To Break or Not to Break: Proving Chiral Symmetry Breaking in QCD with 't Hooft Anomaly Matching

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Based on hep-th/2212.02930 (PRD), 2404.02967 (PLB),
with Luca Ciambriello, Roberto Contino, Andrea Luzio, Marcello Romano

ECT, Aug. 29, 2025

- 2212.02930
  Revisiting the literature, and clarifying the assumptions that lies in the proofs that have been considered, with simplified examples
- 2404.02967
  Presenting our new proof

Apologies for missing important references during the talk, please find the references in our papers.

Infrared phases of QCD

- QCD: (3+1) d $SU(N_c)$ Yang-Mills theory coupled to N_f massless quarks in the fundamental representation
- It is well known that the infrared phases depend on the values of N_c and N_f:

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- QCD: (3+1) d $SU(N_c)$ Yang-Mills theory coupled to N_f massless quarks in the fundamental representation
- It is well known that the infrared phases depend on the values of N_c and N_f:
  1) Infrared free quarks and gluons for $N_f \geq 11N_c/2$
  2) Interacting CFT for $N_f^\star \leq N_f < 11N_c/2$
  3) Chiral symmetry breaking for $2 \leq N_f < N_f^\star$
  4) Gapped with unique vacuum for $N_f = 1$
  5) The θ parameter becomes physical for $N_f = 0$:
  gapped with unique vacuum for generic θ; two degenerate vacua at $\theta = \pi$

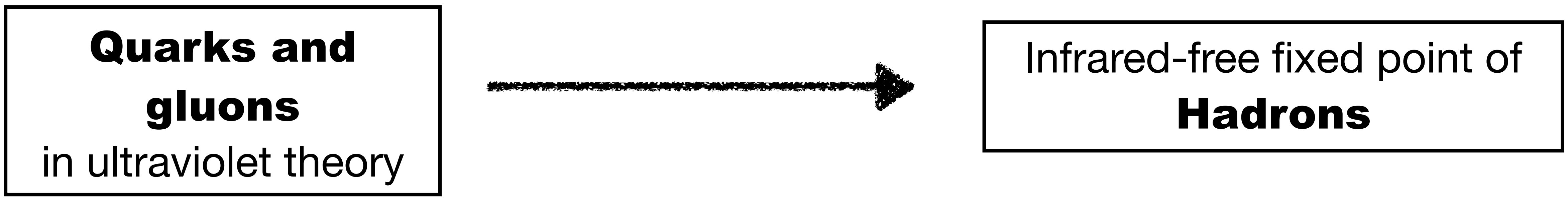
Punchline

- The picture of infrared phases is mainly based on empirical evidences, lattice results and educated guessworks
- Very little has been rigorously/coherently derived, other exotic phases of QCD may be possible

Instead of pinning down the phase diagram for specific N_c and N_f, we establish a general no-go theorem that states the *incompatibility* of unbroken chiral symmetry and color screening.

Color screening

- Characterized by the following RG flow



- The fact that all the hadrons must be color singlets in the low energy theory is conventionally denoted as "confinement". More precisely, in the presence of massless dynamical quarks in the fundamental representation, Wilson lines are screened, hence it needs to be distinguished from the genuine confinement. Perhaps one can call the infrared description "screening confinement".

Implications of the no-go theorem

- Option 1: Assuming that theory flows in the infrared to a fully color-screened, infrared-free phase described by color-singlet hadrons, chiral symmetry must be spontaneously broken.
- Option 2: Conversely, any phase with unbroken chiral symmetry must retain unscreened color charges.

Both consistent with the general wisdom

- Revisiting 't Hooft's cargese lectures in 1979

NATURALNESS, CHIRAL SYMMETRY, AND SPONTANEOUS CHIRAL SYMMETRY BREAKING

G. 't Hooft

- The no-go theorem can be derived from rigorous algebraic equations

Perturbative chiral anomalies of QCD

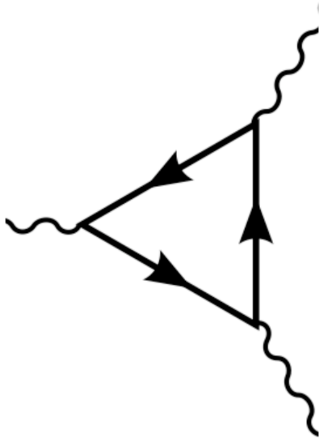
- Consider a QCD-like theory with $G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$

$$q_L \sim (\text{fund., singlet, } 1/N_c)$$

$$q_R \sim (\text{singlet, fund., } 1/N_c)$$

Quarks and gluons
q, g
in UV theory

- The triangle anomalies $[SU(N_f)_{L,R}]^3$ and $[SU(N_f)_{L,R}]^2 U(1)_B$ do not vanish, hence there is obstruction to gauging $G[N_f]$



’t Hooft anomaly matching conditions

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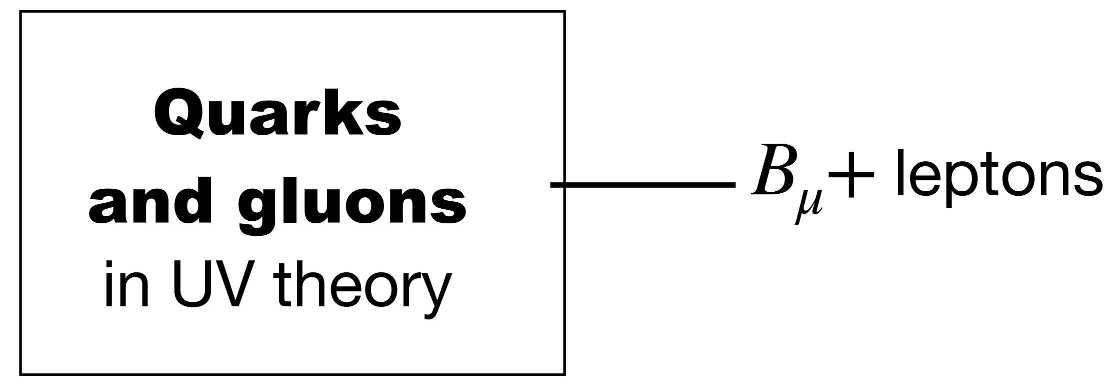
- ’t Hooft: weakly gauging $G[N_f]$ and adding spectator fermions (leptons), which are charged only under $G[N_f]$ but not under color, to cancel the anomalies of quarks



