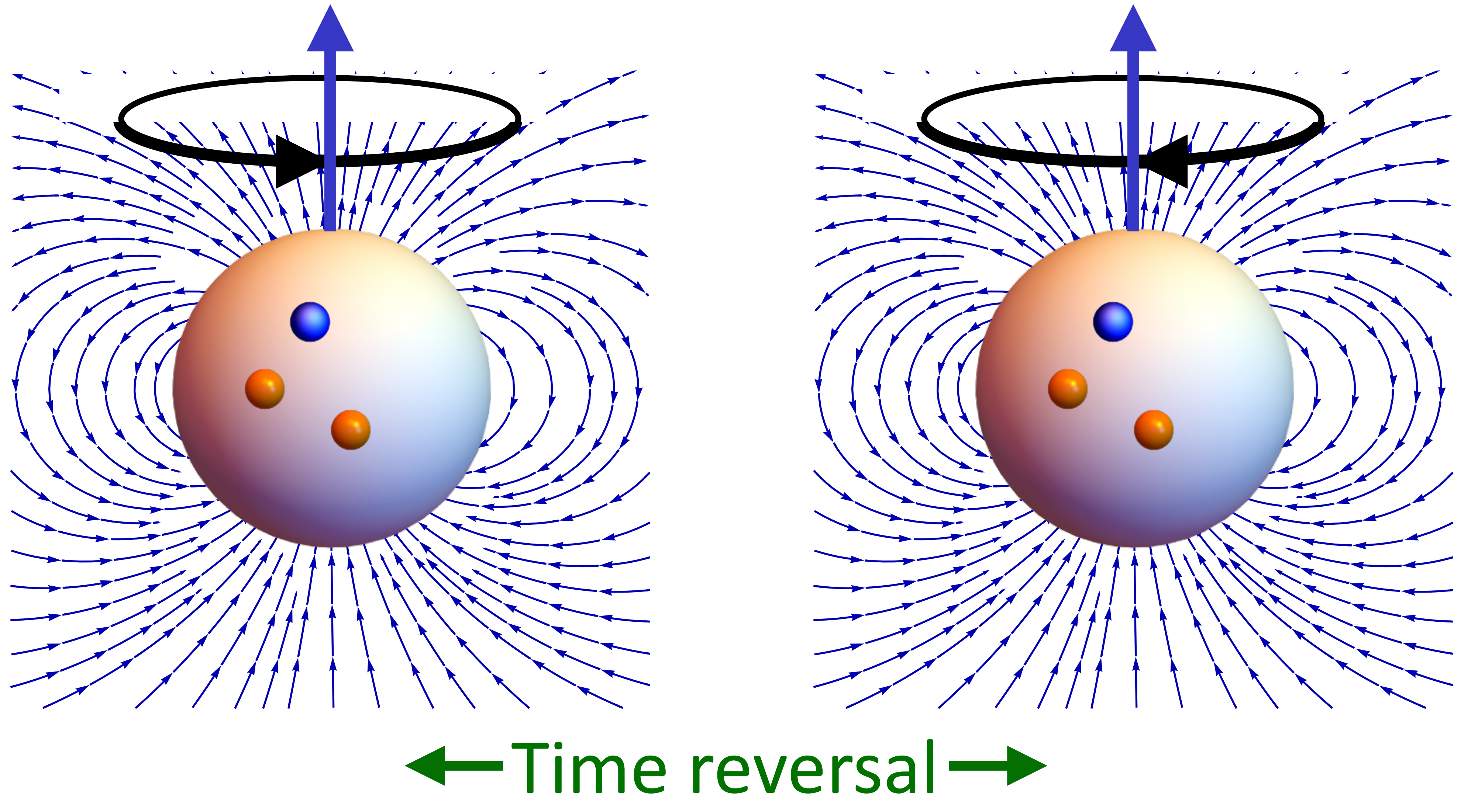
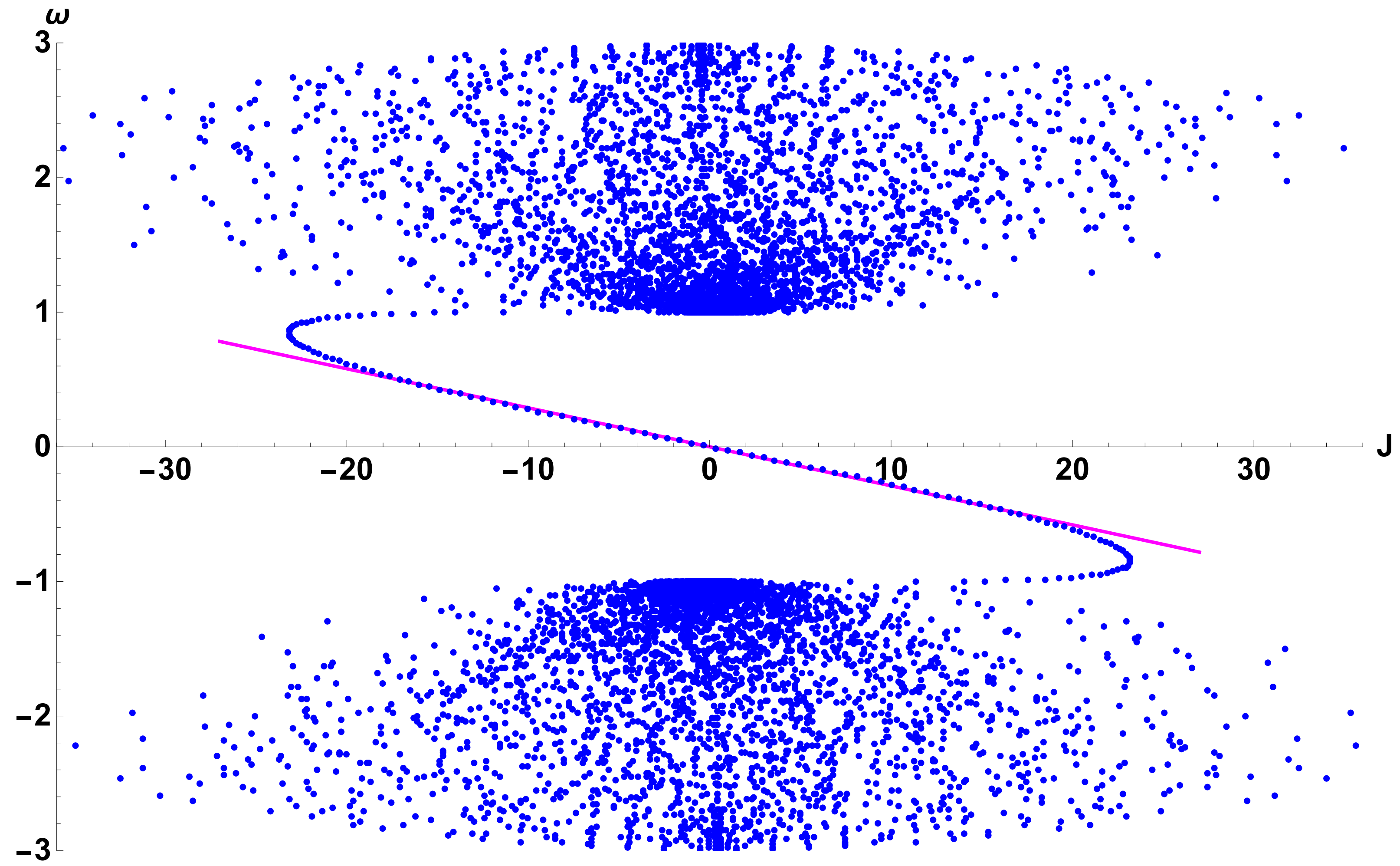
Regulated Chiral Gauge Theory and the Strong CP problem





DB Kaplan: Phys. Rev. Lett. 132 (2024) 141603, arXiv:2312.01494

DB Kaplan, S. Sen: Phys. Rev. Lett. 132 (2024) 141604, arXiv:2312.04012

DB Kaplan, S. Sen: arXiv:2412.02024 (recently revised)

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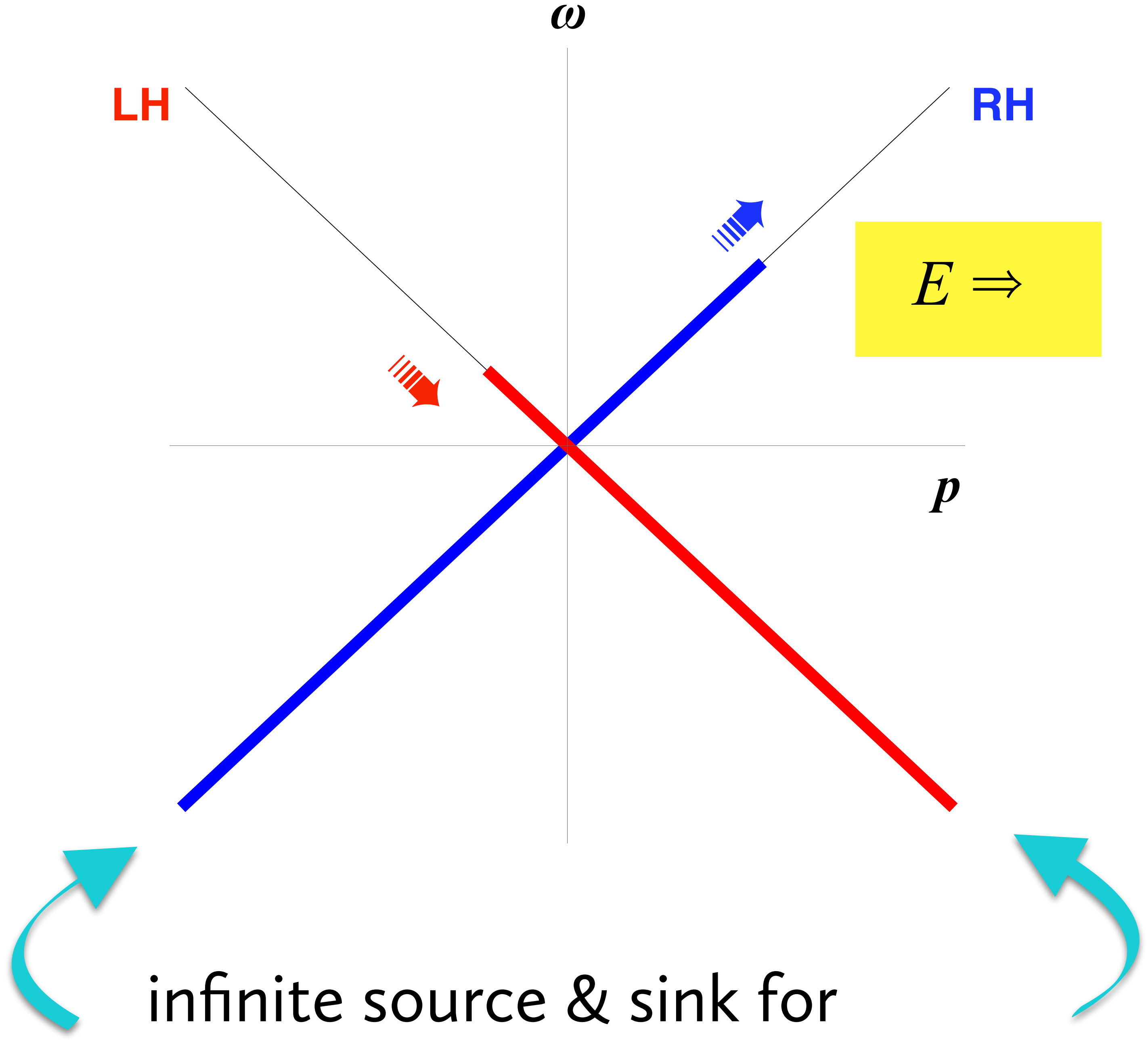
Punchline: stand-alone QCD has a strong CP problem; SM QCD might not.

Anomalies on the lattice



infinite source & sink for fermions

- Heuristic picture for anomalies relies on a “Hilbert Hotel”…not an option for a lattice
- Chiral symmetry must be broken on the lattice?…but in a way that permits chiral gauge theory?

Nielsen-Ninomiya theorem

consider Euclidian fermion action on a lattice:

$$S = \int_{\mathrm{BZ}} \frac{d^d p}{(2\pi)^d} \bar{\Psi}(-p)\widetilde{D}(p)\Psi(p)$$

wanted: massless Dirac fermion with chiral symmetry

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ; ☞ locality
2. $D(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$; ☞ correct continuum limit
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$; ☞ no doublers
4. $\{\Gamma, \tilde{D}(\mathbf{p})\} = 0$. ☞ exact chiral symmetry $(\Gamma = \gamma_5)$

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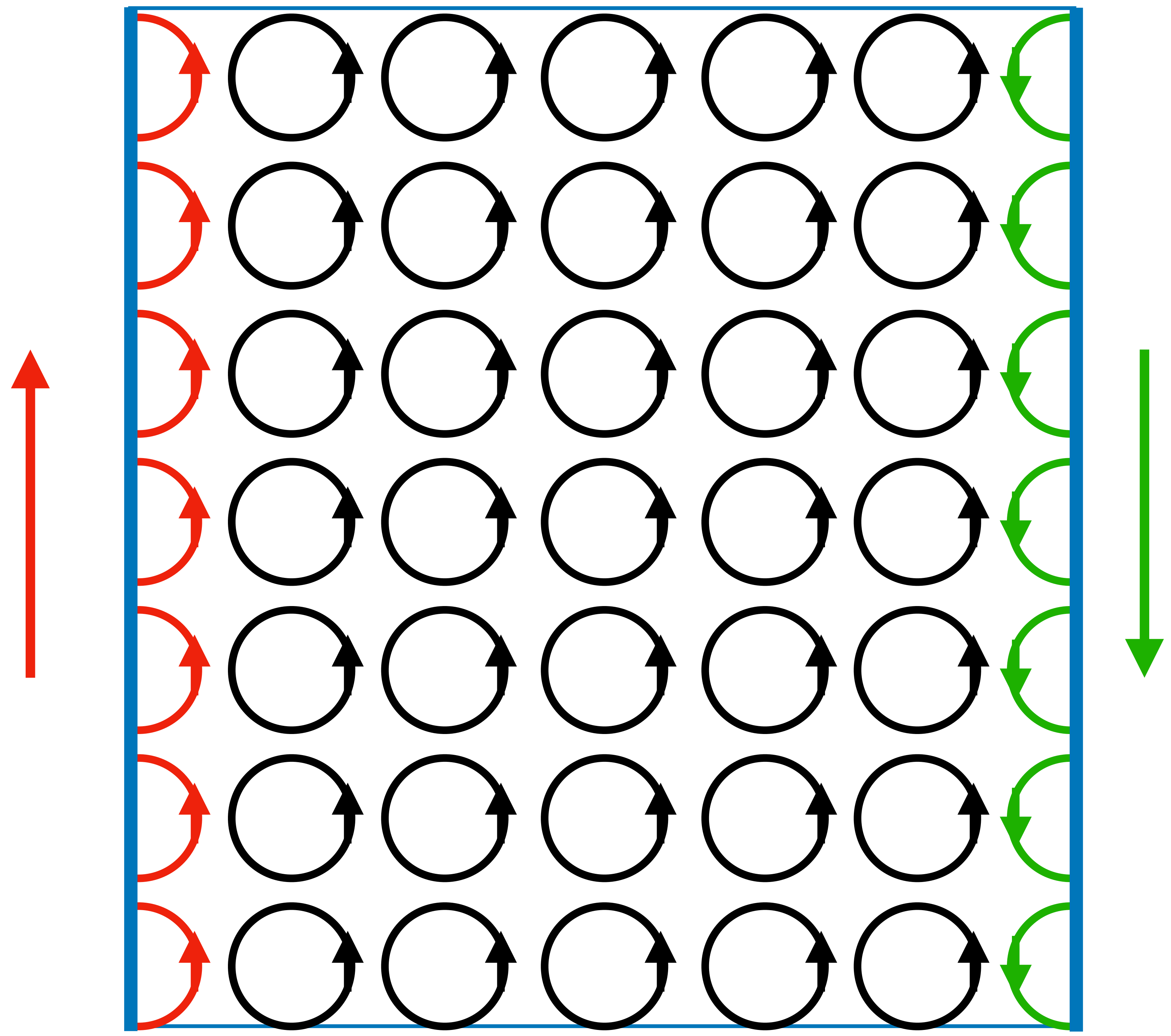
Where else do chiral fermions appear in nature?

Chiral edge states appear naturally in the Integer Quantum Hall Effect:



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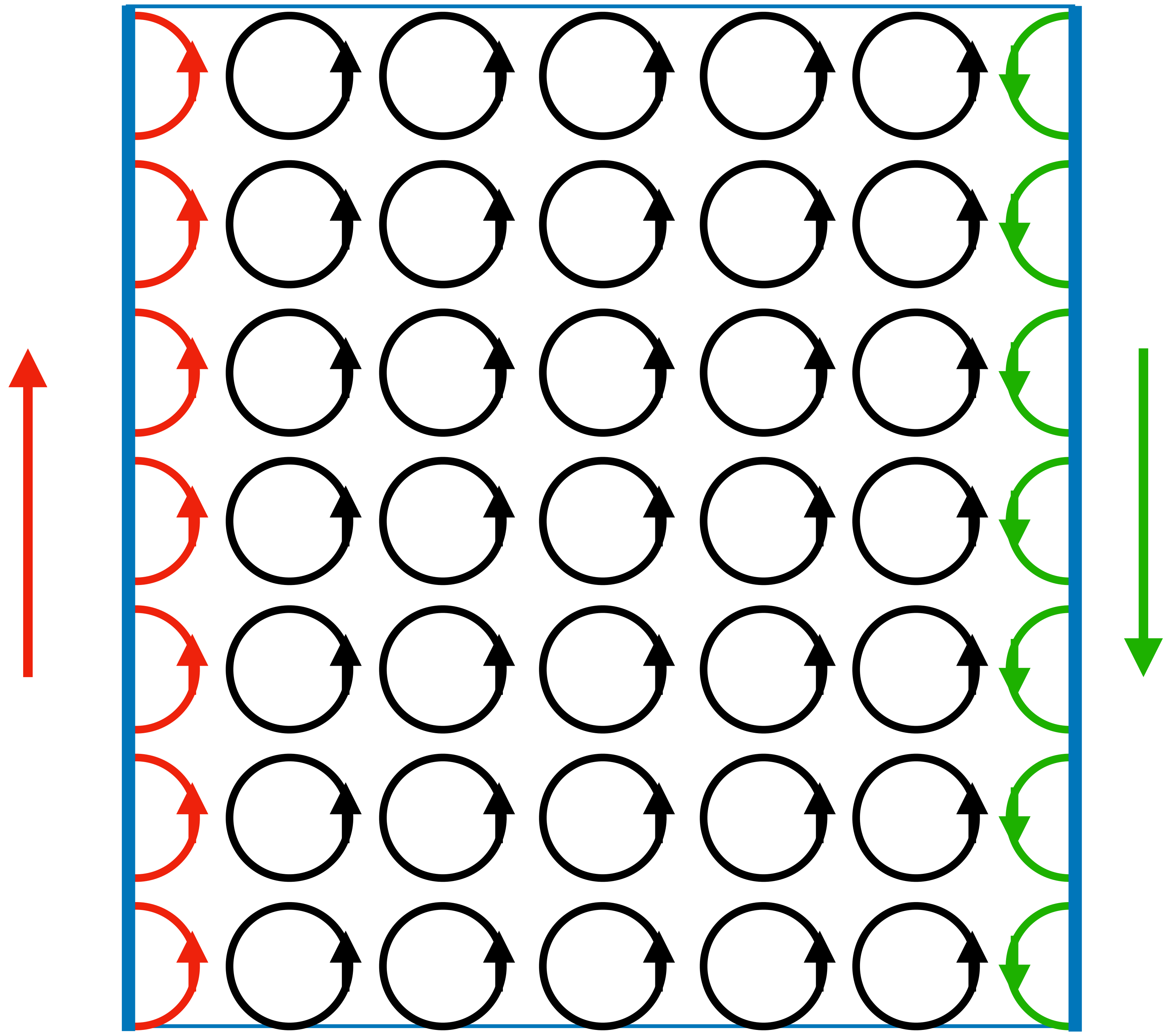

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Analog for Dirac fermions with domain wall mass

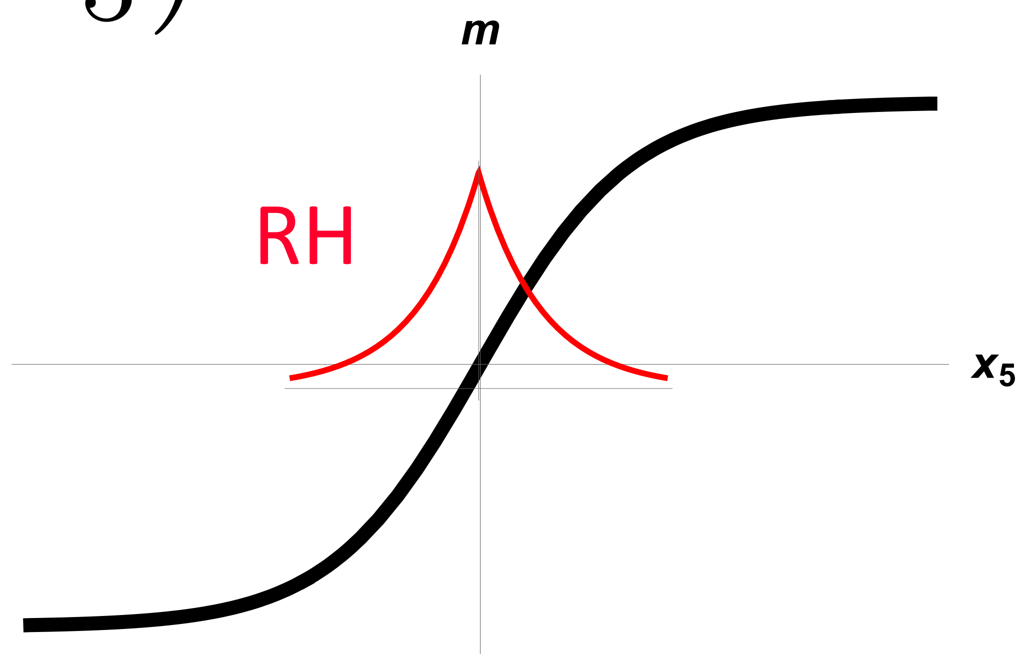
[Jackiw & Rebbi]:

$$\left[\partial\!\!\!/ + \gamma_5\partial_5 + m(x_5)\right]\Psi = 0$$

Has solutions: $\Psi = \phi_\pm(x_5)\chi_\pm$

$$\gamma_5\chi_\pm = \pm\chi_\pm$$

$$\phi_\pm(x_5) = e^{\mp\int^{x_5} m(s)\,ds}$$



Chern-Simons current in the bulk accounts properly for the axial anomaly [Callan & Harvey]

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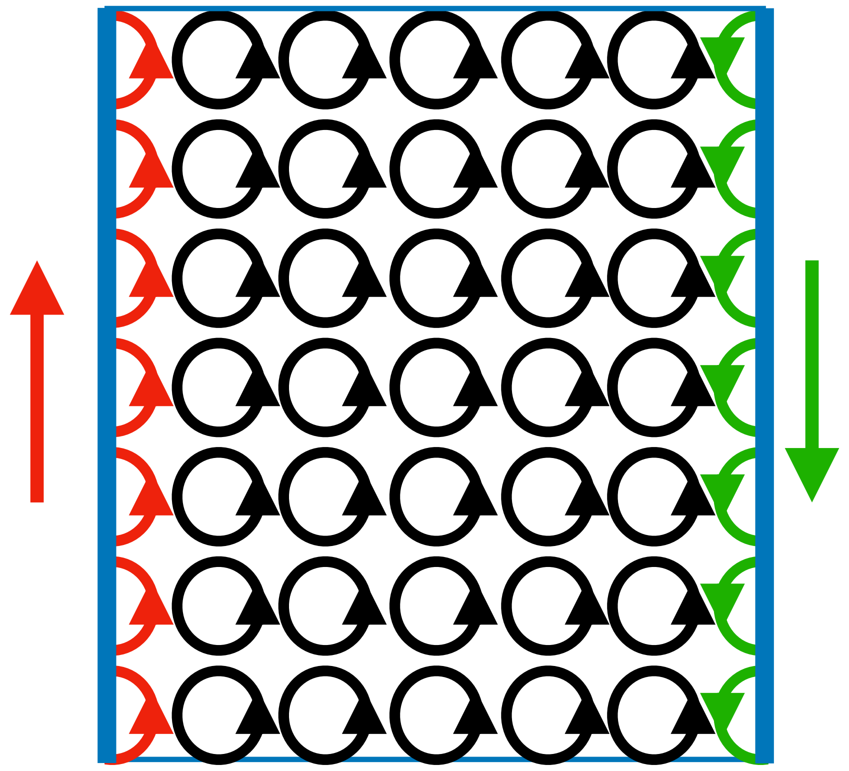

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The QFT analog is the coefficient of the Chern-Simons term obtained by integrating out the massive fermion in a background gauge field.

