Lattice fermions and Floquet insulators

Srimoyee Sen,
Iowa State University

In collaboration with

Raul Briceno,
Berkeley, LBNL

Thomas Iadecola,
Penn State

Lars Sivertsen,
ISU (2019-2024)

ECT* workshop Bridging analytical and numerical methods for quantum field theory

Based on: *Phys.Rev.Lett.* 132 (2024) 13, 136601, *Phys.Rev.Res.* 6 (2024) 1, 013098, e-Print: 2504.10601

Outline

Talk at the intersection of lattice field theory and quantum systems, in particular topological phases in driven systems.

Introduce some of the striking similarities between static quantum systems and lattice.

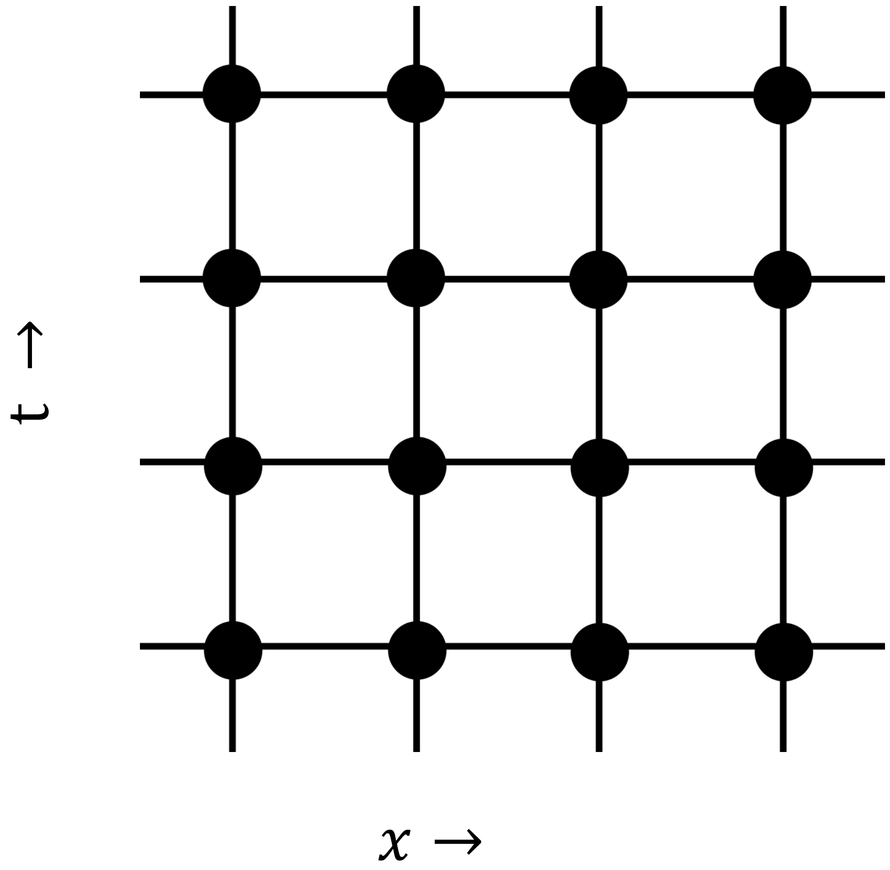
Are there similar ties between lattice field theory and driven quantum systems?

Yes.

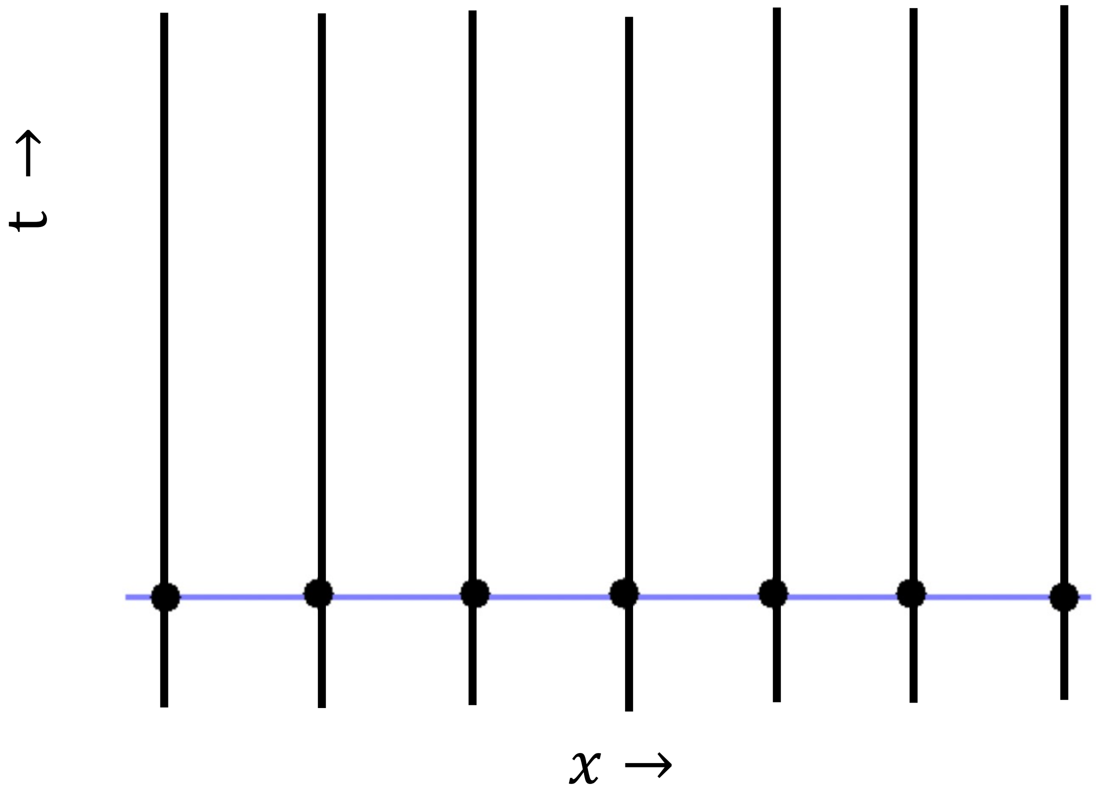
Will motivate where these ties can arise from.

Show an explicit example in one spatial dimension connecting the two.

Straight to the point



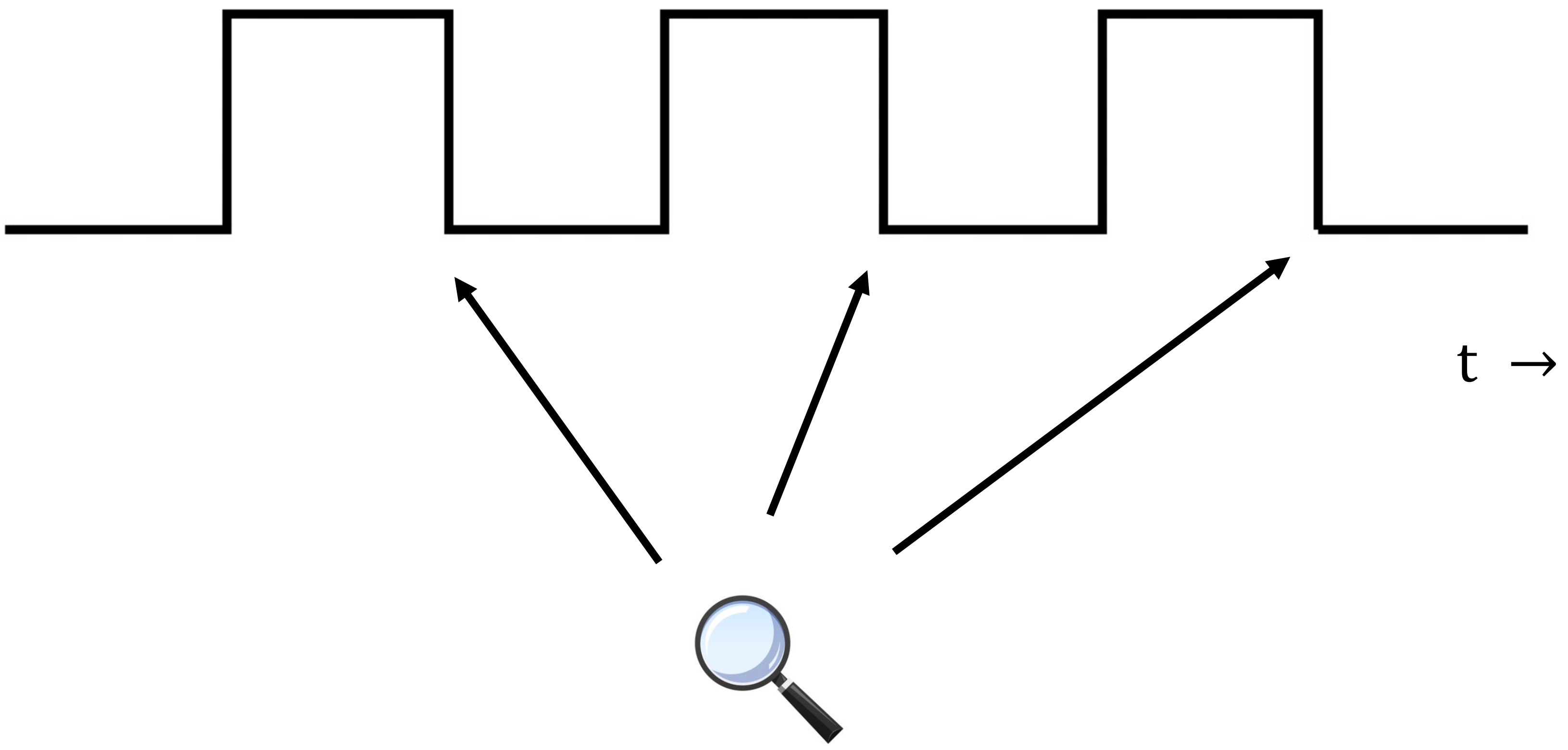
space-time lattice for lattice QFT



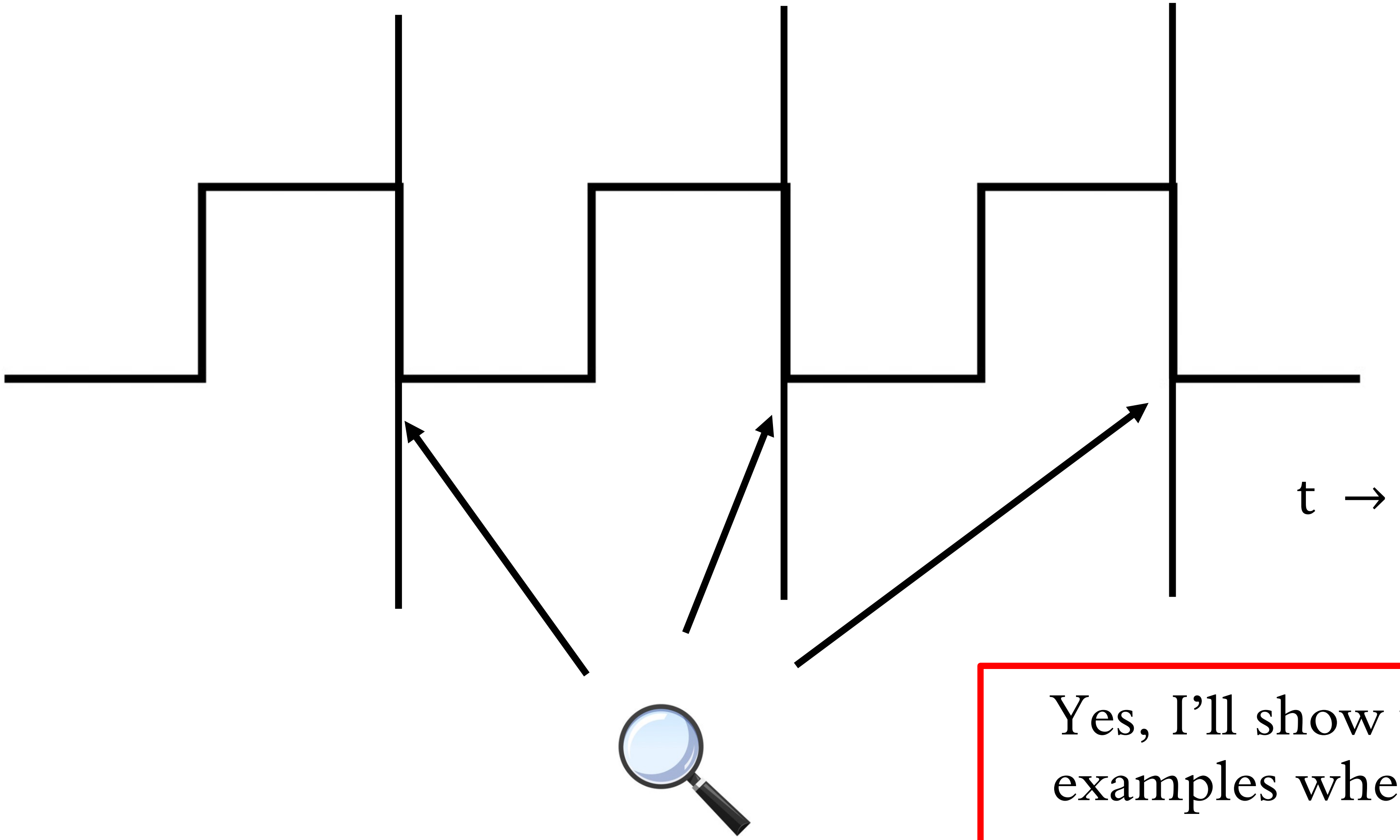
Real world, time is continuous.
Spatial lattice can be found

And yet

Periodically driven quantum systems (Floquet)



Sometimes loosely referred to as discrete time systems (are they really?)



Yes, I'll show you examples where true.

The ties in continuous space-time,static quantum systems (topological insulators) and Dirac fermion

Insulators two types:

1. Uninteresting (For the purpose of this talk) → Trivial, uninteresting bulk and boundary. Gapped bulk and boundary

2. Interesting → Topological, sometimes boring bulk physics but interesting with a boundary. Gapped bulk, gapless boundary.

The ties in continuous space-time

Relativistic fermion with a domain wall in mass:

Massive fermion → Gapped bulk

Domain wall → Boundary

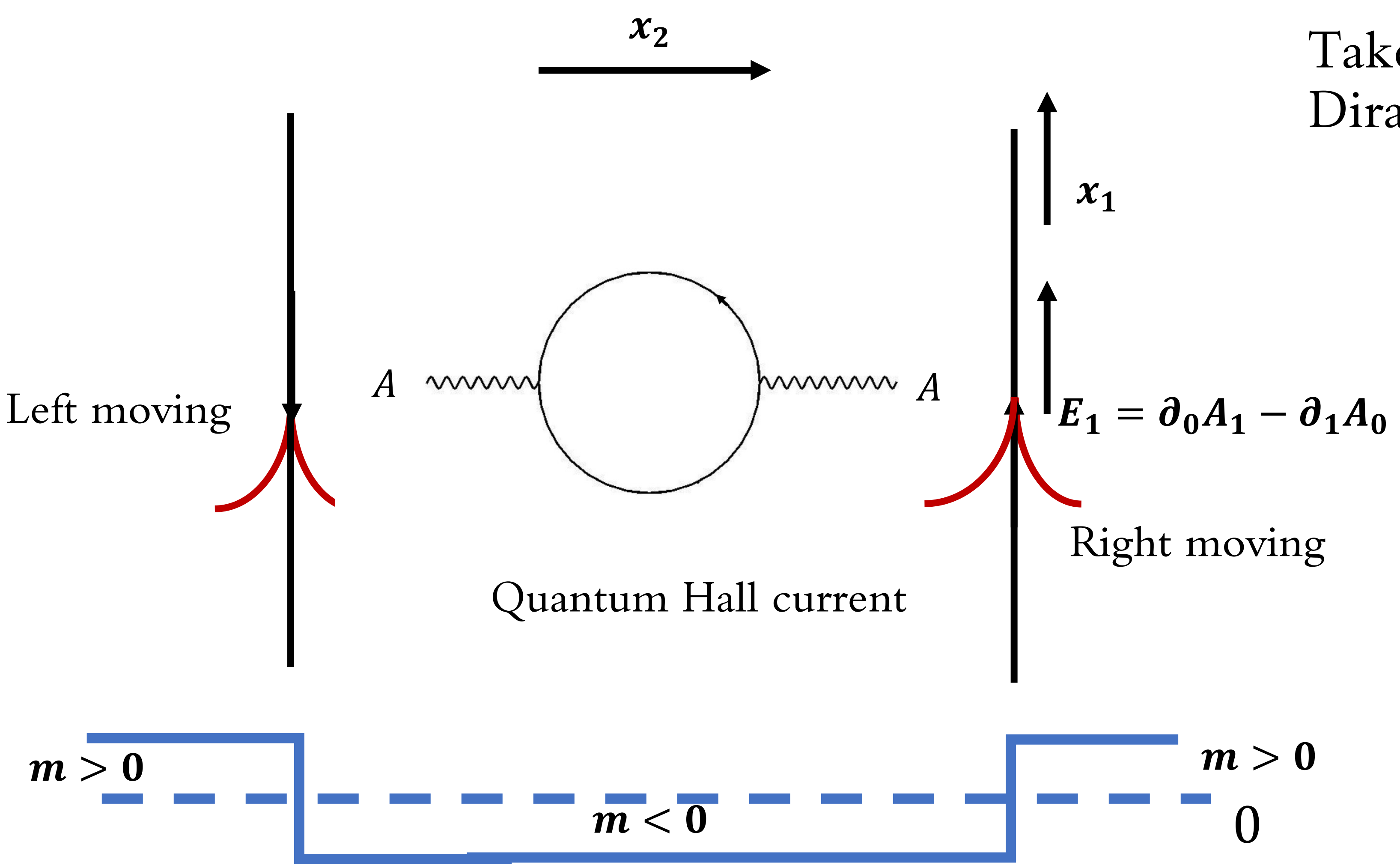
Domain wall is a boundary between boring and interesting

Relativistic fermion with open boundary condition:

Positive mass: Trivial, no boundary modes.