$$\mathscr{A}(q) + \mathscr{A}(L) = 0$$

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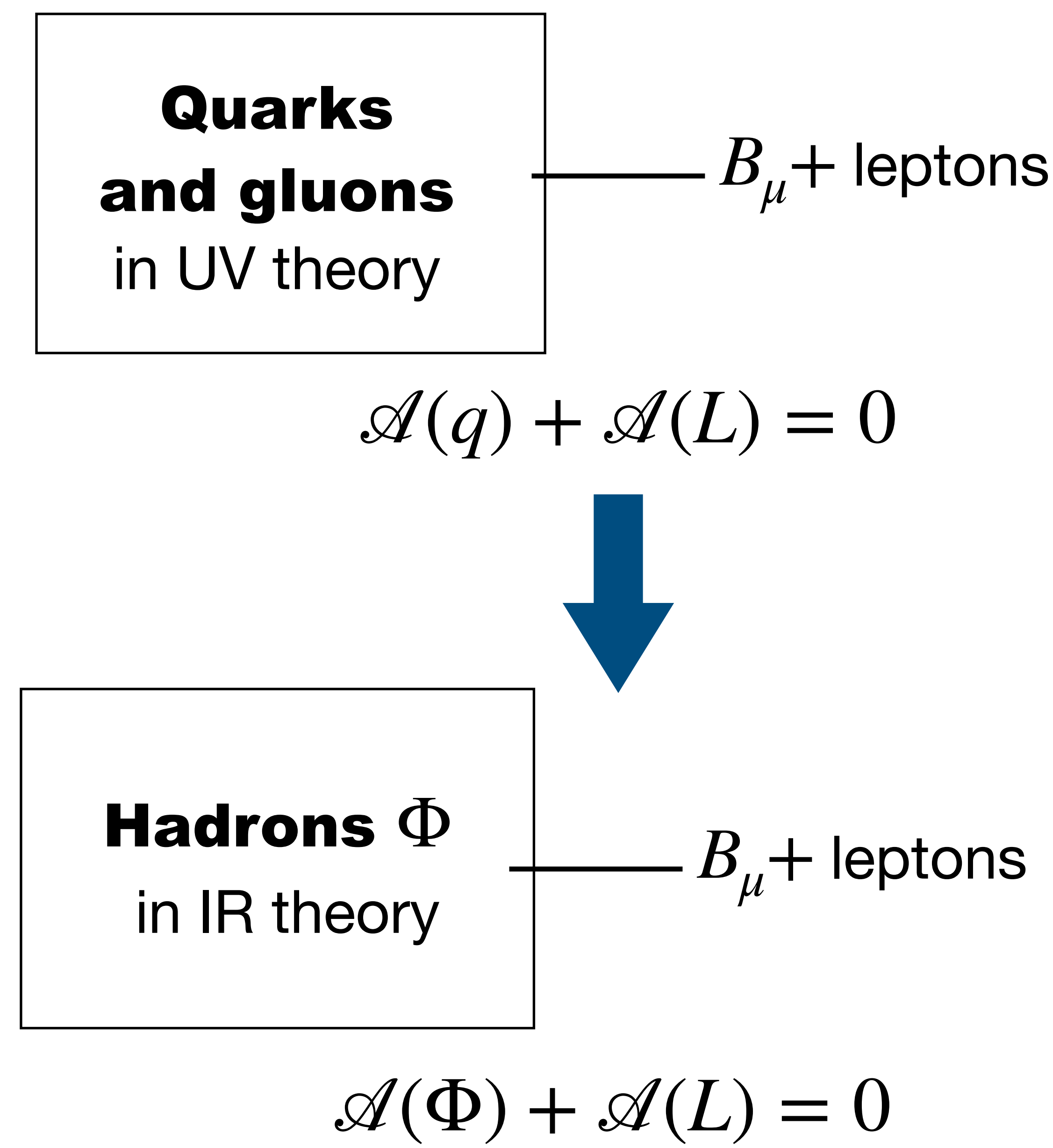
- 't Hooft: weakly gauging $G[N_f]$ and adding spectator fermions (leptons), which are charged only under $G[N_f]$ but not under color, to cancel the anomalies of quarks
- Anomalies match in the UV and IR

$$\mathscr{A}(q) = \mathscr{A}(\Phi)$$



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- However, color-singlet hadrons furnish general irreducible representations under $G[N_f]$, which is *not* under control
- The assumption of unbroken chiral symmetry implies that 't Hooft anomalies are matched by (spin-1/2) massless composite fermions

⟶ It's sufficient to consider odd N_c ; for even N_c all hadrons are bosons

- Anomaly matching conditions (AMC):

$$\sum_{r\in\mathcal{R}[N_f]} \ell(r)\, A_i(r) = N_c\, A_i(r_{q_L})$$

anomaly coefficient (hadrons)

anomaly coefficient (quarks)

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Clearly, 1) all indices must be integers for a physical spectrum

2) the index vanishes when helicities are paired.

3) Nontrivial indices (i.e. $\ell(r) > 1$) imply enhanced symmetry in the infrared.