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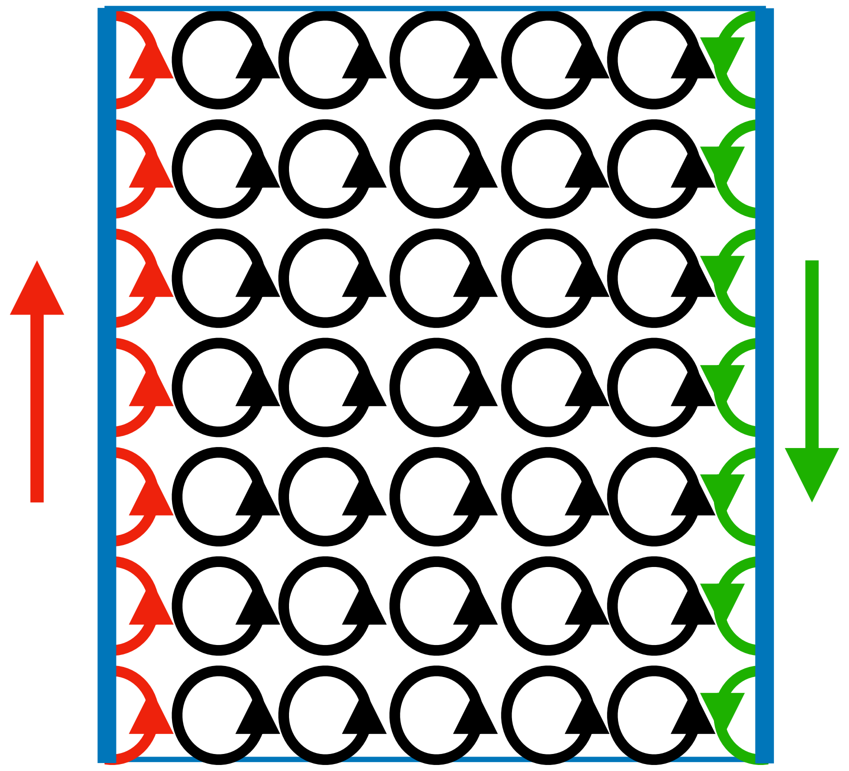

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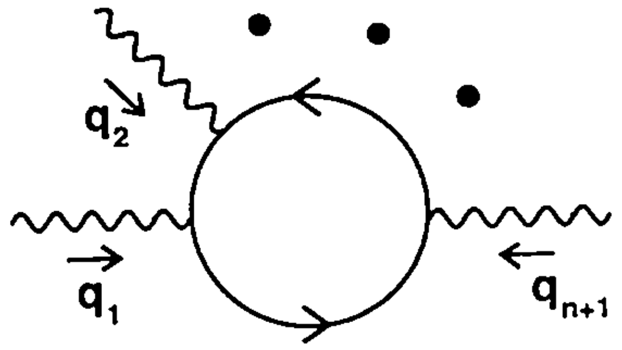
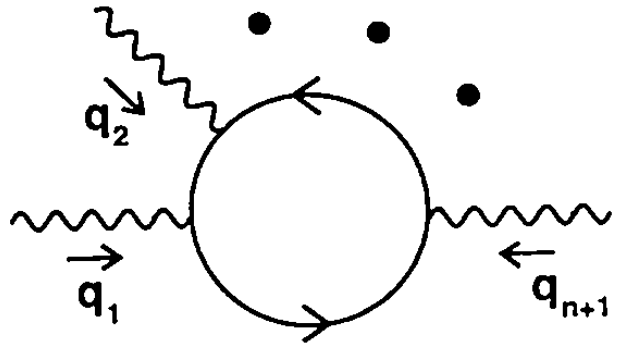
Using Ward identity, Chern-Simons coefficient in d= 2n+1 is proportional to



$$\epsilon_{\mu_1\ldots\mu_d}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\;S(p)\frac{\partial S^{-1}(p)}{\partial p_{\mu_1}}\cdots S(p)\frac{\partial S^{-1}(p)}{\partial p_{\mu_d}}$$

where S(p) is the fermion propagator. When the theory is regulated, this is a winding number for the map $S^{-1}(p)$ from S^d (momentum space) to $S^d = SO(d+1)/SO(d)$



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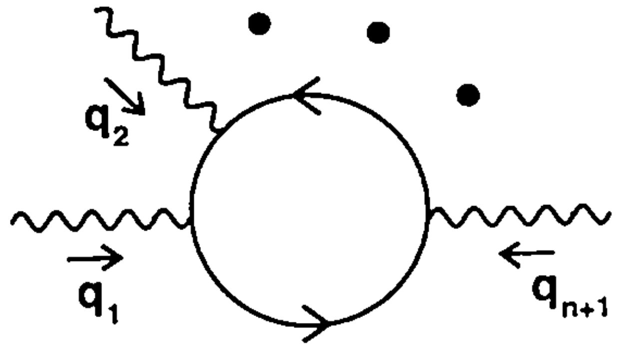
Remarkable fact:

Since the topology is in **momentum/spin space**, topological phases and massless edge states appear at domain wall boundary on an infinite spacetime **lattice**



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E.g. Wilson fermions (DBK 1992; K. Jansen, M. Schmaltz 1993; M. Golterman, K. Jansen, DBK, 1993):

$$\mathcal{D}=\gamma_\mu\partial+M+\frac{r}{2}\Delta$$

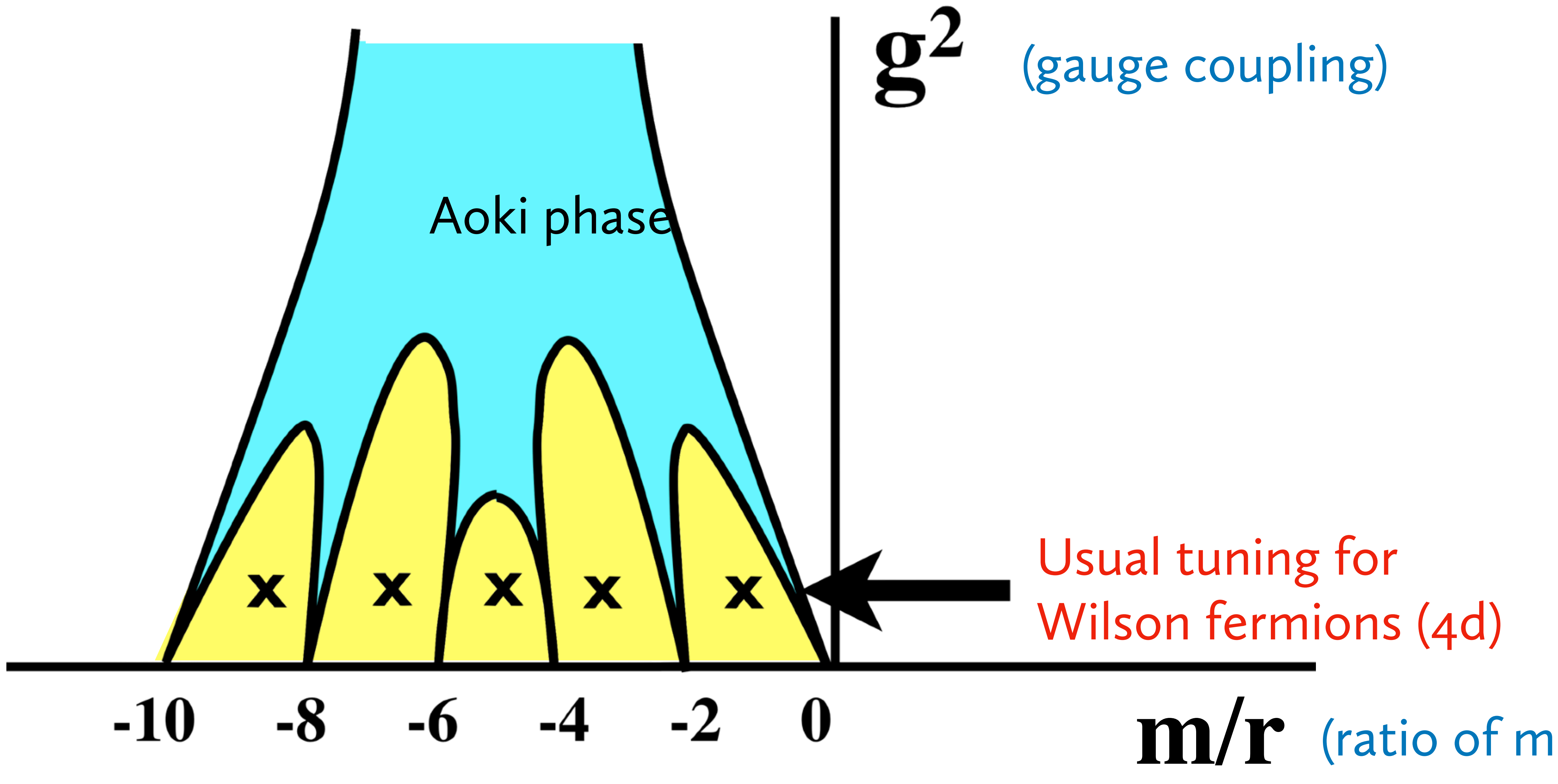
$$\tilde{\mathcal{D}}(p)=M+\sum_\mu\left[i\sin p_\mu\gamma_\mu+\frac{r}{2}(1-\cos p_\mu)\right]$$

$$\partial_\mu\psi(x)=\frac{\psi(x+a\hat{\mu})-\psi(x-a\hat{\mu})}{2a}\;,$$

$$\Delta\psi(x)=\frac{\psi(x+a\hat{\mu})-2\psi(x)+\psi(x-a\hat{\mu})}{a^2}$$

Nontrivial topological phases for $0<\dfrac{M}{r}<2d$ with phase boundaries at $\dfrac{M}{r}=0,2,\ldots,2d$

Phase diagram for QCD with Wilson fermions in 5d Euclidian spacetime



S Aoki, Prog Th Phys 122 (1996) 179

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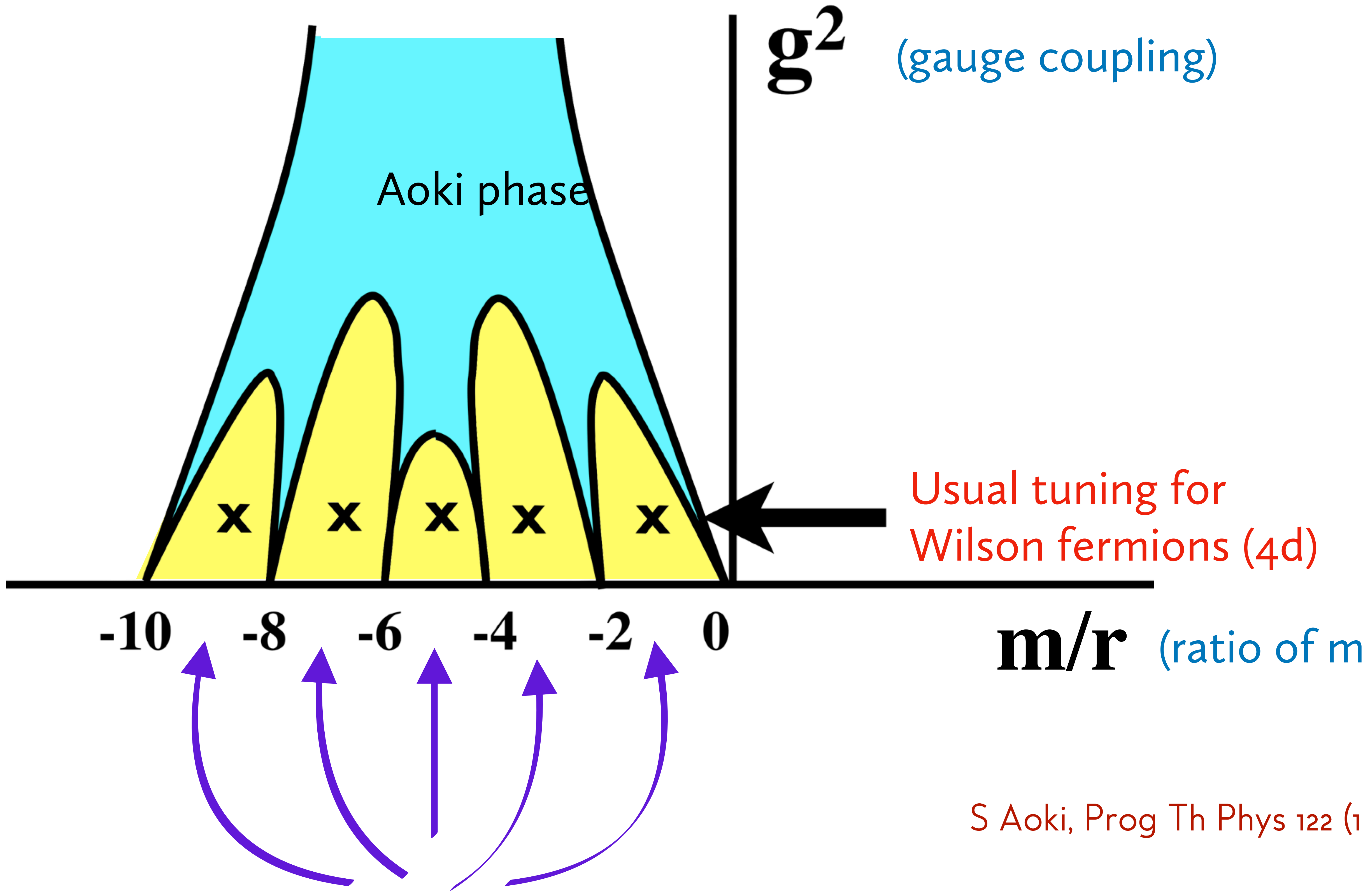

Topological phases —
where to sit for chiral DWFs

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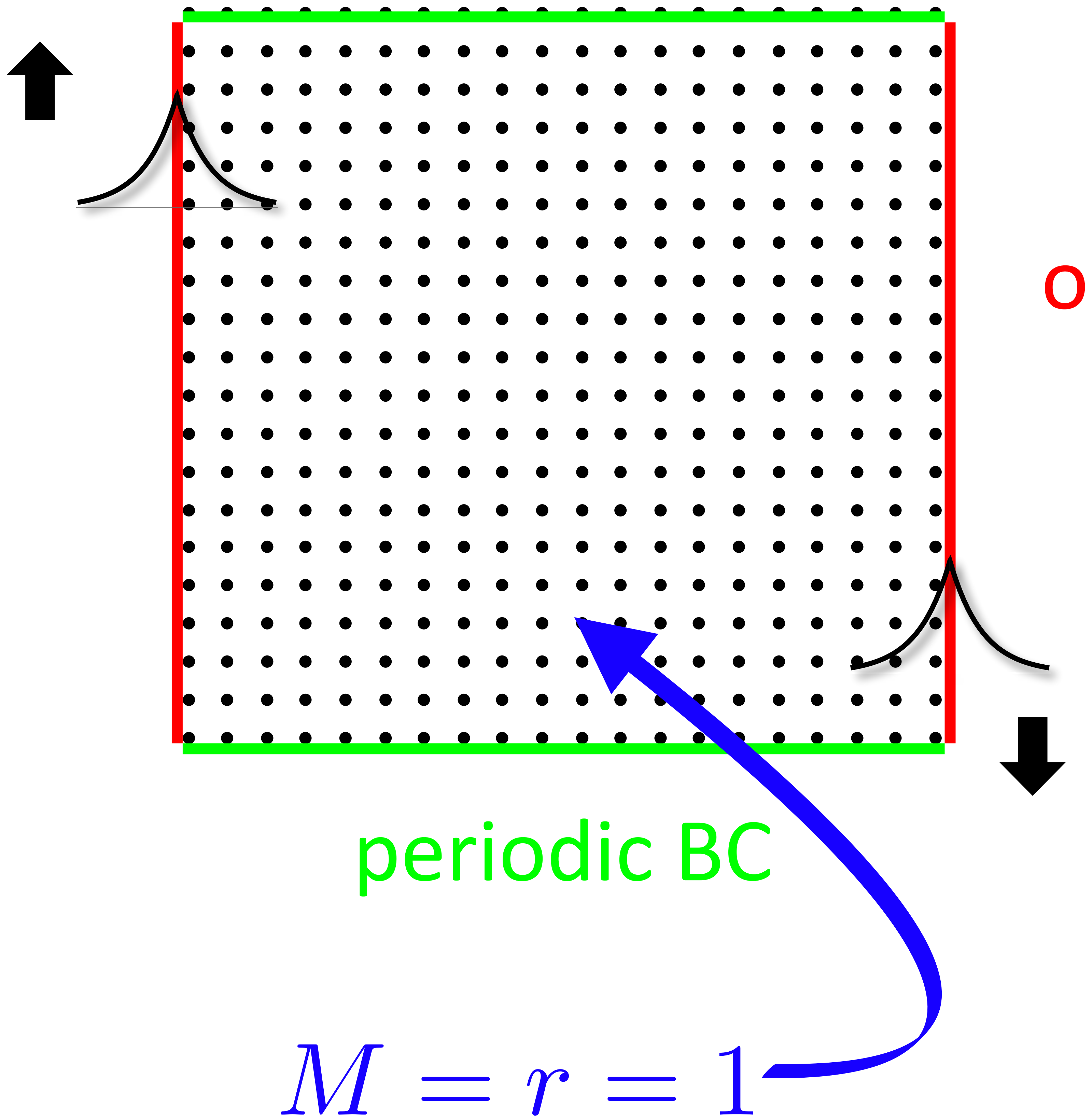


periodic BC

RH Weyl

open BC (ψ=0) (Y. Shamir, 1993)