Negative mass: Topological, boundary modes etc.

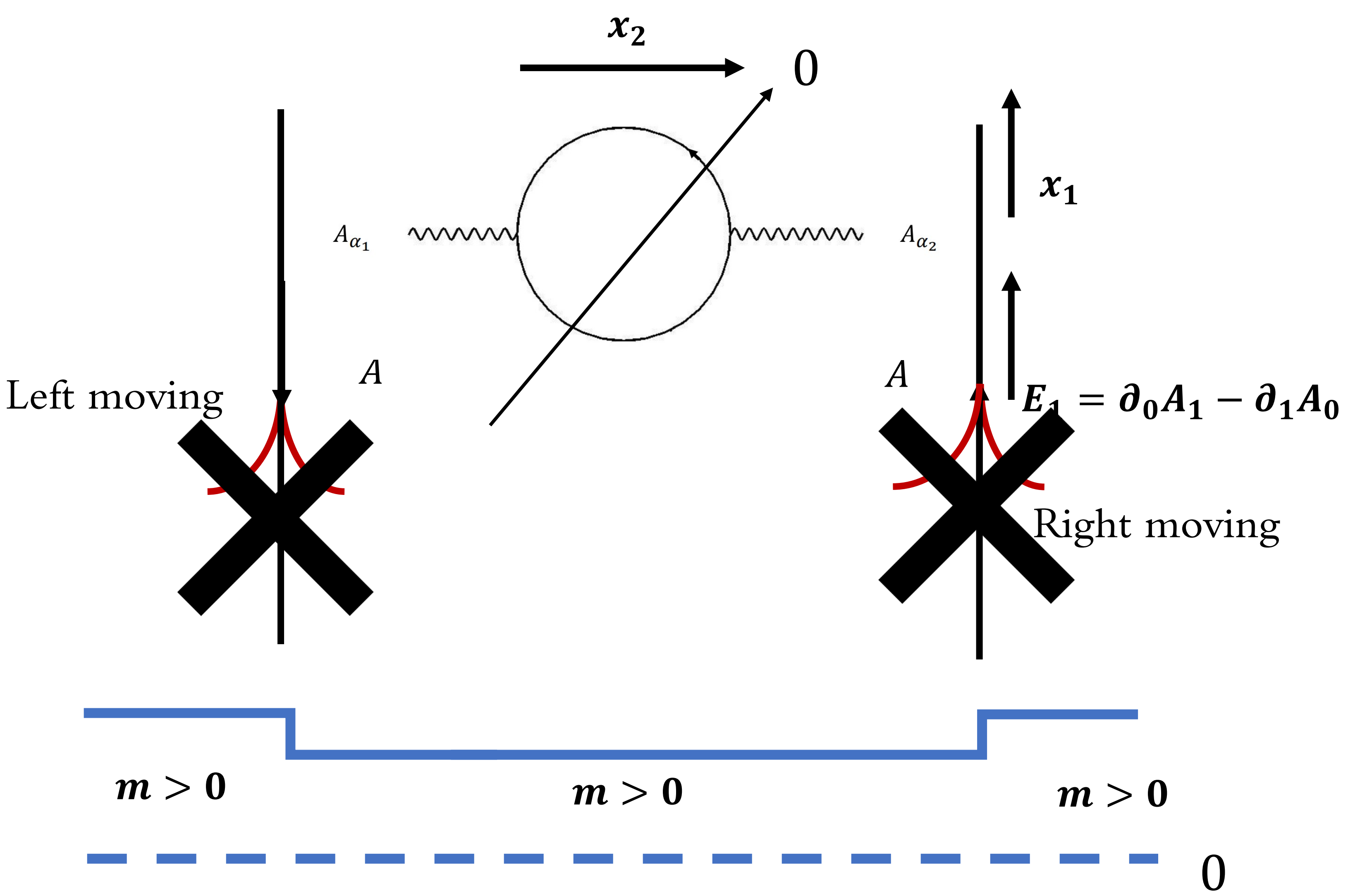
Relativistic fermion: Quantum Hall Effect(QHE)(Callan-Harvey 1984)



Take a $2+1$ dimensional Dirac fermion

Step function in Dirac mass m in x_2 (domain walls)

Transition to boring (non-topological)

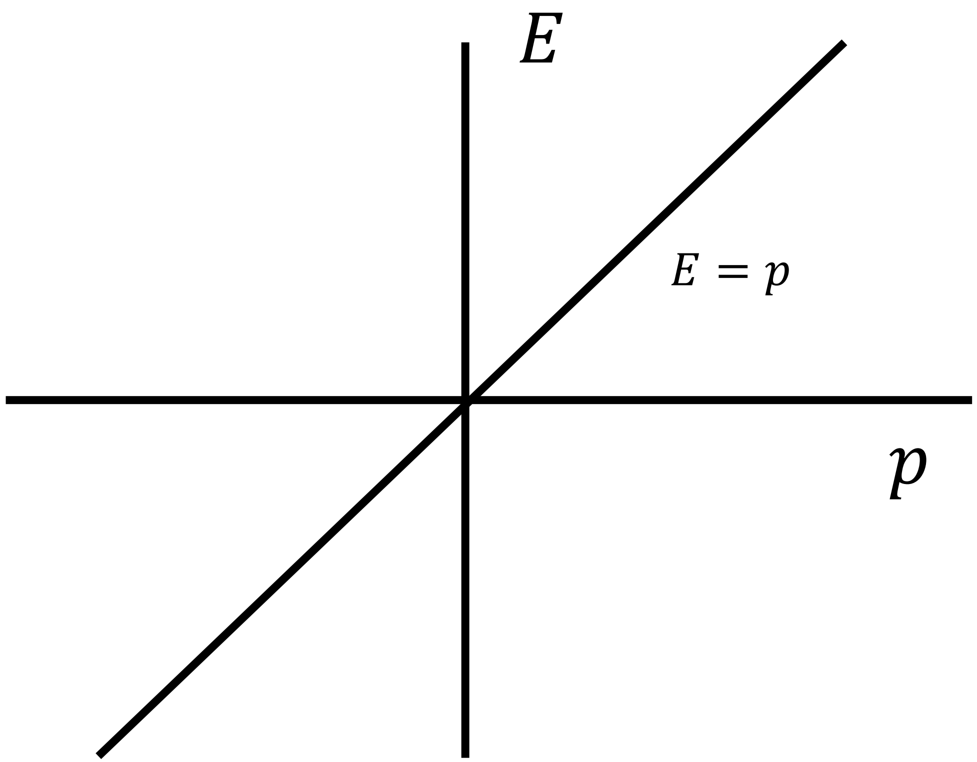


Some interesting points about lattice

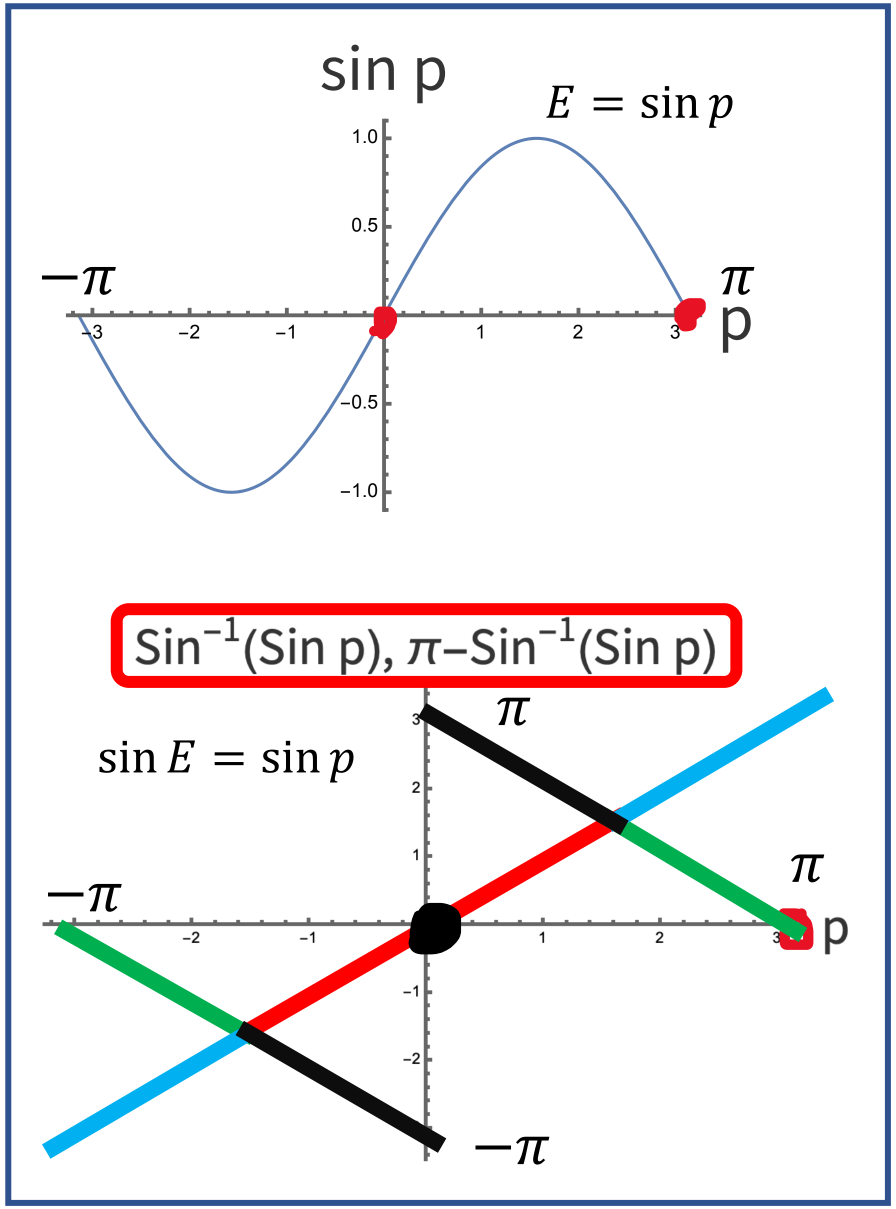
- Discretization of space-time or space alone lead to new features (topological phases) in both.

- Set of new features in the latter, a subset of the former.

Brilliuoin zones(Weyl) discrete space and time



Continuum



Lattice

Weyl Fermion modes lattice : $(0,0), (0,\pi), (\pi,0), (\pi,\pi)$

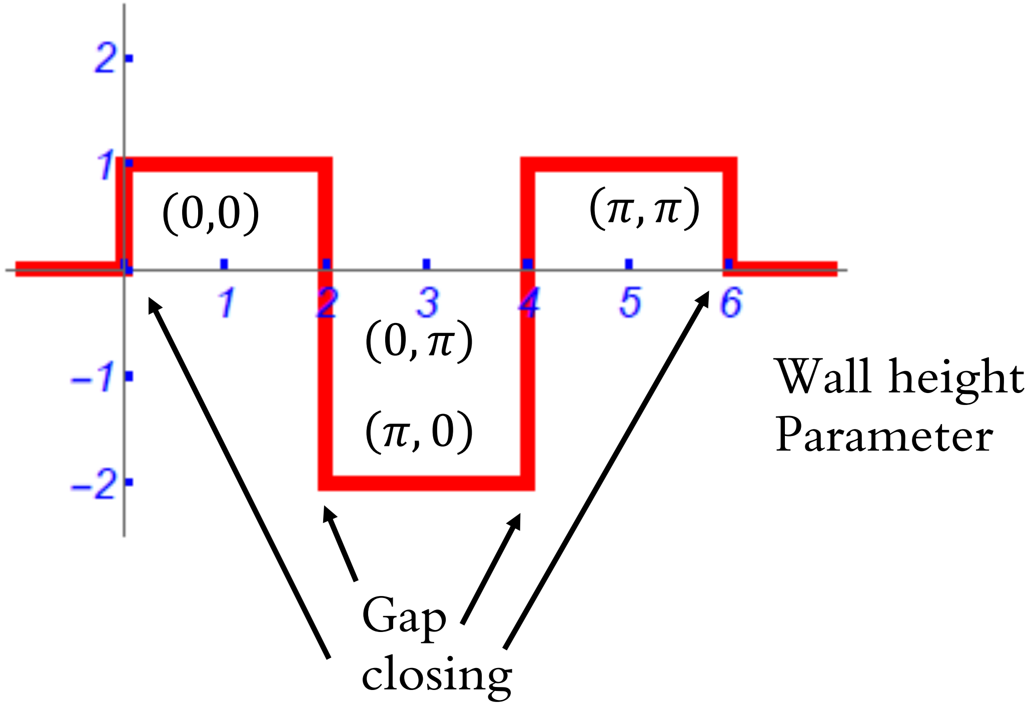
$+ \quad - \quad - \quad +$

Discrete space discrete time domain wall setup

Weyl Fermion modes:

$(0,0), (0,\pi), (\pi,0), (\pi,\pi)$

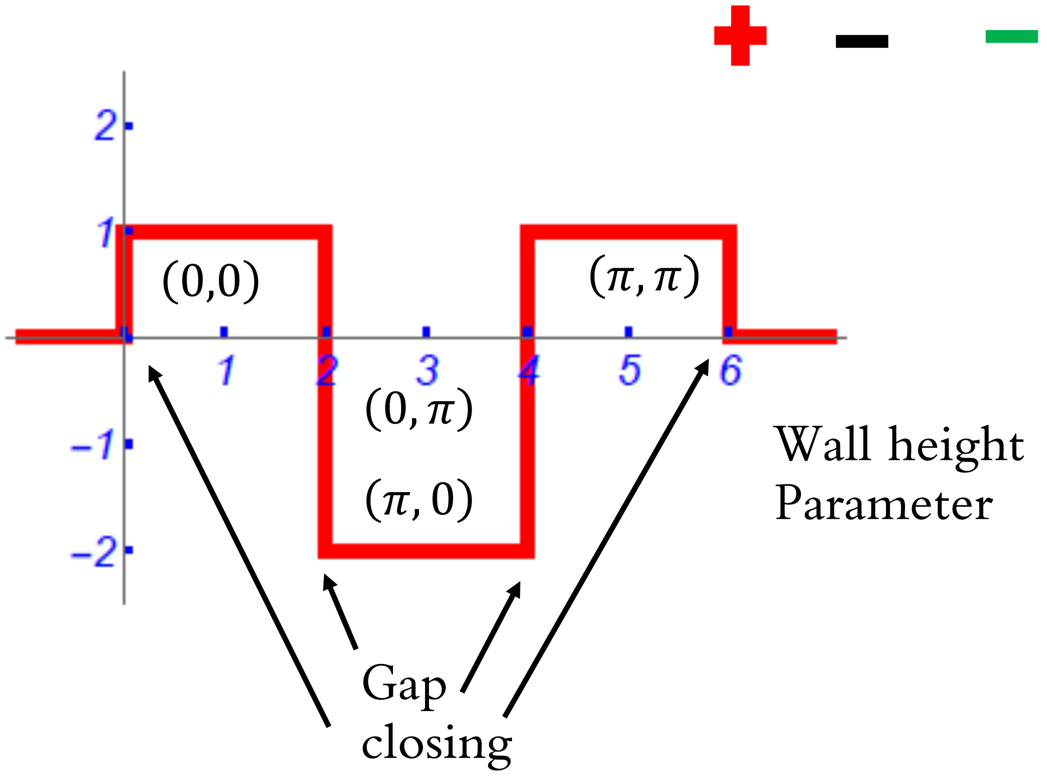
\+ − − +



Wilson-Dirac model with a domain wall
(Kaplan, Jansen, Golterman, 1992; Jansen Schmaltz 1992)

Discrete time vs continuous time

Weyl Fermion modes: $(0,0), (0,\pi), (\pi,0), (\pi,\pi)$



Wilson-Dirac with a domain wall
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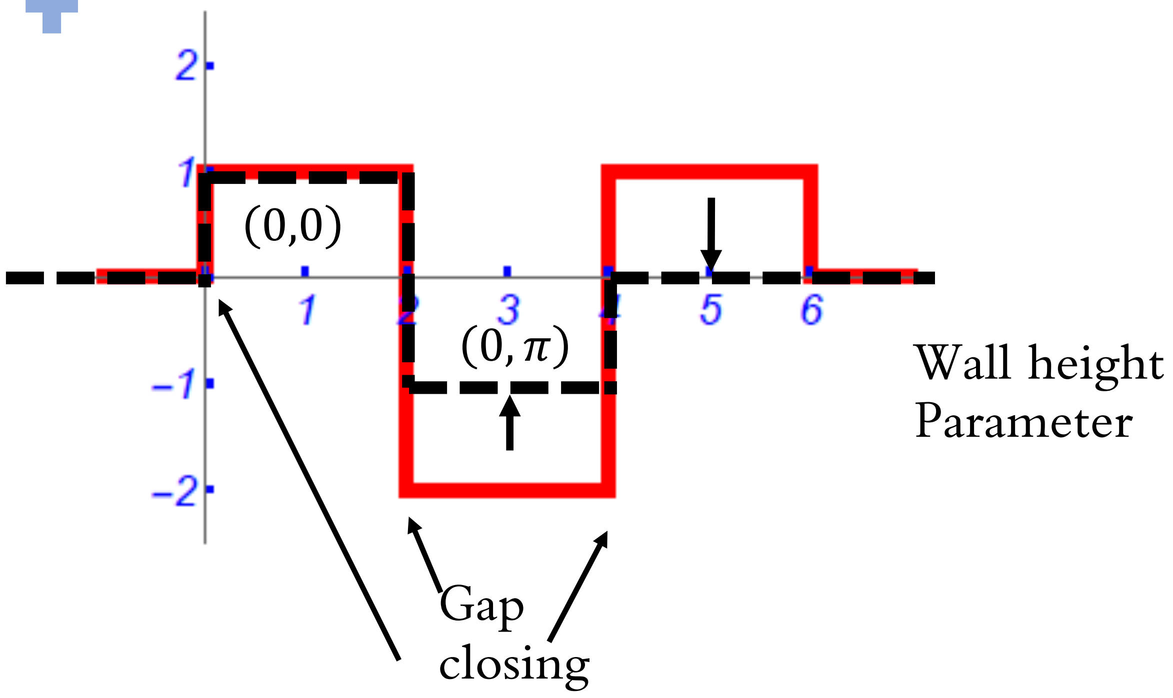


BHZ model (Wilson-Dirac with continuous time) with a domain wall (Similar to TKNN)

Curious case of Floquet insulators (free fermion)

What's needed for this talk :

Continuous time but periodically driven.