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> plus their parity conjugates. There are no sets of composite particles that are consistent with the anomaly condition when n is a multiple of 3, because the $SU(n)_L \times SU(n)_L \times U(1)_V$ anomaly in this case is a multiple of 3 for any color-singlet states, while the elementary quarks in QCD give an anomaly of unity. But there are an infinite number of possible sets of massless composite fermions that do satisfy the anomaly conditions, for all values of n that are not multiples of 3.

[Preskill, Weinberg, *Phys.Rev.D* 24 (1981) 1059]

$N_c = 3$

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- The challenge is to prove the AMC equations do not have integer solutions for *any* spectrum of color-singlet hadrons and for *any* N_c and N_f
- For special N_c and N_f, there is hope to prove

$$\tfrac{1}{2}(n+2)(n+3)\ell_a + \tfrac{1}{2}(n-2)(n-3)\ell_b + (n^2-3)\ell_c + n(n+2)\ell_d$$
$$+n(n-2)\ell_e + \tfrac{1}{2}n(n+1)\ell_f + \tfrac{1}{2}n(n-1)\ell_g = 1\,. \qquad (22.5.8)$$

There is no problem in satisfying Eq. (22.5.7), but note that if n is a multiple of three then for all values of the ℓs, each term on the left-hand side of Eq. (22.5.8) is also a multiple of three, which makes it impossible to satisfy this condition. We conclude in particular that *the* $SU(3)_L \times SU(3)_R \times U(1)_V$ *symmetry of quantum chromodynamics with three flavors of massless quarks must be spontaneously broken.* This result is not

[Weinberg, QFT textbook vol.2, section 22.5]

Prime factor —

In $QCD[N_c, mp]$*, where* p *is a prime factor of* N_c *and* m *a positive integer, there exist no integral solutions of the* $[SU(mp)_{L,R}]^2U(1)_V$ *AMC. Therefore,* χSB *must occur in* $QCD[N_c, mp]$ *if the theory confines.*

This result is valid for any spectrum of color-singlet composite fermions, see the proof in 2404.02967

Additional constraints needed

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— The intuition is to deform the massless theory with quark masses and keep track of the symmetries. This is another probe which is allowed only in vectorlike theories.

Persistent Mass Conditions

- PMC states that bound states with massive constituents (and with nonzero $U(1)_{H_i}$ charges) are massive
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- PMC implies that the vectorlike part of $G[N_f]$ cannot be spontaneously broken (i.e., the Vafa-Witten theorem)
- Relatedly, Weingarten's hadrons mass inequalities [Phys. Rev. Lett. 51, 1830 (1983)] suggest the presence of massless pseudo-scalars interpolated by the pseudo-density operator $\bar{q}_i \gamma_5 q_j$, which look like Nambu-Goldstone bosons. However, such an argument is not conclusive, since it can happen that

$$\langle 0|\bar{q}\gamma^5 q|\pi\rangle \neq 0 \qquad \langle 0|\bar{q}_i\gamma^5\gamma^\mu q_j|\pi\rangle = 0$$

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$$\det(\not{D} + m) = \prod_{\lambda}(-i\lambda + m) = \prod_{\lambda>0}(\lambda^2 + m^2) \cdot m^{n_0} > 0$$

2) The bound on the quark propagator in the background of gauge fields (with some technicalities on smearing)

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$$|S_{\Delta}^{A}(x, y; m)| \leq \alpha(\Delta, m)\ e^{-m|x-y|}$$

It follows that the gauge-averaged quark propagator satisfies the bound:

$$\langle q(x)\bar{q}(y)\rangle = \frac{1}{Z}\int \mathcal{D}A_{\mu}\left[\det(\not{D} + m)\right]^{N_f} e^{-S_{\mathrm{YM}}[A]} S_{\Delta}^{A}(x, y; m)$$

$$\leq \alpha(\Delta, m)\ e^{-m|x-y|}$$

- Let m be the quark mass of one flavor, and ϵ that of the others. Let $B(x)$ be a composite operator, it follows that

$$|\langle B(x)^{\dagger} B(y)\rangle| \leq e^{-(n_H \cdot m + n_L \cdot \epsilon)|x-y|}$$

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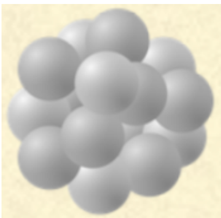
- At large distances, $\langle B(x)^\dagger B(y)\rangle \sim e^{-M|x-y|}$
- Bound on masses of hadrons versus its quark constituents $M \geq n_H m + n_L \epsilon$
- Taking the limit $\epsilon \to 0$
  - $M \geq n_H m > 0$ for $n_H > 0$
  - $G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ reduces to $G[N_f,1] = SU(N_f-1)_L \times SU(N_f-1)_R \times U(1)_B \times U(1)_{H_1}$

    (Notice that the hadron must be charged nontrivially under $U(1)_{H_1}$ to ensure $n_H > 0$.)

Chiral symmetry and PMC equations



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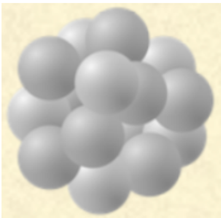

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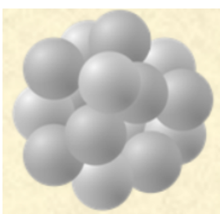
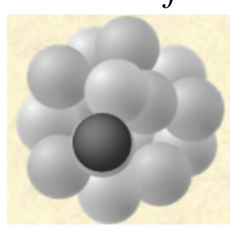
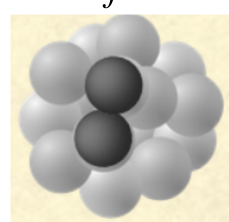
 $+$  $+$  $+ \cdots$

