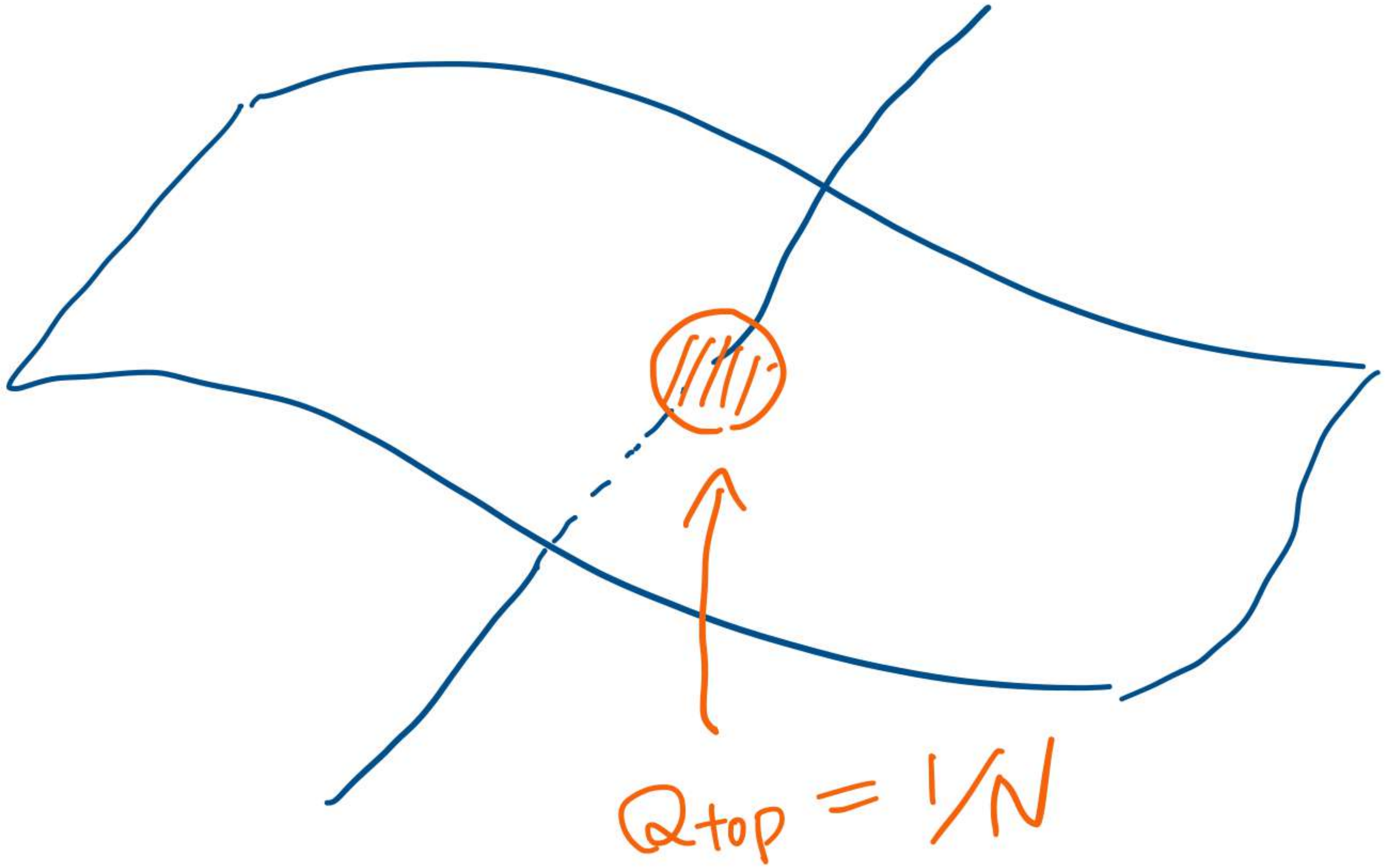
but the zero mode has the peaks at each constituents.

How does the fractionalization happen in 4d? [* speculative!]

4d confinement vacua have percolated center-vortex worldsheets.

(cf. Del Debbio, Faber, Greensite, Olejnik '96, Kovacs, Tomboulis '97, ⋯)

⇒ Those vortex sheets should have point-like intersections.



Such intersections have $1/N$ fractional Q_{top}! [Engelhardt, Reinhardt 2000, Cornwall 2000]