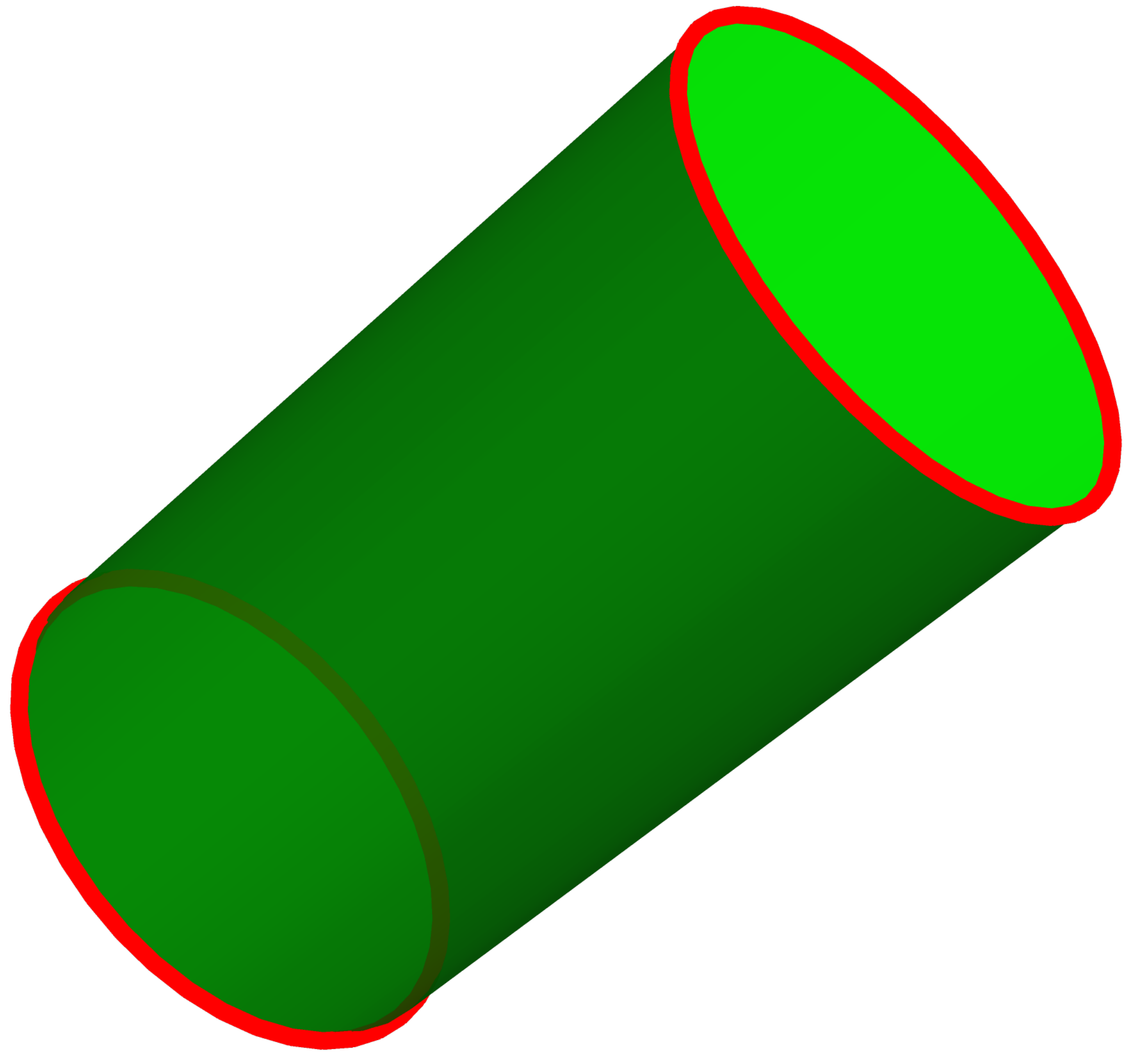
$M = -\infty$ $M = -\infty$

LH Weyl

periodic BC

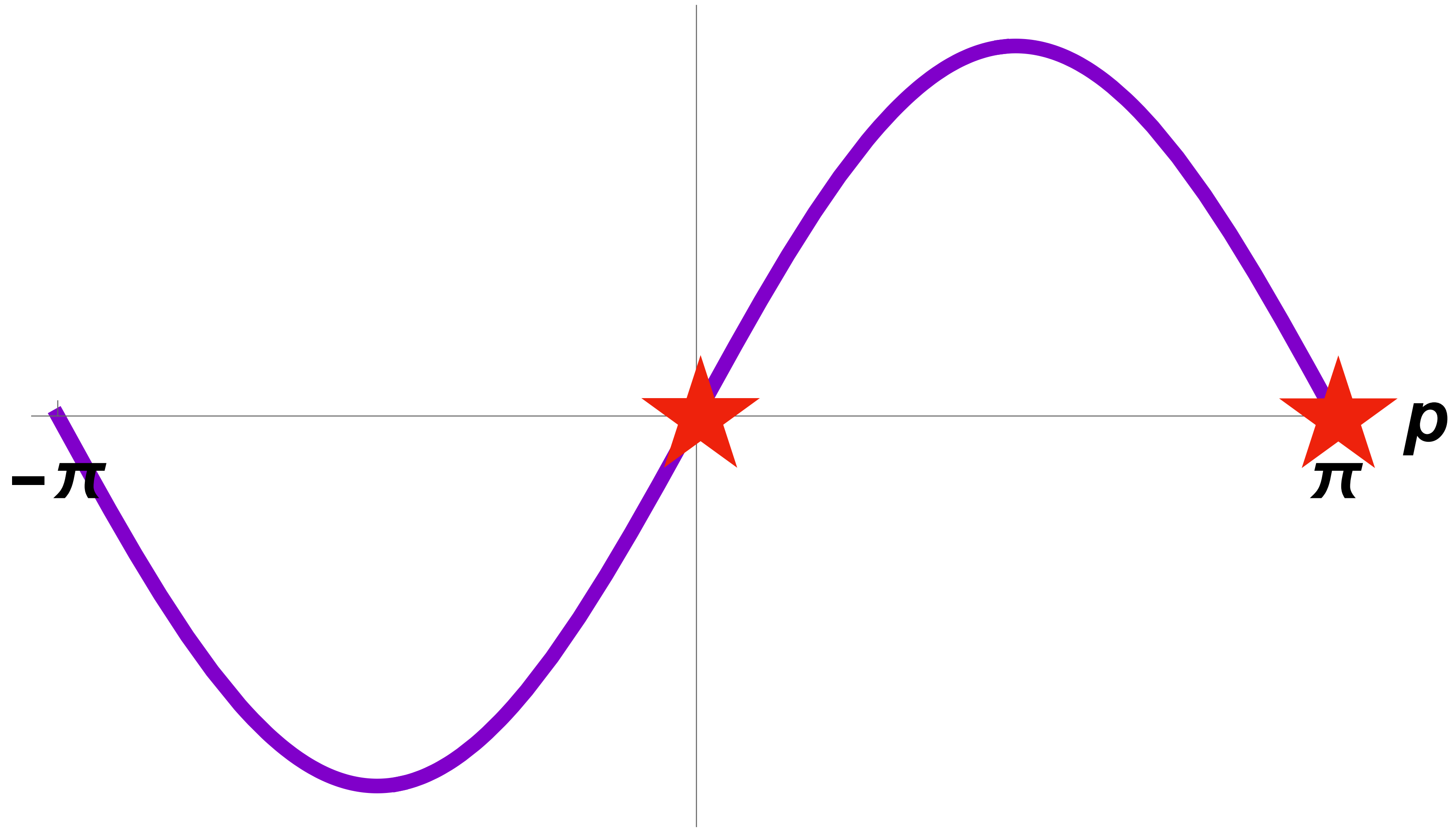
$M = r = 1$

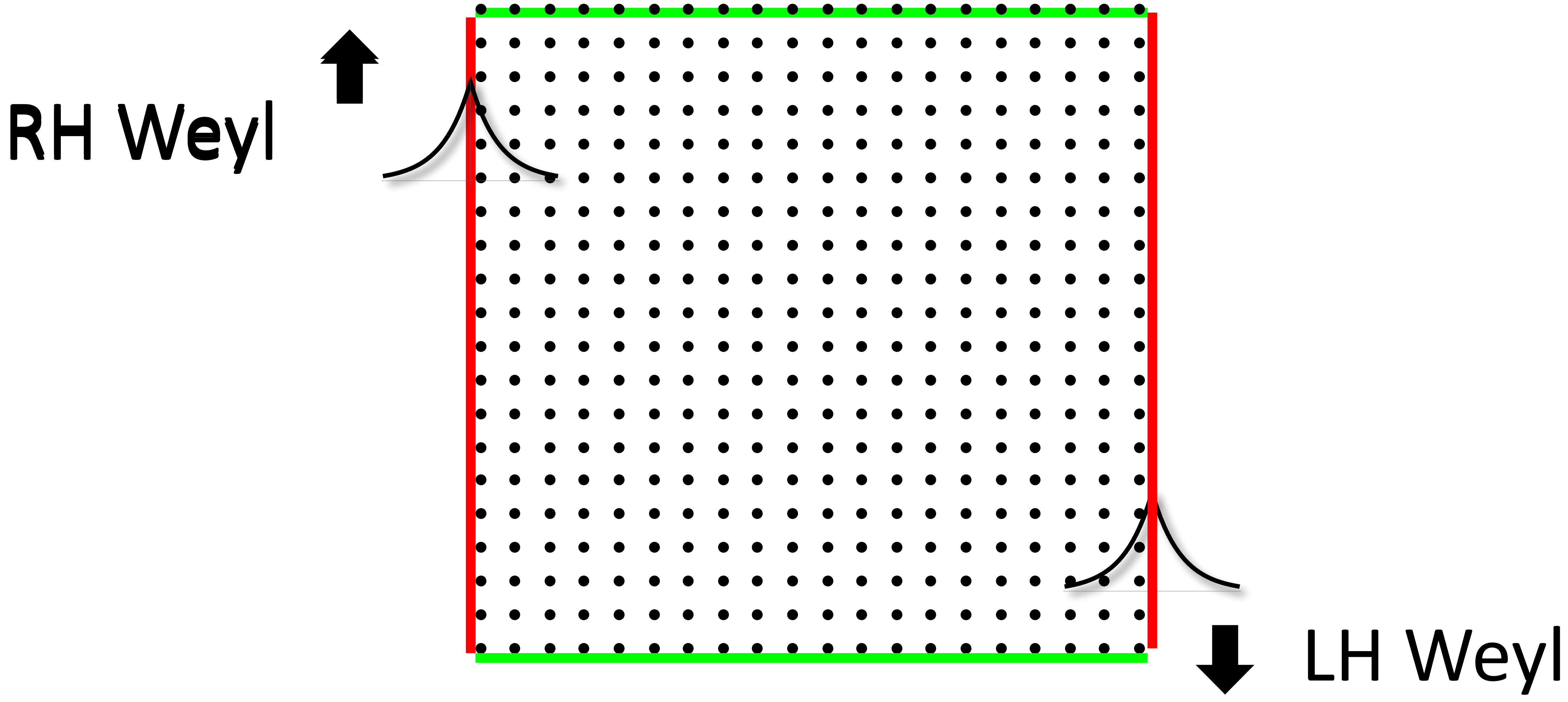
Obtain *almost* massless RH & LH Weyl fermions… mass $\propto e^{-2ML}$



Lattice has topology of an open cylinder with two boundaries

Won't there be doubled copies of fermions on each wall?





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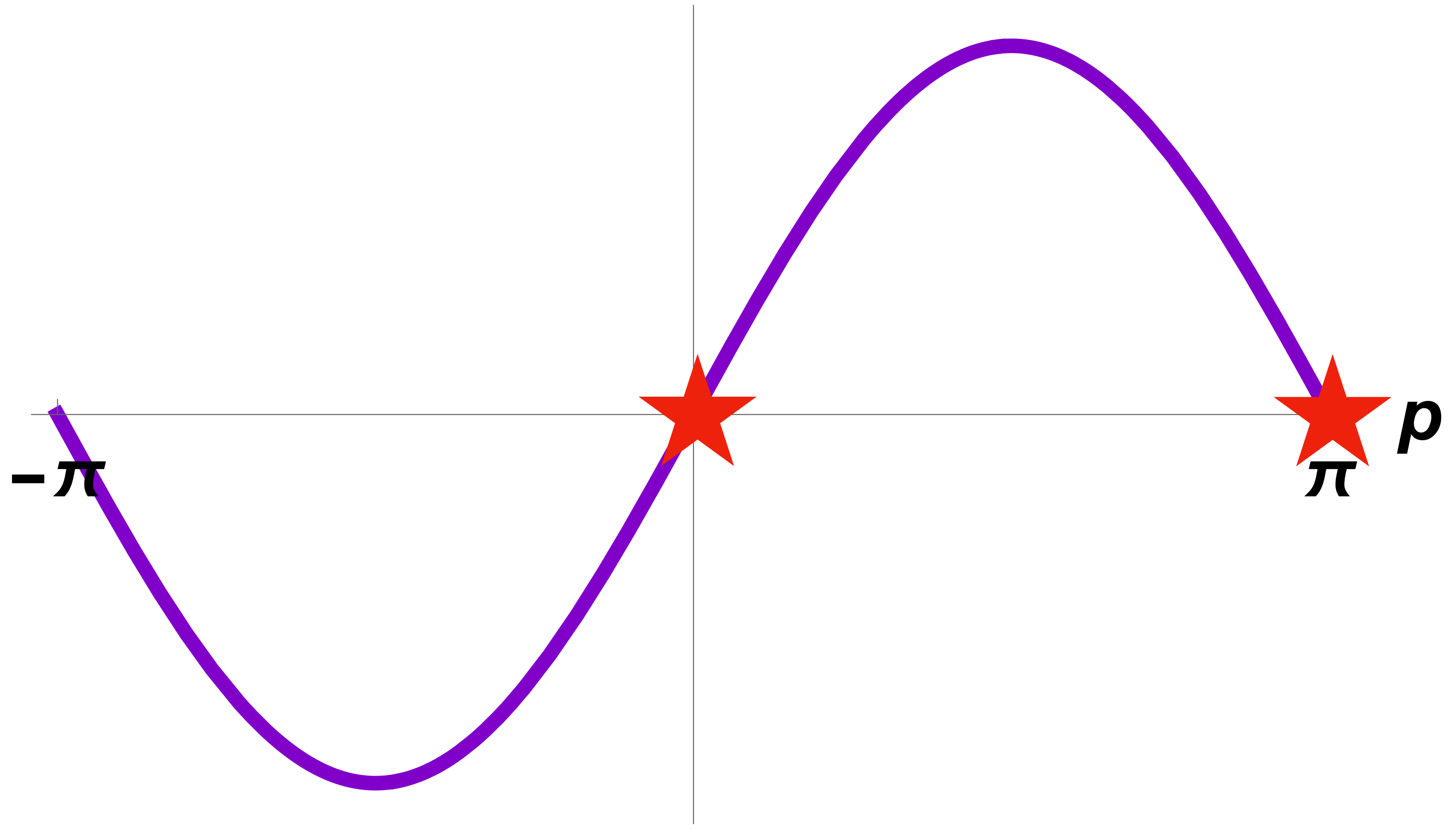

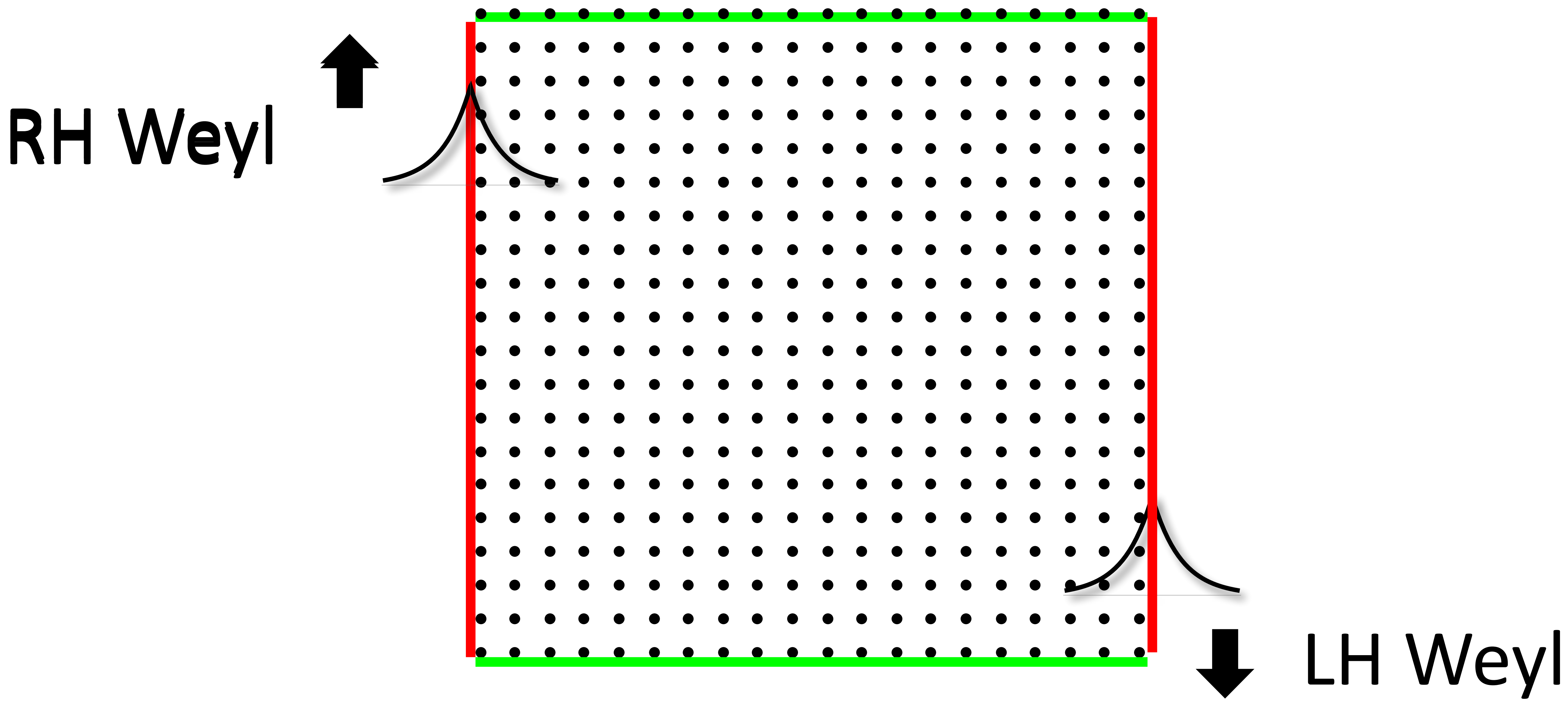

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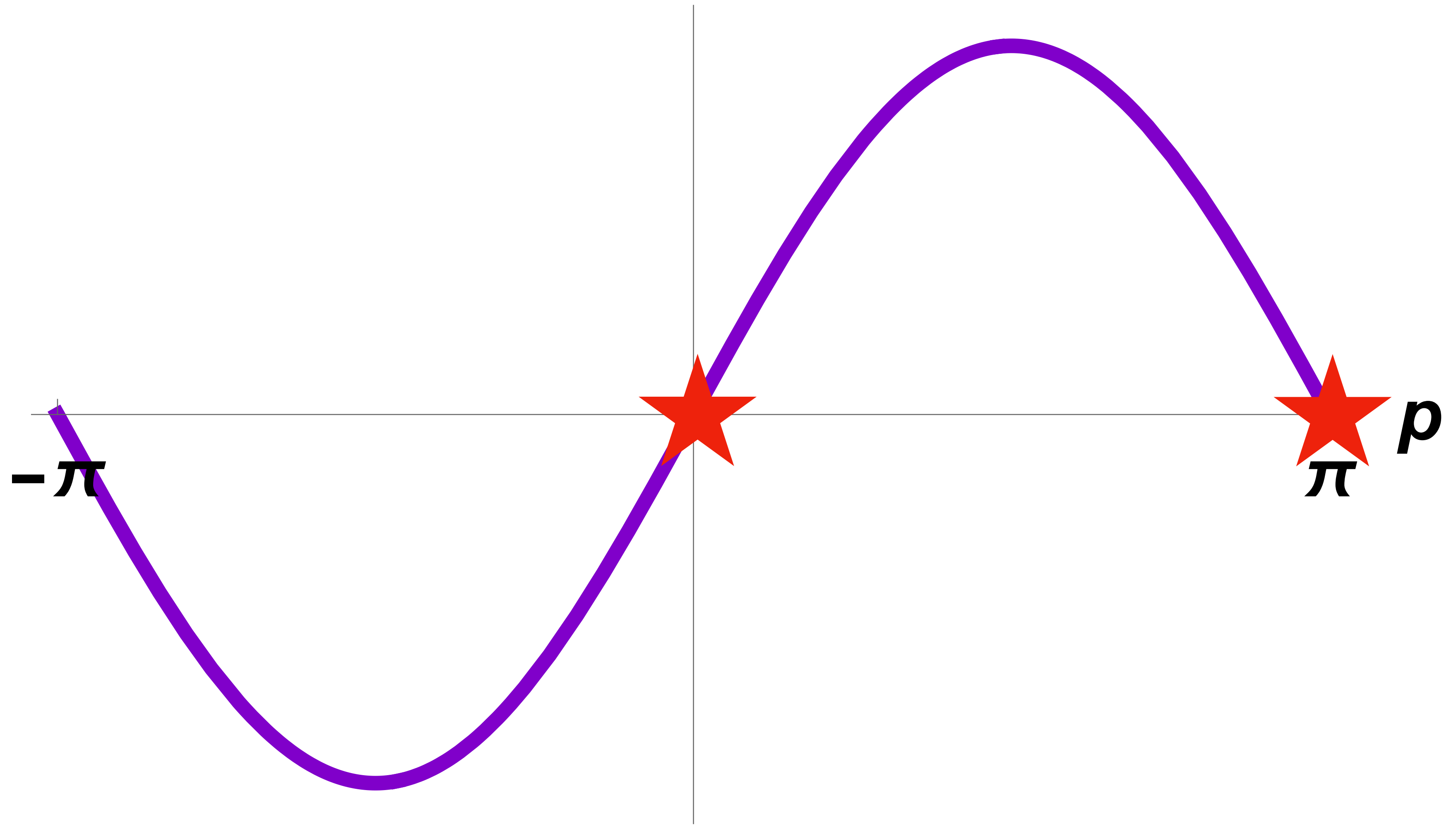

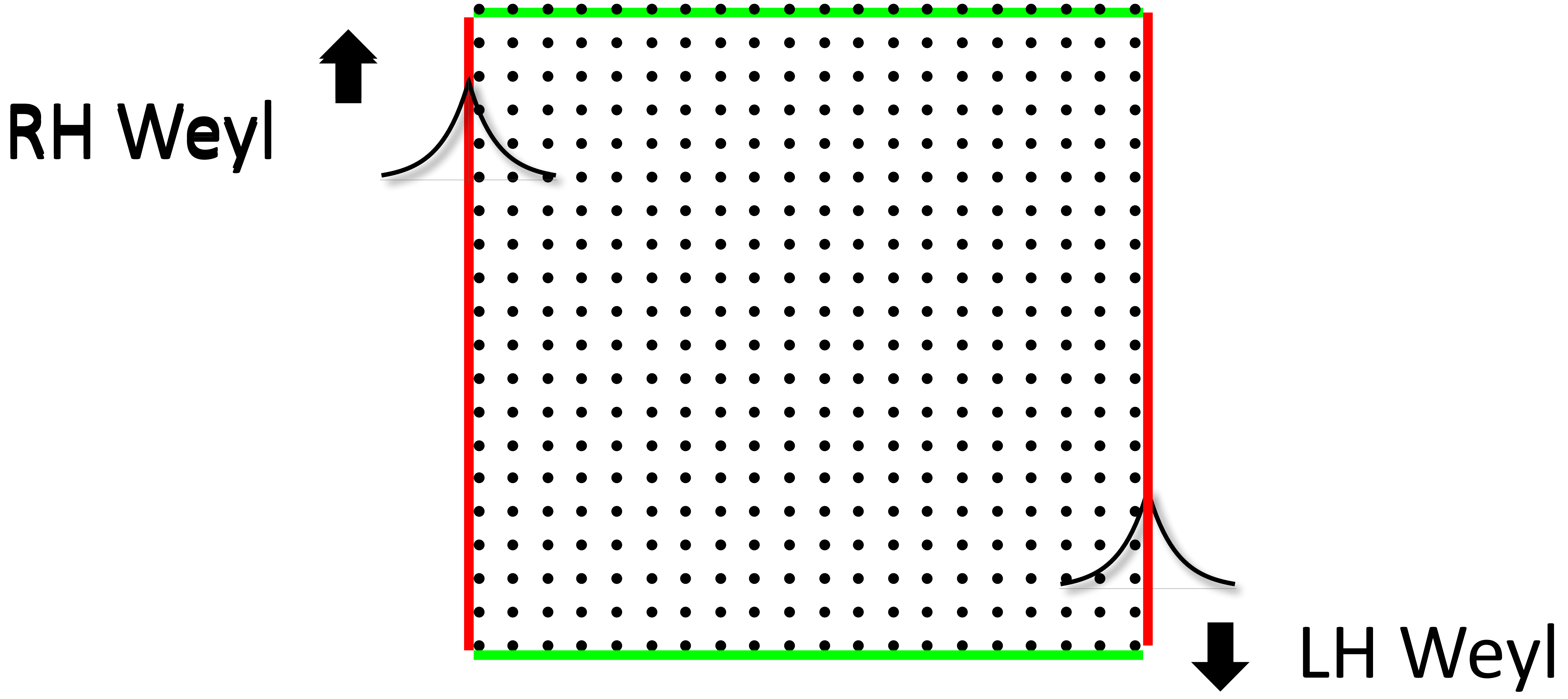

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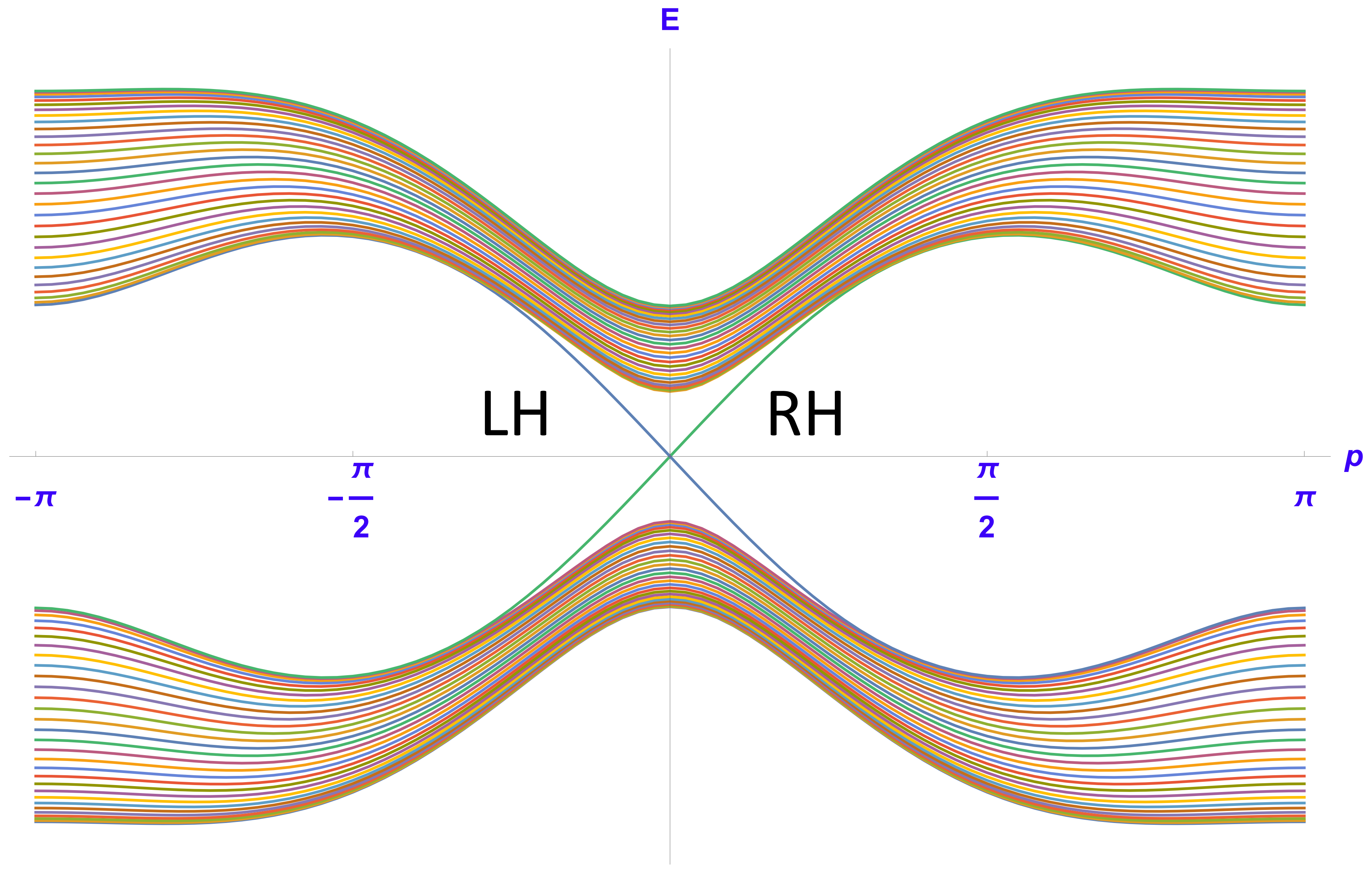

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Not a simple path from here to chiral gauge theory

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Consider instead edge states on manifold with a **single** boundary.