Can exhibit novel phases: similar to undriven case.

Topological transition associated with gap closing.

Curious case of Floquet insulators

What's needed for this talk :

Continuous time but periodically driven.

Can exhibit novel phases: similar to undriven case.

Topological transition associated with gap closing.

What does this mean? Energy is not conserved.

Curious case of Floquet insulators

Driving a Hamiltonian over period T.

Observe the system at integer multiples of T.

Define quasi energy:

Time evolution operator $U_F(T)$. Quasi energy is the $\frac{i}{T}\log U_F(T)$.

Conserved.

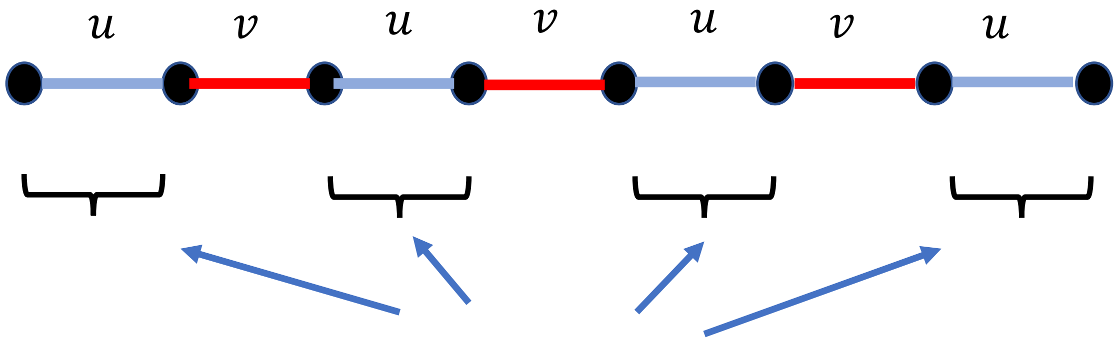
Curious case of Floquet insulators

Identify phase boundaries by considering gap closing in quasi energy.

Interestingly, we observe boundary modes of quasi energy: $\frac{\pi}{T}$.

Reminiscent of time doublers in lattice field theory.

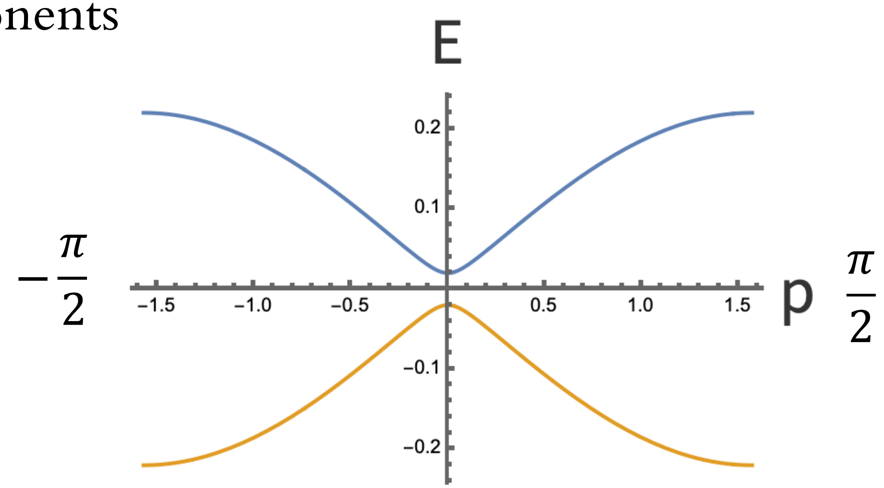
SSH model(very similar to lattice staggered fermion: Dirac)



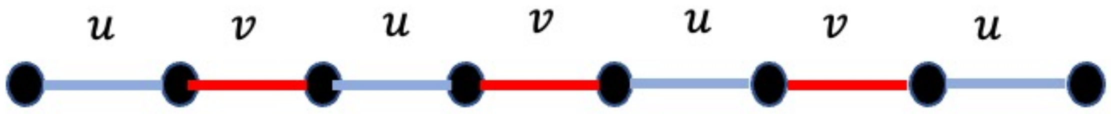
spatial lattice

unit cell,
two sites –> two components
of Dirac fermion

Periodic boundary (PBC) condition
leads to Dirac dispersion



SSH model: Static topological Hamiltonian



PBC: $E(p) = \pm\sqrt{(u-v)^2 + 4uv\sin^2 p}$

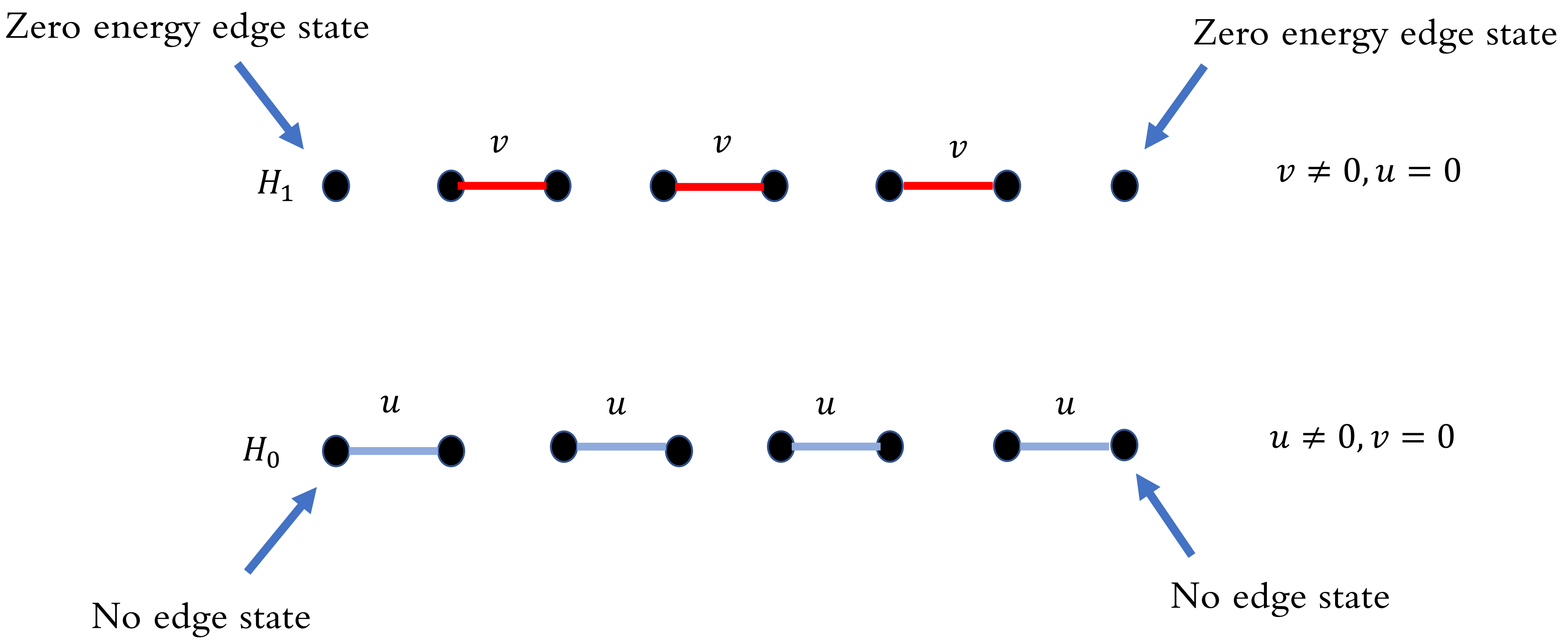
$u - v$ is Dirac mass.

$v - u > 0$: topological phase with zero energy edge mode for OBC (open boundary)

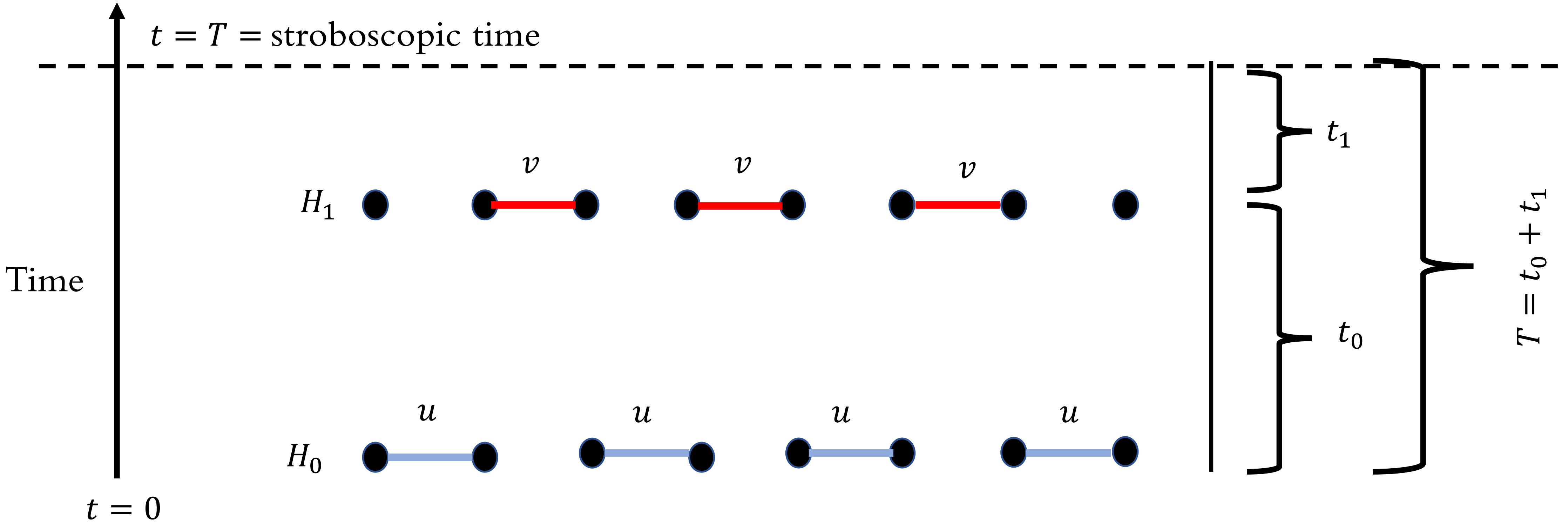
$u - v > 0$: non-topological phase with zero energy edge mode for OBC (open boundary)

Open boundary is the same as having mass defect

Edge states (with OBC)



Driven SSH model

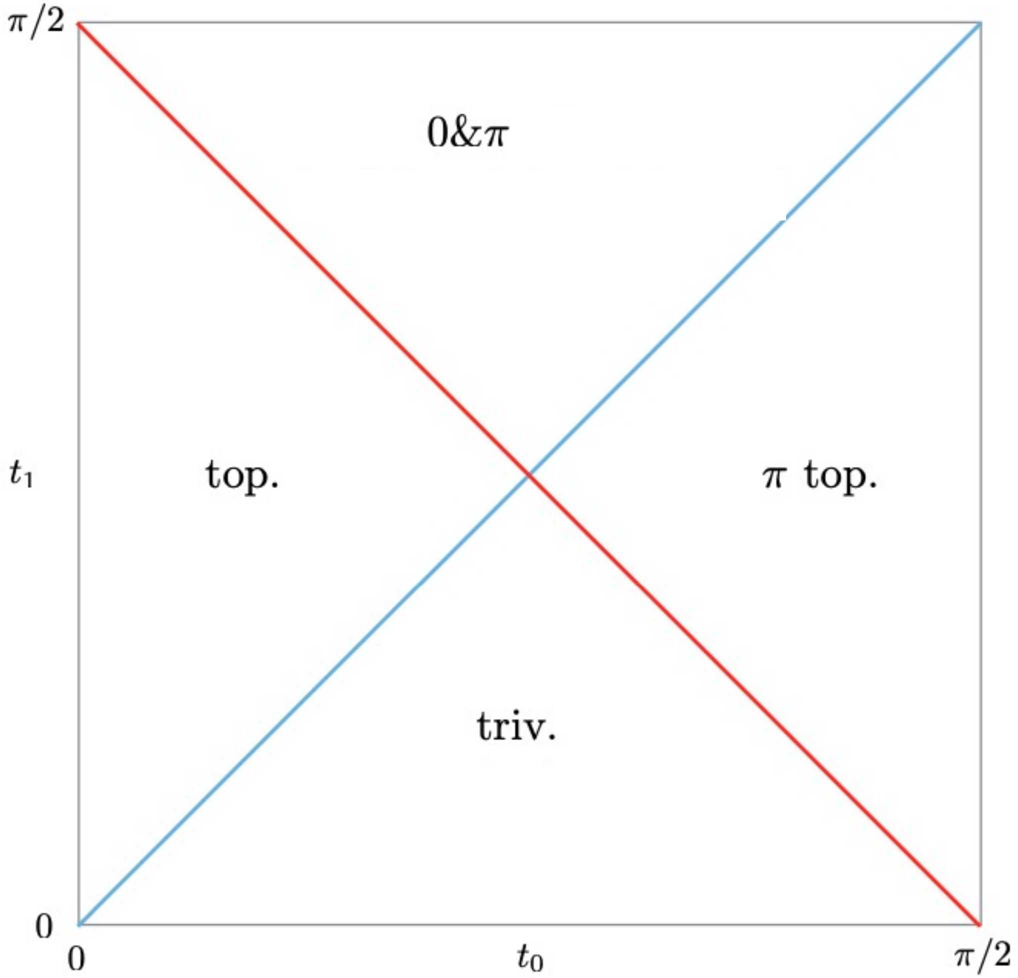


$$U_F(T) = e_0^{-iH_1 t_1} e^{-i\,H_0 t_0} \equiv e^{-i\,H_F T}$$

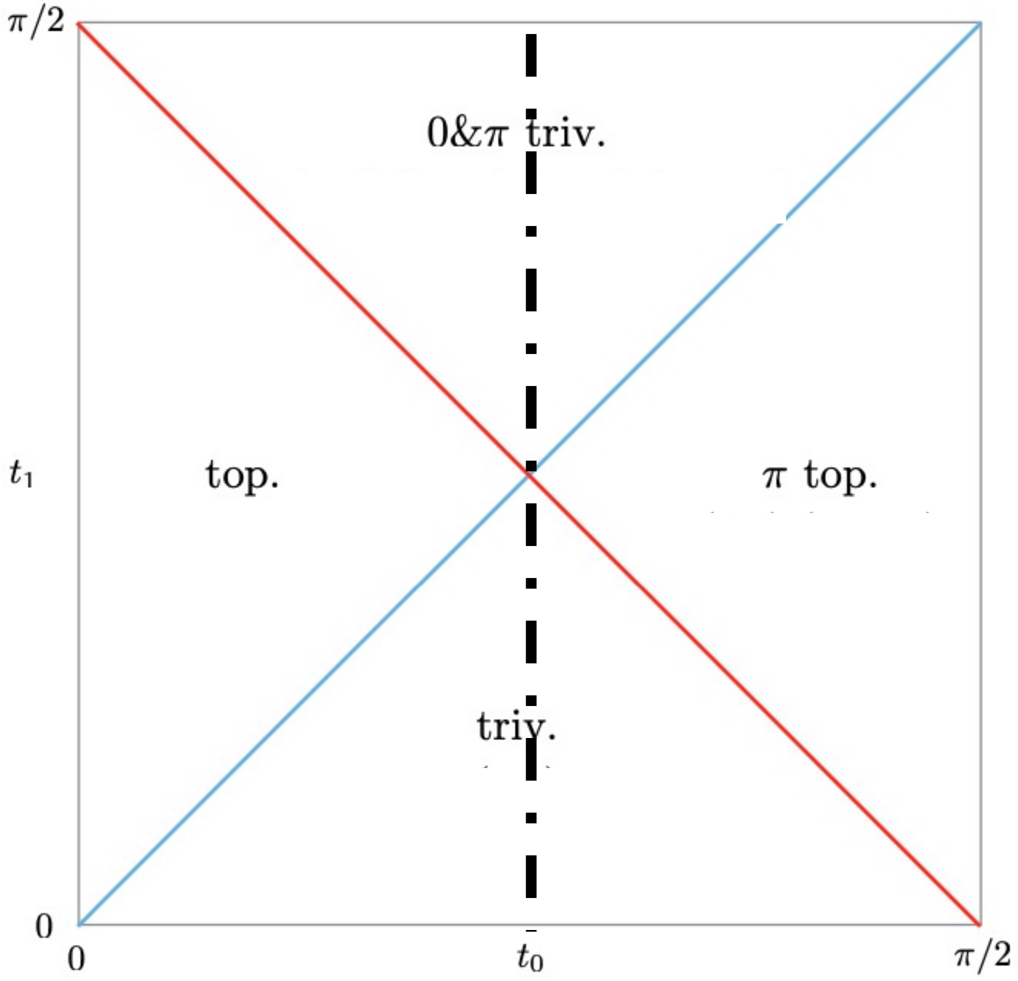
Get the quasi-energy ϵ by taking a log

Inspired by Keyserlingk and Sondhi, Majorana model

Quasi-energy Phase diagram (PBC, OBC)