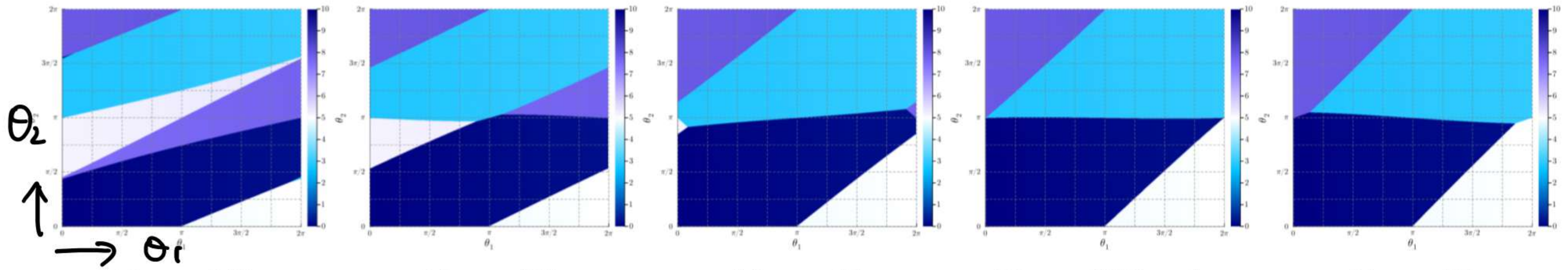
Application to QCD-like theories

- QCD (Anti-Sym / Sym) [YT, Ünsal 2205.11339]

  Fractional instanton always has $Q_{top} = \frac{1}{N}$, but the fermion zero-mode fractionalizes as $e^{i\left(\frac{N\mp 2}{N}\eta' - \frac{\theta}{N}\right)}$ $\Rightarrow$ $N\mp 2$ chiral-broken vacua.

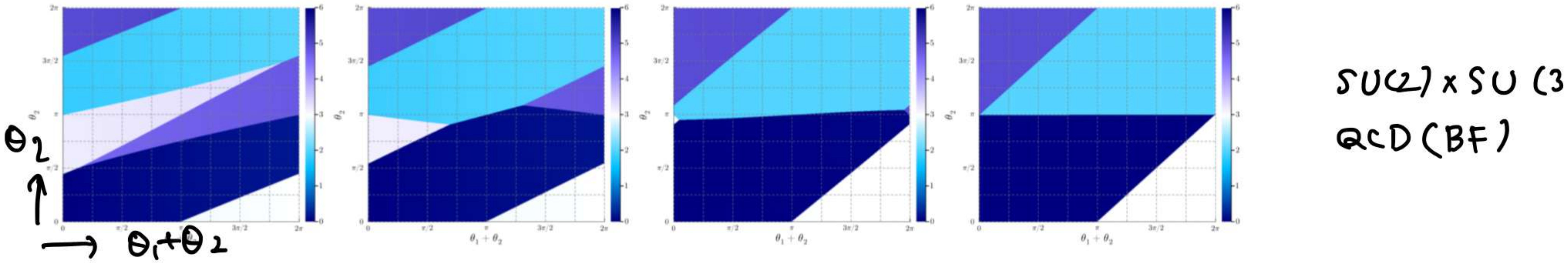
- Karasik-Komargodski non-SUSY duality cascade for QCD (Bi-Fund).

  [Y. Hayashi, YT, H. Watanabe 2404.16803]
  [A. Karasik, Z. Komargodski 1904.09551]



(a) $m\mu = 0.02$ (b) $m\mu = 0.06$ (c) $m\mu = 0.2$ (d) $m\mu = 0.263 \approx m^*\mu$ (e) $m\mu = 0.3$

SU(2)×SU(5) QCD(BF)



(f) $m'\mu \approx 0.02182$ (g) $m'\mu = 0.08$ (h) $m'\mu = 1.2$ (i) $m'\mu = 100$

SU(2)×SU(3) QCD(BF)

Adiabatic continuity conjecture

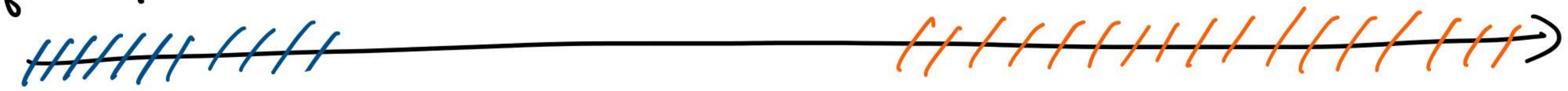
$NL\Lambda \lesssim 1$

- Weakly-coupled
- Many expected features of 4d dynamics are obtained

smoothly connected ?

$L\Lambda \sim O(1)$

- Confinement w/ strongly-coupled dynamics
- Difficult to solve



size of T^2

(cf.
T^4 : twisted Eguchi-Kawai model (Gonzalez-Arroyo, Okawa '83, '10, ...)
$\mathbb{R}\times T^3$: van Baal '80s, Garcia-Perez, Gonzalez-Arroyo ... '90s, Yamazaki, Yonekura '17, Cox, Poppitz, Wandler '21
$\mathbb{R}^3\times S^1$: Davies, Hollowood, Khoze '99, Ünsal, Yaffe, Shifman, Poppitz, ... ~'07)

Claim

Suitable choice of (N, P) achieves the continuity for pure YM w/ $N \gg 1$.

Evidence for $N=2$ $\Rightarrow$ Bergner, Gonzalez-Arroyo, Soler 2505.10396 & Talk on Monday

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SU(N) YM on $\mathbb{R}^2 \times T^2$ w/ minimal 't Hooft flux $p=1$

$L < \frac{1}{N\Lambda}$

- $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)}$ is preserved at the classical level
- $\mathbb{Z}_N^{(1)}$ is restored at the semi-classical analysis.