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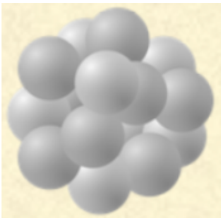

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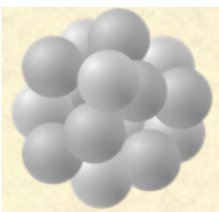
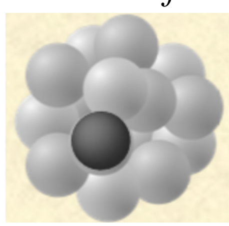
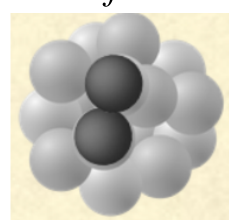
 $+$  $+$  $+ \cdots$

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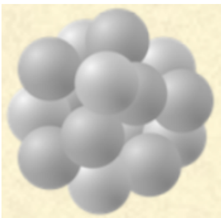

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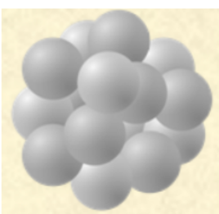
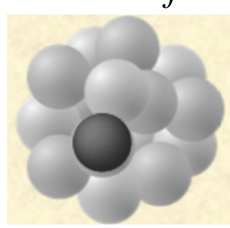
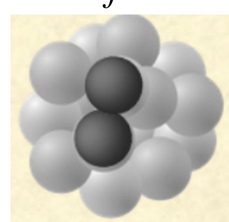
 +  +  + $\cdots$

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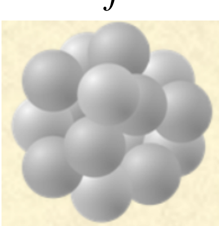
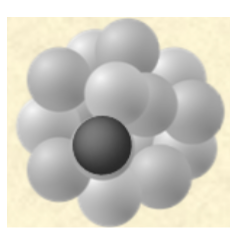
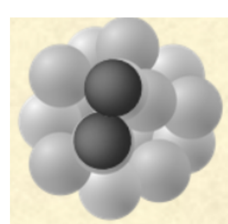
 +  +  + $\cdots$

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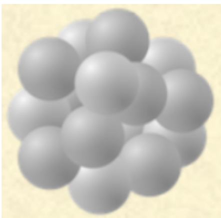

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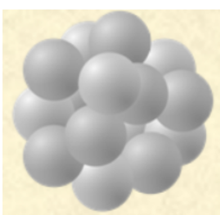
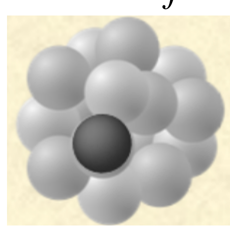
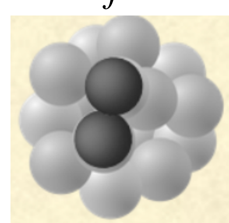
 +  +  + $\cdots$

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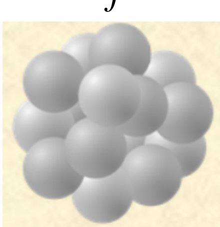
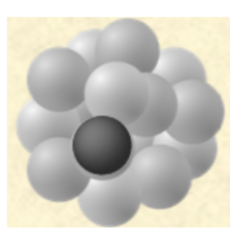
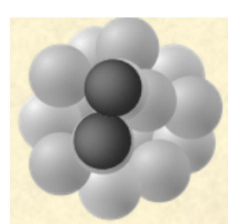
 +  +  + $\cdots$

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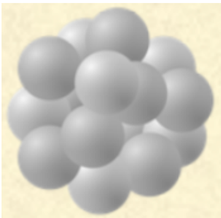
$$0 = \ell(r_2) = \sum_{\hat{r}_1} \ell(\hat{r}_1)\, \kappa(\hat{r}_1 \to r_2) = \sum_{\hat{r}_1} \left(\sum_r \ell(r)\, \kappa(r \to \hat{r}_1) \right) \kappa(\hat{r}_1 \to r_2)$$

for each irrep with $H_1 = 0,\ H_2 \neq 0$

Chiral symmetry and PMC equations



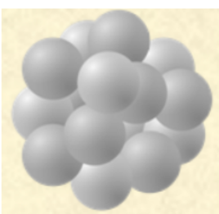
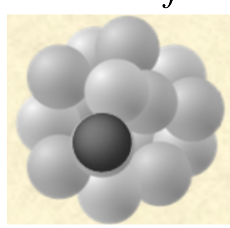
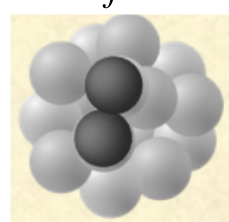
Massless, irrep of

$G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$

$\ell(r) \longleftarrow$ AMC

$m_1 > 0$

Irreps of $G[N_f, 1] = SU(N_f - 1)_L \times SU(N_f - 1)_R \times U(1)_{H_1} \times U(1)_B$

 +  +  + $\cdots$

$H_1 = 0$ $\quad$ $H_1 \neq 0$ (massive)

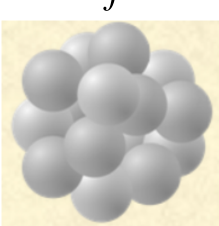
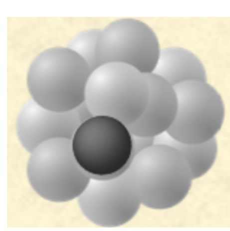
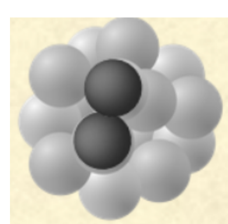
$$0 = \ell(r_1) = \sum_r \ell(r)\,\kappa(r \to r_1)$$

for each irrep with $H_1 \neq 0$

$m_2 \neq m_1 > 0$

Irreps of

$G[N_f, 2] = SU(N_f - 2)_L \times SU(N_f - 2)_R \times U(1)_{H_1} \times U(1)_{H_2} \times U(1)_B$