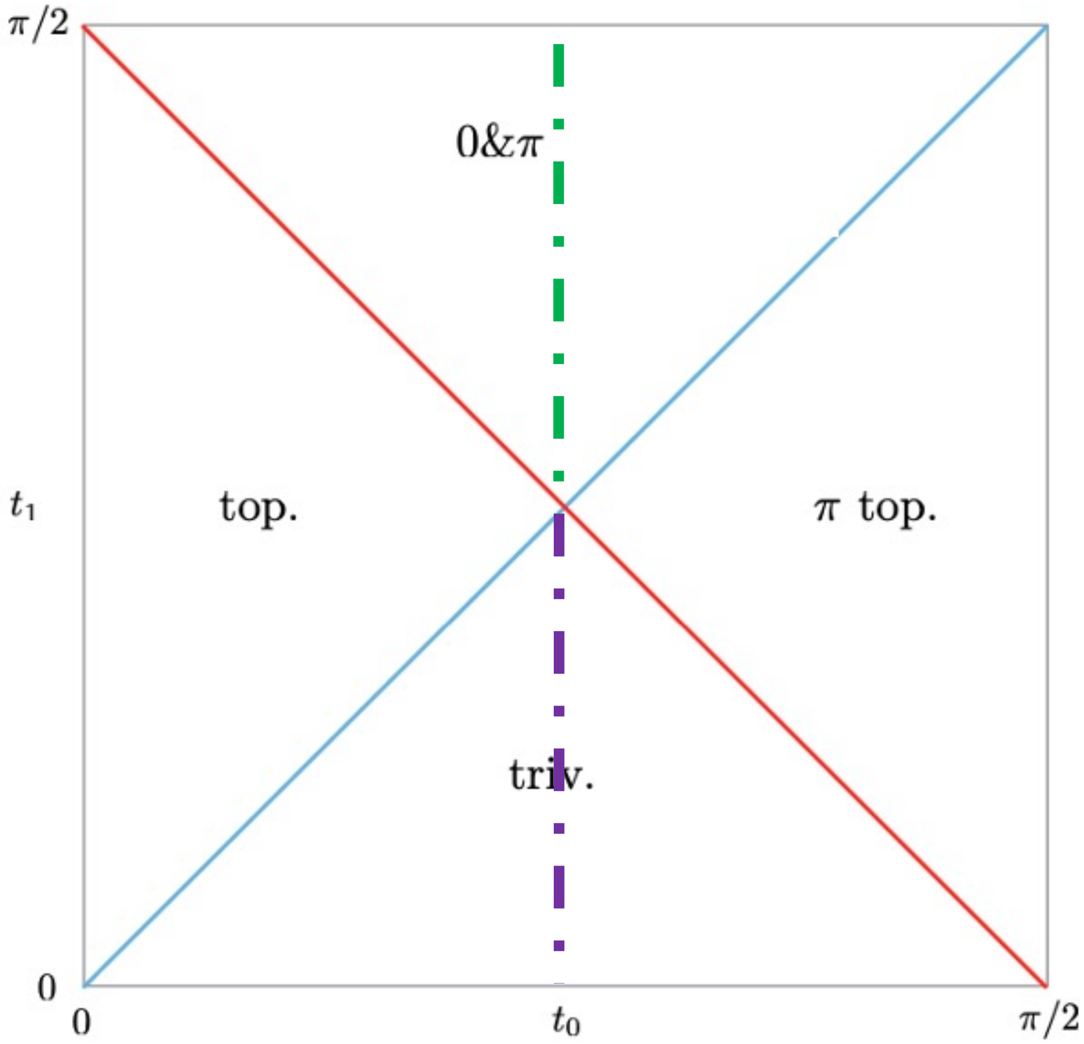


Energy eigenvalues with PBC



Along that line energy eigenvalues come in pairs.

i.e. ϵ and $\frac{\pi}{T} - \epsilon$ appear together.



The two regions indistinguishable with PBC. They have the same eigenvalues

Reflection symmetric about $t_1 = \frac{\pi}{4}$.

With OBC, one has zero and pi modes. The other has none.

Appears mappable to a discrete time lattice Hamiltonian.

What is this Hamiltonian like?

The zero eigenvalue map

Zeroes of $(i\partial_t - H_F)$ to the zeroes discrete time operator $(i\nabla_t - H)$

$\downarrow$

$$p_0 - \epsilon_{H_F} = 0$$

ϵ_{H_F} are eigenvalues of H_F

$\downarrow$

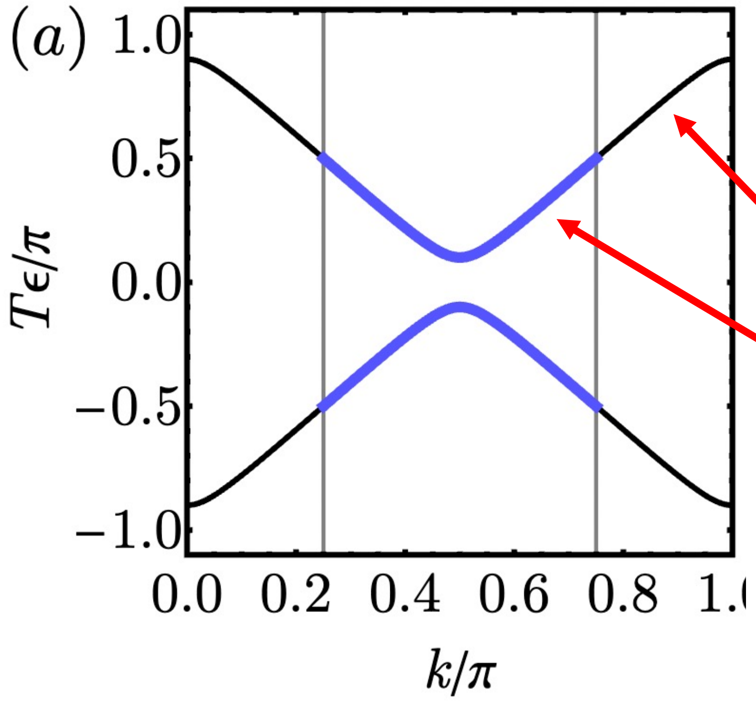
$$\sin p_0 T - \epsilon_H T = 0$$

ϵ_H are eigenvalues of H

Clearly, H has to have half the dimensions as that of H_F.

What is H ?

The Floquet spectrum on the symmetric line (PBC)



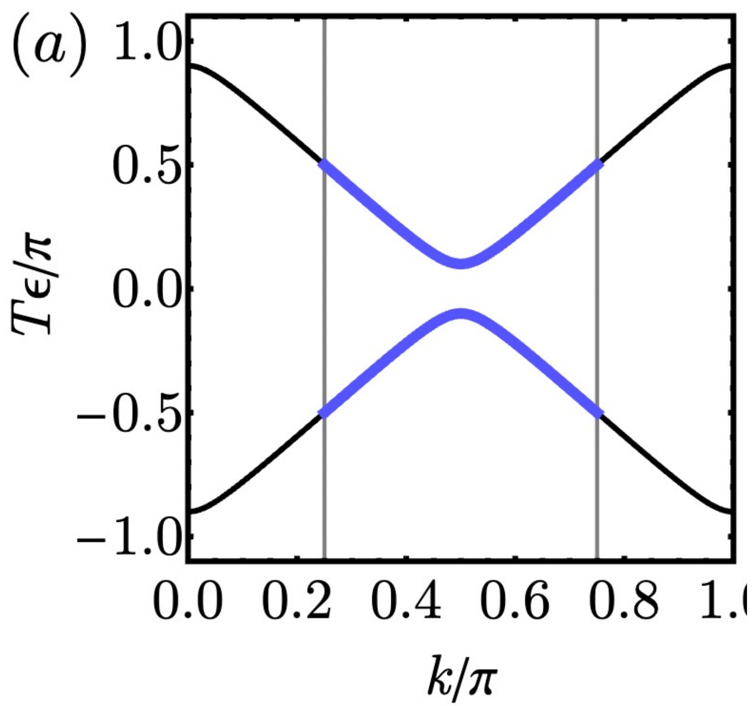
Obtained by evolving the Hamiltonians under PBC

for the two different sets of u and v for time t_0 and t_1.

The blue and black regions are π paired with each other.

Discard half

Discard half of the eigenvalues of H_F. Which ones?



Keep the blue line, discard the black one.

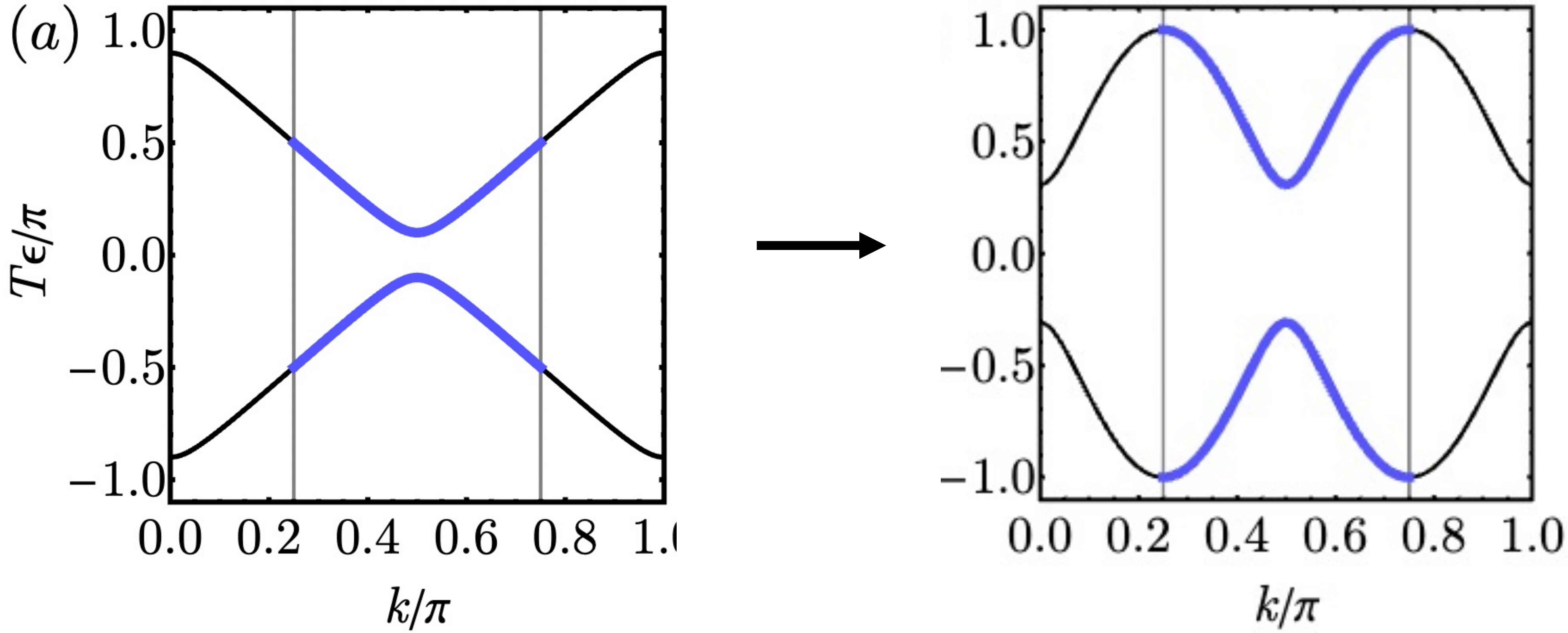
If we solve the discrete time Schroedinger equation with these eigenvalues for energy, what solutions do we obtain?

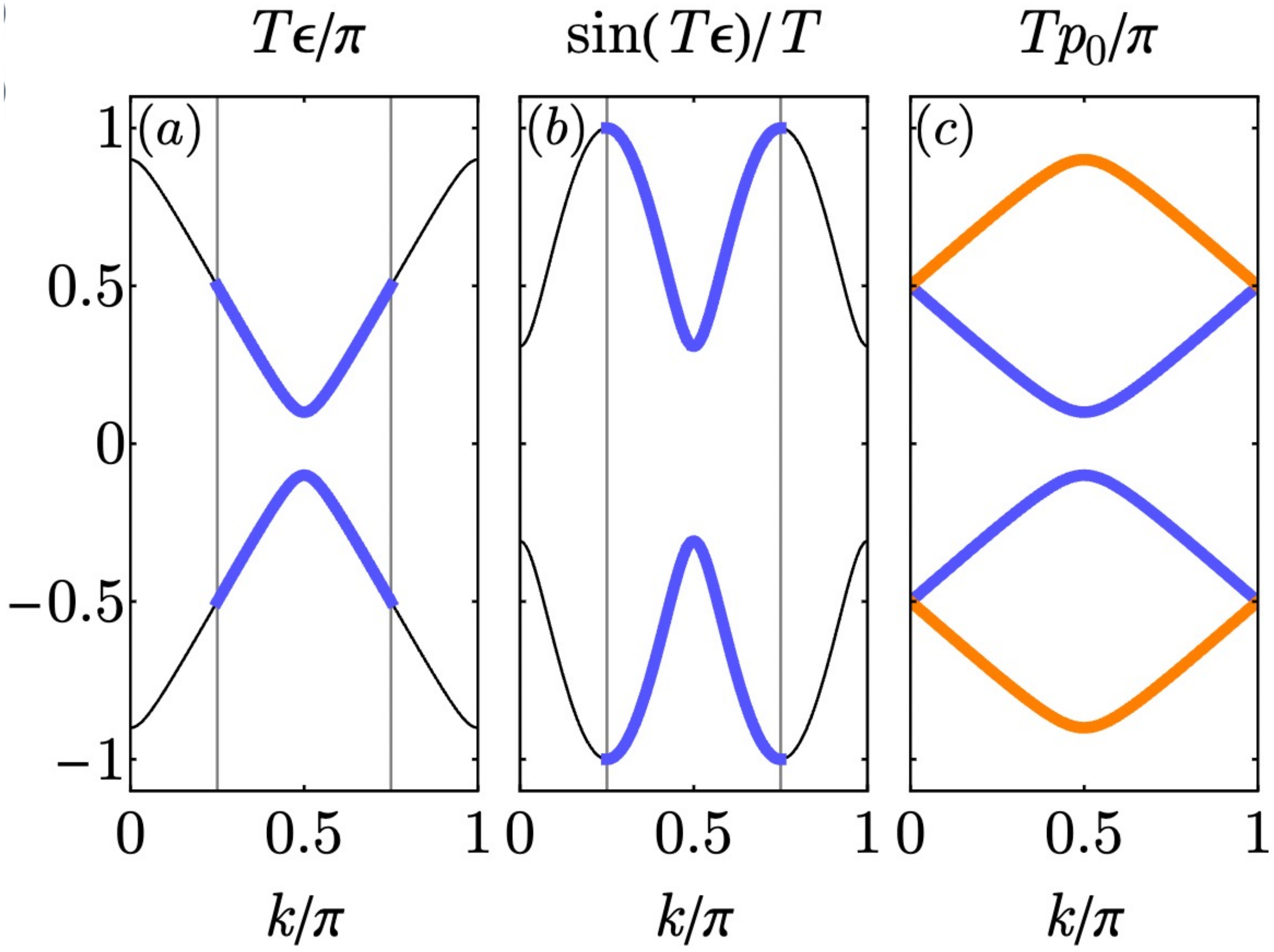
$$\sin(Tp_0) = T\,\epsilon$$

$$\Rightarrow\ p_0 = \frac{1}{T}\sin^{-1} T\epsilon \quad \text{and} \quad p_0 = \frac{\pi}{T} - \frac{1}{T}\sin^{-1} T\epsilon$$