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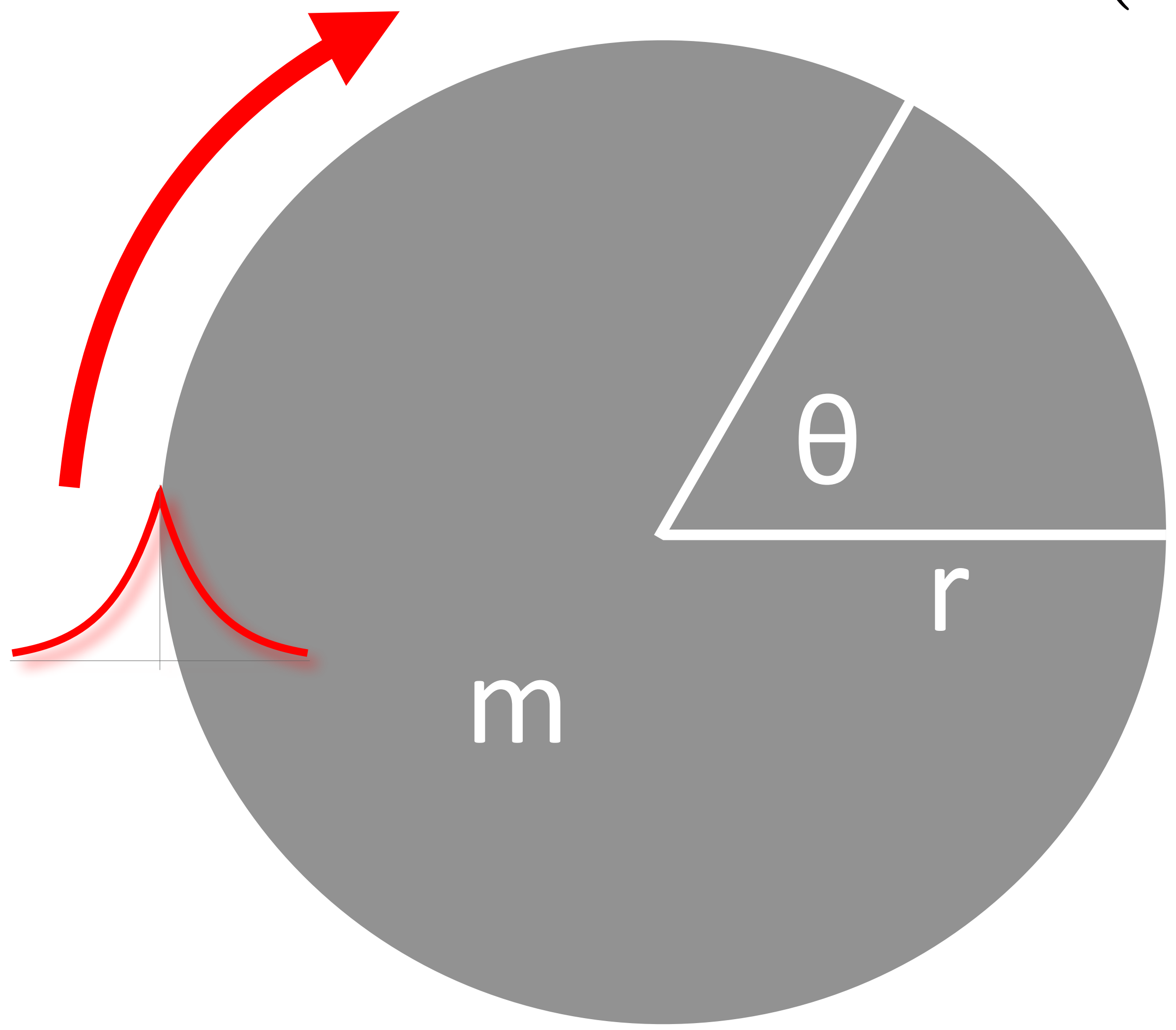
Dirac fermion in d+1 continuum dimensions:

$$m(r) = \begin{cases} m & r < R \\ -M & r > R \end{cases}$$

$$-M \to -\infty$$



Appearance of chiral fermions at topological phase boundaries is a **robust** phenomenon

A strip with two boundaries produced LH + RH chiral edge states.

Consider instead edge states on manifold with a **single** boundary.

Dirac fermion in d+1 continuum dimensions:

$$m(r) = \begin{cases} m & r < R \\ -M & r > R \end{cases}$$

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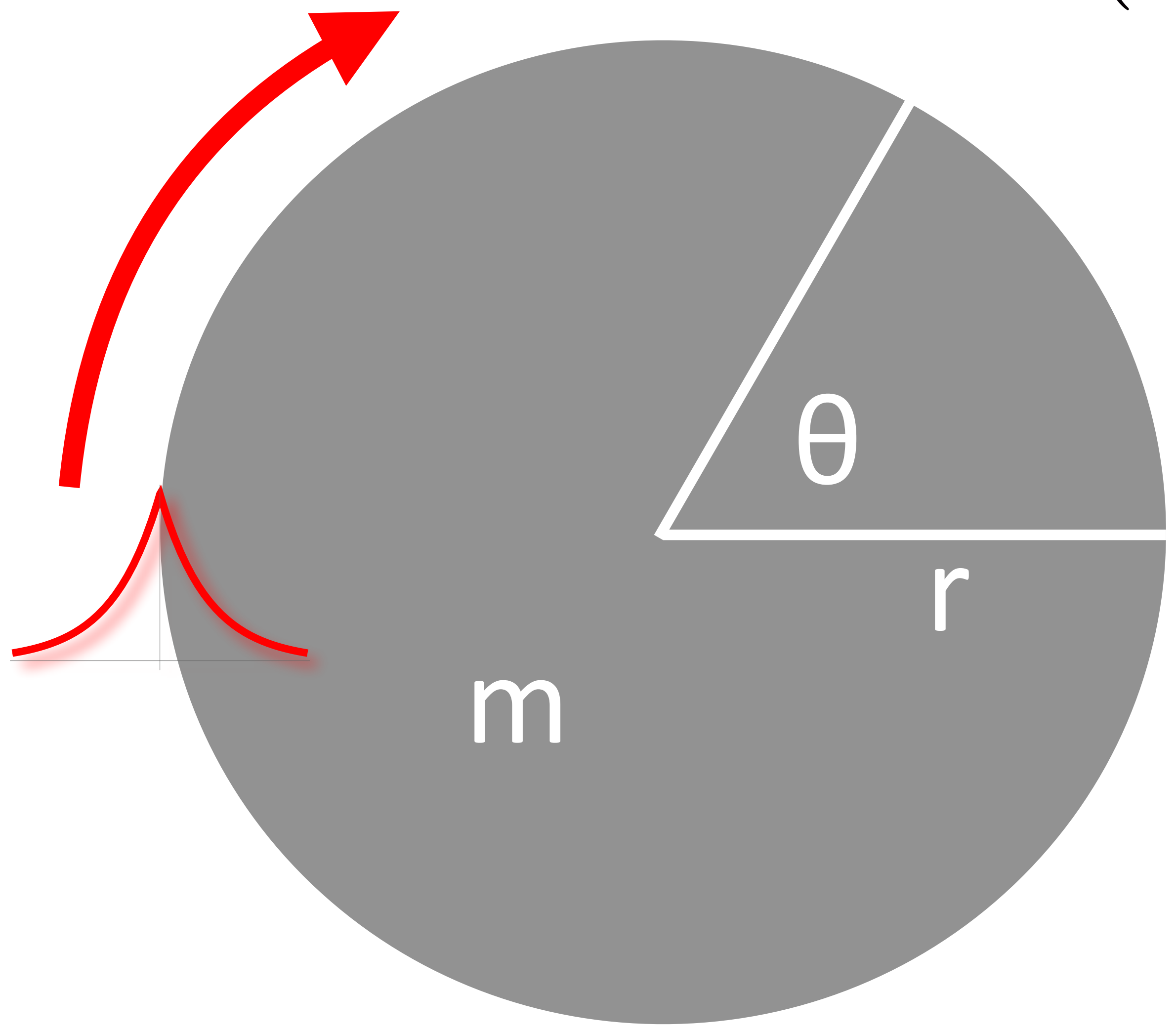

Shouldn't this have a single Weyl fermion edge state?

Which must be exactly massless?

Which can be realized with Wilson fermions on a lattice?

Lots of (wrong) reasons for why this shouldn’t work…so it took 30 years to check it out.

Moral: Think less, calculate more

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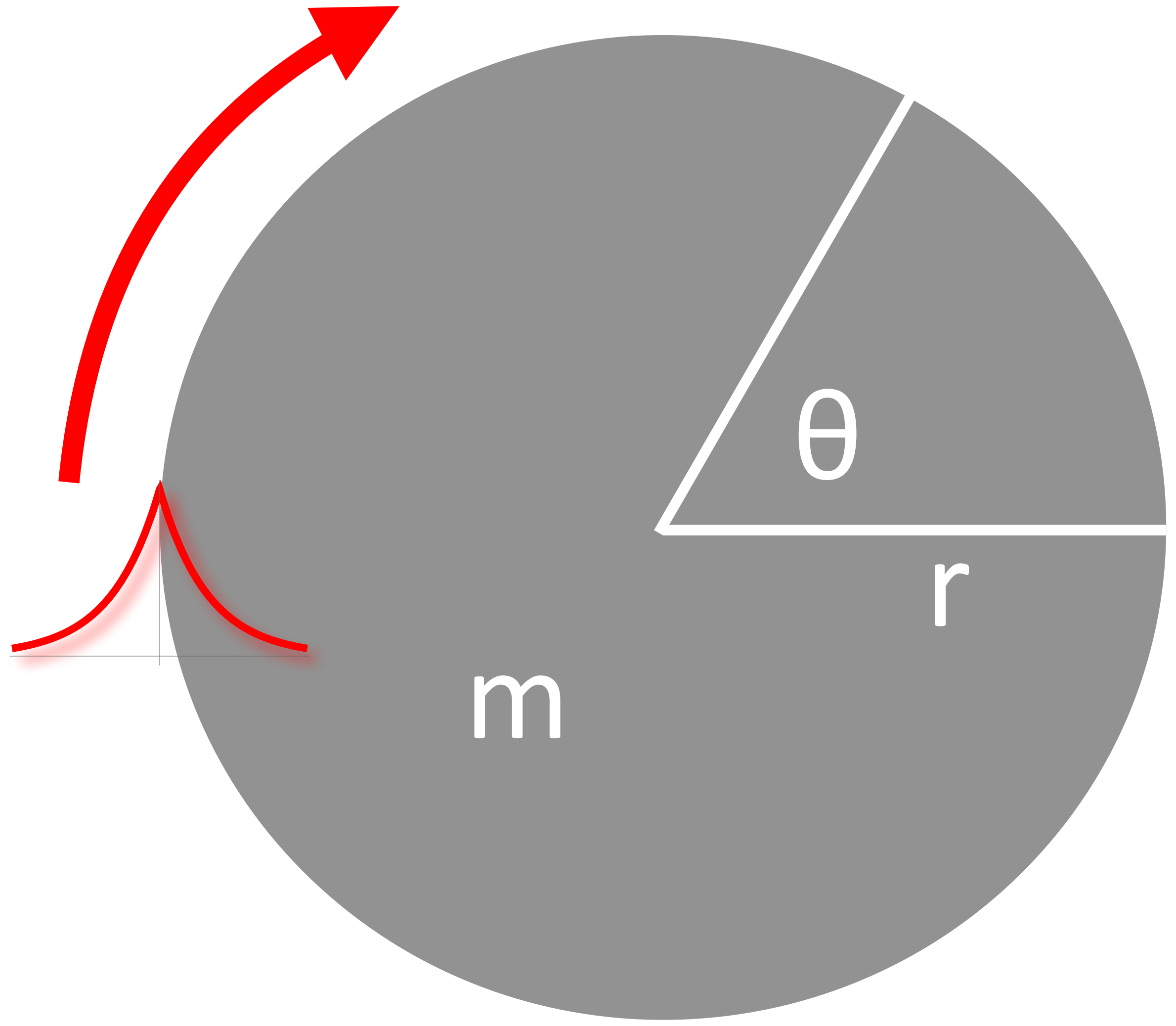

Solve the Dirac equation with this mass profile

(DBK: Phys. Rev. Lett. 132 (2024) 141603, arXiv:2312.01494)

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Furthermore, the same physics works on the lattice…physics is like the continuum annulus with R' = a = lattice spacing. Thus mirror states to not have a continuous spectrum as $R \to \infty$, invalidating assumption of Nielsen-Ninomiya

Weyl edge state?
Look at 1+1 dispersion relation

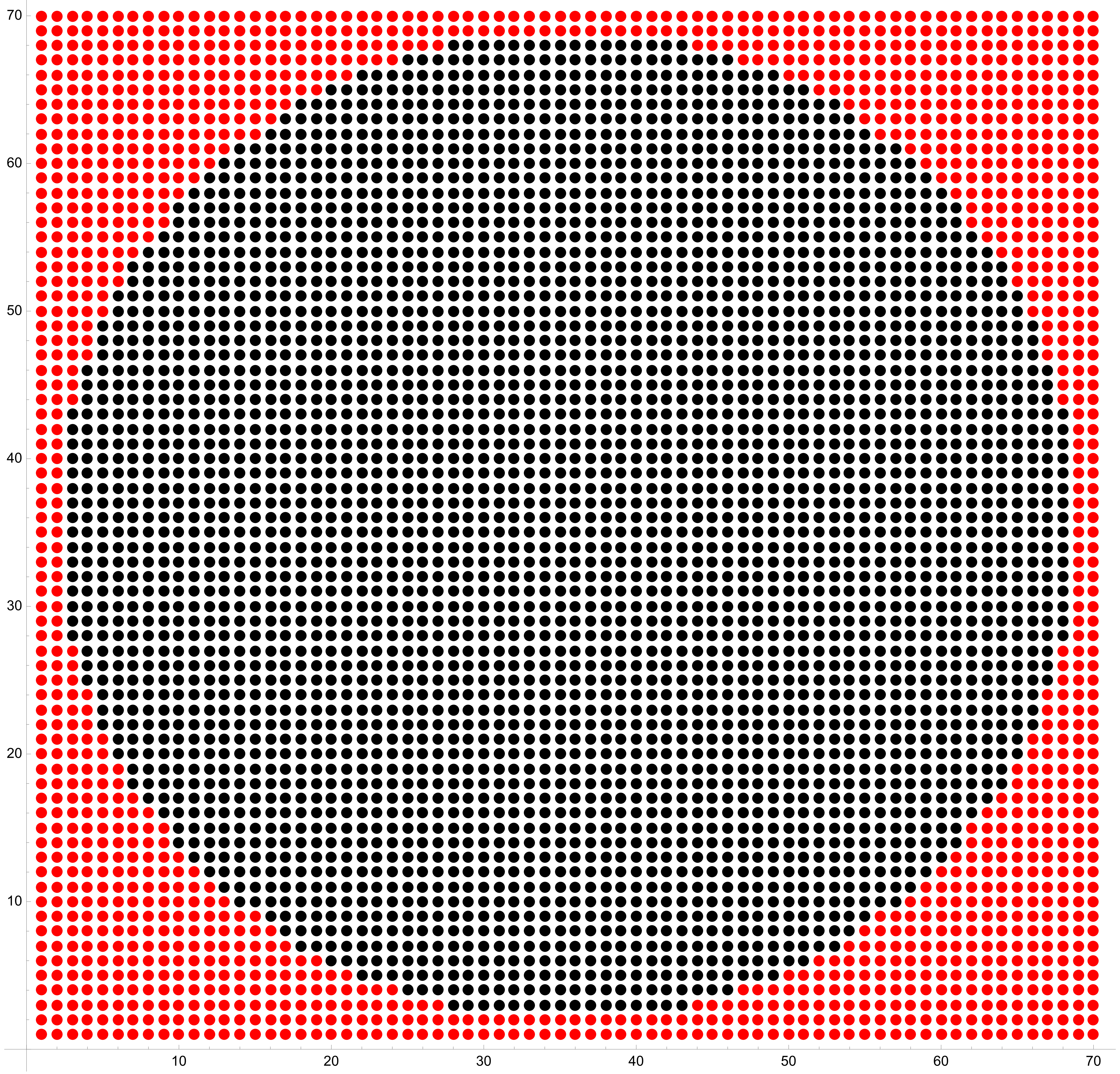
Work on a lattice disc with open BC

$$P_R = \begin{cases} 0 & x^2+y^2 \geq R^2 \\ 1 & x^2+y^2 < R^2 \end{cases}$$

$$H_{\text{disc}} = P_R\, H_{L\times L}\, P_R$$

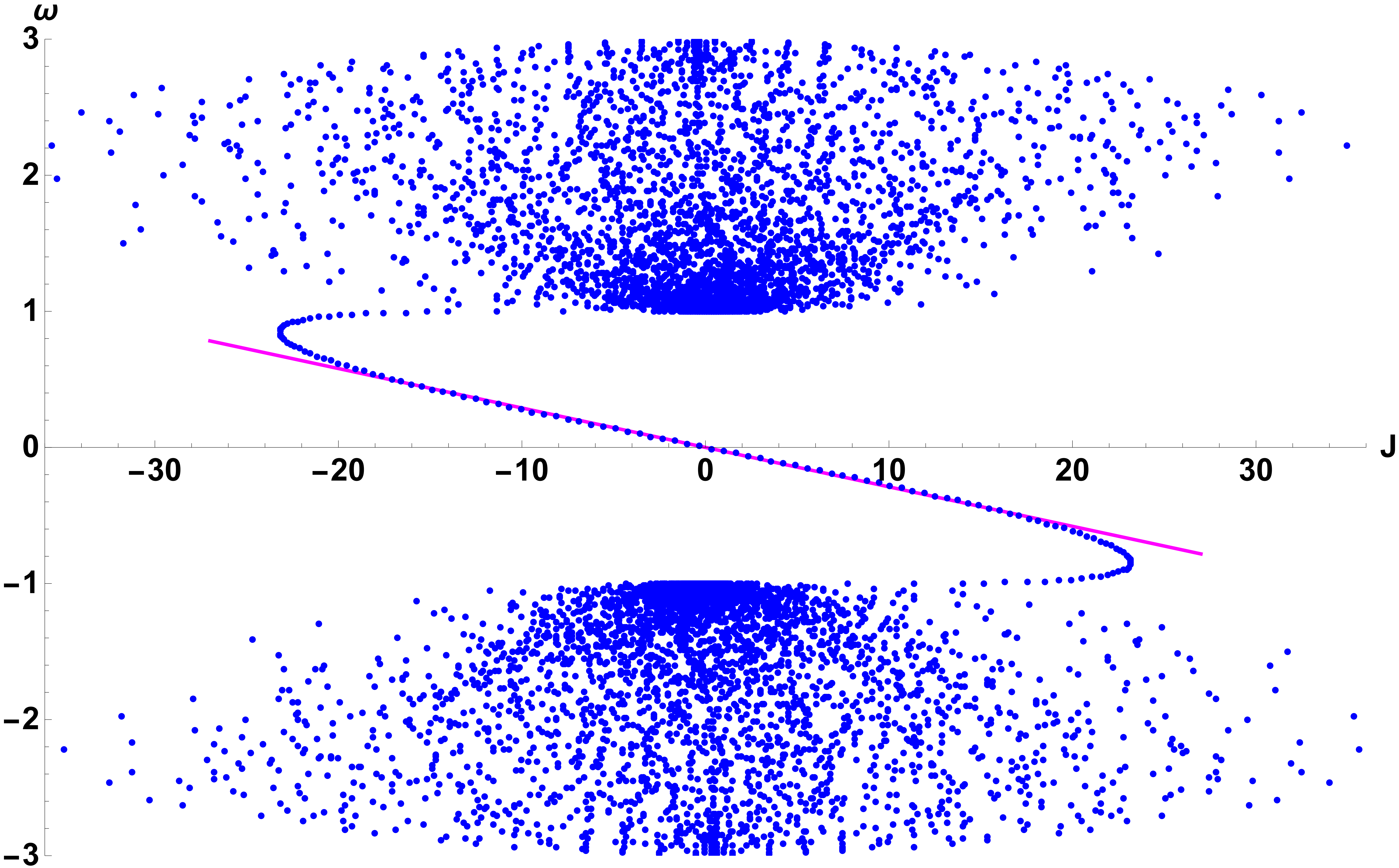
We took L=70, R = 34.