$L \gtrsim \Lambda^{-1}$

- Confined, i.e. $\mathbb{Z}_N^{(1)}$ is unbroken

size of T^2

$\Rightarrow$ {
- Same Symmetry / Anomaly (Kinematics)
- Same Realization of Symmetry (Dynamics)

<u>Q</u>. Why do we concern about continuity?

Isn't the continuity the most natural scenario?

Large-N tachyonic instability ($p=1$)

- 't Hooft flux preserves the $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)}$ symmetric vacuum (= twist eater)

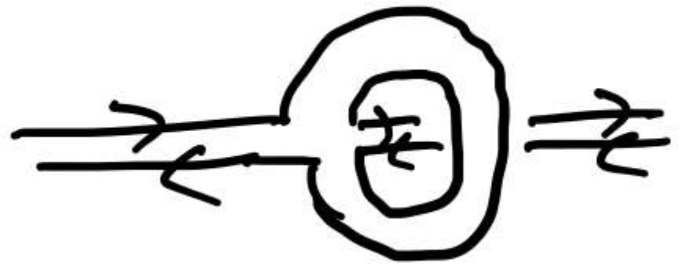
$$P_3 P_4 = e^{\frac{2\pi i}{N}} P_4 P_3 \Rightarrow \begin{cases} P_3 = S \propto \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \\ P_4 = C \propto \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \end{cases} \quad \text{(up to gauge trans.)}$$

- Comparison of the classical potential for $\mathbb{R}^2 \times \underbrace{(1\times 1 \text{ lattice})}_{\approx T^2}$.

$$\begin{cases} S\Big|_{\substack{\text{twist eater} \\ (P_3, P_4) = (S, C)}} = 0 . \\ S\Big|_{\substack{\text{center-broken} \\ P_3, P_4 \propto \mathbb{1}_N}} = \dfrac{1}{g^2}\left\{ \mathrm{tr}\left(\mathbb{1} - e^{\frac{2\pi i}{N}} \mathbb{1}\right) + \text{c.c.} \right\} \sim O\left(\dfrac{1}{\lambda_{tH}}\right) \end{cases}$$

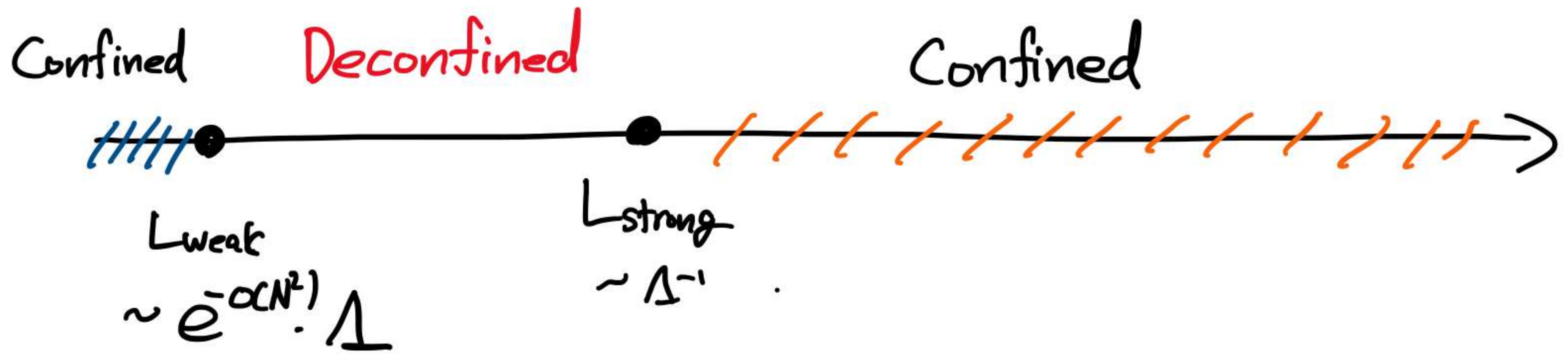
Unless $\lambda_{tH} \lesssim \frac{1}{N^2}$, the twist eater may be destabilized by $O(N^2)$ fluctuations.

▶ Guralnik, Helling, Landsteiner, Lopez '02 : Tachyonic instability via non-planar self-energy at $\lambda_* \sim N^{-1}$.



▶ Bietenholz, Nishimura, Susaki, Volkholz '06 : 1st-order deconfinement transition at $\lambda_* \sim N^{-2}$ for $(N = 2k+1,\ p = k)$.

Phase diagram for $p=1$



- Weakly-coupled confinement is separated from strongly-coupled confinement

Two proposals to eliminate the intermediate deconfined phase

We test it for $\mathbb{R}^2 \times T^2$.

① Suitably chosen N-dependent twist p (Gonzalez-Arroyo, Okawa '10)

- Chamizo, Gonzalez-Arroyo : $(N, p) = (F_{k+2}, F_k)$ should work

  Fibonacci seq.