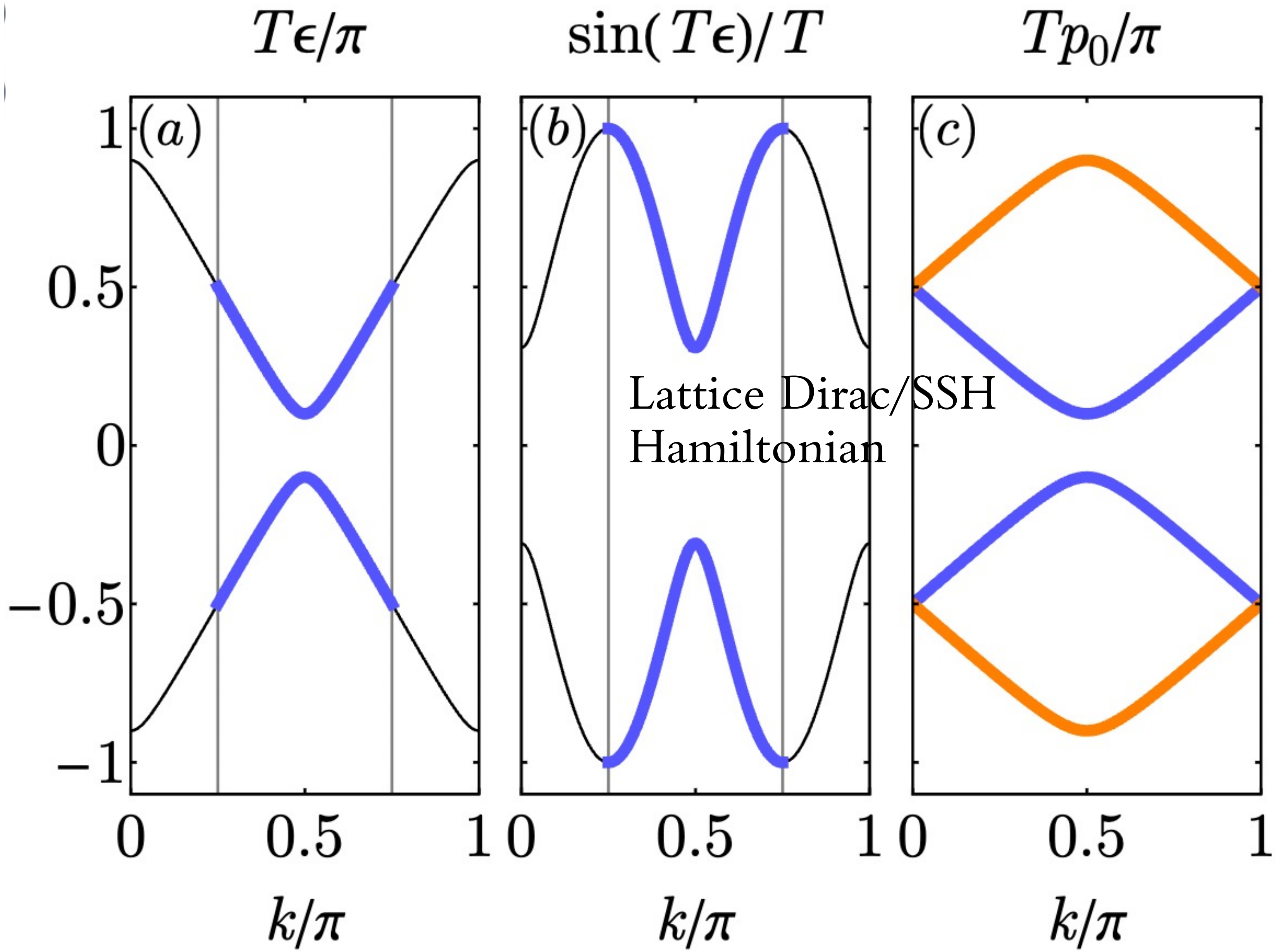
Not what we want. We need to solve $\sin(Tp_0) = \sin T\epsilon$

The sine transformed eigenvalues (PBC)



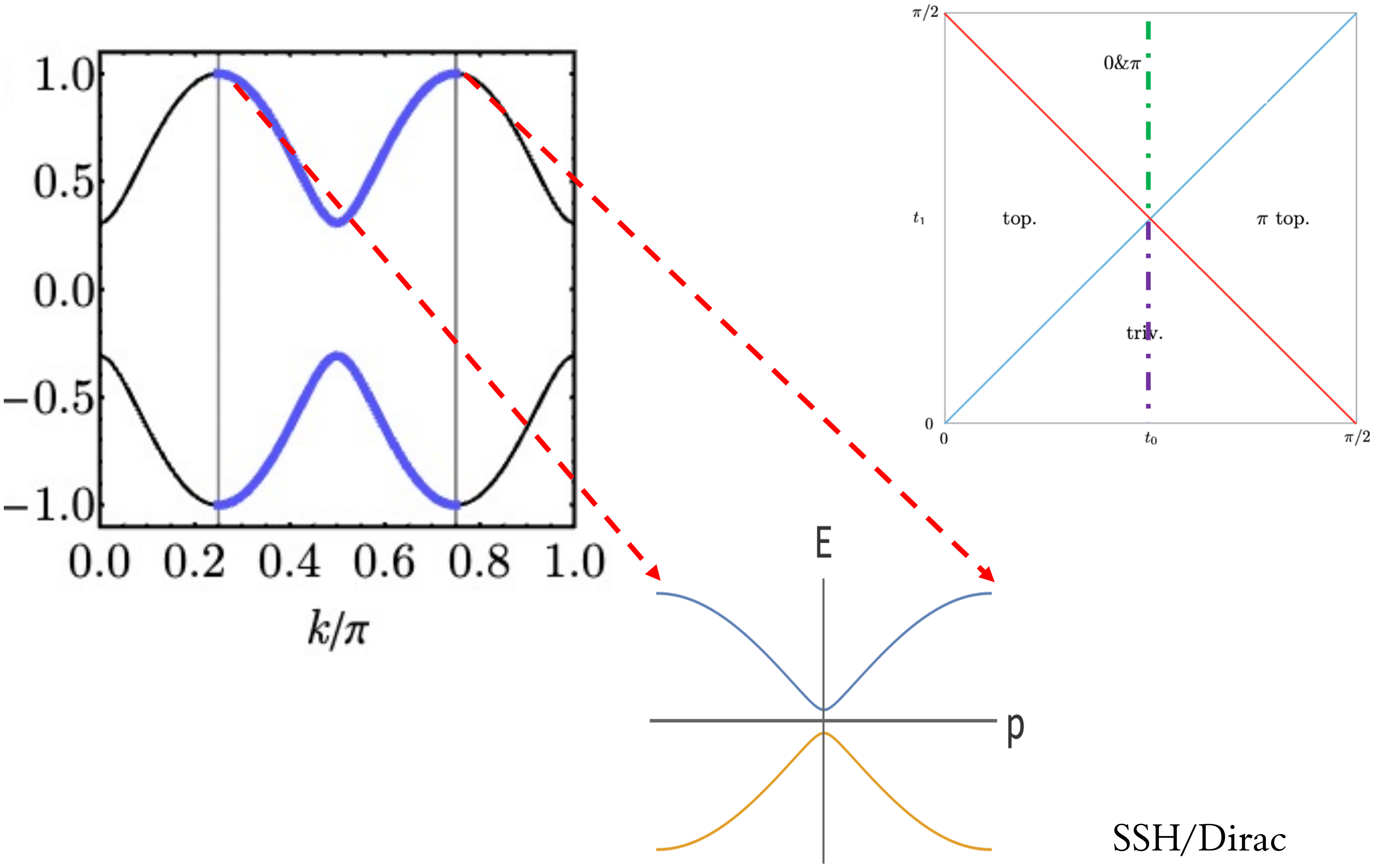


This works. But more is true..



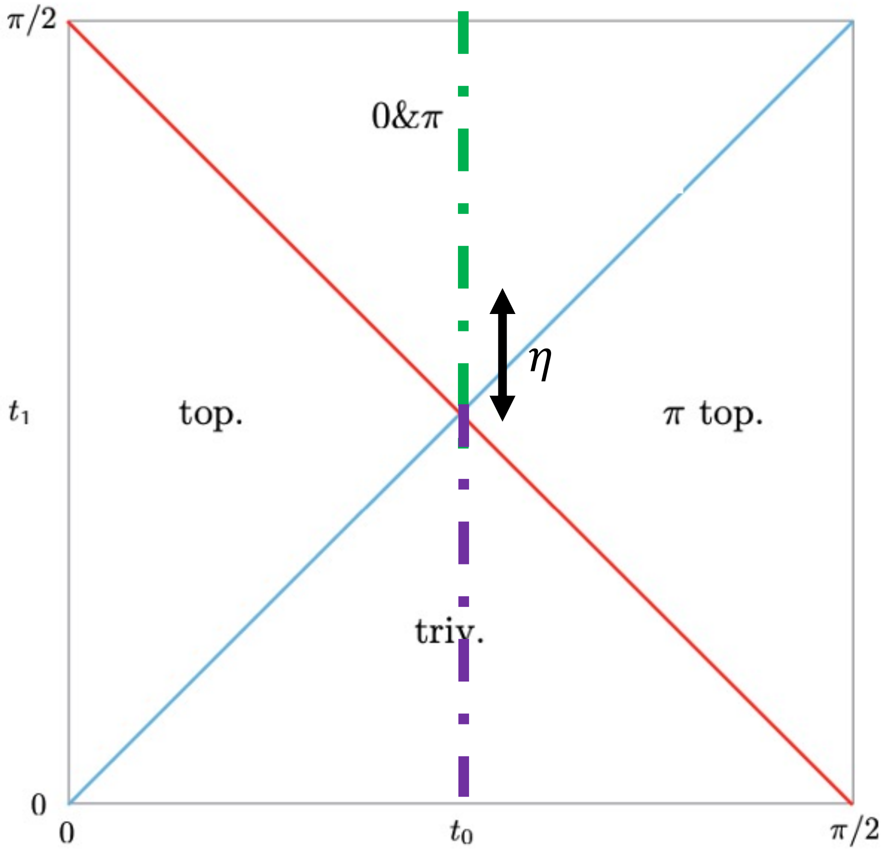
Lattice Dirac/SSH Hamiltonian

Surprise



1.0
0.5
0.0
−0.5
−1.0
0.0 0.2 0.4 0.6 0.8 1.0
k/π
$\pi/2$
$0\&\pi$
t_1
top.
π top.
triv.
0
0
t_0
$\pi/2$
E
p
SSH/Dirac

Two choices



Axes in units of T

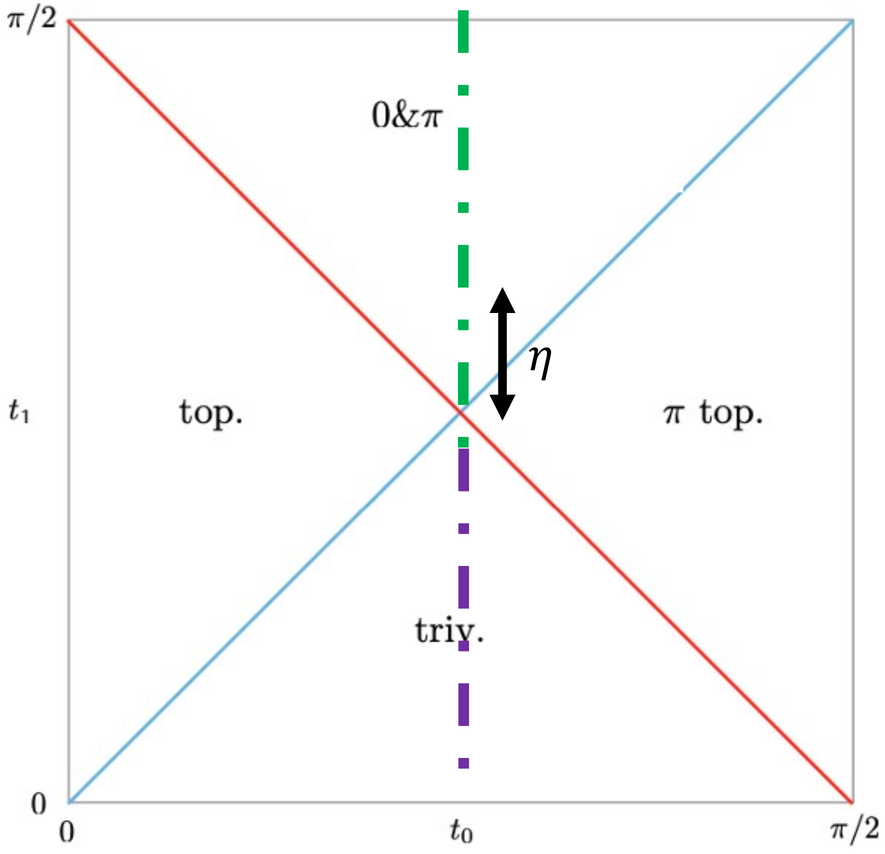
SSH:

$$u = \frac{1 + \sin 2\eta}{2T} \qquad v = \frac{1 - \sin 2\eta}{2T}$$

or

$$u = \frac{1 - \sin 2\eta}{2T} \qquad v = \frac{1 + \sin 2\eta}{2T}$$

The appropriate assignment



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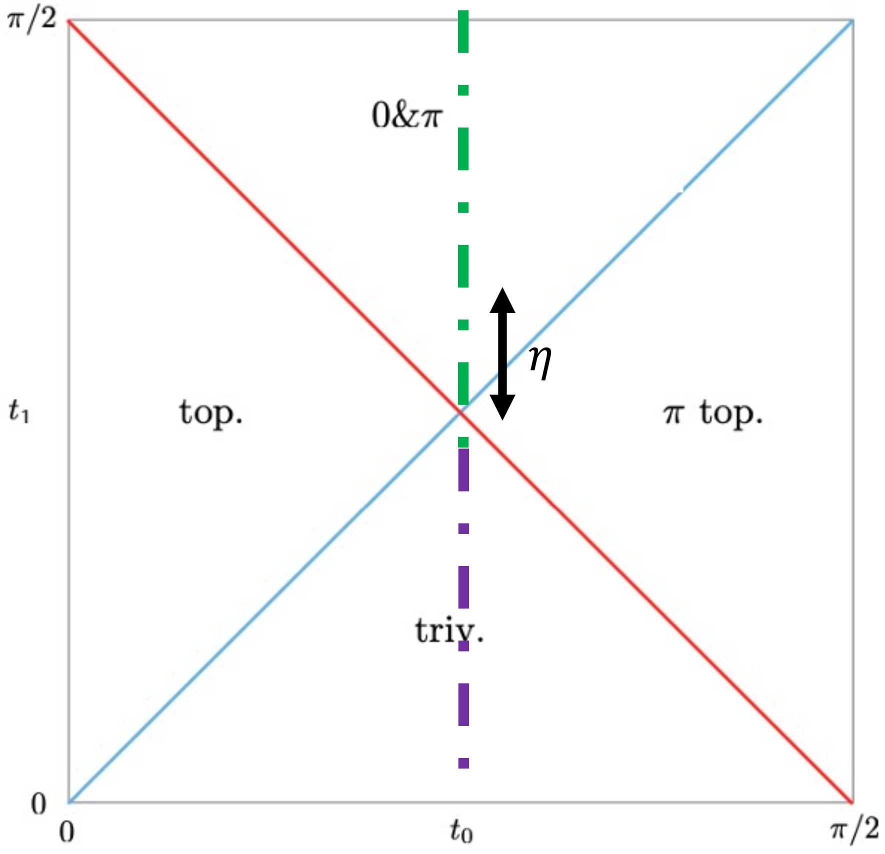