 +  +  + $\cdots$

$H_2 = 0$ $\quad$ $H_2 \neq 0$ (massive)

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for each irrep with $H_1 = 0,\ H_2 \neq 0$

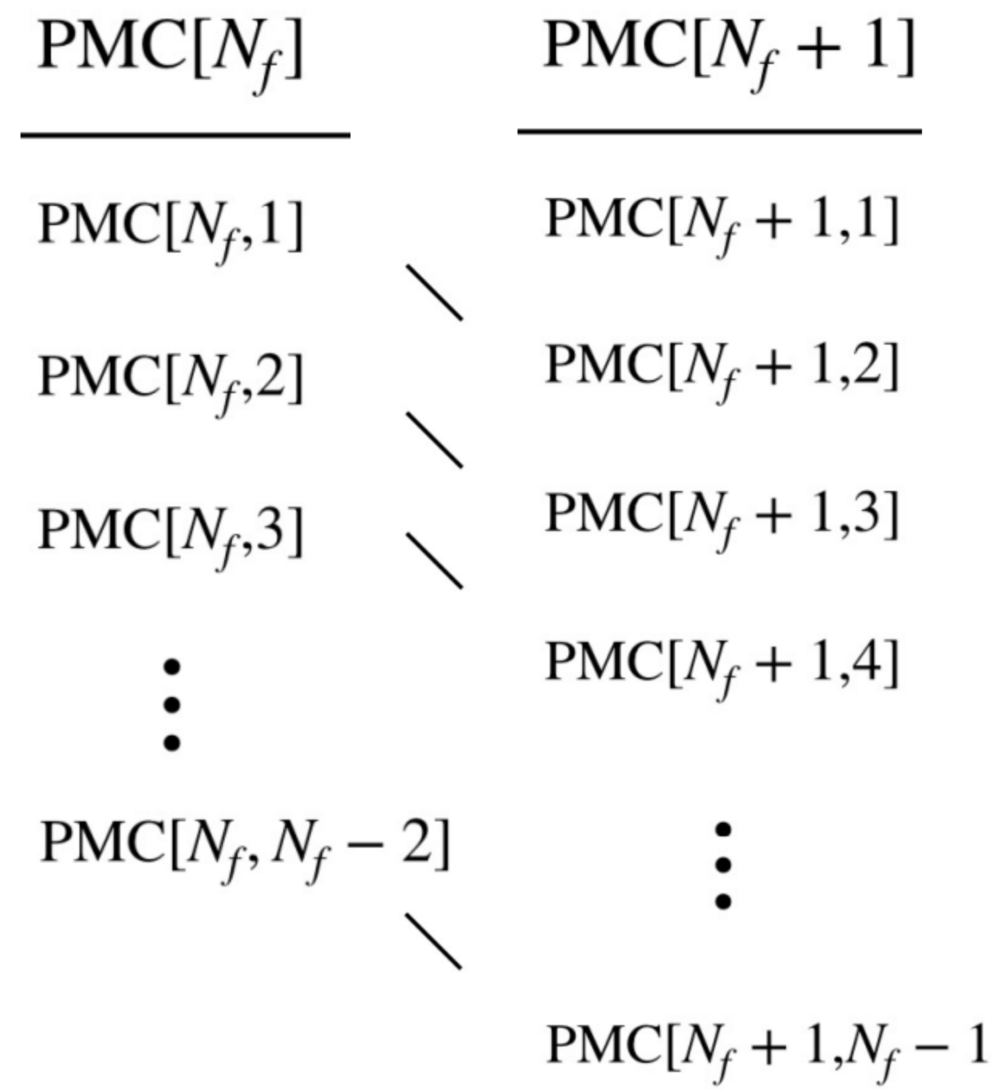
An observation on PMC equations

$$\begin{array}{c} \mathrm{PMC}[N_f] \\ \hline \mathrm{PMC}[N_f,1] \\ \mathrm{PMC}[N_f,2] \\ \mathrm{PMC}[N_f,3] \\ \vdots \\ \mathrm{PMC}[N_f, N_f - 2] \end{array}$$

An observation on PMC equations

$\mathrm{PMC}[N_f]$	$\mathrm{PMC}[N_f + 1]$
$\mathrm{PMC}[N_f, 1]$	$\mathrm{PMC}[N_f + 1, 1]$
$\mathrm{PMC}[N_f, 2]$	$\mathrm{PMC}[N_f + 1, 2]$
$\mathrm{PMC}[N_f, 3]$	$\mathrm{PMC}[N_f + 1, 3]$
$\vdots$	$\mathrm{PMC}[N_f + 1, 4]$
$\mathrm{PMC}[N_f, N_f - 2]$	$\vdots$
	$\mathrm{PMC}[N_f + 1, N_f - 1]$