  [Analysis is done for 3d YM on $\mathbb{R} \times T^2$].

② Introduce massive adjoint fermions (Azeyanagi, Hanada, Ünsal, Yacoby '12).

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- $\mathbb{R}^2 \times T^2$ semiclassics w/ N-dependent twist & large-N adiabatic continuity

Criterion for adiabatic continuity

$$\underbrace{\mathbb{Z}_N^{(1)}}_{\boxed{\text{center-vortex gas restores it}}} \times \underbrace{\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)}}_{\boxed{\text{stabilized for the classical vacuum}}} \text{ in 2d} \Leftarrow \mathbb{Z}_N^{(1)} \text{ in 4d}$$

- Both mechanisms should not be destroyed by $O(N^2)$ quantum fluctuations.

<u>Our interpretation of Chamizo, Gonzalez-Arroyo ('16)</u>

$\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(0)}$ is stabilized against quantum fluctuations for $(N, p) = (F_{k+2}, F_k)$ for $k \gg 1$.

<u>Idea</u> $\mathbb{Z}_N \xrightarrow{N \to \infty} U(1)$ is spontaneously broken to a subgroup $\mathbb{Z}_{n'} \subset U(1)$

if $\frac{p}{N} \xrightarrow{N, p \to \infty} \frac{p'}{n'} \in \mathbb{Q}$. (Candidate of classical config: [Appendix of our paper.]

$$P_3 = [S_{(n')}^{-p'} \otimes \mathbf{1}_{M \times M}] \oplus S_{(L)}^{-k}$$

$$P_4 = \underbrace{[C_{n'}^{-1} \otimes \mathbf{1}_{M \times M}]}_{n'M \times n'M} \oplus \underbrace{C_{(L)}^{-1}}_{L \times L}$$

w/ $M, L \sim O(N)$)

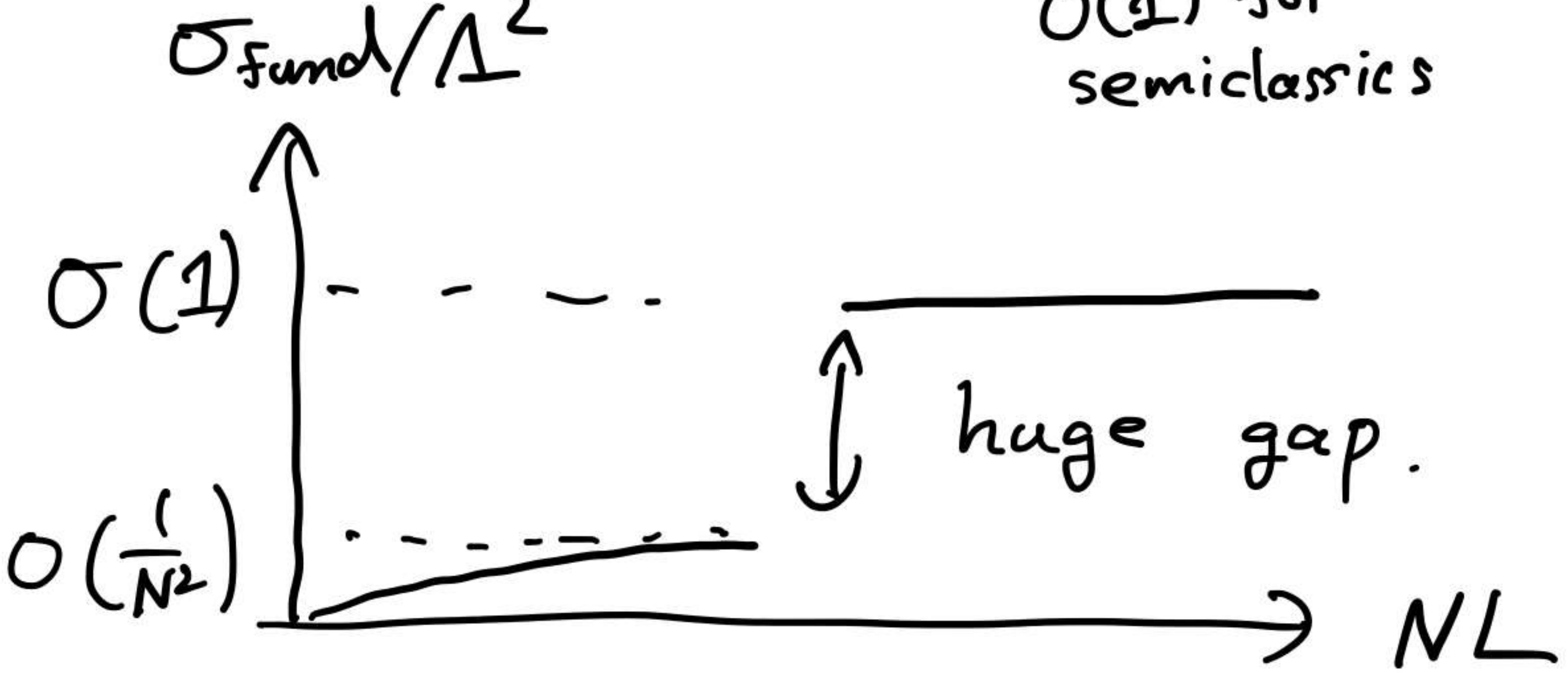
$\Rightarrow \frac{p}{N}$ should converge to an irrational number that is "far from" $\mathbb{Q}$.