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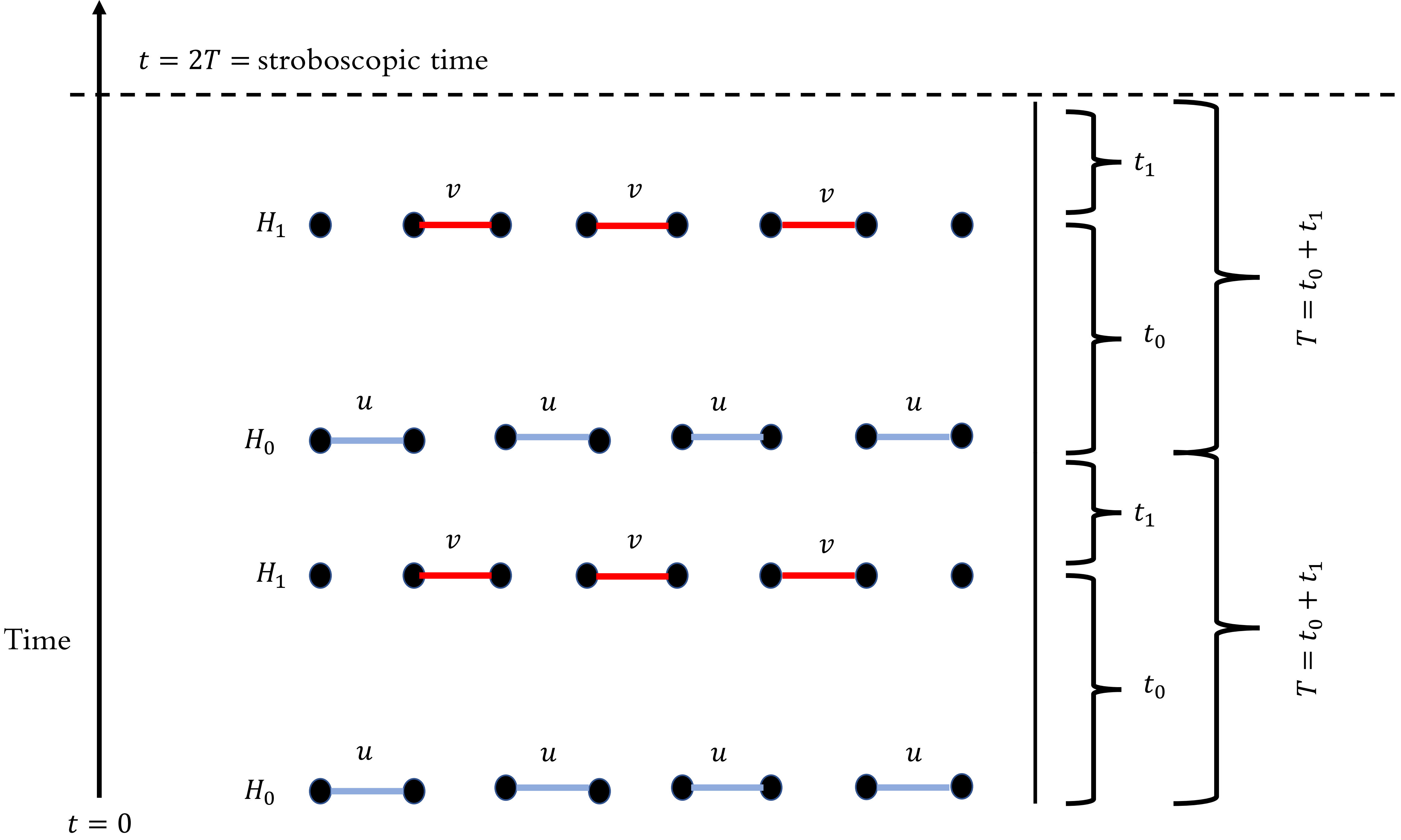
Topological to non-topological transition

Utilize pairing to get two flavors?

Increase the stroboscopic time by a factor of two.

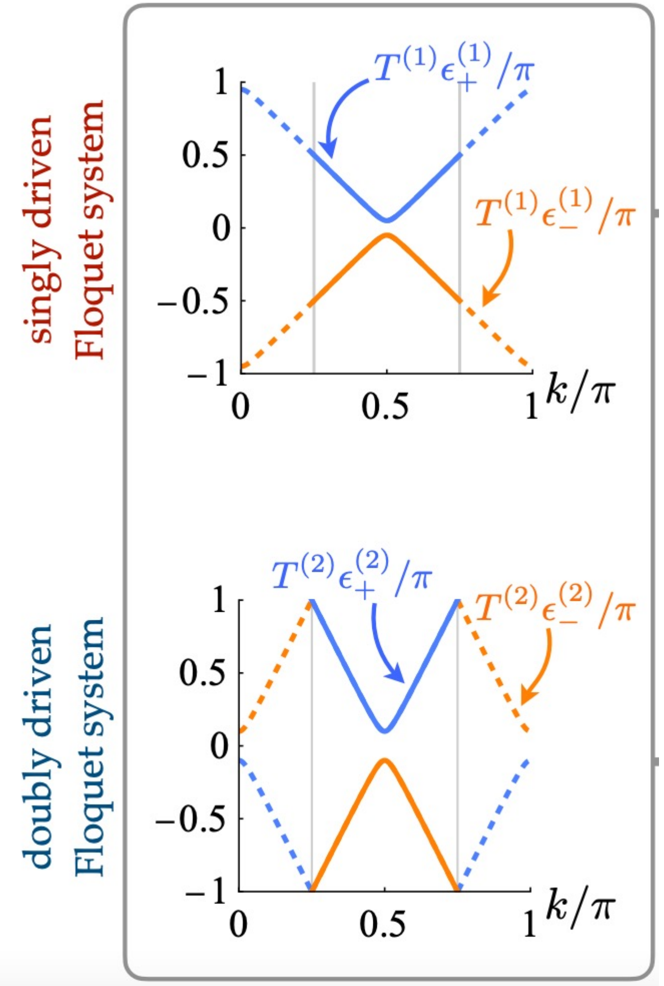
$$U_F(T) = e^{-i\,H_1 t_1} e^{-i\,H_0 t_0} \equiv e^{-i\,H_F T}$$ Previous drive

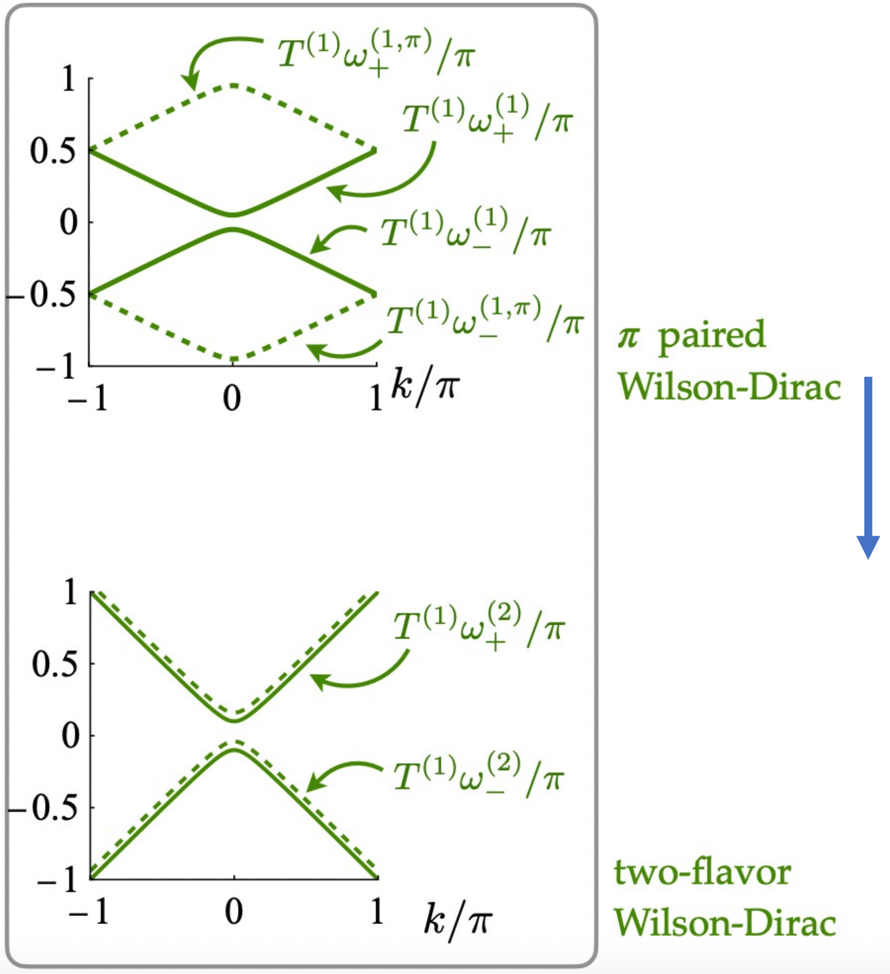
$$U(2T) = U_F\,(T)^2 = e^{-i\,H_1 t_1} e^{-i\,H_0 t_0} e^{-i\,H_1 t_1} e^{-i\,H_0 t_0} \equiv e^{-i\,H_{F,2} 2T}$$ New drive



t = 2T = stroboscopic time
Time
t = 0
H1
H0
H1
H0
v
u
t1
t0
T = t0 + t1

What this achieves





Achieved by staggering in the time direction

What follows

A map between correlation functions on the two sides.

Incorporating interactions in this correspondence.

Observables match in the long wavelength limit in both free and interacting theory in perturbative expansion.

Summary

Periodically driven systems are sometimes termed as discrete time systems in a loose way.

We make this comparison concrete.

The target Hamiltonian H ends up being a topological Hamiltonian itself !!

We find a that the Floquet transition maps to a topological transition of a static Hamiltonian with discrete time.

π pairing can be used to replicate fermion doubling and construct a two-flavor theory.

You can simulate one in terms of the other.

Interactions can be incorporated in the correspondence and the equivalence holds in the long wavelength limit.

Further work

- There is a way to make this comparison off the $t_0 = \frac{\pi}{4}$ line as well. See *Phys.Rev.Res.* 6 (2024) 1, 013098
- We found a similar correspondence connecting 2+1 dimensional Floquet systems with 2+1 dimensional discrete time fermion action formulation. (*SciPost Phys.Core* 8 (2025) 035)

Open questions

- Can we extend the correspondence for interacting theories beyond the long wavelength limit?

- What about strongly coupled theories?

- Floquet systems are sought after in quantum circuits due to their tunability in designing target Hamiltonian. Here, the same system can be tuned to simulate a single flavor theory and a two-flavor theory.

Possible applications?

Big picture

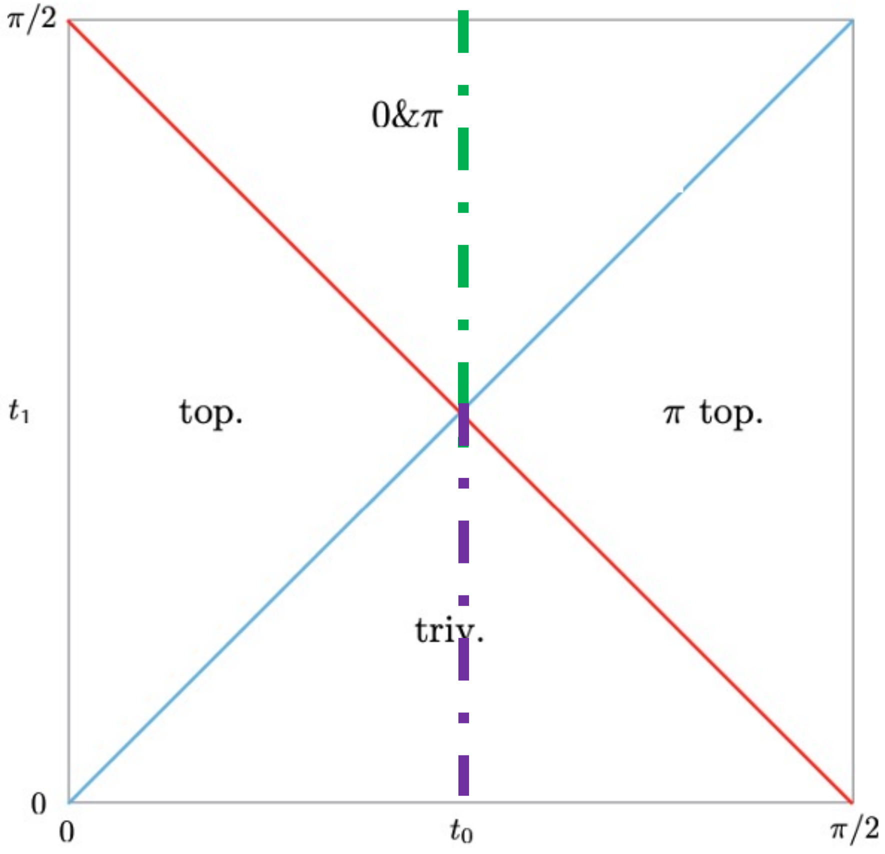
Many common threads between different areas of physics.
Sometimes recognized in hindsight.

The ties of topological phases and fermion field theories go deep.

We understand these ties for equilibrium, free and some interacting theories.

Exploring such ties between periodically driven systems and lattice field theorymay provide new perspective in both.

Two choices from PBC



Dirac Hamiltonian with a negative mass and a positive mass: indistinguishable with PBC.

The green and the purple region in this diagram: indistinguishable with PBC.

Green → Negative mass

Purple → Positive mass

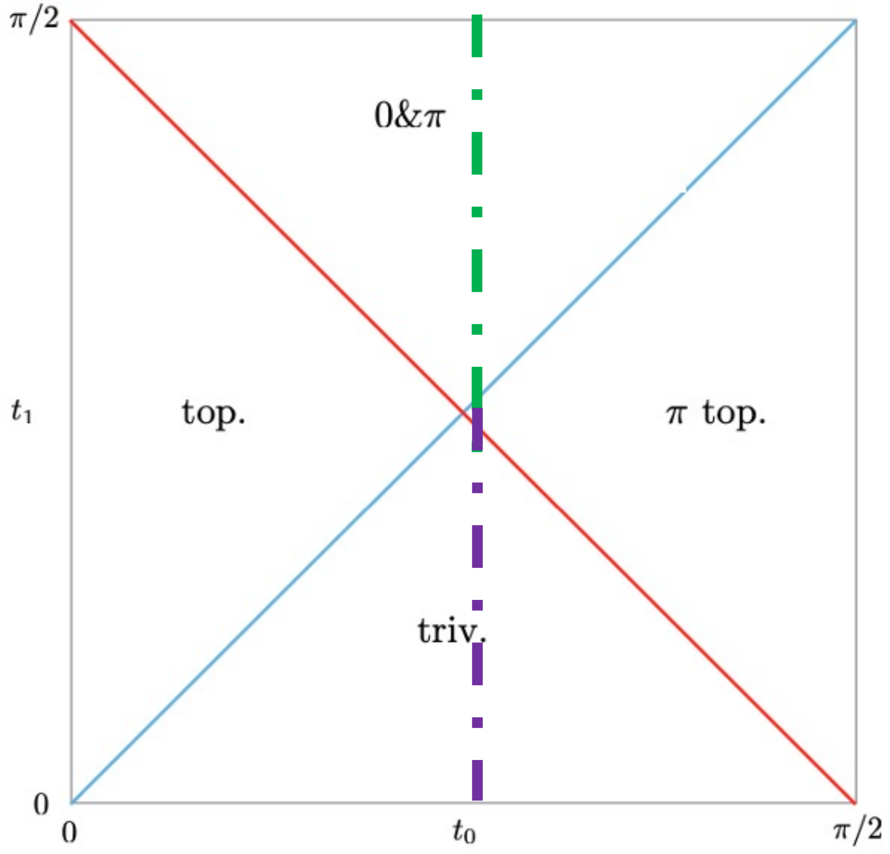
✓

Green → Positive mass

Purple → Negative mass

✓

Only one choice from OBC



Green region of the diagram: zero and π edge states
Purple region of the diagram: No edge states

Dirac Hamiltonian negative mass: zero energy edge state (with discrete time, also π modes)
Dirac Hamiltonian positive mass: No edge state

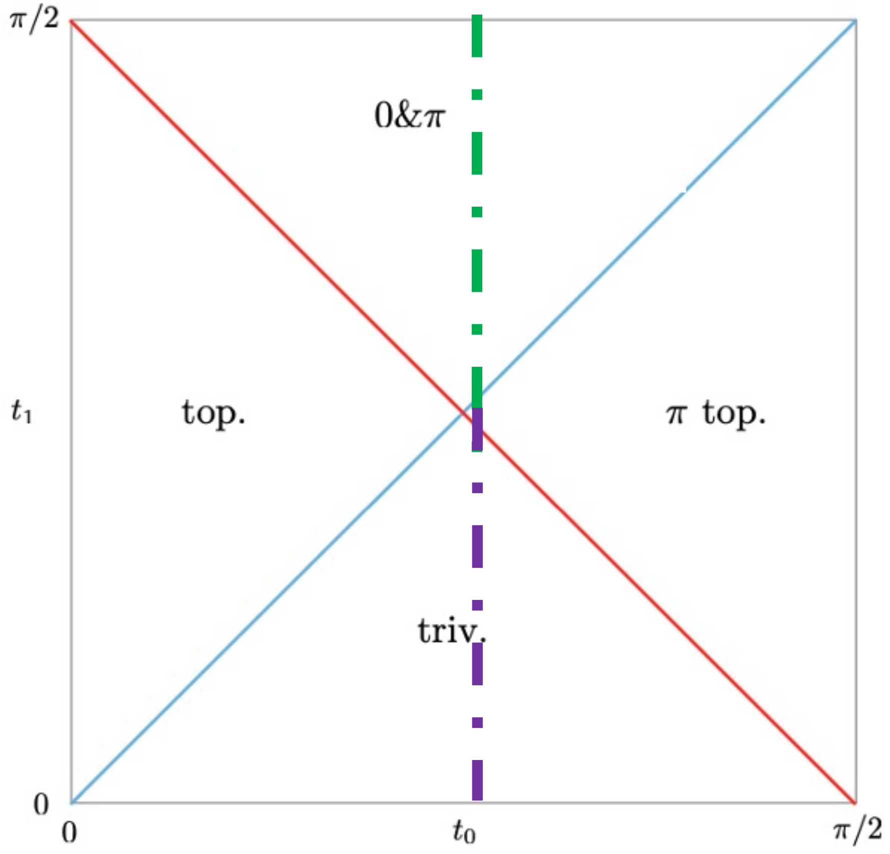
Green →	Negative mass
Purple →	Positive mass