If you want E vs p for the edge state, plot E vs J



DBK, S. Sen: Phys. Rev. Lett. 132 (2024) 141604 arXiv:2312.04012

Energy eigenvalue ω_n



Angular momentum $\langle\psi_n|\hat{J}|\psi_n\rangle$

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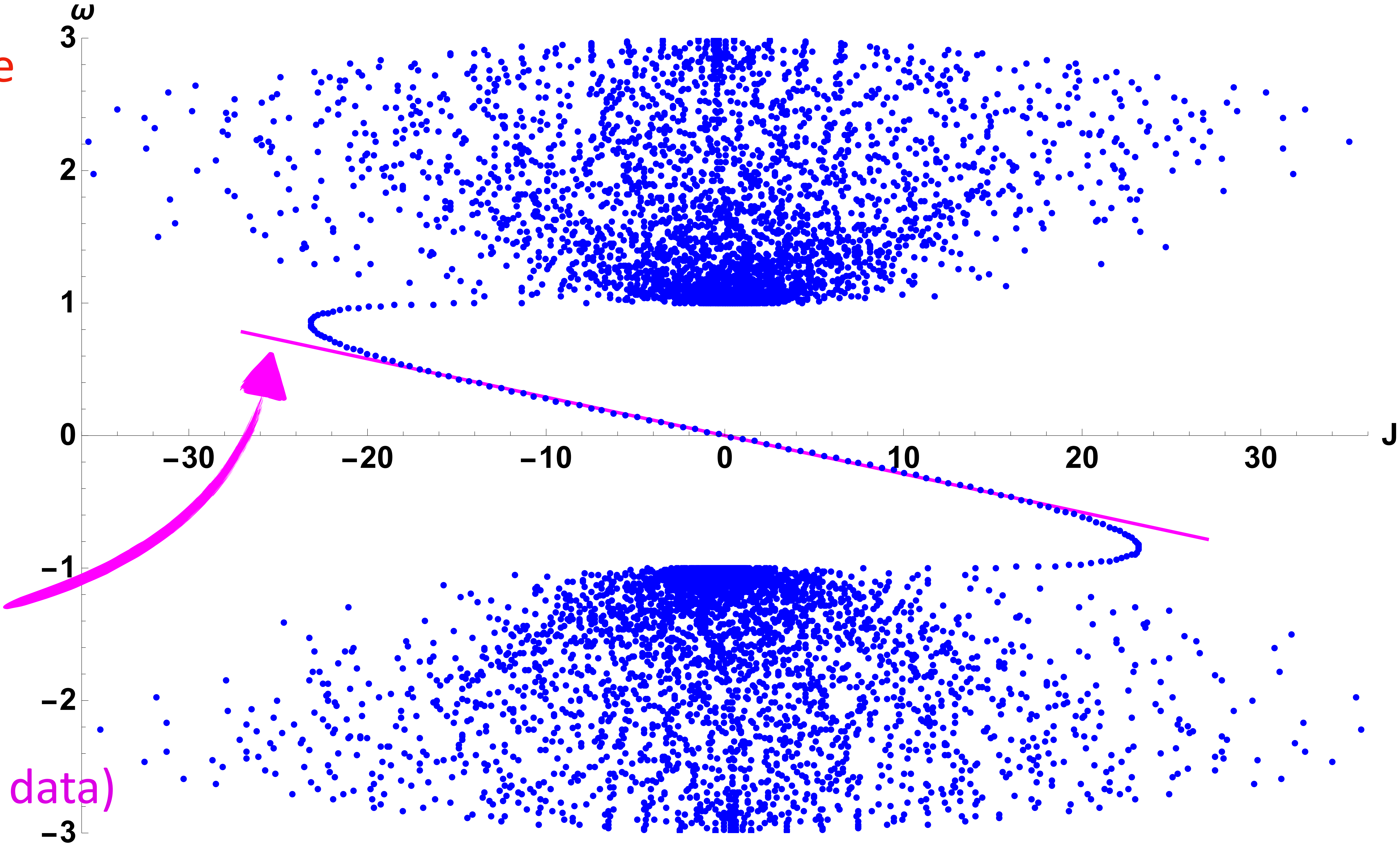

Angular momentum

$\langle \psi_n | \hat{J} | \psi_n \rangle$

$$\omega = -\frac{J}{R}$$

(Not a fit to data)

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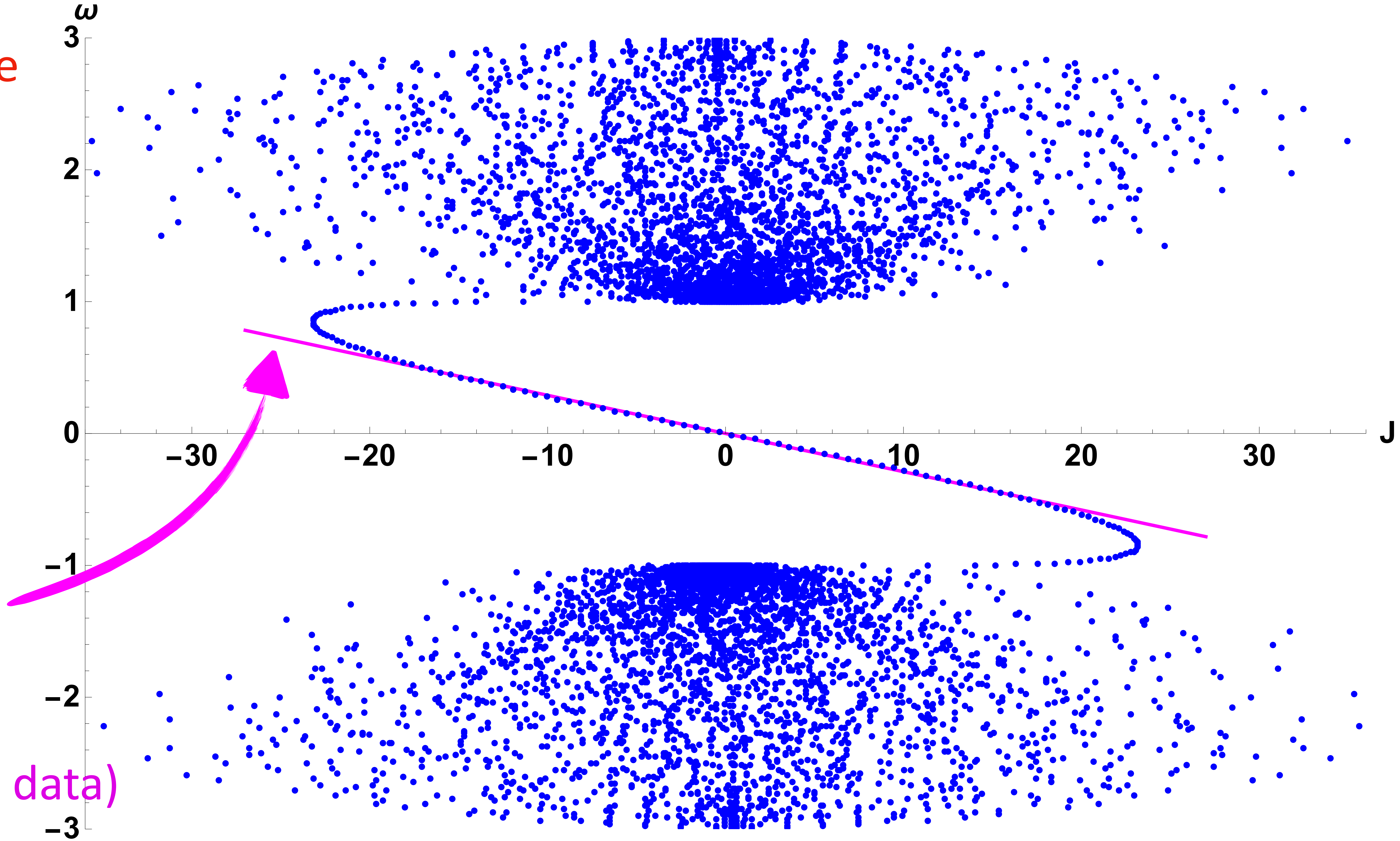

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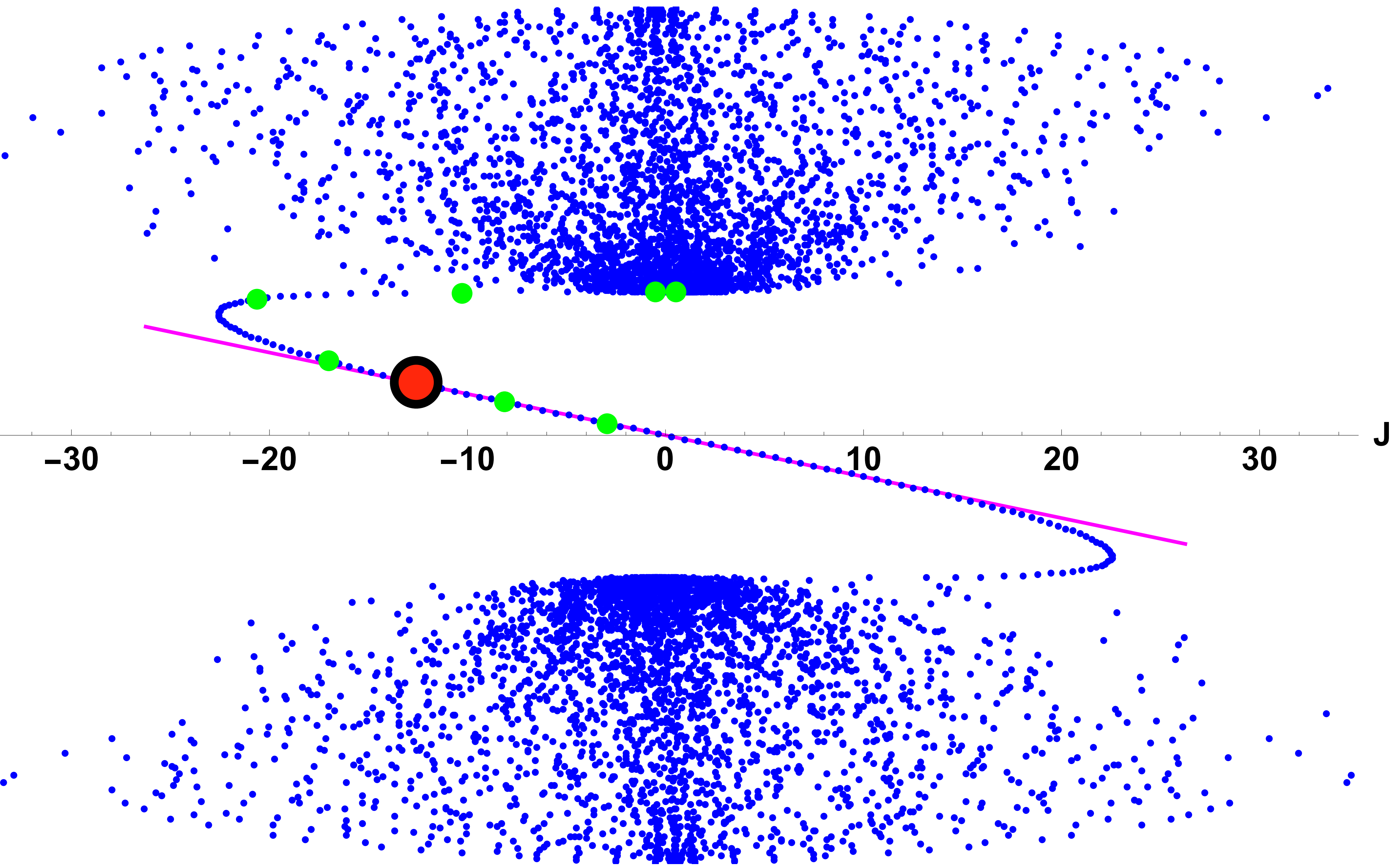
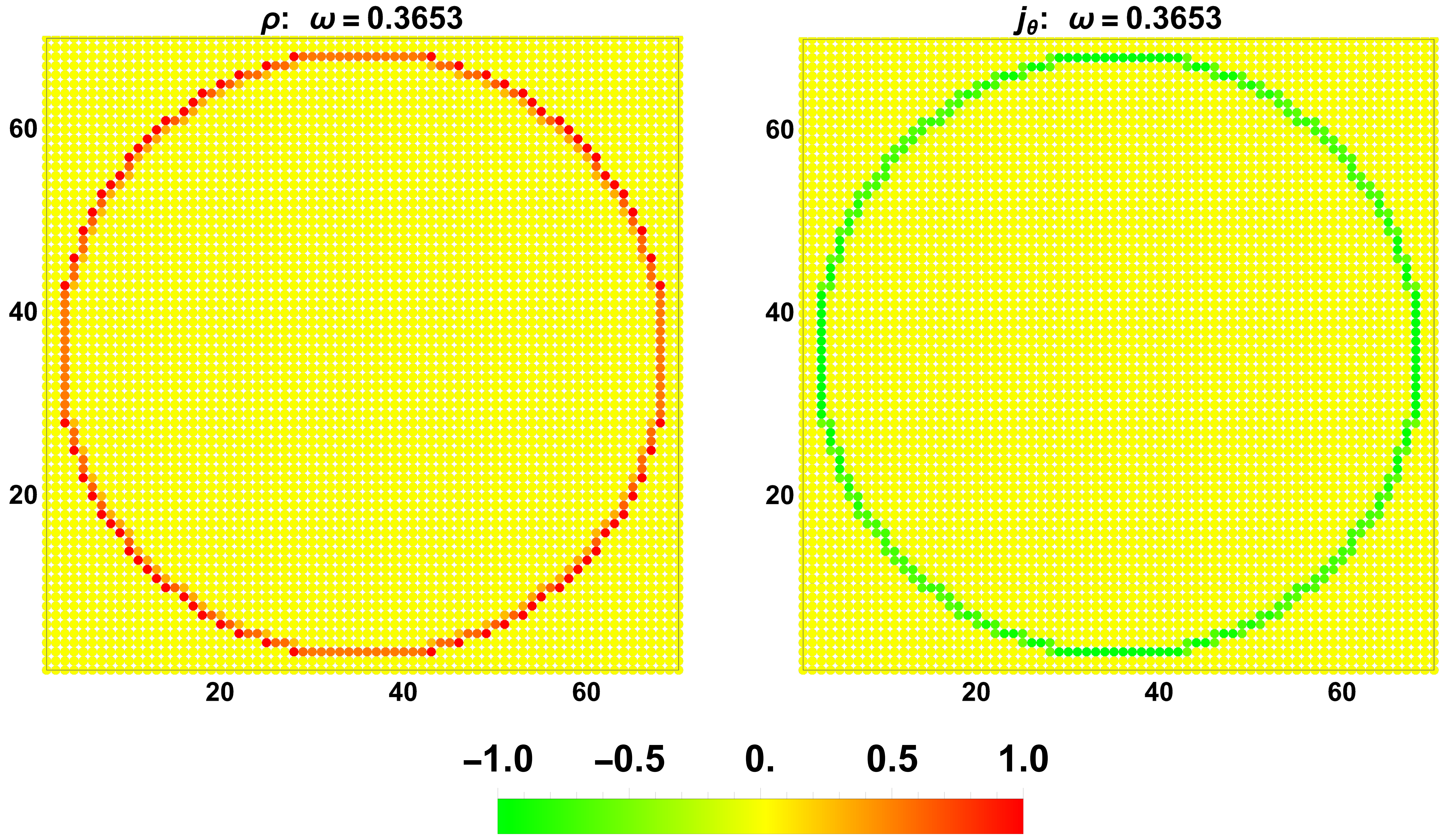
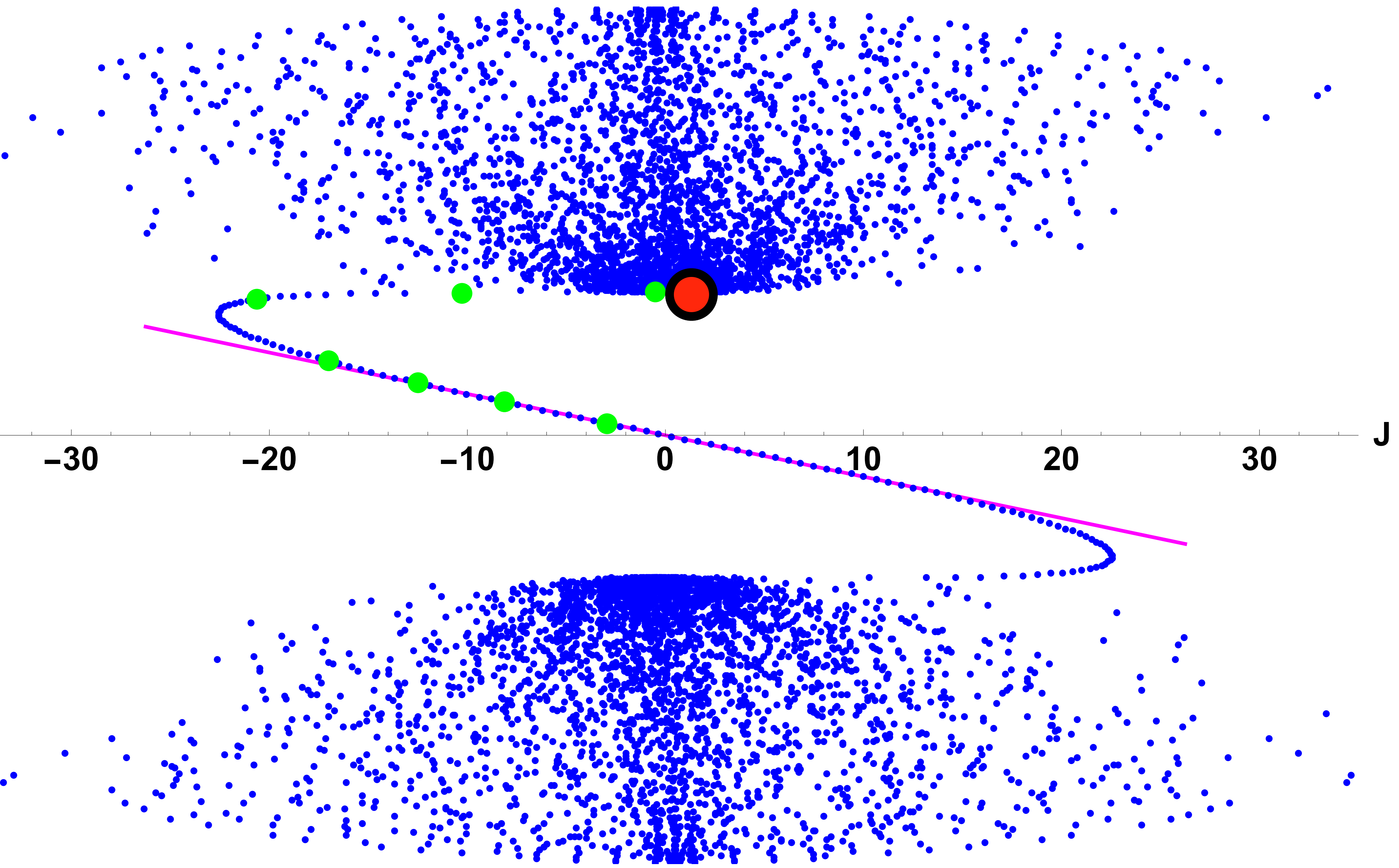
Nielsen-Ninomiya would have you believe this is not possible for sensible system



charge density ρ

current density jθ





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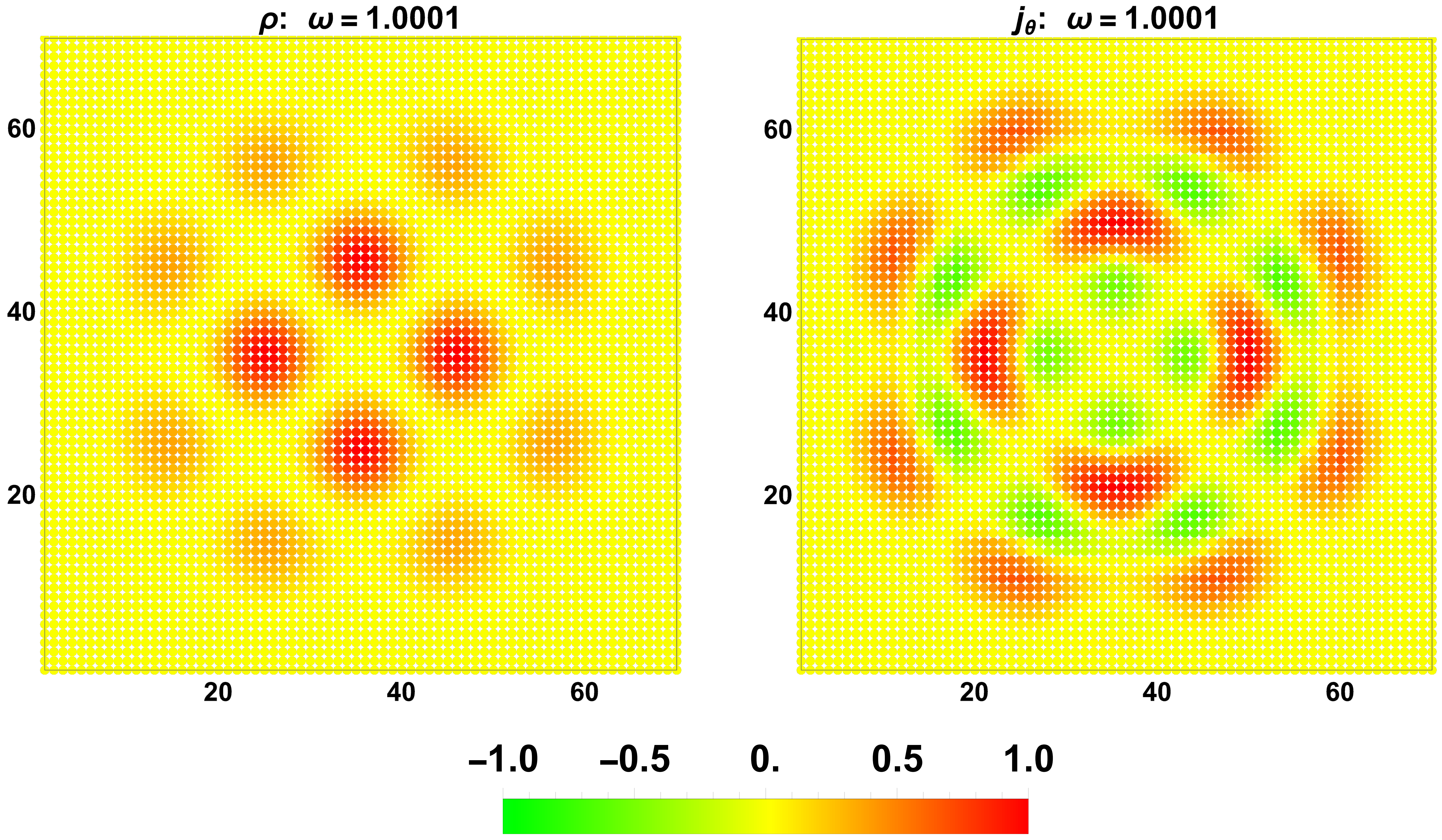
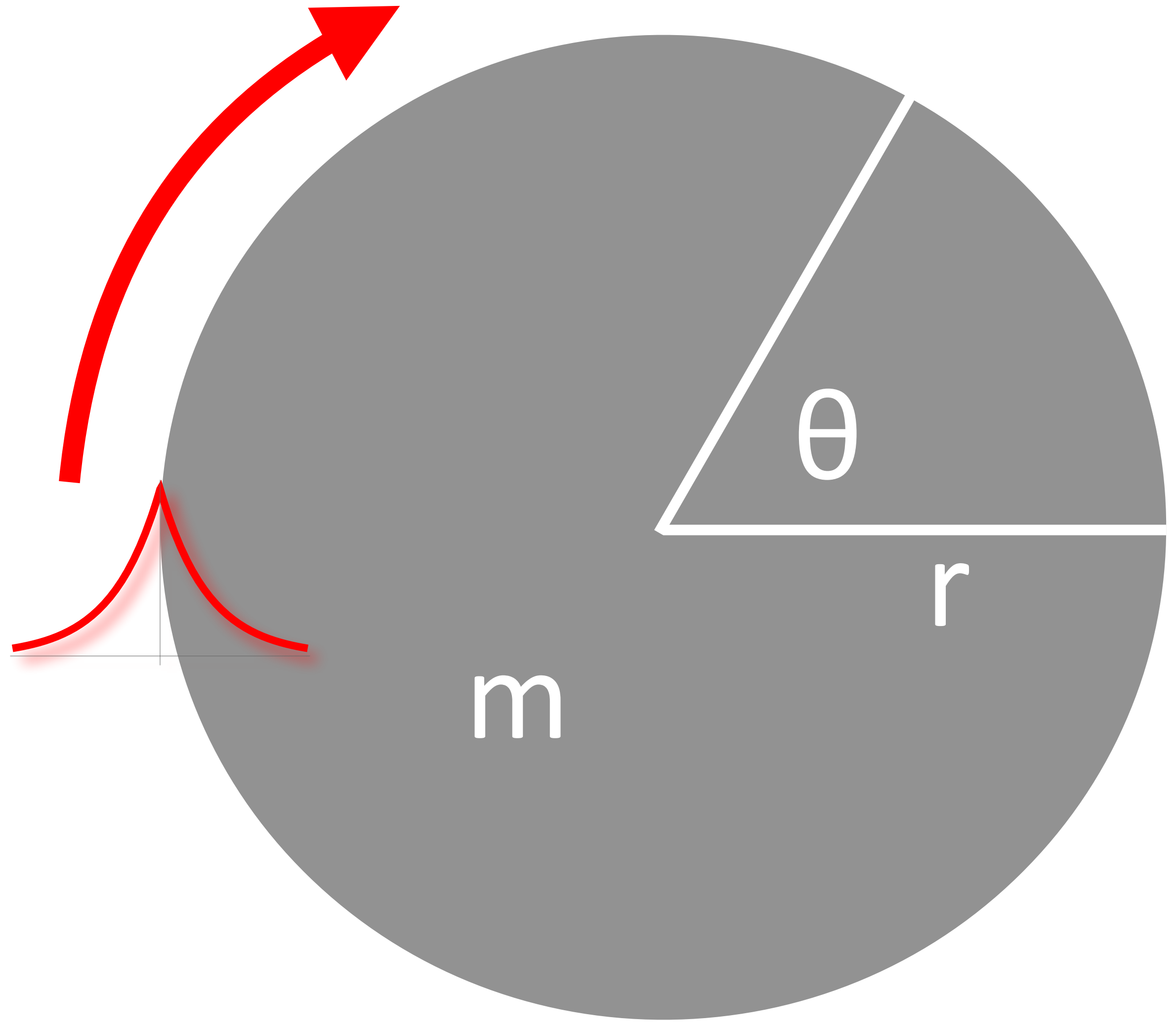