Large-N $(\mathbb{Z}_N^{(1)})_{2d}$ breaking for $p=1$

Let us check if $\mathbb{Z}_N^{(1)}$ is also stabilized for $(N, P) = (F_{k+2}, F_k)$.

- If $p=1$, dilute gas approximation of center-vortices gives

$$\sigma_{\text{fund}} \sim \Lambda^2 \cdot \underbrace{(NL\Lambda)^{\frac{5}{3}}}_{O(1)\text{ for semiclassics}} \cdot \underbrace{\sin^2\left(\frac{\pi}{N}\right)}_{\to 0 \text{ as } N\to\infty}$$



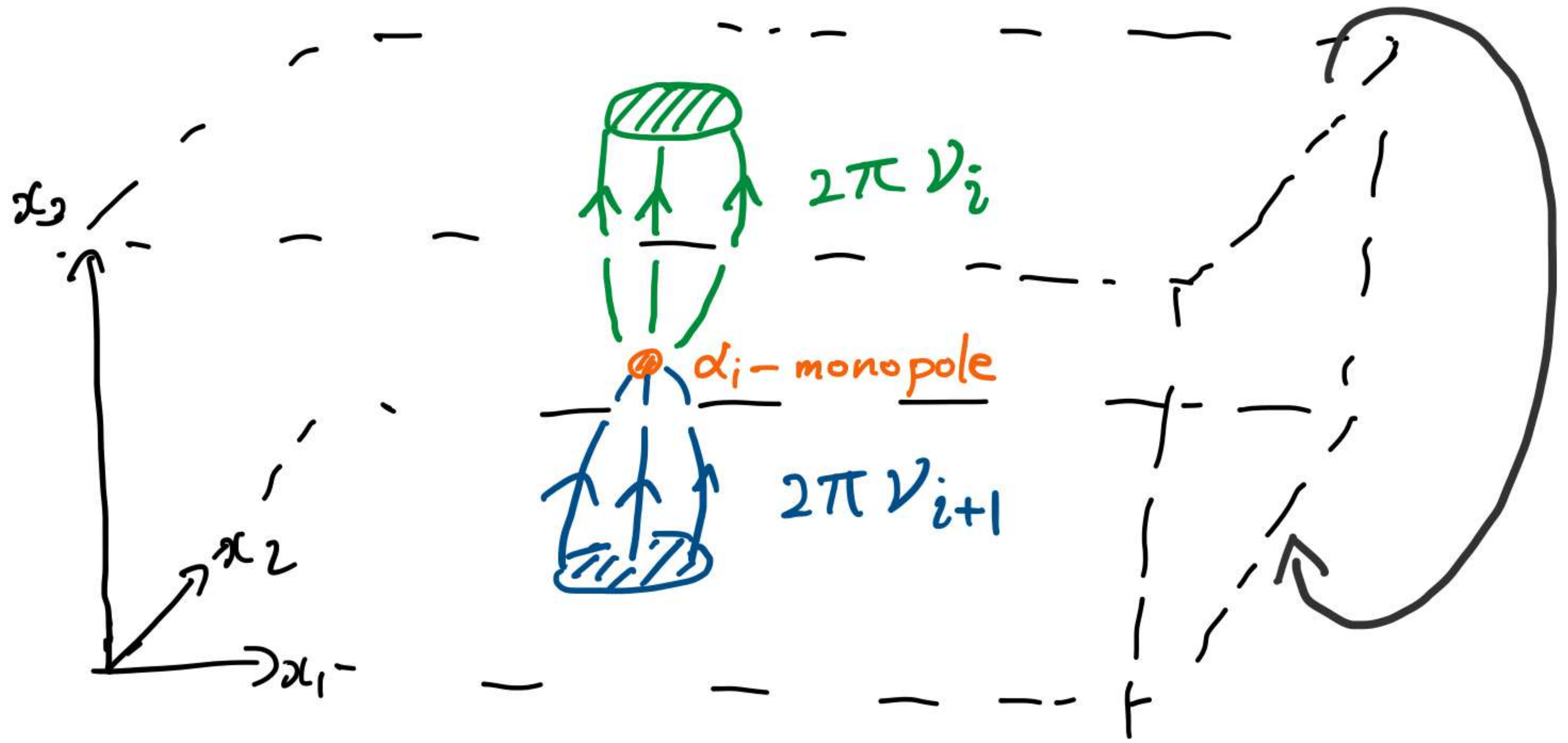
$\Rightarrow$ For suitable P, how can we close this gap?