✓

Green →	Positive mass
Purple →	Negative mass

✗

Only one choice from OBC



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Dirac Hamiltonian positive mass: No edge state

Green →	Negative mass
Purple →	Positive mass

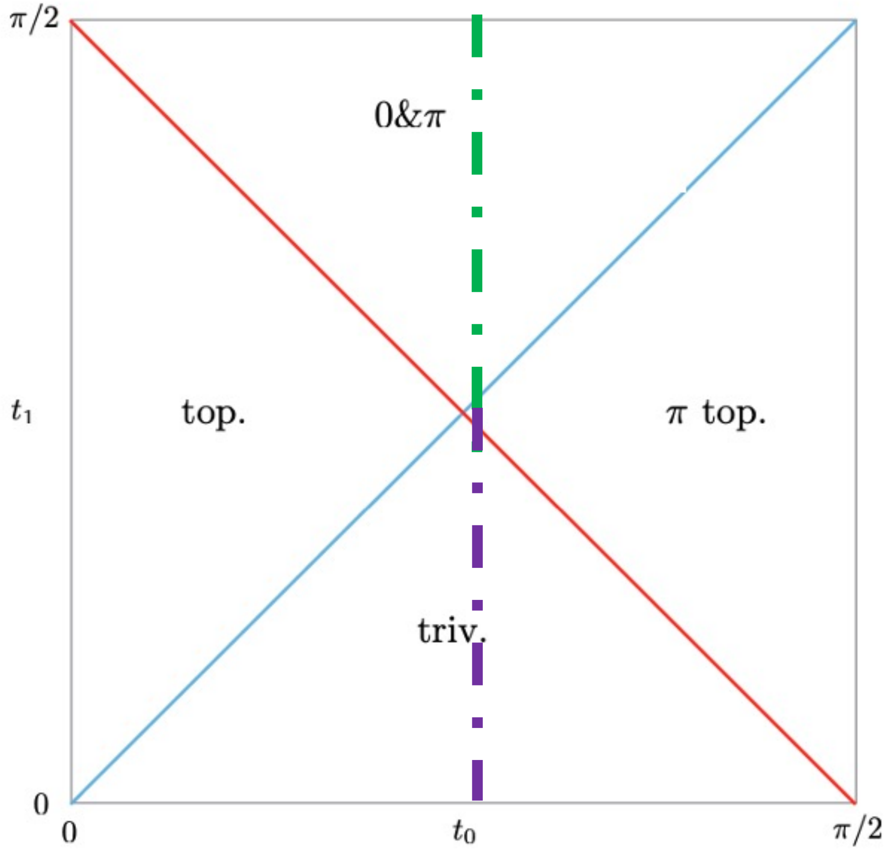
✓

Green →	Positive mass
Purple →	Negative mass

✗

So, we have been able to map a Floquet phase transition to an equilibrium topological transition!

Only one choice from OBC



Axes in units of T

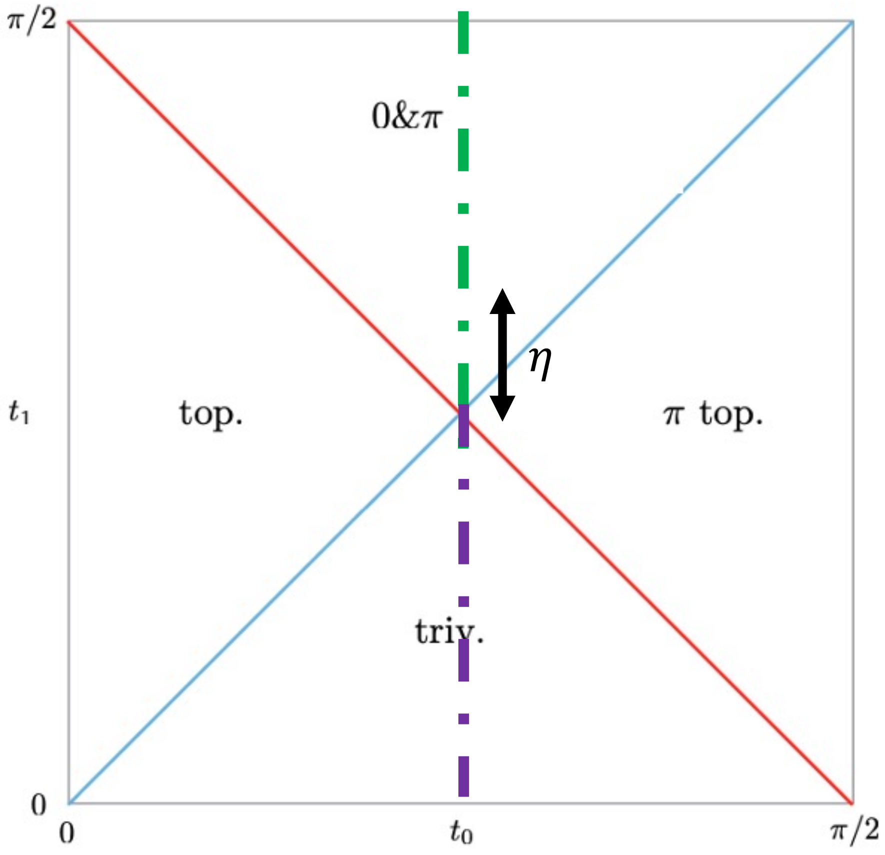
Green region of the diagram: Dirac Hamiltonian negative mass
Purple region of the diagram: Dirac Hamiltonian positive mass

Discretization of time leads to boundary zero mode and π mode for the Dirac Hamiltonian in the negative mass region

Discretization of time in the positive mass region: No zero mode, no π mode

A Floquet phase transition got mapped to a static topological to non-topological transition

Two choices



Axes in units of T

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or

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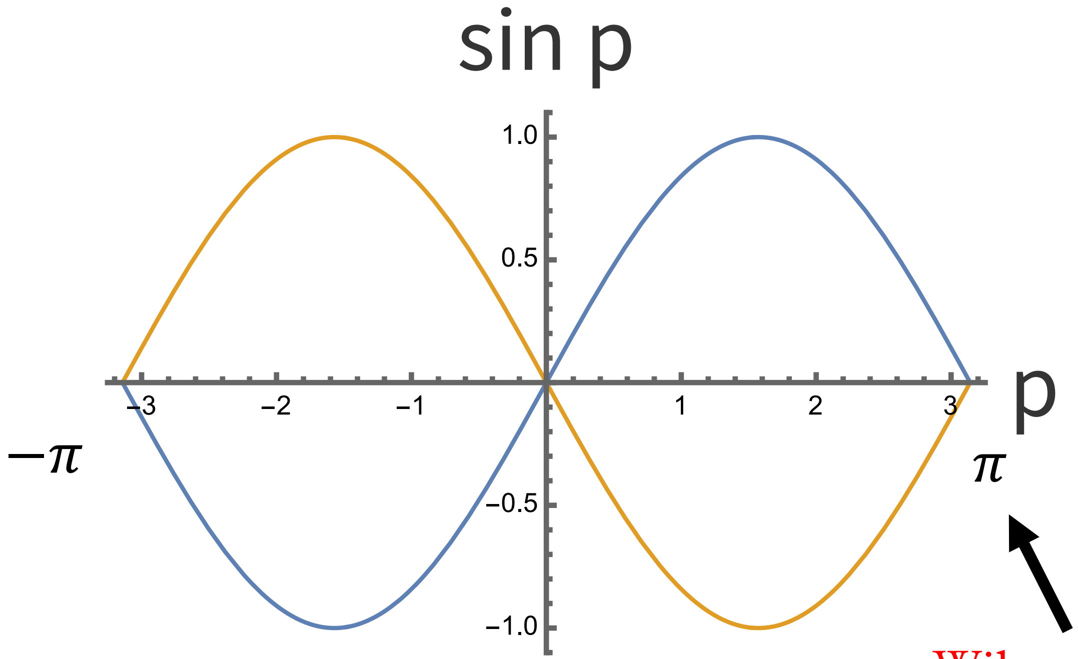
Wilson term for Dirac



Wilson term removes this.

Add a momentum dependent mass term

Single particle Hamiltonian:

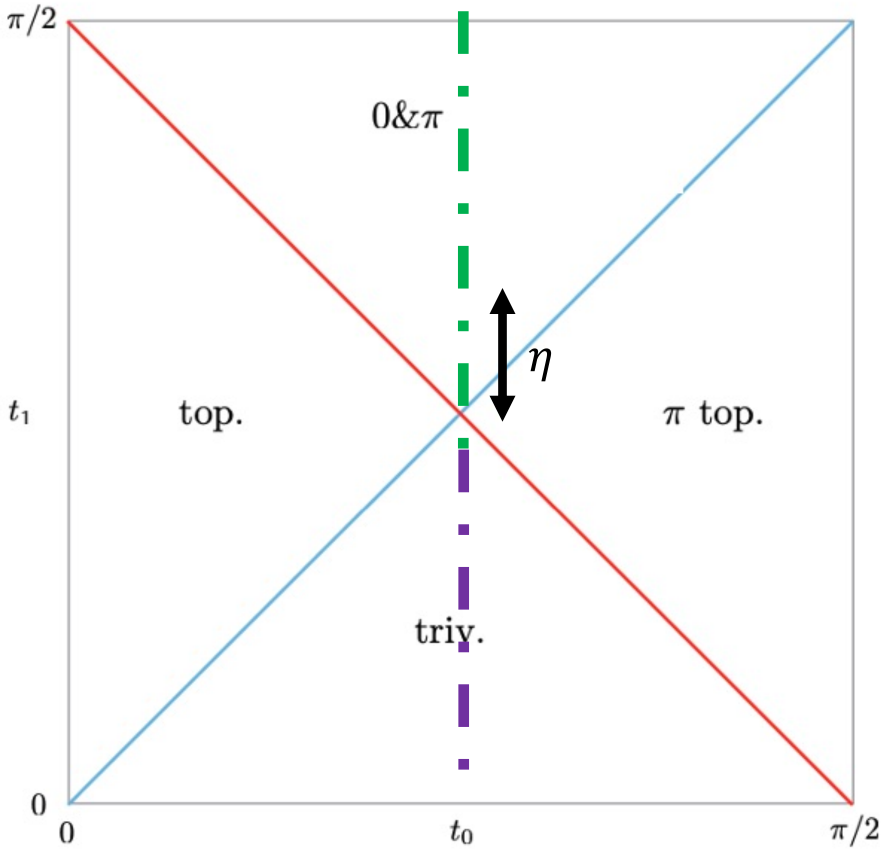
$$H = -i\gamma^1\nabla_1 + m + \frac{R}{2}\nabla \qquad \nabla \to (1-\cos p)$$

Wilson term.

∇_1= Symmetric finite difference in space

∇= symmetric discrete spatial Laplacian

The appropriate assignment



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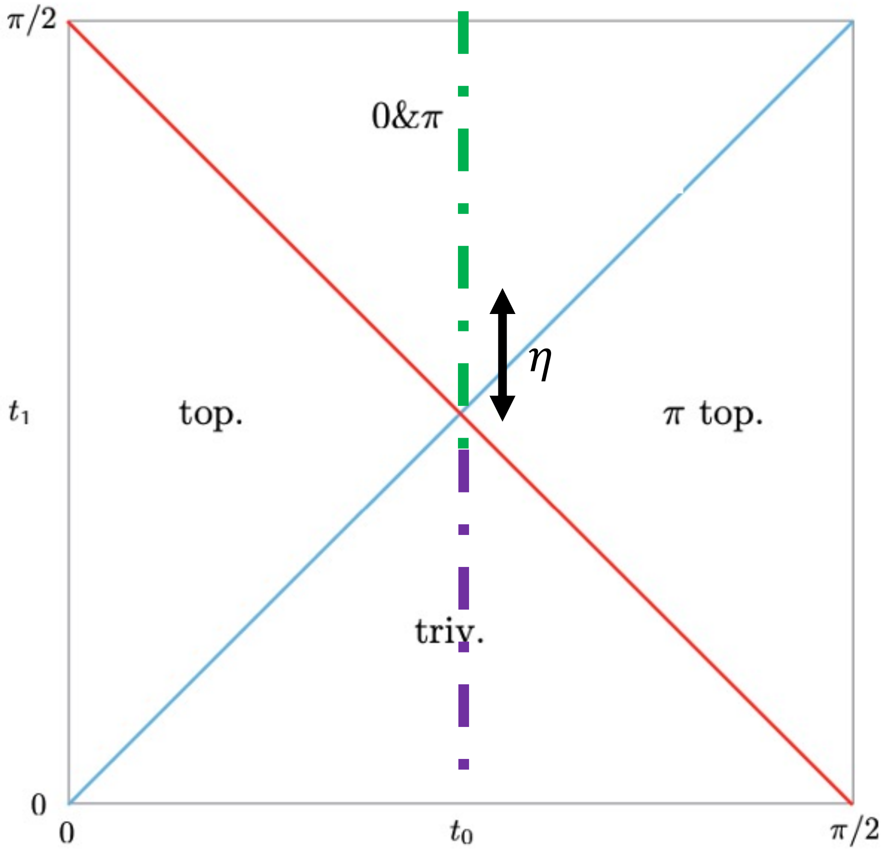

Axes in units of T

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Topological to non-topological transition

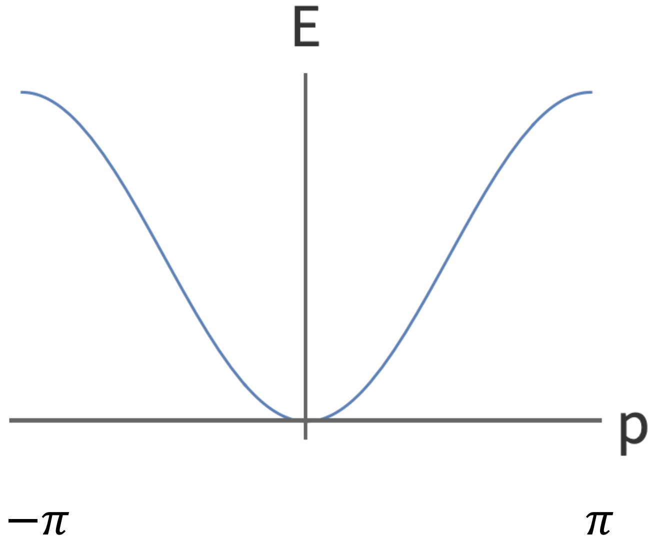
Static topological Hamiltonian: Wilson-Dirac

Hamiltonian of the form

$$\mathrm{H}_{\mathrm{WD}} = \gamma_0(-i\,\gamma_1\nabla_1 + m - \tfrac{R}{2}\nabla_1^2)$$

$\frac{m}{R} > 0$: non-topological phase, no edge states with OBC

$\frac{m}{R} < 0$: topological phase, edge state with OBC

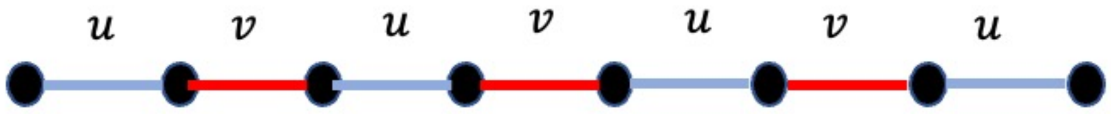


$$E_k = \sqrt{\sin^2 p + (R(1-\cos p) + m)^2}$$

PBC

Domain wall between the two hosts domain wall fermion.

Static topological Hamiltonian: SSH model (also Dirac)



PBC: $E(p) = \pm\sqrt{u^2 + v^2 - 2uv\cos 2p}$

$v - u$ is Dirac mass.