An observation on PMC equations



The PMC equations connected by the diagonal lines can be identified,

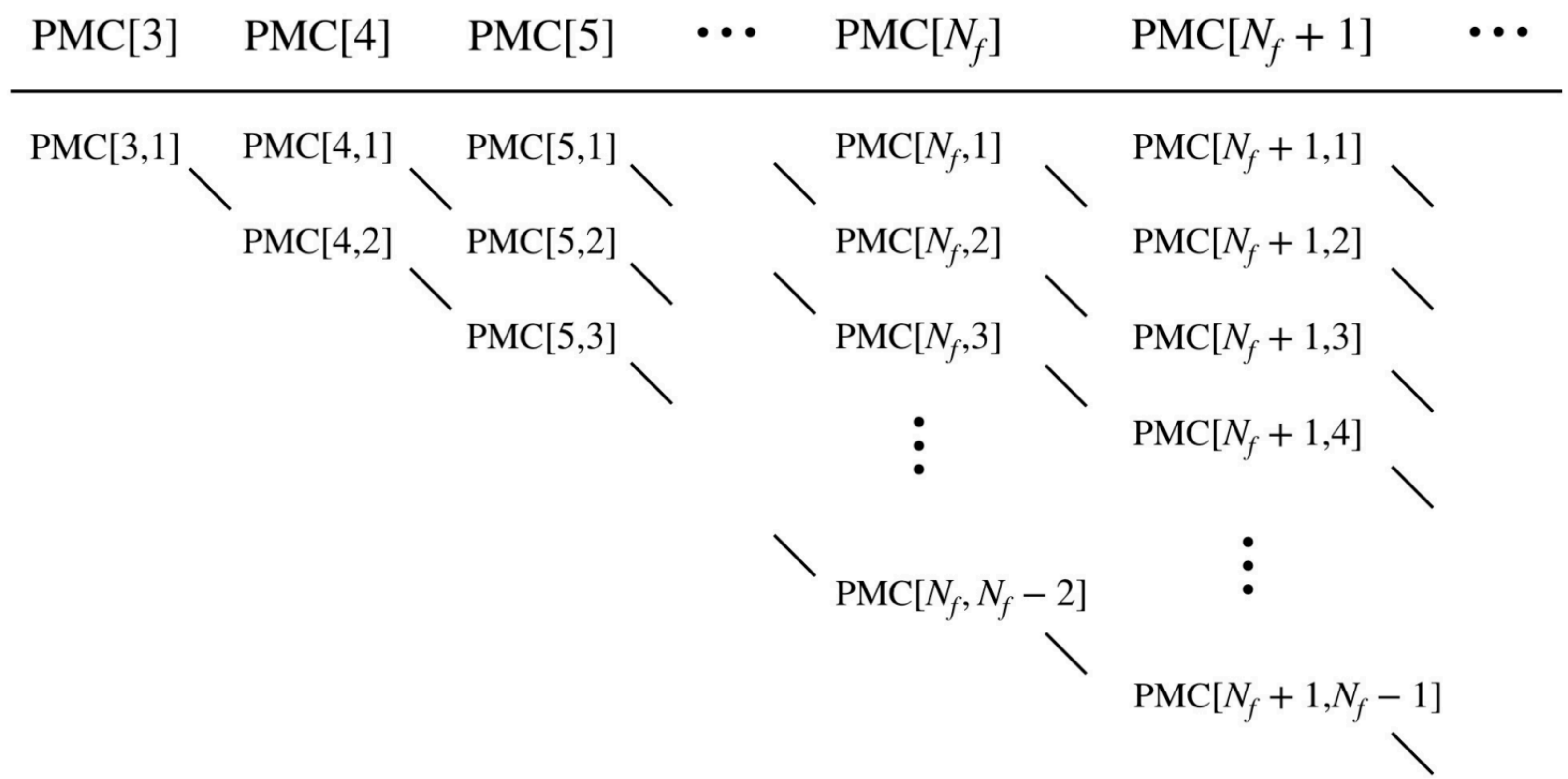
An observation on PMC equations

$\mathrm{PMC}[N_f]$	$\mathrm{PMC}[N_f+1]$
$\mathrm{PMC}[N_f,1]$	$\mathrm{PMC}[N_f+1,1]$
$\mathrm{PMC}[N_f,2]$	$\mathrm{PMC}[N_f+1,2]$
$\mathrm{PMC}[N_f,3]$	$\mathrm{PMC}[N_f+1,3]$
$\vdots$	$\mathrm{PMC}[N_f+1,4]$
$\mathrm{PMC}[N_f,N_f-2]$	$\vdots$
	$\mathrm{PMC}[N_f+1,N_f-1]$

The PMC equations connected by the diagonal lines can be identified, since each irrep of $G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ can be identified with that of $G[N_f+1,1] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_{H_1}$ with zero $U(1)_{H_1}$ charge, vice versa.

The coherent view of PMC

- Coherent structure of PMC for theories with different N_f while N_c is fixed:

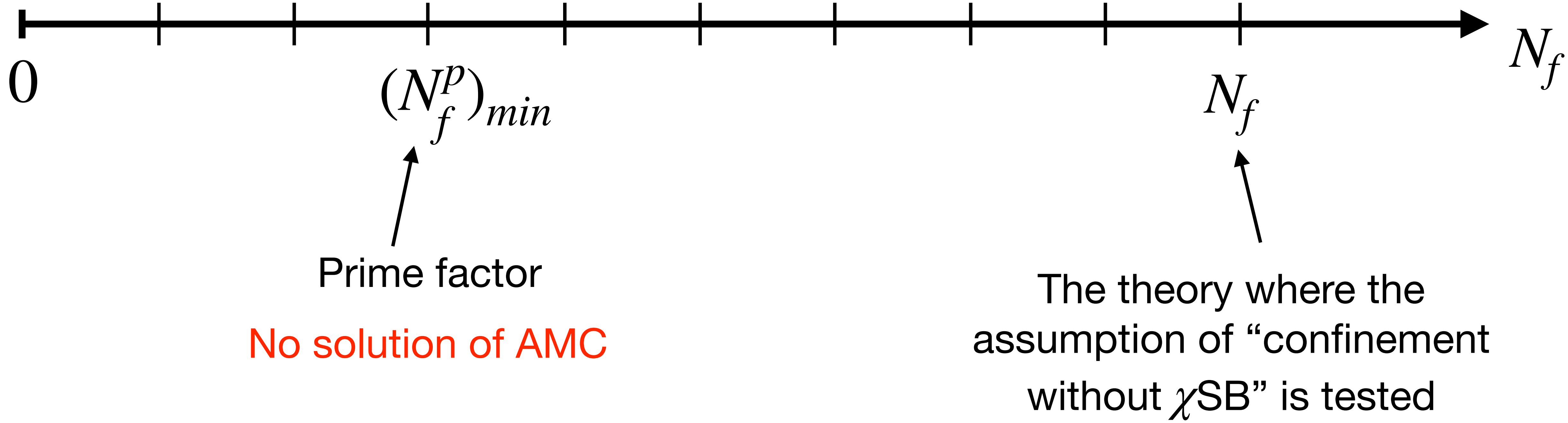


- Summarizing in one line, we have the identifications
$\mathrm{PMC}[N_f, i] \sim \mathrm{PMC}[N_f - 1,\ i - 1]$