Back to the *continuum*.

$$-M \to -\infty$$



Equivalent to finite disk with BC

$(1+\gamma_5)\psi(R)=0$, $\gamma_5 \equiv \hat{r}\cdot\vec{\gamma}$

- Add 5d background gauge field B_k, $k=1,\ldots,5$.
- Look at physics below gap *m* (integrate out massive fermion modes)
- To tame divergences, include a Pauli-Villars field*, same BC but mass -*m*

* Role of PV field is crucial — it compactifies momentum space, required for topological interpretation, quantized Chern-Simons coefficient, anomaly inflow…

Integrate out massive bulk fermion + PV field with background 5d gauge field B_k

$$\int d\chi\, d\bar\chi\, \frac{\Delta[B_k]}{\Delta^*[B_k]}\, e^{-S(\chi,\bar\chi,A_\mu)}$$

$$A_\mu(x) = B_\mu(x,r)\Big|_{r=R}$$

= 4d boundary gauge field

χ = Weyl boundary mode with 4d action

Δ = bulk fermion contribution to fermion determinant

Δ* = Pauli-Villars contribution to fermion determinant

- Pauli-Villars has canceled the real part of the fermion contribution to log[Δ/Δ*]
- The remaining imaginary part is proportional to the η-invariant of the bulk Dirac operator = (regulated) sum of λ/|λ| … from:

$$\lim_{m\to\infty} \mathrm{Im}\left[\ln\frac{\lambda+im}{\lambda-im}\right] = \pi\frac{\lambda}{|\lambda|}$$

- in perturbation theory, η-invariant = Chern-Simons operator

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Boundary theory that is free of gauge anomalies is described by partition function that only depends on boundary values of the gauge fields

$$\int d\chi\, d\bar{\chi}\, \boxed{\frac{\Delta[B_k]}{\Delta^*[B_k]}}\, e^{-S(\chi,\bar{\chi},A_\mu)} \longrightarrow e^{i\phi[A_\mu(x)]}$$

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It appears that we can weight by the 4d Yang-Mills action, integrate over the boundary gauge fields, and have a path integral for an chiral gauge theory:

$$\int dA\, e^{-S_{YM}[A]} \int d\chi\, d\bar\chi\, e^{i\phi[A]}\, e^{-S(\chi,\bar\chi,A_\mu)}$$

Proposal for defining chiral gauge theory: phase of fermion measure determined from bulk physics, automatically fails if boundary gauge theory is anomalous

$$\int dA\, e^{-S_{YM}[A]} \int d\chi\, d\bar{\chi}\, e^{i\phi[A]}\, e^{-S(\chi,\bar{\chi},A_\mu)}$$

This passes a critical common sense test:

Q: “What would go wrong if we tried to regulate a 4d theory that suffered from gauge anomalies?”

A: “It would not look like a 4d gauge theory ($\eta[B_k]$ depends on 5d gauge fields)”

Two important ingredients:

- Existence of robust chiral edge states
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- When anomalies cancel, bulk fermions contribute phase that only depends on boundary gauge field

  Not automatic…

The lattice is regulated; momentum space is a torus. No need for a Pauli-Villars field from the point of view of requiring finite results & well defined topology.

$$\Delta[B] = \det D_w(B)$$

D_W = Wilson operator with open BC

However:

- Δ includes both bulk and edge contributions
- The bulk contribution to the fermion determinant Δ[B] is not a pure phase… the real part of log[Δ[B]] contributes to a bulk 5d Yang-Mills operator, for example, which will give a 5d Coulomb law between boundary charges instead of 4d.

We also need to cancel bulk contribution to Re[log[Δ[B]] for the theory to look 4d.

- Must not remove boundary fermion contribution
- Must not change imaginary part, which already correctly encodes anomalies

Proposal:

$$\det D_w \to \frac{\det D_w}{\sqrt{\det\left({D_w}^{\dagger} D_w + \mu^2 \delta_{r,R}\right)}}$$

boundary mass term, avoids light PV edge states

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No. There can be topological obstructions to doing so.

A concrete proposal to continuing boundary gauge field A_μ into the bulk:

$B_k(x,r)$ solves the 5d Euclidian Yang-Mills equations subject to BC

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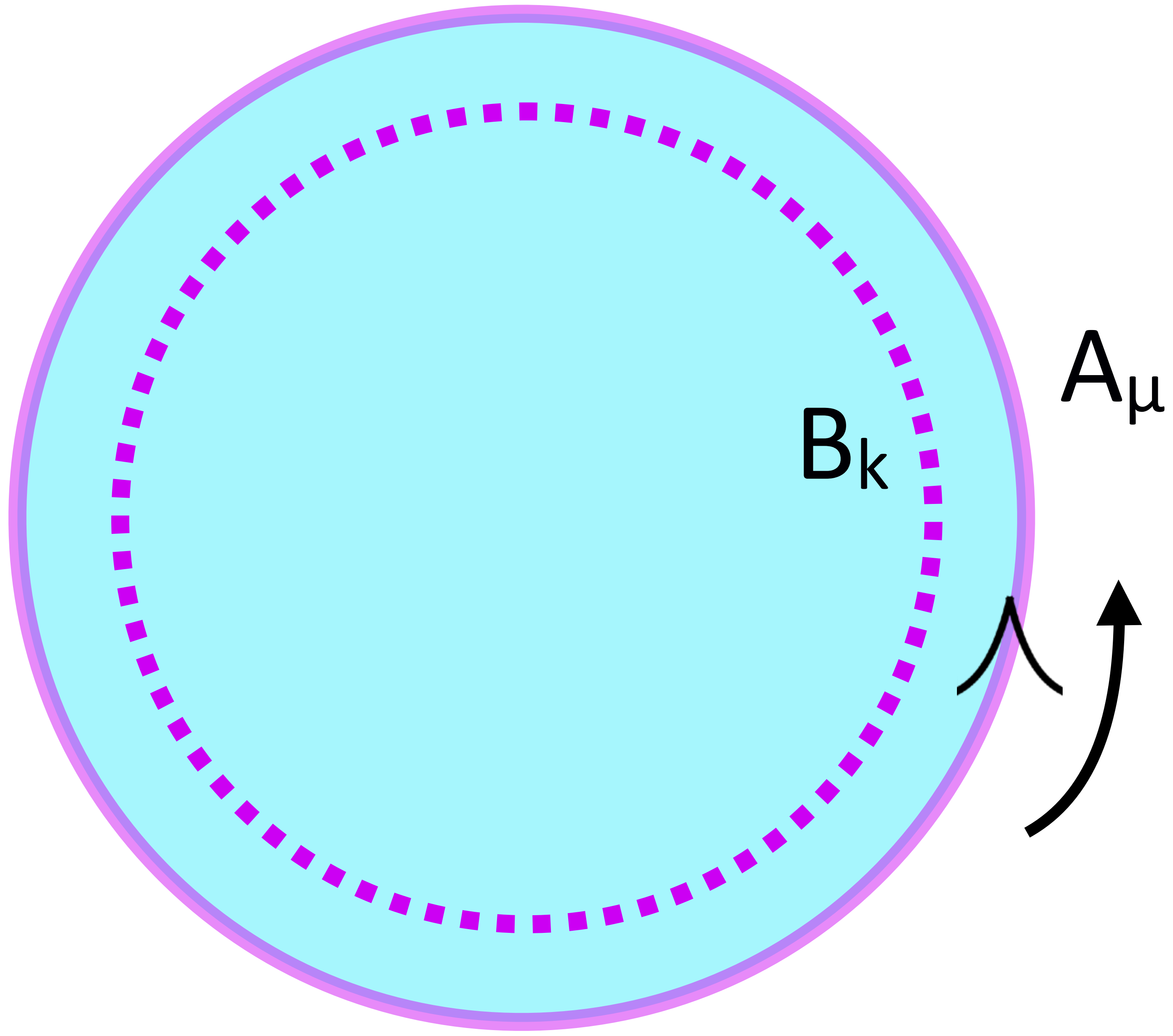

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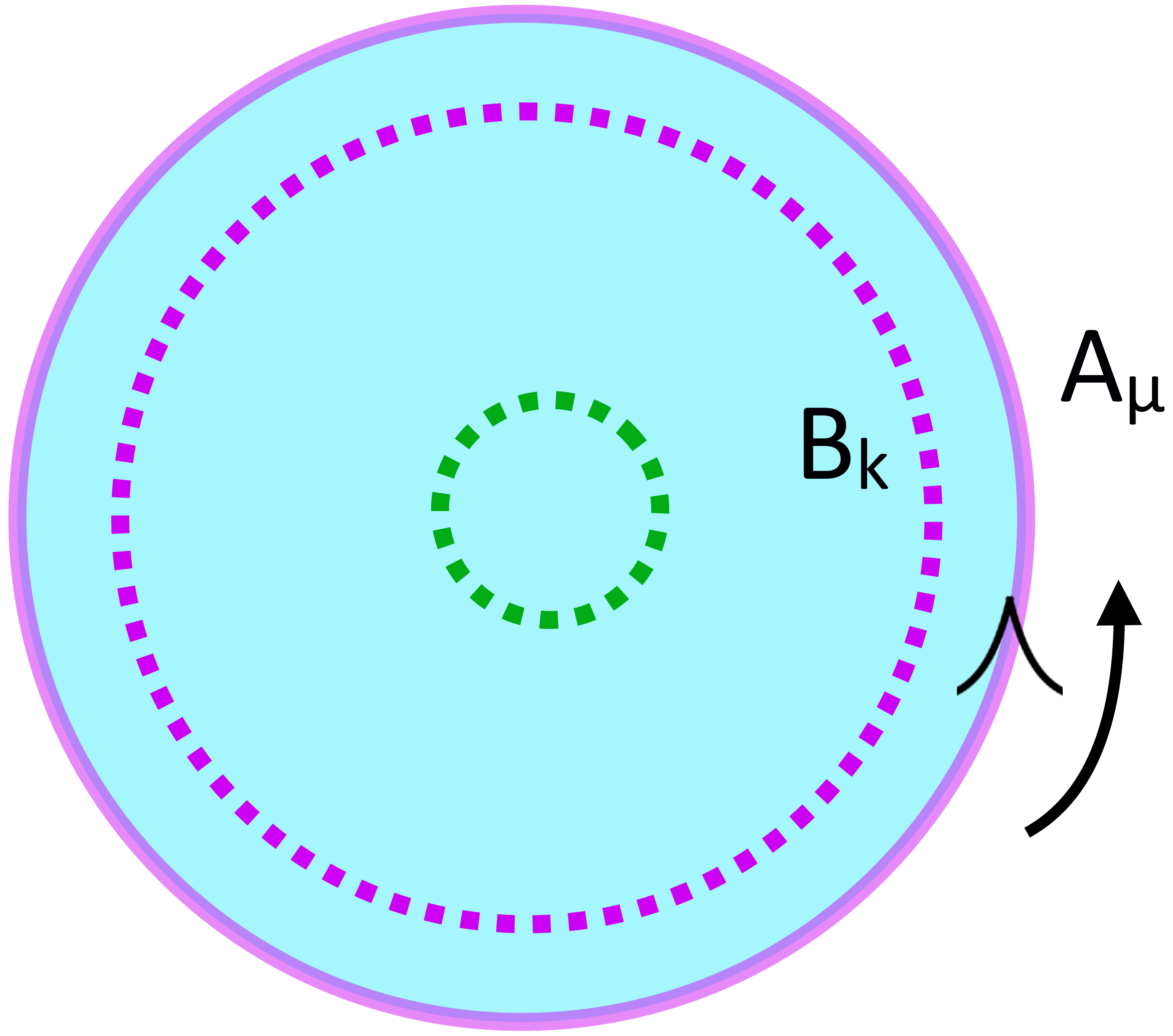

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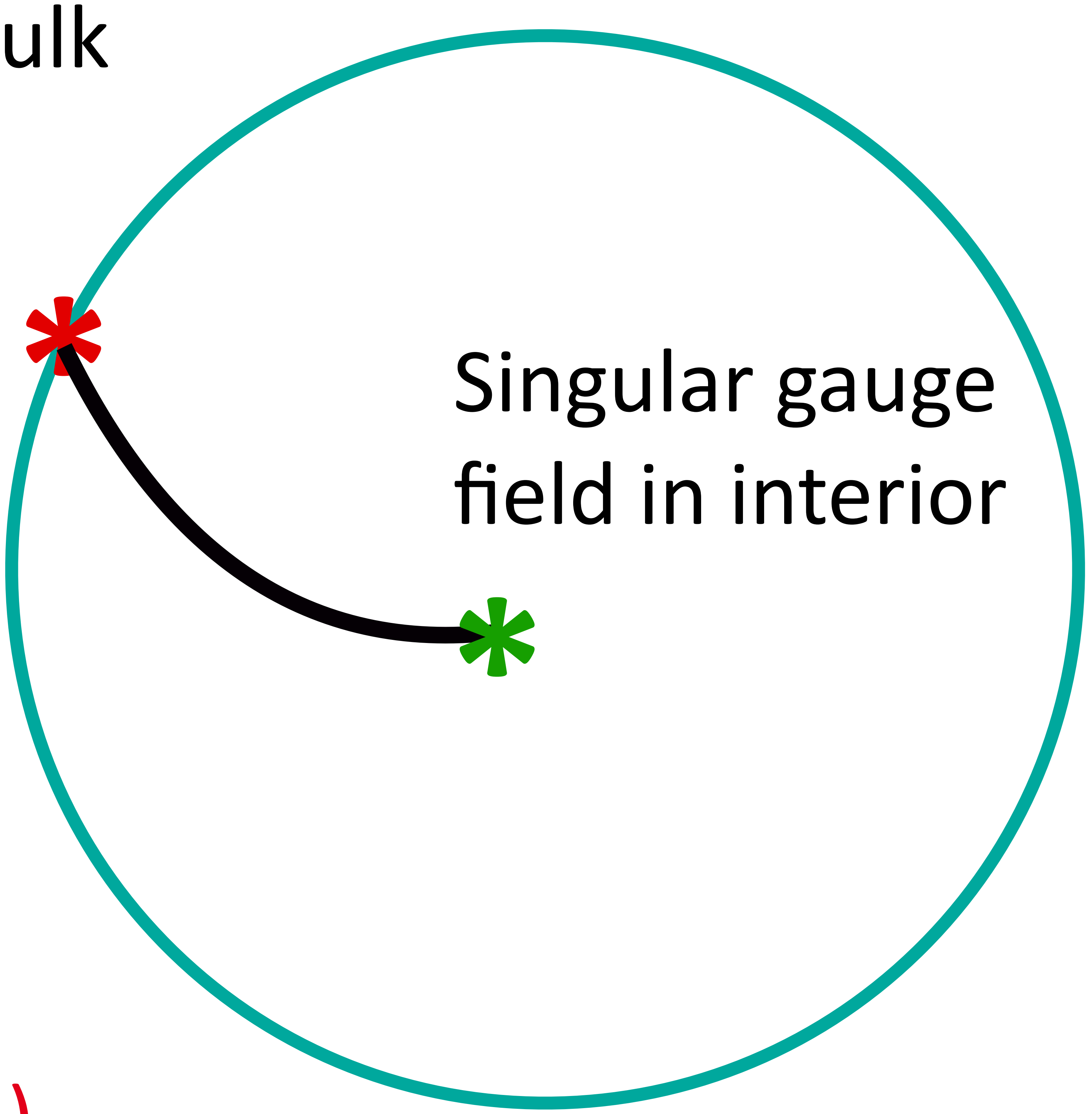
But suppose A_μ has nontrivial topology… as one contracts interior 4d surface, winding number must change discontinuously —> ensures that B_k has a singularity in the bulk

One instanton in boundary theory, continued into bulk



The inability to define interior gauge field smoothly is related to two objections to chiral boundary proposal:

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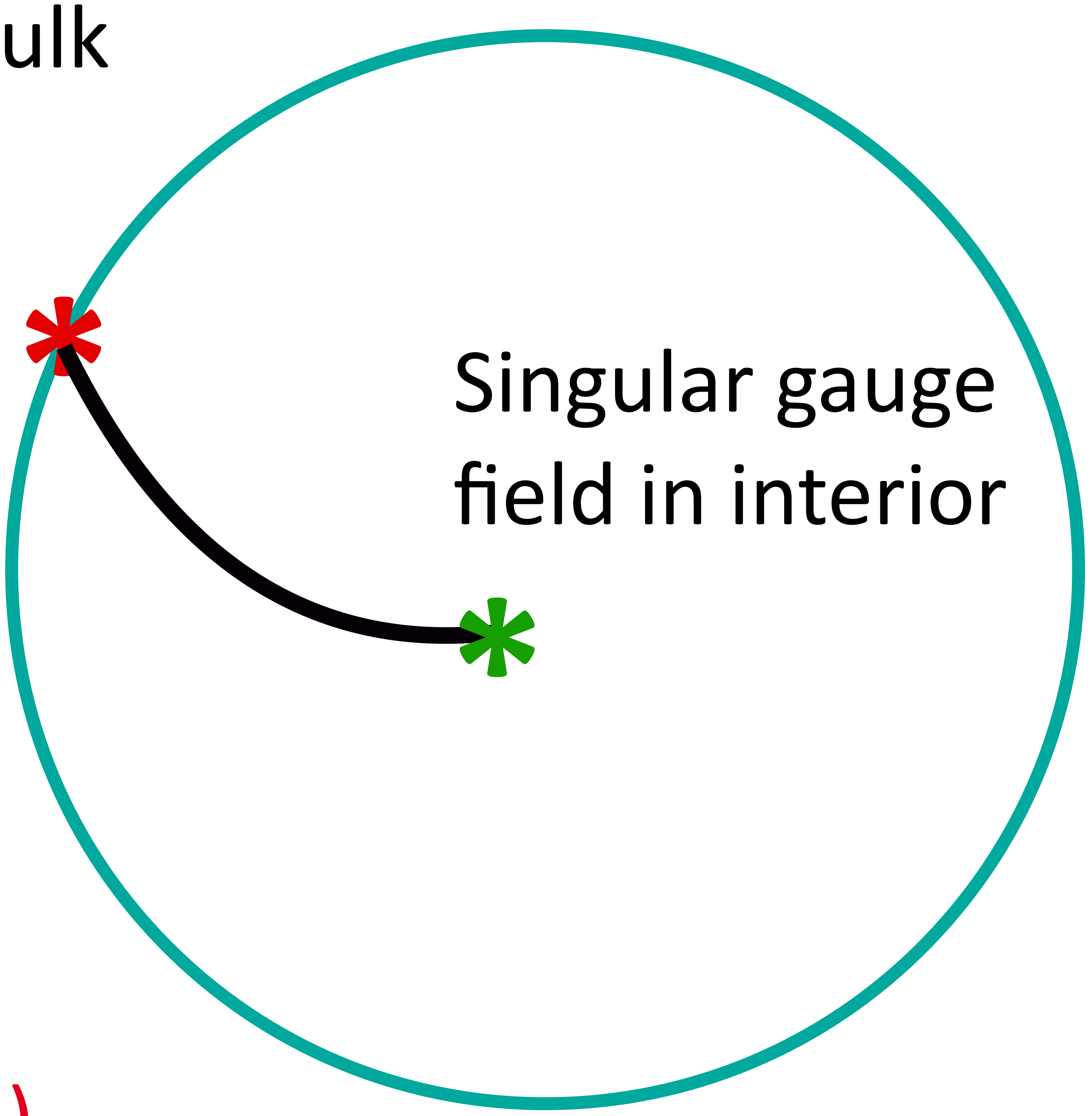

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To understand the implications, consider simple case where boundary theory is supposed to look like N_f=1 QCD

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- Massive η’ meson, even in limit of zero quark mass
- Possible θ term and strong CP violation

Heuristic picture of η' physics:

If the U(1)$_A$ were *only* spontaneously broken

- η' would be the Nambu-Goldstone boson
- U(1)$_A$ realized as shift symmetry η' -> η' + f

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If the $U(1)_A$ were *only* spontaneously broken + explicitly broken by small complex quark mass $M_q e^{i\theta}$

- $U(1)_A$ realized as approximate shift symmetry η’ -> η’ + f
- η’ would be the pseudo Nambu-Goldstone boson, mass proportional to $\sqrt{M_q}$
- The angle θ appears…but can be shifted away by η’ -> η’ + θ f … no CP violation if one ignores the anomaly

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta'\partial_\mu\eta' - M_q\cos\left(\frac{\eta'}{f} - \theta\right) + \ldots$$

Anomaly enters through index theorem: quark zeromodes associated with nonzero winding number

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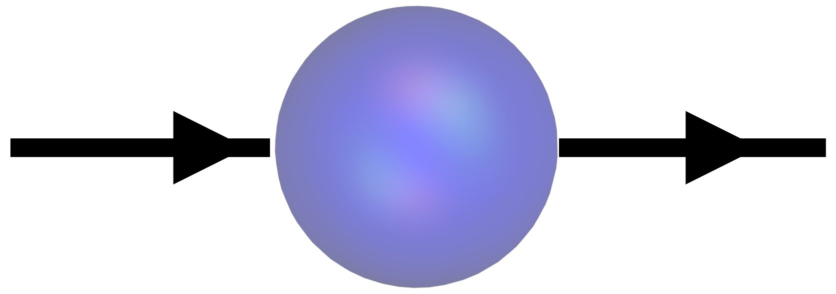