Center vortex on $\mathbb{R}^2 \times \underbrace{T^2}_{\text{flux}}$ = KvBLLY monopole instanton $(p=1)$

$SU(N)$ gauge field on $\mathbb{R}^3 \times S^1$ w/ nontrivial holonomy : N fundamental monopoles (Lee, Yi '97, Lee, Lu '98, Kraan, van Baal '98)

[$\Rightarrow$ 3d semiclassics by Ünsal, ... since 2007]

α_i - monopole emits the magnetic flux $2\pi\alpha_i = 2\pi(\nu_i - \nu_{i+1})$.



$\mathbb{Z}_N$-twisted b.c. (= 't Hooft flux on T^2)

$2\pi\nu_i \mapsto 2\pi\nu_{i+1}$

[Hayashi, YT 2405.12402]

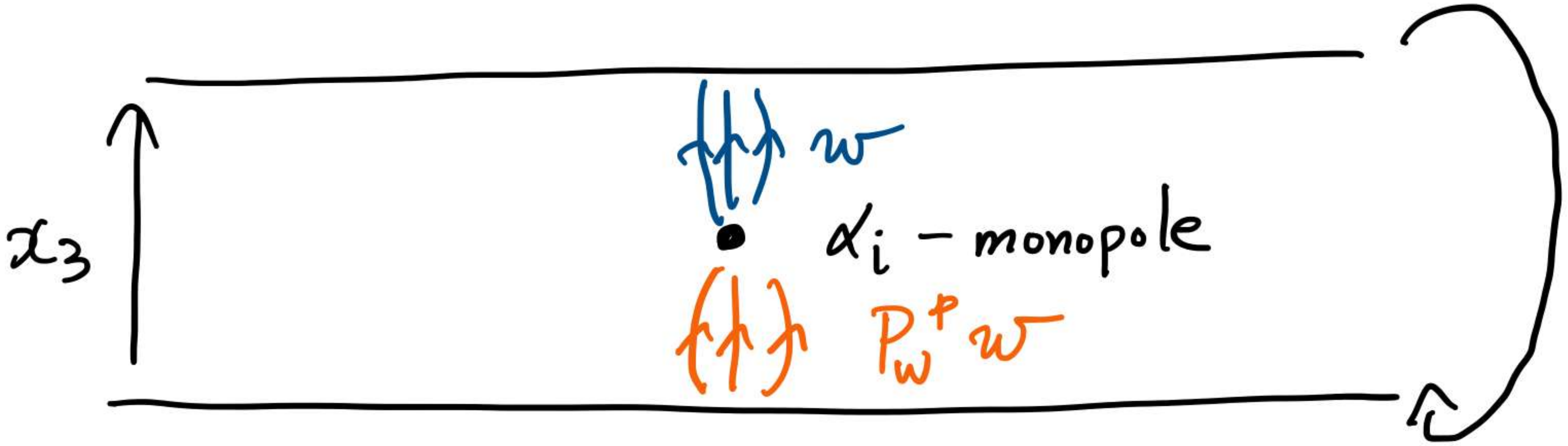
$\mathbb{Z}_N$-twisted b.c. gives the perturbative gap $\frac{2\pi}{NL_3}$ $\Rightarrow$ Magnetic flux localizes.

$$\boxed{\text{Monopole} = \text{Junction of the center vortex}}$$

(cf. Ambjorn, Giedt, Greensite '99, de Forcrand, Pepe '00)

Center-vortex fractional instanton for general (N,p)

Magnetic flux $\vec{w}$ of self-dual center vortex must satisfy



't Hooft twist:

$$\vec{\nu}_i \mapsto \vec{\nu}_{i+p} = P_w^p \vec{\nu}_i$$

$$\vec{w} = \vec{\nu}_i + \vec{\nu}_{i+p} + \cdots + \vec{\nu}_{i+(q-1)p}$$

with $q \in \mathbb{Z}_N$ s.t. $q \cdot p = 1 \mod N$

$$-)\ P_w^p \vec{w} = \vec{\nu}_{i+p} + \cdots + \vec{\nu}_{i+(q-1)p} + \underbrace{\vec{\nu}_{i+pq}}_{= \vec{\nu}_{i+1}}$$

$$\vec{w} - P_w^p \vec{w} = \vec{\nu}_i - \vec{\nu}_{i+1} \left(= \vec{\alpha}_i \right)$$

$\Rightarrow$ $\frac{1}{N}$-fractional instanton carries $\frac{q}{N}$-flux.