Our proof

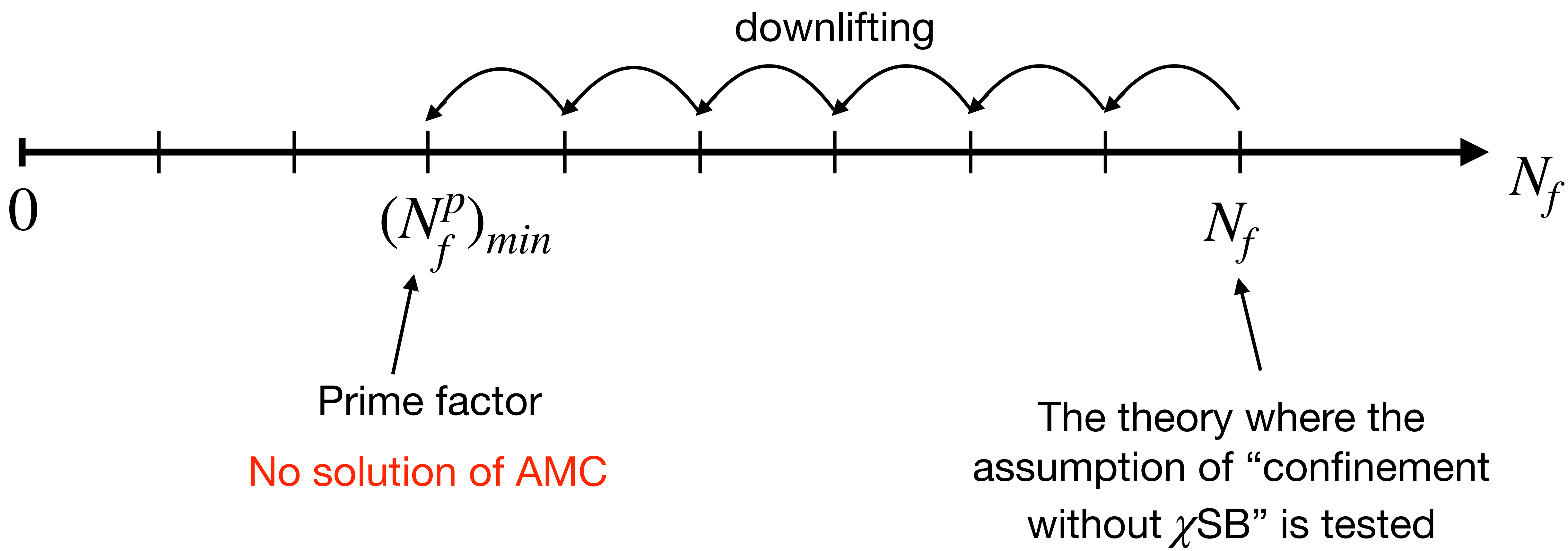
based on induction and contradiction

QCD with $N_c = 3$



- Suppose the assumption is true for N_f, there exist integral solutions to AMC[N_f] & PMC[N_f].

QCD with $N_c = 3$



- Suppose the assumption is true for N_f, there exist integral solutions to AMC[N_f] & PMC[N_f].
- **Downlifting**: constructing integral solutions of AMC[N_f - 1] & PMC[N_f - 1] from those of AMC[N_f] & PMC[N_f]. Eventually, we find integral solutions for AMC[3] & PMC[3].

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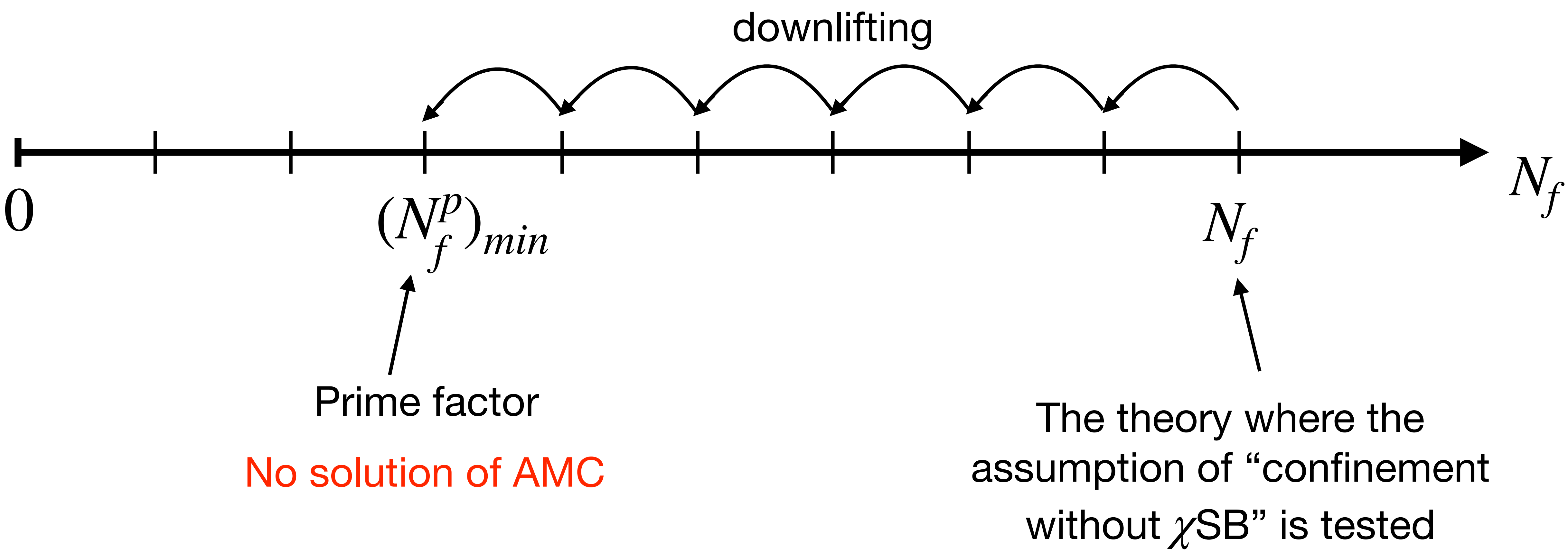