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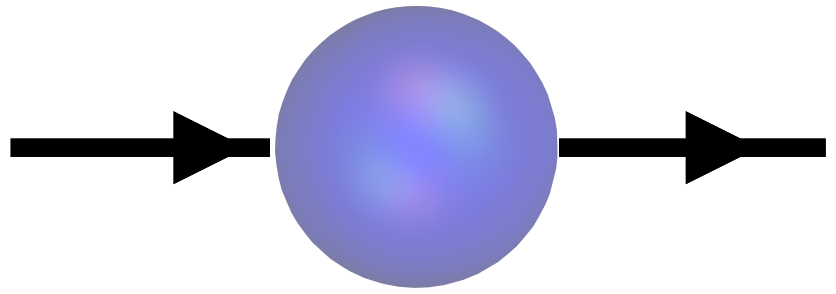

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summing over all unique instanton & anti-instanton positions exponentiates effective vertex and contributes $U(1)_A$ - violating term to action

$$e^{-S_{\text{inst.}}} = \sum_{n,\bar{n}=0}^{\infty} \frac{\left(\Lambda \int \bar{q}_R q_L(x)\,dx\right)^n}{n!} \frac{\left(\Lambda \int \bar{q}_L q_R(y)\,dy\right)^{\bar{n}}}{\bar{n}!} = e^{\Lambda \int (\bar{q}_R q_L(x) + \bar{q}_L q_R(x))\,dx}$$

(Λ = QCD mass scale not computable in instanton model)

match to η’ effective theory: $\bar{q}_R q_L(x) \to \frac{\Sigma}{2} e^{i\eta'(x)/f}$, $\bar{q}_L q_R(x) \to \frac{\Sigma}{2} e^{-i\eta'(x)/f}$, $\frac{\Lambda\Sigma}{f^2} = M_{\eta'}^2$,

Obtain: $$\mathcal{L} = \frac{1}{2}\partial_\mu \eta' \partial_\mu \eta' - M_\eta'^2 f^2 \cos\frac{\eta'}{f} - M_q \cos\left(\frac{\eta'}{f} - \theta\right) + \ldots$$

large anomaly contribution

small quark mass contribution

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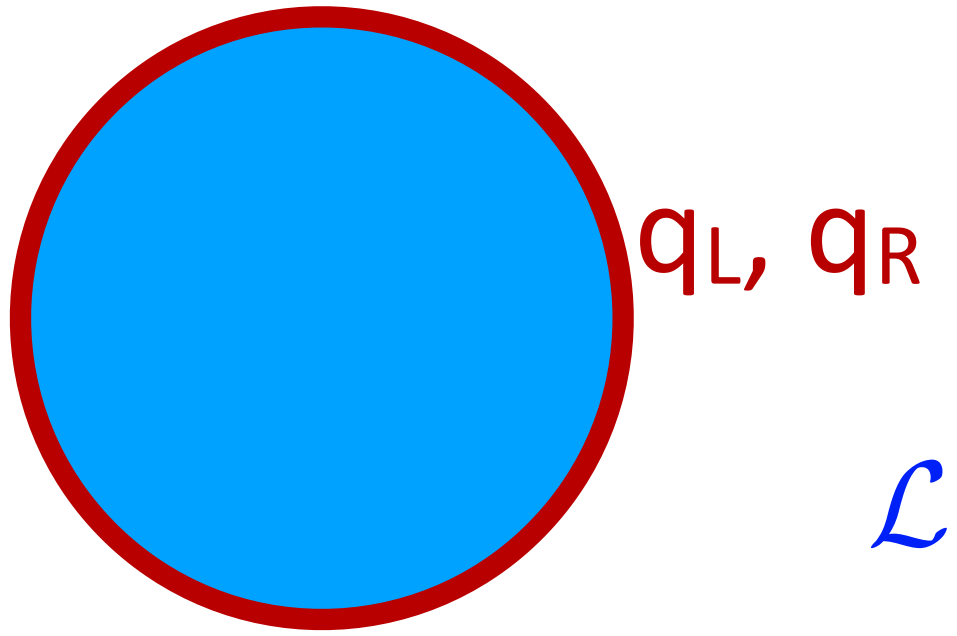
Can we achieve this physics from the 5d chiral boundary theory proposal?

Golterman & Shamir (continuum version):

Consider $N_f = 1$ QCD on the boundary (1 LH + 1 RH Weyl fermion)



From 5d theory:

$$\mathcal{L} = \bar{\psi}^+(\not{D} + m)\psi^+ + \bar{\psi}^-(\not{D} - m)\psi^- + \text{ PV}$$

$$P_\pm \psi^\pm(R) = 0$$

This has exact U(1) x U(1) symmetry.

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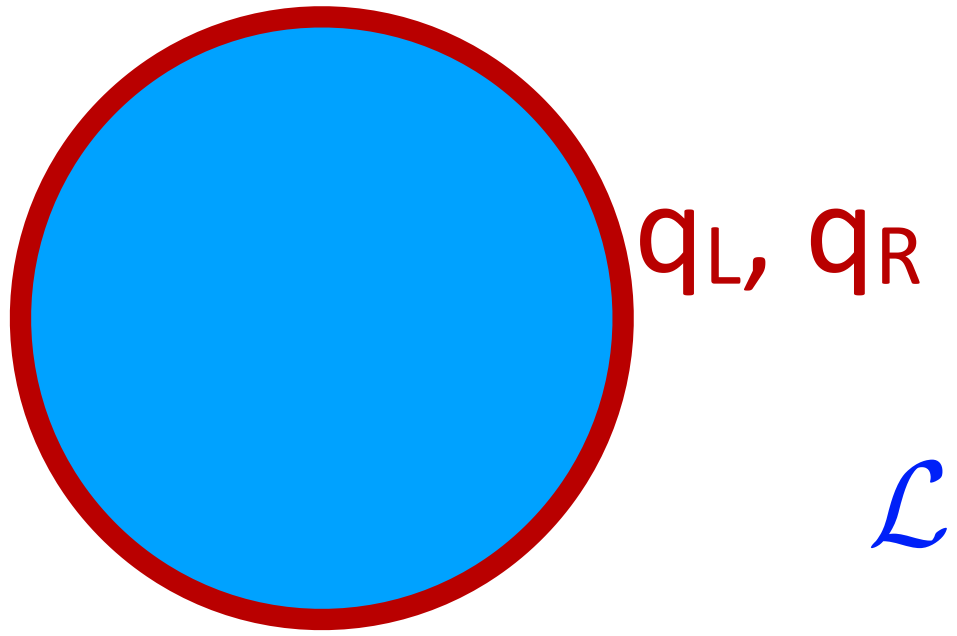

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Callan-Harvey anomaly in-flow argument: put in source for conserved current corresponding to

$$\psi^\pm \to e^{\mp i\alpha}\psi^\pm$$

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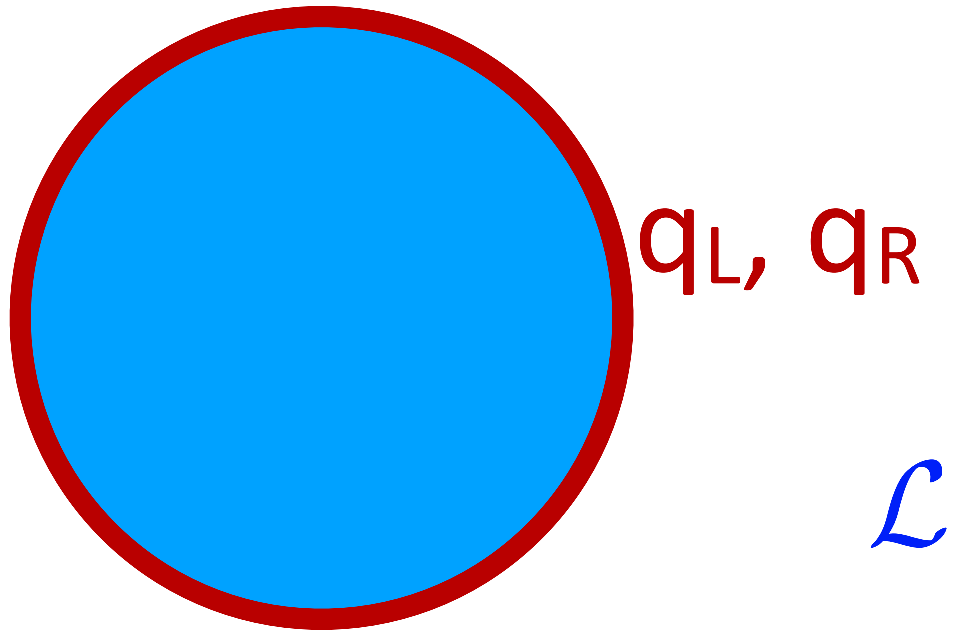

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$$\mathcal{L} = \bar{\psi}^+(\not{D} + m)\psi^+ + \bar{\psi}^-(\not{D} - m)\psi^- + \text{ PV} \qquad P_\pm \psi^\pm(R) = 0$$

This has exact U(1) x U(1) symmetry.

Callan-Harvey anomaly in-flow argument: put in source for conserved current corresponding to

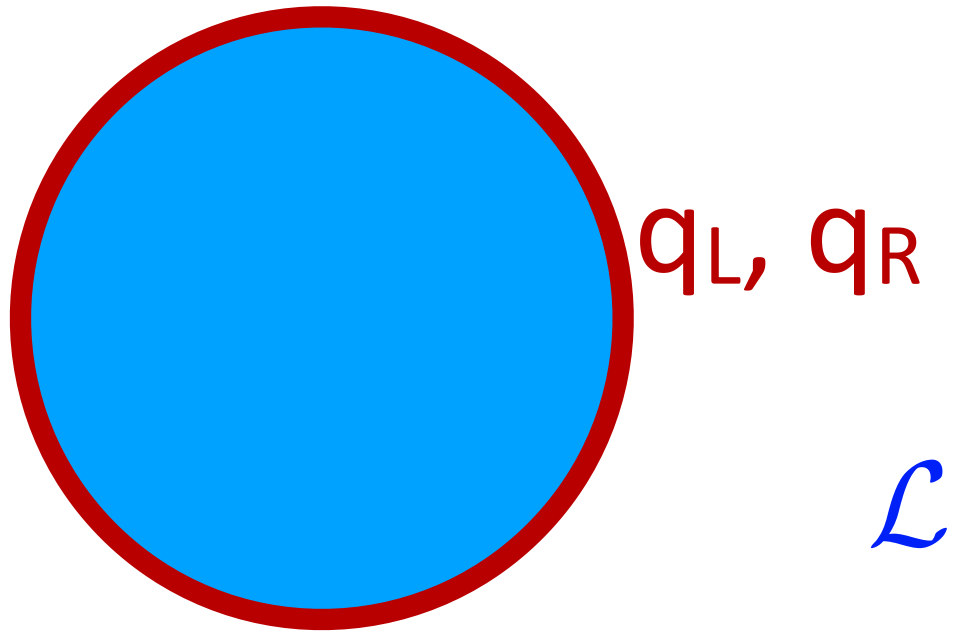
$$\psi^\pm \to e^{\mp i\alpha}\psi^\pm$$

varying action w.r.t. source gives the correct anomalous WT identity for $U(1)_A$ current in the boundary theory:

$$\partial_\alpha \, \bar{q}\gamma^\alpha\gamma_5 q = \frac{1}{16\pi^2}\epsilon_{5\mu\nu\rho\sigma}\text{Tr}\, F_{\mu\nu}F_{\rho\sigma}\Big|_{r=R}$$

Golterman & Shamir (continuum version):

Consider N$_f$ = 1 QCD on the boundary (1 LH + 1 RH Weyl fermion)



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good!
anomaly in-flow!

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…but our formula for the effective theory after integrating out the bulk fermions without a source looked like:

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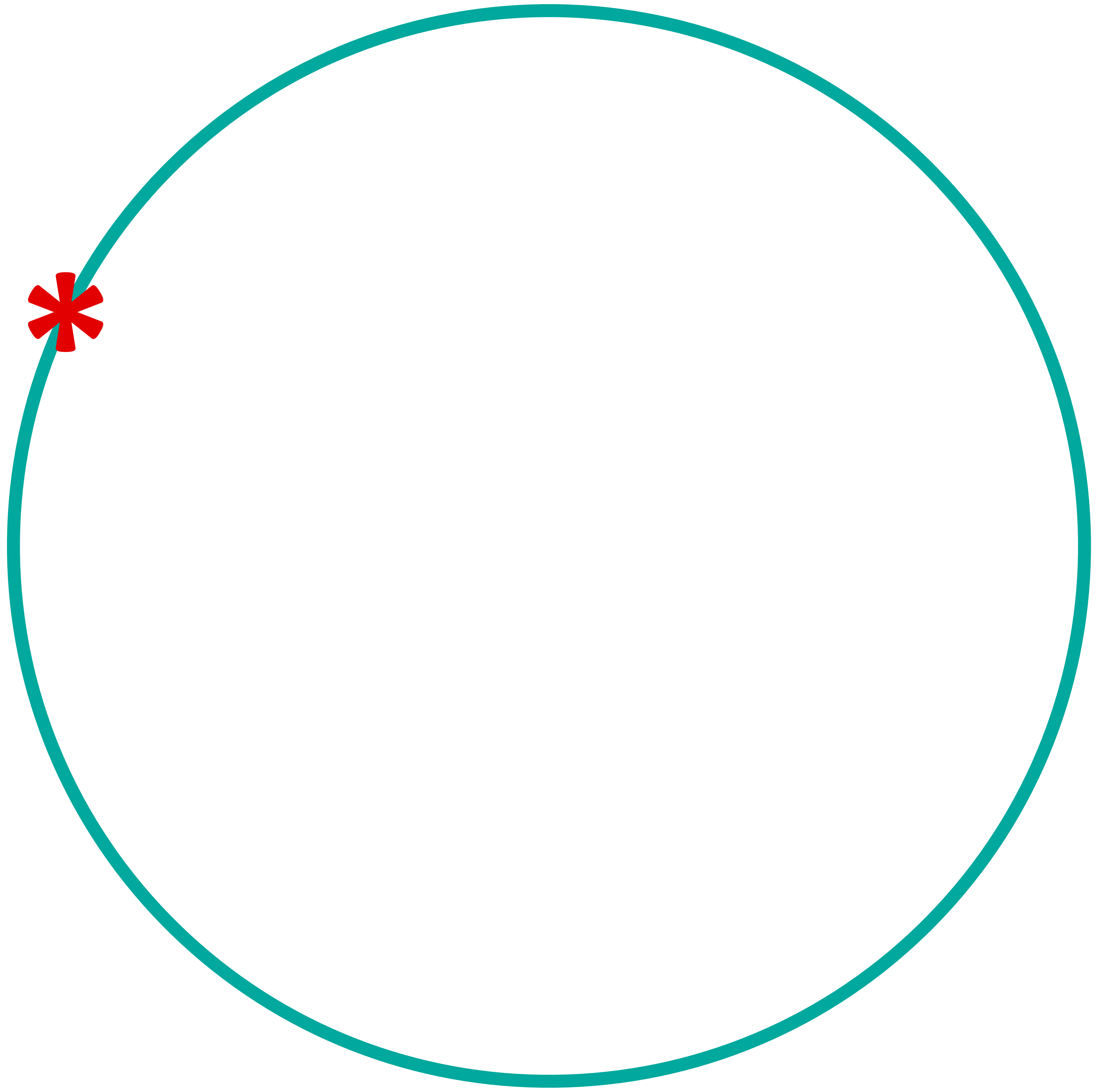
Our conclusion:

- the true boundary theory one obtains must actually have an exact $U(1)_A$ symmetry
- when $U(1)_A$ spontaneously breaks there is not a massless NGB
- Furthermore, the theory does not exhibit strong CP violation
- The existence of bulk gauge field singularities and bulk fermion zeromodes play central role

One instanton in boundary theory, continued into bulk



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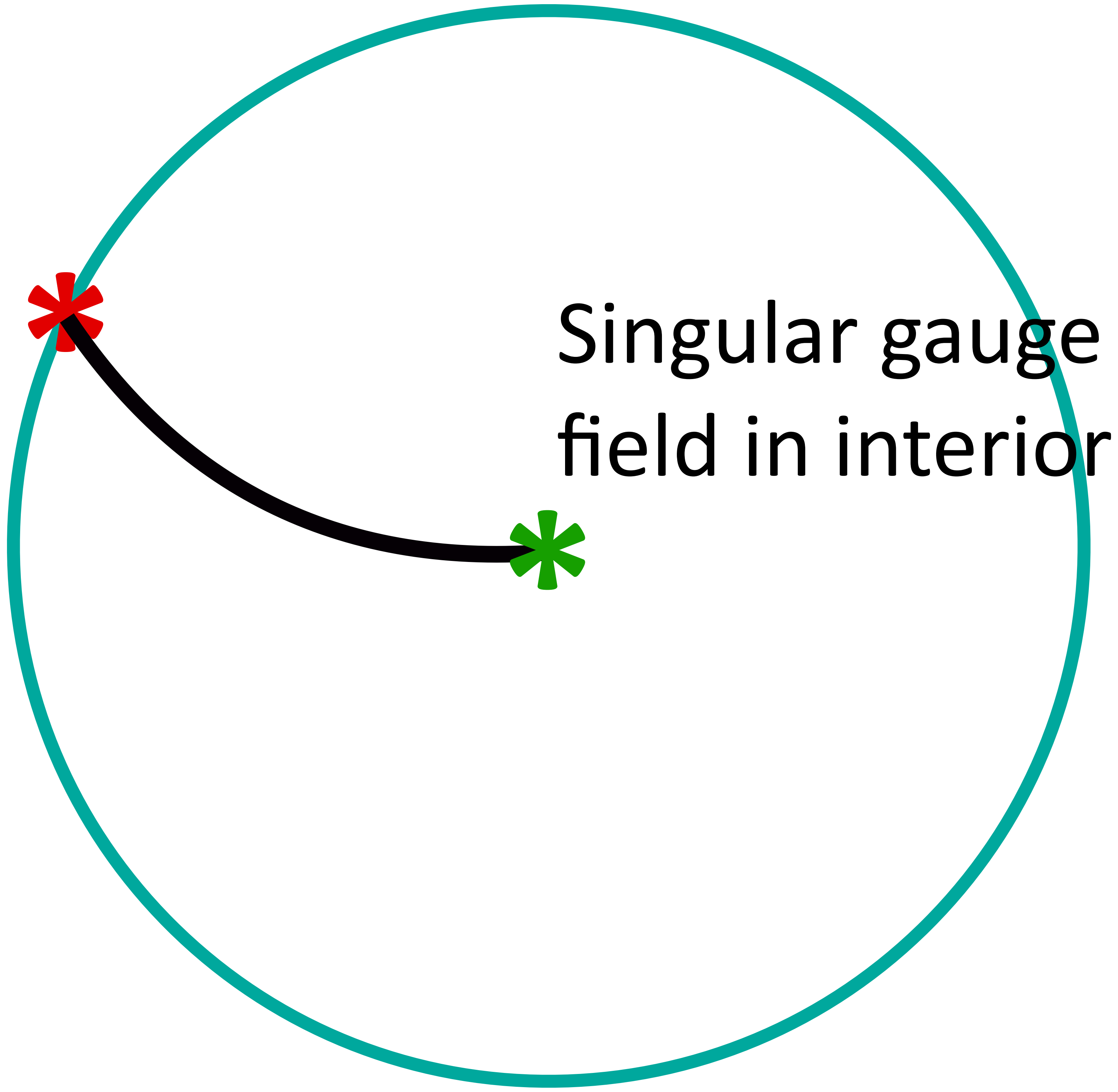

Singular gauge field in interior

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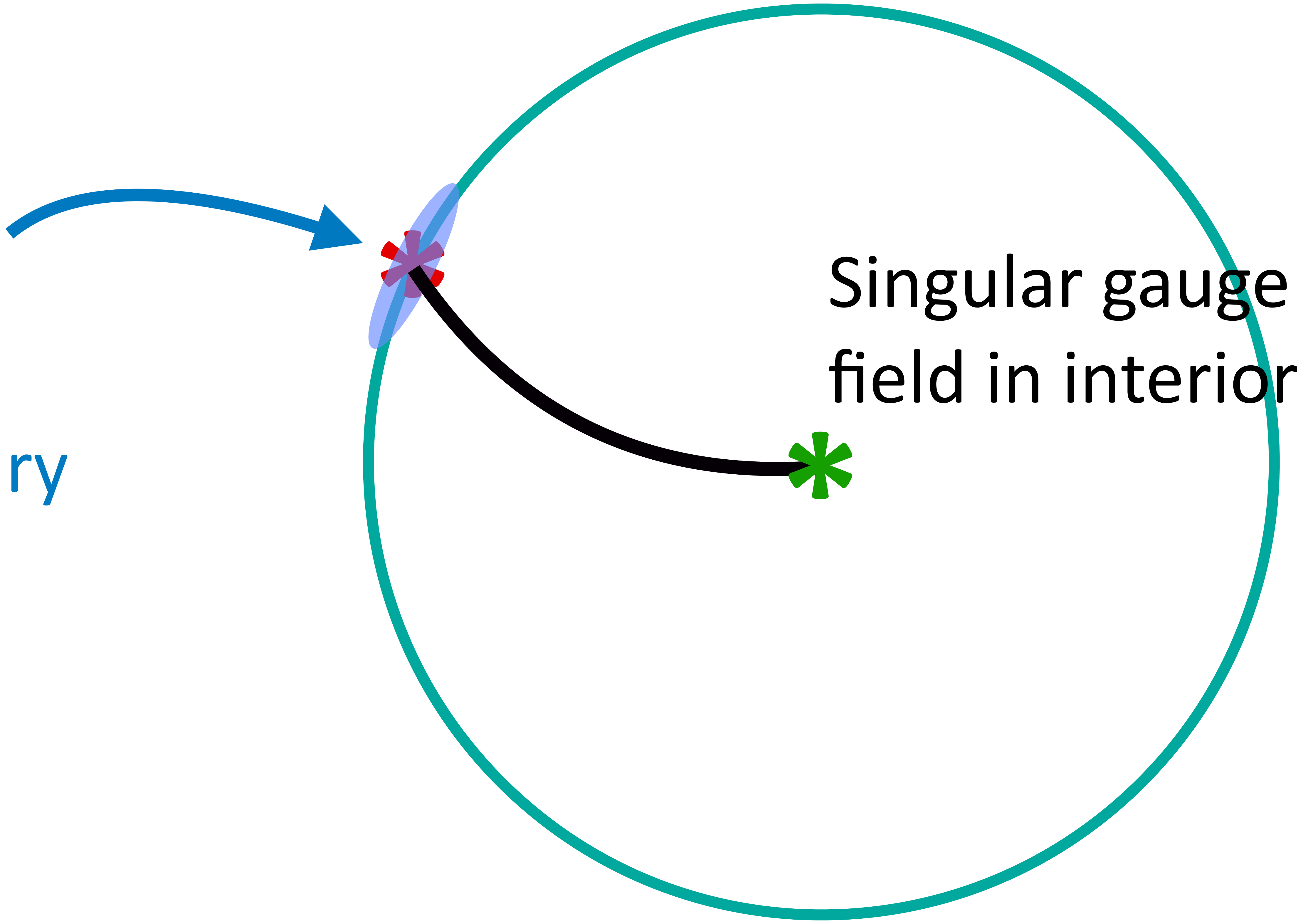