Partition funcion on $\underset{\to \mathbb{R}^2}{M_2} \times T^2$ & θ-dependence

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

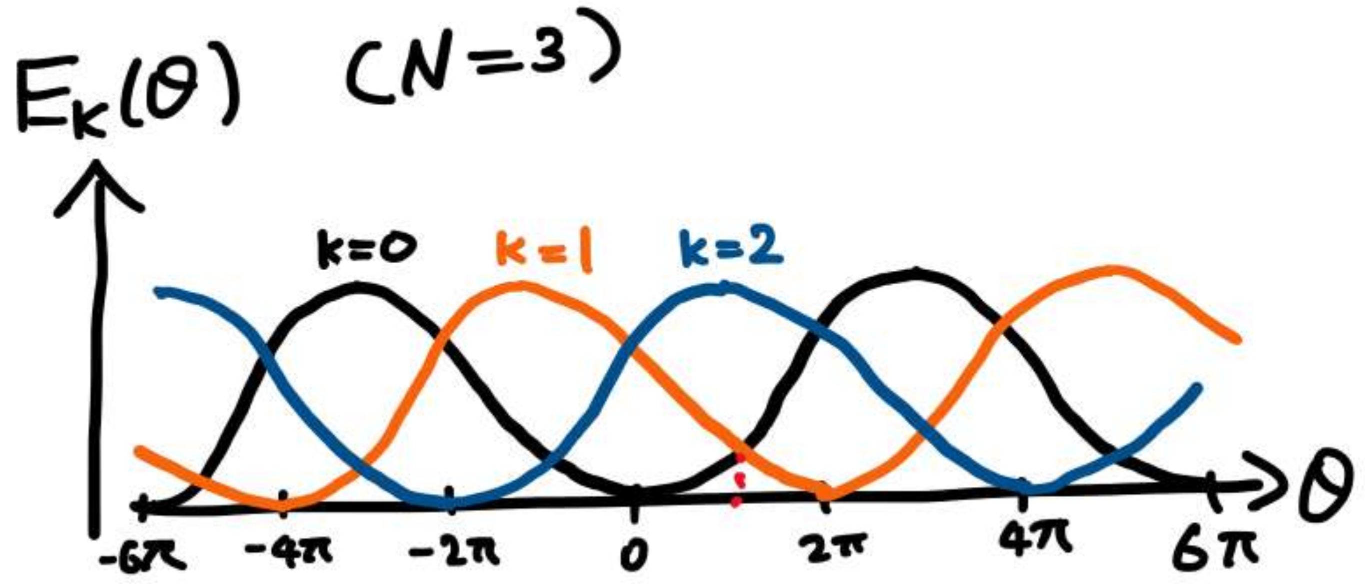
we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}_{\text{vortex}}\Big)^{n} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}_{\text{anti-vortex}}\Big)^{\bar{n}}$$

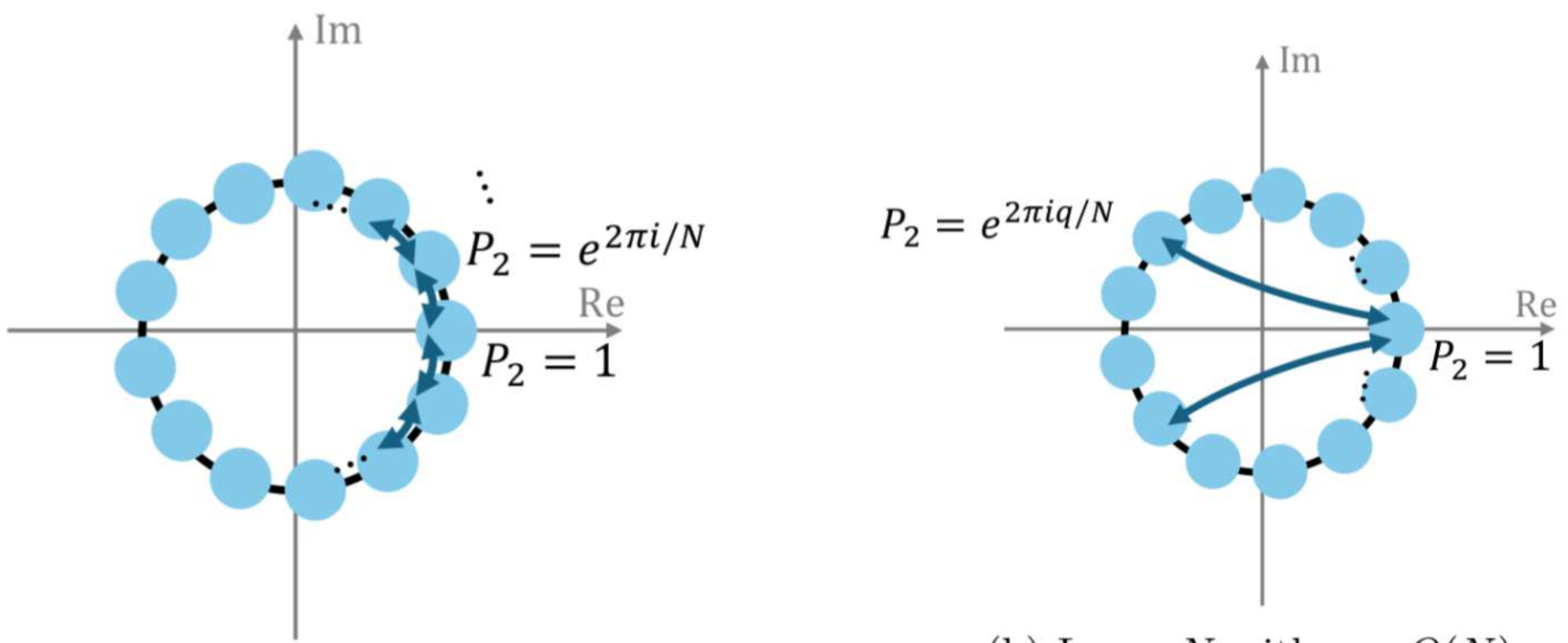
$$= \sum_{k=0}^{N-1} \exp\Big[-V\big(\underbrace{-2K e^{-\frac{8\pi^2}{g^2N}} \cos\big(\tfrac{\theta-2\pi k}{N}\big)}_{E_k(\theta)}\big)\Big]$$