- Suppose the assumption is true for N_f, there exist integral solutions to AMC[N_f] & PMC[N_f].
- **Downlifting**: constructing integral solutions of AMC[N_f - 1] & PMC[N_f - 1] from those of AMC[N_f] & PMC[N_f]. Eventually, we find integral solutions for AMC[3] & PMC[3].
- However, there isn't any integral solution for AMC[3]. Contradiction!

Downlifting

Let $\{\ell(r)\}$ *be a solution of* $AMC[N_f] \cup PMC[N_f]$*; then* $\{\tilde{\ell}(r')\}$ *is a solution of* $AMC[N_f - 1] \cup PMC[N_f - 1]$ *for*

$$\tilde{\ell}(r') \equiv \sum_{r \in \mathcal{R}[N_f]} \ell(r)\, k\,(r \to r') \quad \forall\, r' \in \mathcal{R}[N_f - 1]\,.$$

- Let us start with $\{\ell(r)\}$, which solves AMC[N_f] & PMC[N_f].
- Giving mass to one flavor, decomposing the irreps $r \in G[N_f]$ to $r' \in G[N_f, 1]$.

$$\text{where} \quad G[N_f, 1] = SU(N_f - 1)_L \times SU(N_f - 1)_R \times U(1)_B \times U(1)_{H_1}$$
$$G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$$

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- Giving mass to one flavor, decomposing the irreps $r \in G[N_f]$ to $r' \in G[N_f,1]$. The index of each r' is calculable from that of r:

$$\ell(r') \equiv \sum_{r\in\mathcal{R}[N_f]} \ell(r)\ k\,(r \to r')$$

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$$G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$$

- For r' with $H_1 = 0$:

  — Their indices solve PMC[N_f, i] with $2 \leq i \leq N_f - 2$ by decomposition step by step. (Notice their indices are not set to zero by PMC[N_f,1].)

- Let us start with $\{\ell(r)\}$, which solves AMC[N_f] & PMC[N_f].
- Giving mass to one flavor, decomposing the irreps $r \in G[N_f]$ to $r' \in G[N_f, 1]$. The index of each r' is calculable from that of r:

$$\ell(r') \equiv \sum_{r \in \mathcal{R}[N_f]} \ell(r)\, k\,(r \to r')$$

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$$G[N_f, 1] = SU(N_f - 1)_L \times SU(N_f - 1)_R \times U(1)_B \times U(1)_{H_1}$$
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  - Their indices solve PMC[N_f, i] with $2 \le i \le N_f - 2$ by decomposition step by step. (Notice their indices are not set to zero by PMC[N_f,1].)
  - With PMC[$N_f - 1,\ i - 1$] $\sim$ PMC[N_f, i], their indices solve PMC[$N_f - 1,\ i - 1$] where $2 \le i \le N_f - 2$, which are all the PMC equations for the theory of $N_f - 1$ flavors.

Hence, we have shown that the downlifted indices given by the ansatz successfully solves PMC[$N_f - 1$].

Next, we show the same ansatz also solves AMC[$N_f - 1$].

- Let us evaluate anomaly coefficients of $SU(N_f)_{L,R}$ on the $SU(N_f-1)_{L,R}$ Lie subalgebra. Following the rule of decomposition, we have

$$A(r) = \sum_{\text{All } r'} k(r \to r')\ A(r')$$

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- Plugging this equation into AMC[N_f] and switching the order of sums, we have

$$\begin{aligned} A_{\text{UV}} &= \sum_{r \in \mathcal{R}[N_f]} \ell(r) \left(\sum_{\text{All } r'} k(r \to r')\ A(r') \right) \\ &= \sum_{\text{All } r'} \left(\sum_{r \in \mathcal{R}[N_f]} \ell(r)\ k(r \to r') \right) A(r')\ . \end{aligned}$$

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- PMC[N_f,1] imply the sum in the parenthesis in the second line vanishes unless for r' with zero $U(1)_{H_1}$ charge; therefore

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- This equation can be viewed as AMC[N_f-1], where the indices are given by the ansatz.

Summary and outlook

- Pros: Our proof crucially relies on the coherent structure of PMC for QCD theories of different N_f while N_c is fixed, which however does not involve further assumptions on the putative hadron spectrum.

- Cons: We cannot prove for N_f smaller than the smallest prime factor of N_c. For instance, $N_c = 3,\ N_f = 2$.

- Question: What we studied may be viewed as the “zero-form” QCD, which only involves local data relevant for particles. How solid our result is given the possible impact of nonlocal data of higher symmetries? Which QCD??

Thank you! Comments welcome!