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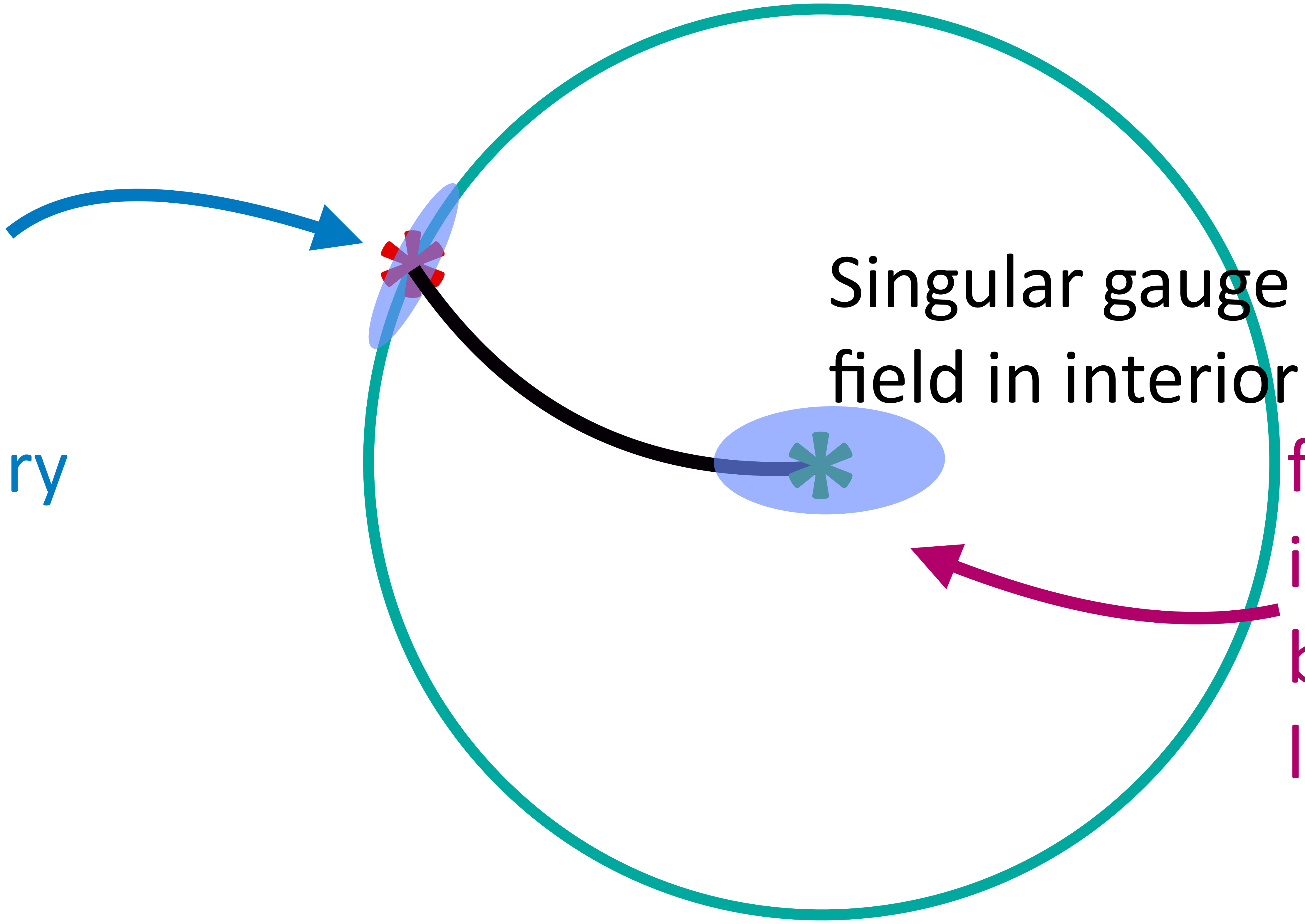

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fermion zeromode in bulk (Aoki et al.) Will be exponentially localized due to gap.

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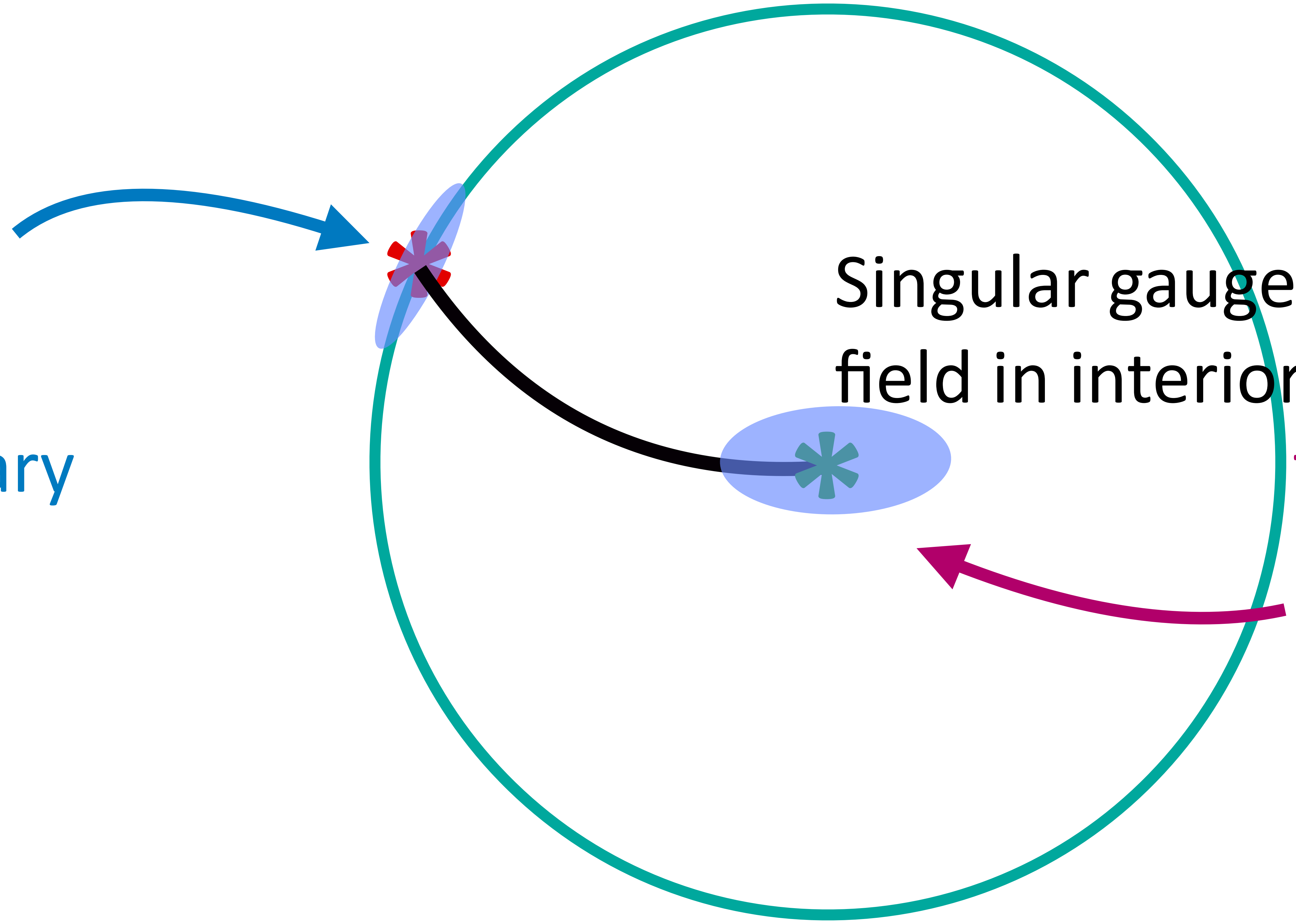

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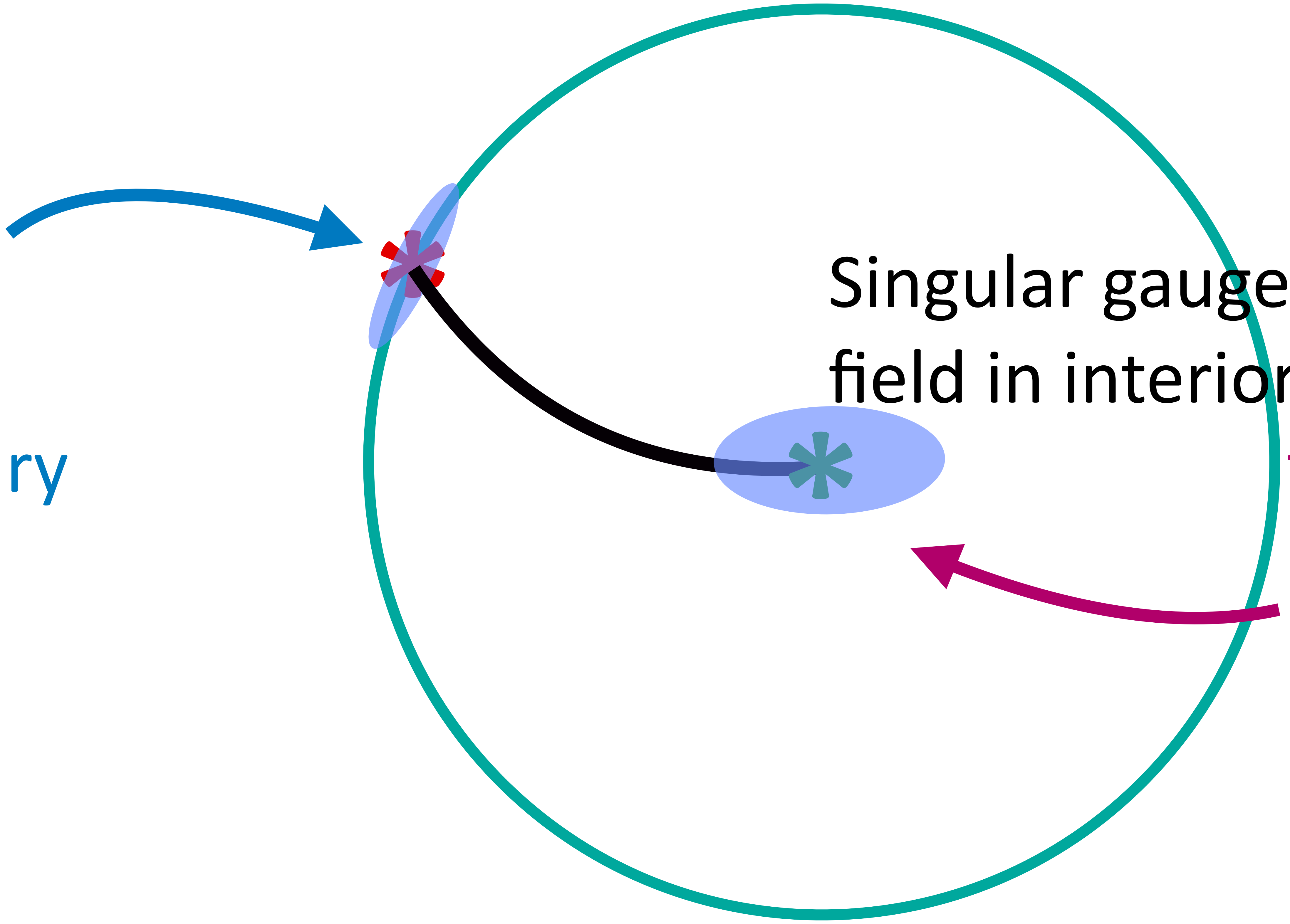

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Callan-Harvey analysis assumed all bulk fermions gapped, integrated them out.
Not true in the presence of nontrivial topology.

To redo the ’t Hooft analysis including bulk fermion zeromodes make assumption about gauge field flow into the interior (“annealing flow”):

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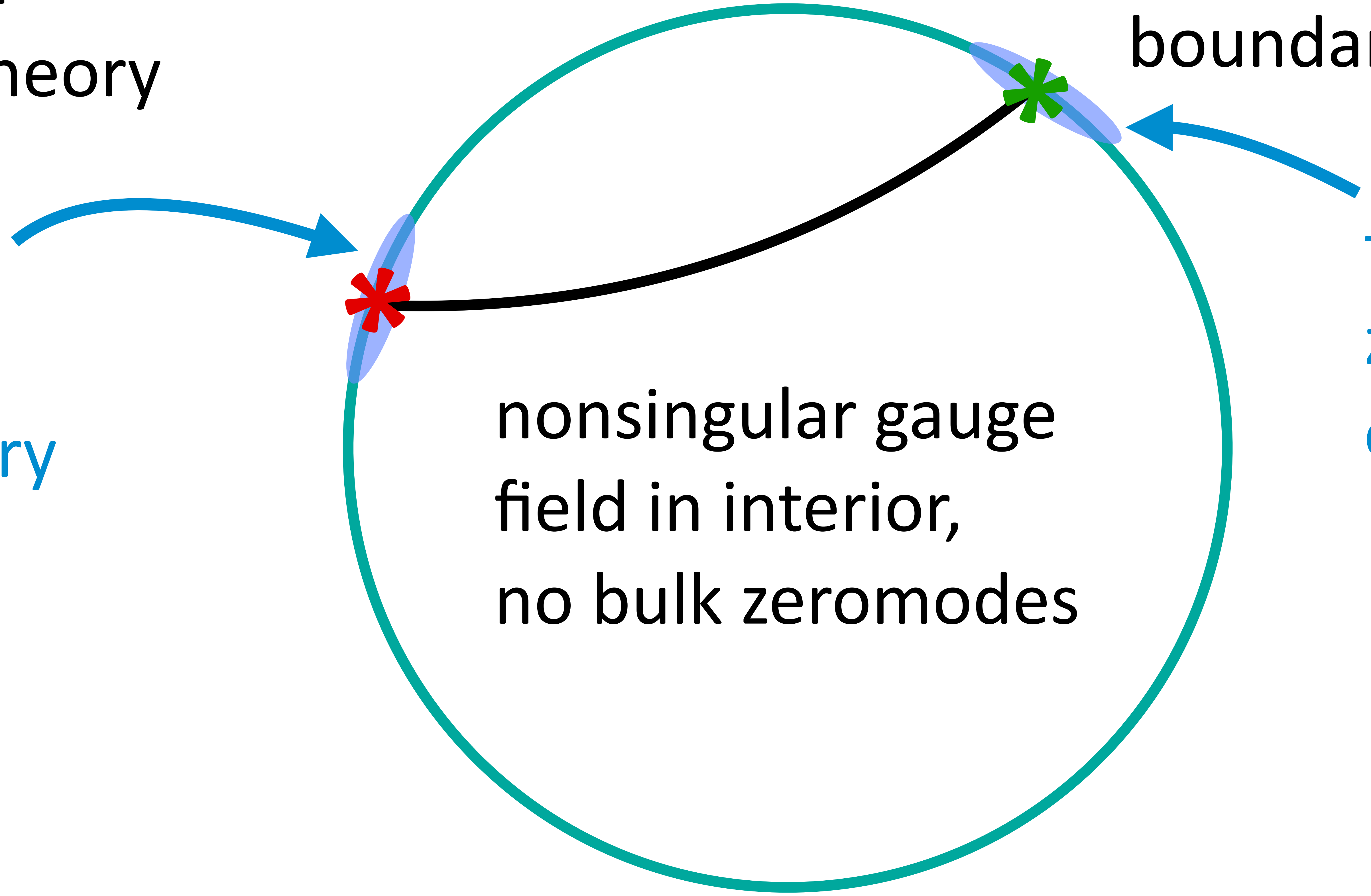

Instanton in
boundary theory

fermion
zeromode
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Anti-Instanton in
boundary theory

fermion
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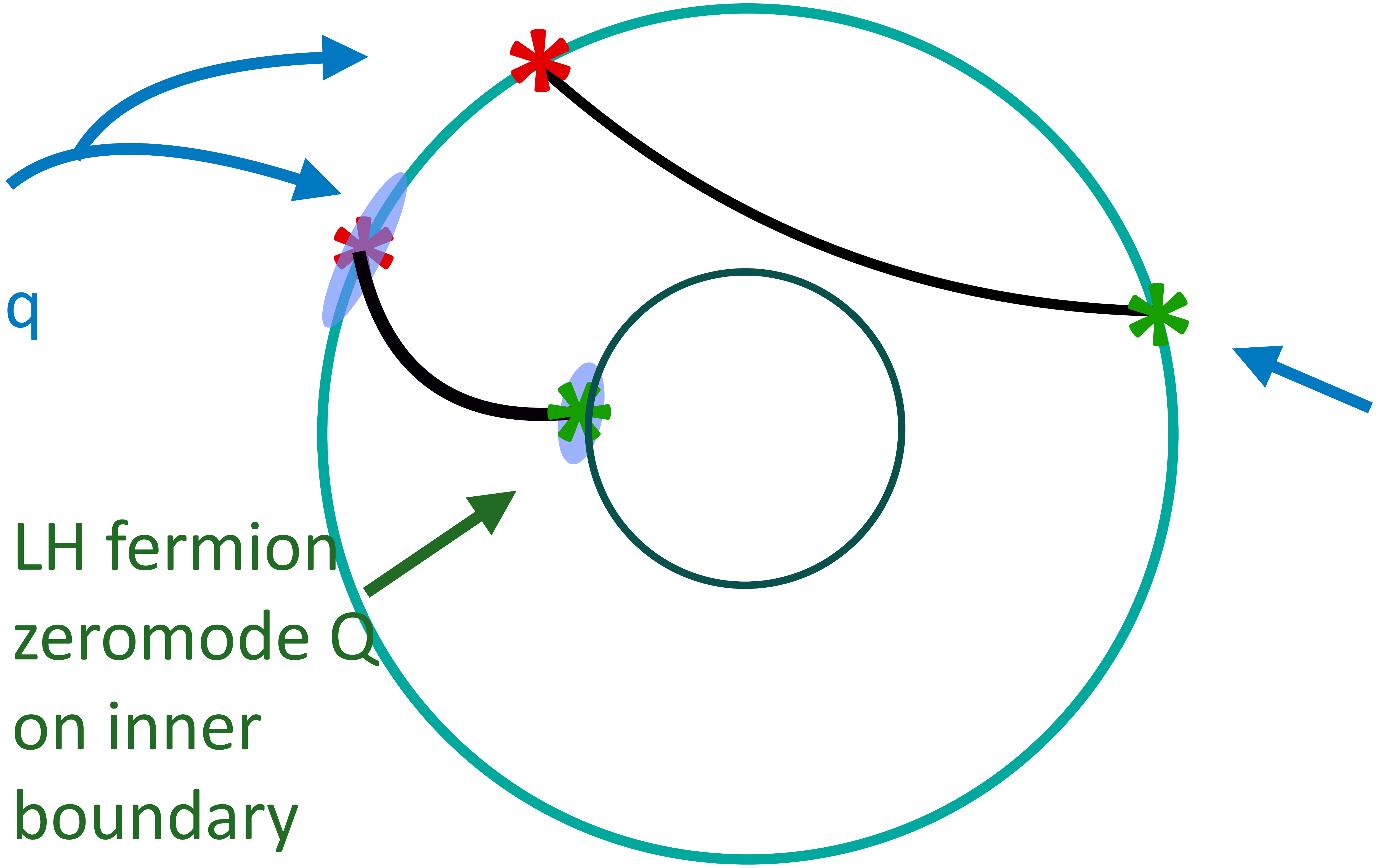
nonsingular gauge
field in interior,
no bulk zeromodes

To redo the ’t Hooft analysis including bulk fermion zeromodes simplify analysis by considering annulus, with no singularities in the bulk:



RH fermion zeromodes q on outer boundary

LH fermion zeromode Q on inner boundary

LH fermion zeromode q on outer boundary

Define ’t Hooft instanton vertices

$$\mathcal{O} = \Lambda \int_V \frac{d^4x}{V}\, \bar{q}_R q_L \ , \quad \bar{\mathcal{O}} = \Lambda \int_V \frac{d^4x}{V}\, \bar{q}_L q_R \ , \quad X = \Lambda \int_{V'} \frac{d^4y}{V'}\, \bar{Q}_R Q_L \ , \quad \bar{X} = \Lambda \int_{V'} \frac{d^4y}{V'}\, \bar{Q}_L Q_R \ ,$$

Sum the instanton contributions:

- n/n̄ instantons/anti-instantons on outer boundary
- |n-n̄| instantons or anti-instantons on inner boundary

$$\begin{aligned} e^{-\tilde{S}_{\rm int}} &= \sum_{n,\bar{n}} e^{i(n-\bar{n})\theta} \frac{(V\mathcal{O})^n}{n!} \frac{(V\bar{\mathcal{O}})^{\bar{n}}}{\bar{n}!} \Big((V'X)^{(n-\bar{n})}\Theta(n-\bar{n}) + (V'\bar{X})^{(\bar{n}-n)}\Theta(\bar{n}-n) + \delta_{n,\bar{n}} \Big) \\ &= Z_1 + Z_2 + Z_3 \end{aligned}$$

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Each term in sum is invariant under the exact 5d U(1)$_A$:

$$\mathcal{O} \to e^{2i\alpha} O \ , \quad \bar{\mathcal{O}} \to e^{-2i\alpha} \bar{O} \ , \quad X \to e^{-2i\alpha} X \ , \quad \bar{X} \to e^{2i\alpha} \bar{X}$$

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So only the 3rd sum Z_3 without X operators is experimentally accessible to us. This comes entirely from contributions where $n = \bar{n}$; we see topological fluctuations, but net topology is zero.

$$Z_3 \equiv e^{-\widetilde{S}_{\text{inst.}}} = \sum_{n=0}^{\infty} \frac{\left(\Lambda \int \bar{q}_R q_L(x)\, dx\right)^n}{n!} \frac{\left(\Lambda \int \bar{q}_L q_R(y)\, dy\right)^n}{n!} = I_0\left(2\Lambda V \sqrt{\overline{\bar{q}_R q_L}\;\; \overline{\bar{q}_L q_R}}\right)$$

$$\widetilde{S}_{\text{inst.}} \xrightarrow{V\to\infty} V\left[-2\Lambda\sqrt{\overline{\bar{q}_R q_L}\;\;\overline{\bar{q}_L q_R}} + O\left(\frac{\ln V}{V}\right)\right]$$

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Now assume chiral symmetry breaking and match to the η’ Lagrangian as before

$$\begin{aligned}\mathcal{L}_{\rm inst.} &= -VM_{\eta'}^2 f^2\sqrt{\overline{\left(e^{i\eta'/f}\right)}\;\;\overline{\left(e^{-i\eta'/f}\right)}}\,. \\ &= \tfrac{1}{2}\partial_\mu\eta'\partial_\mu\eta'(x) + M_{\eta'}^2 f^2\left[-1 + \tfrac{1}{2}\left(\eta'(x) - \overline{\eta'}\right)^2 + O(\eta'^4)\right]\end{aligned}$$

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- …yet at the same time, the η’ is heavy and there is no axion
- Perhaps QCD embedded in SM is not equivalent to standard LQCD at nontrivial topology?

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To do: beyond the lattice

- If a Hamiltonian formulation is possible, there will be a dynamical Minkowski spacetime version of the theory… it will be weird, given that only 4d gauge fields are dynamical. Can one construct a cosmological model for 5d BSM physics?