$E_k(\theta)$: Ground-state energy densities

$\Rightarrow$
- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



Dilute gap approximation of $\frac{q}{N}$-flux center-vortex instantons for σ_{string}

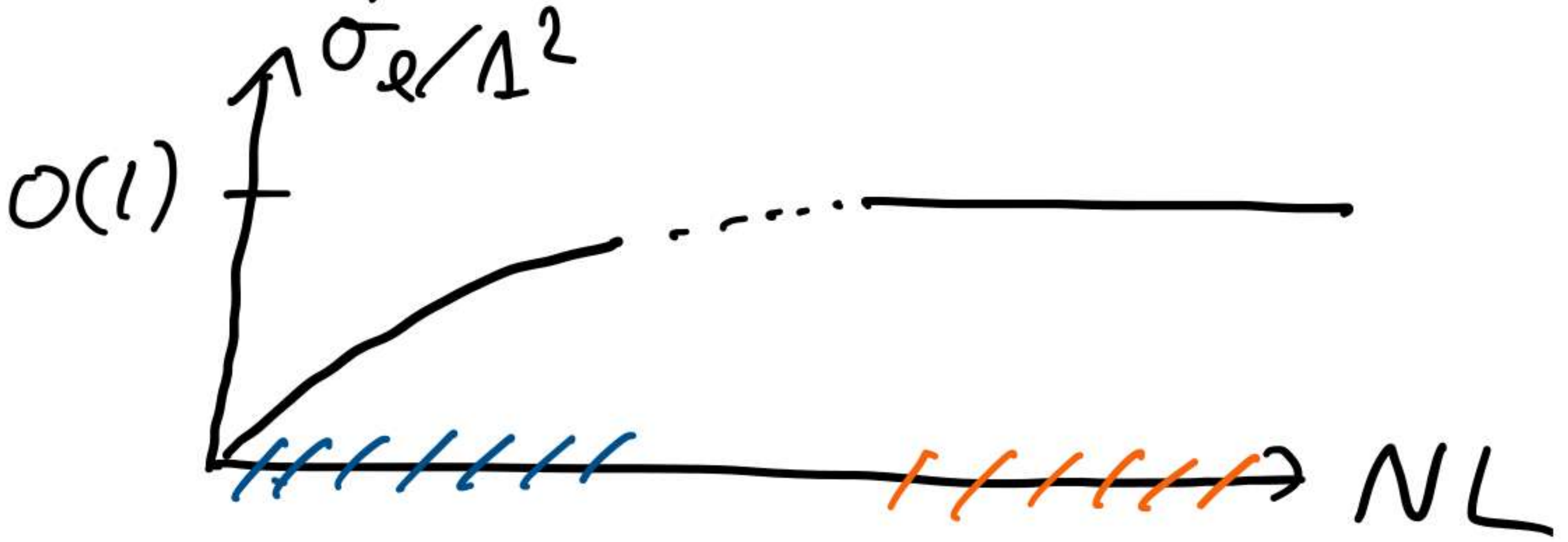


(a) Large N with $p = q = 1$

(b) Large N with $q \sim O(N)$

$$\Rightarrow \quad \sigma_{\ell\text{-string}} \sim \Lambda^2 \underbrace{(NL\Lambda)}_{O(1)}{}^{\frac{5}{3}} \cdot \sin^2\left(\frac{\pi q}{\underbrace{N}_{O(1)}} \ell\right)$$

For $\ell \sim O(1)$,



Continuity is now reasonable!

Summary

Studying 4d gauge theories on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux is useful to investigate some aspects of confinement.

- Center vortex can be used for analytically well-controlled semiclassics.
- Center vortex in this setup $= \frac{1}{N}$ fractional instanton
  $=$ KvBLLY monopole instanton
- Fresh perspective on $U(1)_A$ & η'.
  - $\frac{1}{N}$ fractionalization of Kobayashi-Maskawa-'t Hooft vertex
- Other fermions (QCD(Sym/ASym/BF)) can also be studied. (YT, Ünsal '22; Hayashi, YT, Watanabe '23, '24)
  $\Rightarrow$ Large-N orbifold equivalence.
- Gonzalez-Arroyo, Okawa, Chamizo proposal $(N, p) = (F_{k+2}, F_k)$ works nicely for large-N adiabatic continuity.