$v - u > 0$: topological phase with zero energy edge mode for OBC

$u - v > 0$: non-topological phase with zero energy edge mode for OBC

Discrete space-time

If you start with a Weyl fermion in 1+1 and then discretize

+ chirality: $\{0,0\}, \{\pi, \pi\}$

- chirality: $\{0, \pi\}, \{\pi, 0\}$

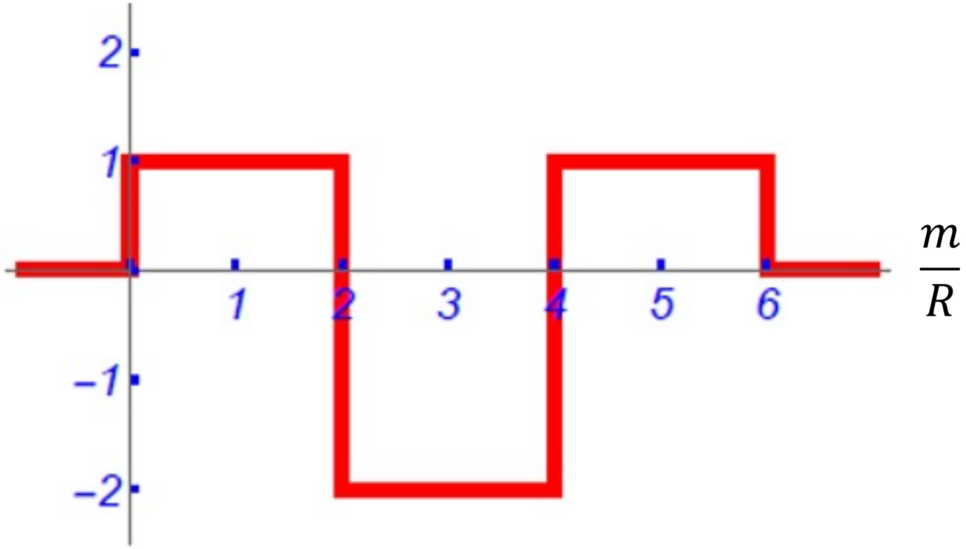
This is unwanted in lattice QFT! We want a single massless flavor. So, we add a Wilson term to shift the mass $m + R(1 - \cos p_0) + R(1 - \cos p_1)$ to kill all doublers.

Play with Wilson term and you can kill one or some doublers.

Cool thing about doubling

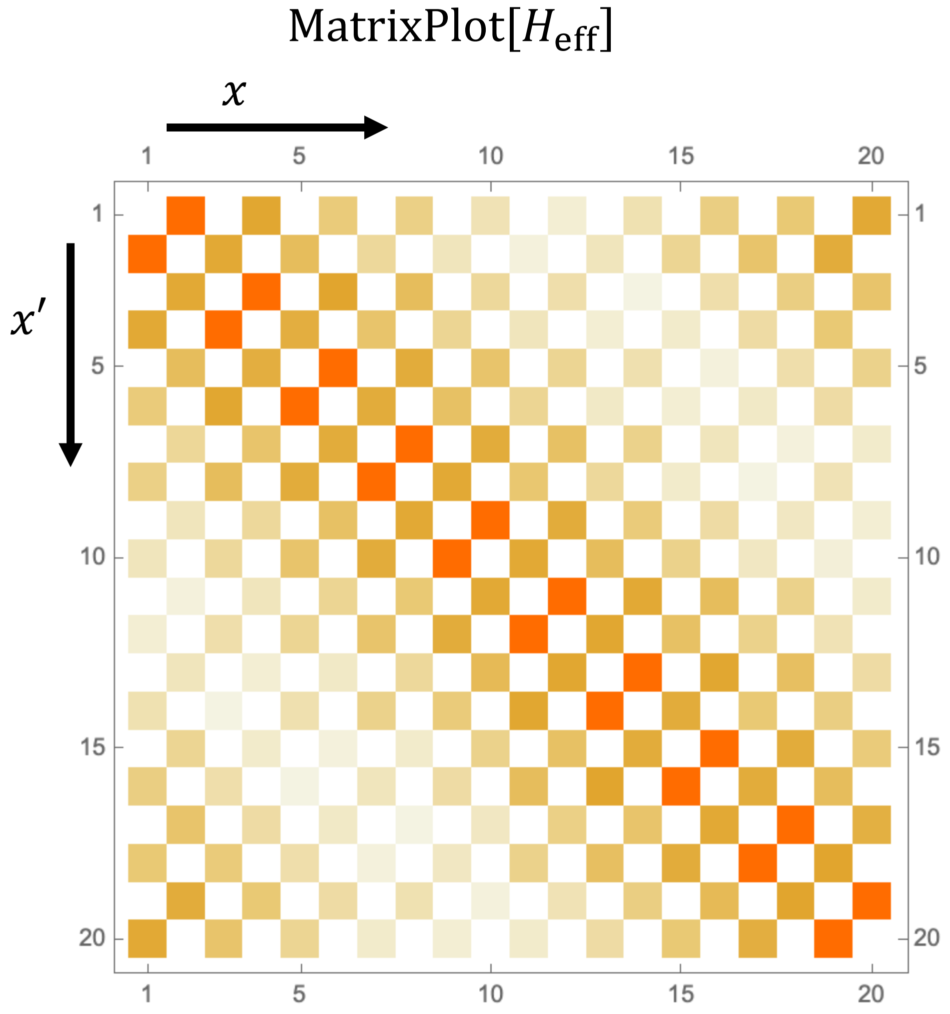
In 3D with a domain wall in the third direction, you can get chiral edge states on the wall.

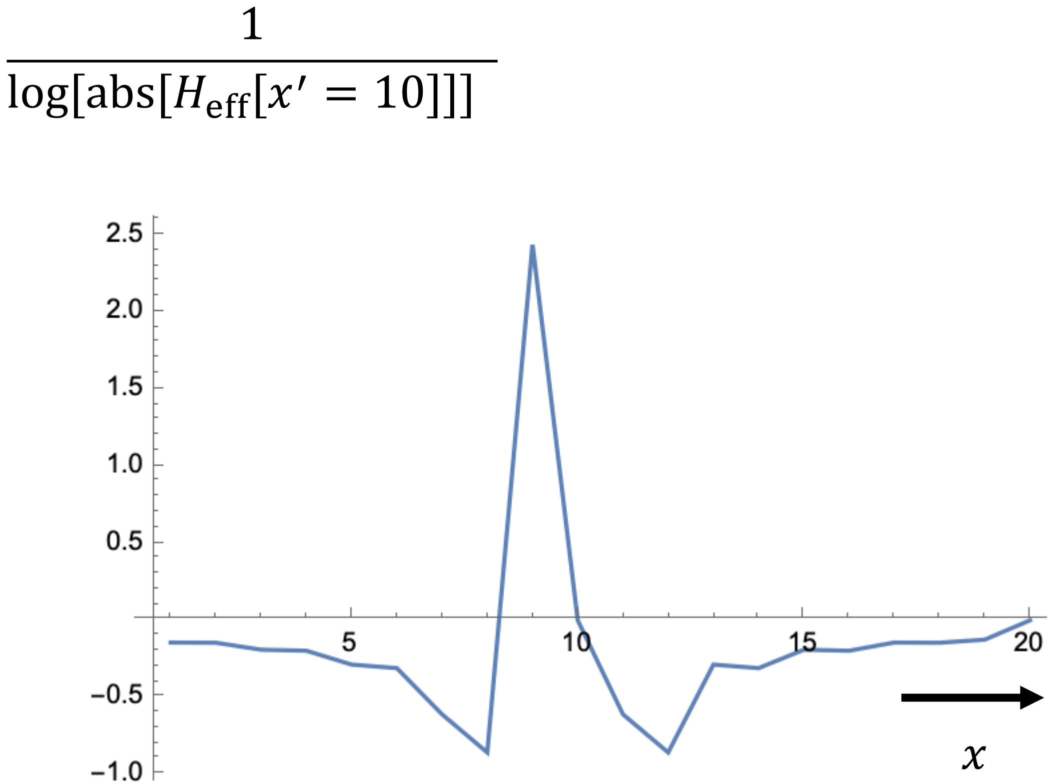
Different doublers can appear simultaneously.



Edge mode chirality with a domain wall

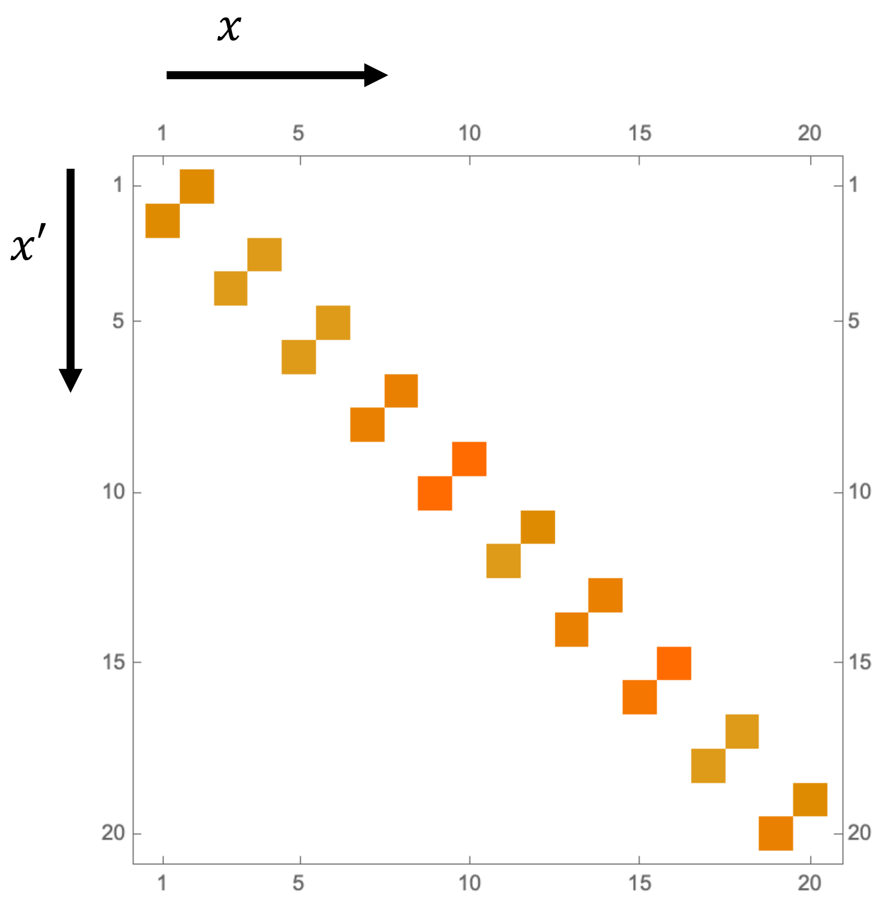
Floquet Hamiltonian in position space (local)



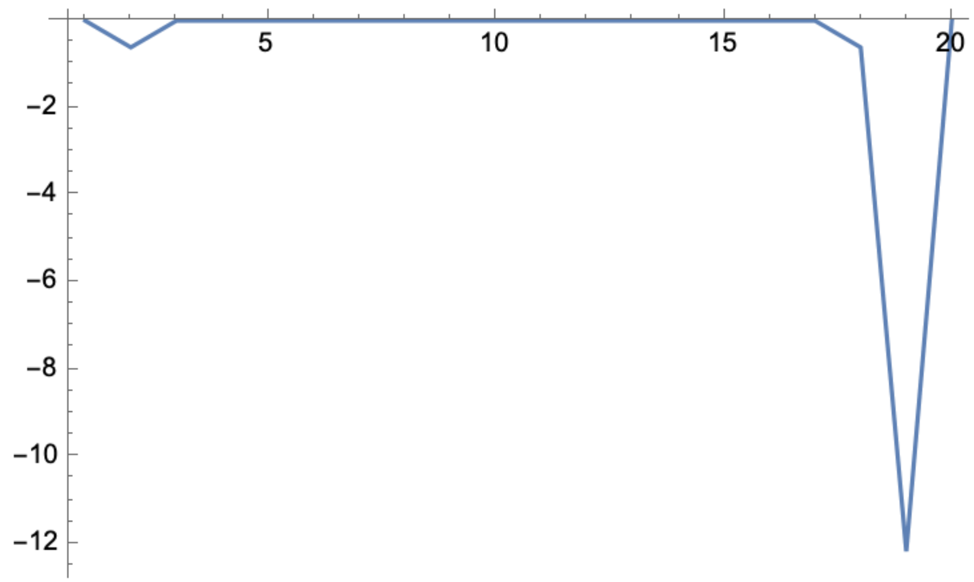


Sine of the Floquet Hamiltonian in position space (ultralocal)

MatrixPlot[sin($T\ H_{\mathrm{eff}}$)]



$$\frac{1}{\log[\mathrm{abs}[H_{\mathrm{eff}}[x' = 10]]]}$$



Domain wall in 1+1 D

$$(\gamma_0 p_0 + \gamma_1 \nabla_1 + m + R\nabla_1^2)\psi(p_0, x_1) = 0$$

Continuous time edge mode

Zero for $p_0 = 0$

Solve this to get normalizable transverse profile

$$(\gamma_0 \sin(p_0) + \gamma_1 \nabla_1 + m + R\nabla_1^2)\psi(p_0, x_1) = 0$$

Discrete time edge mode

Zero for both $p_0 = 0$ and $p_0 = \pi$.

Takeaway: there is no time doubling for continuous time systems.

Domain wall in 1+1 D

$$(\gamma_0 p_0 + \gamma_1 \nabla_1 + m + R\nabla_1^2)\psi(p, x_1) = 0$$

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Discrete time edge mode

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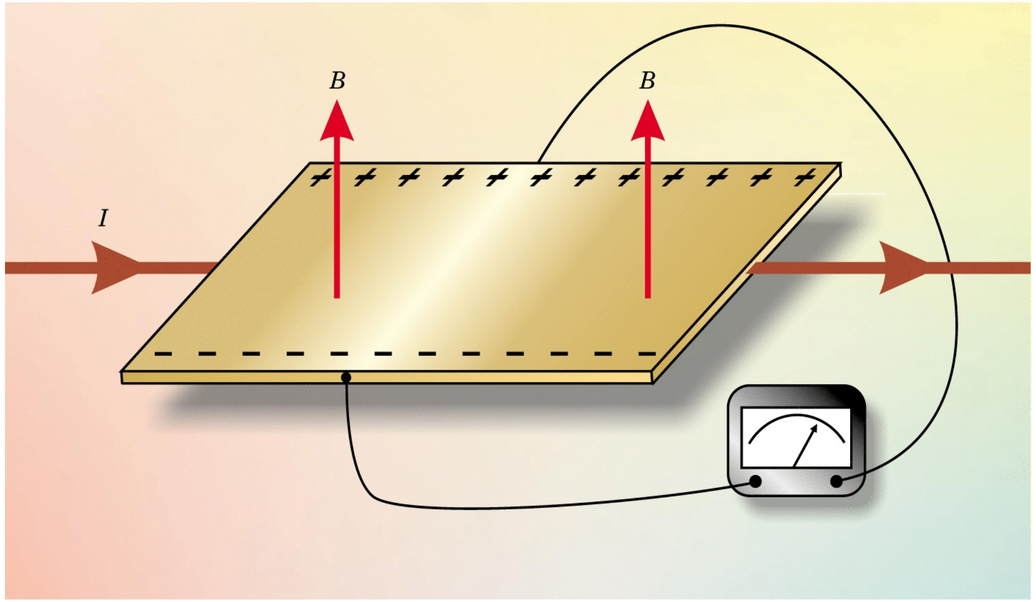
Takeaway: there is no time doubling for continuous time systems.

But, something curious happens for periodically driven systems.

Dirac fermion and Quantum Hall Effect(QHE)



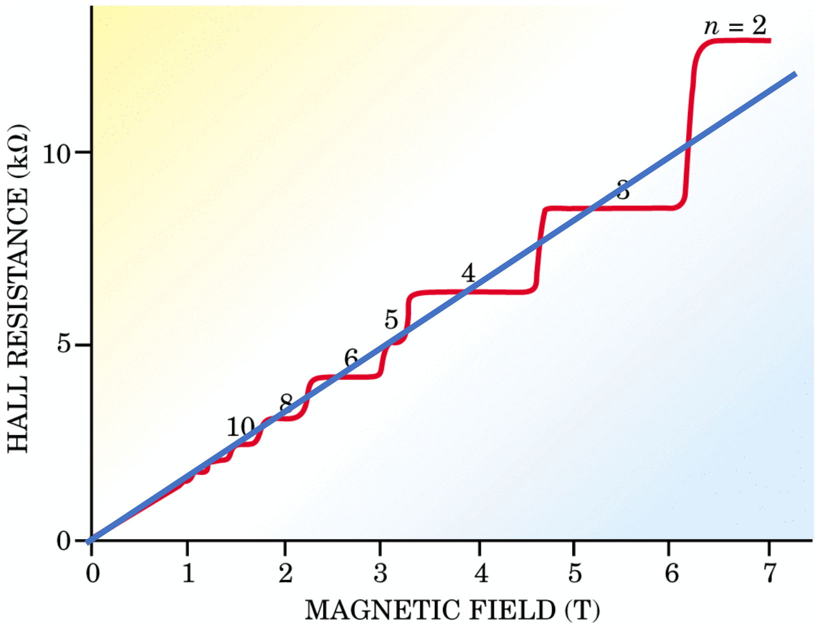
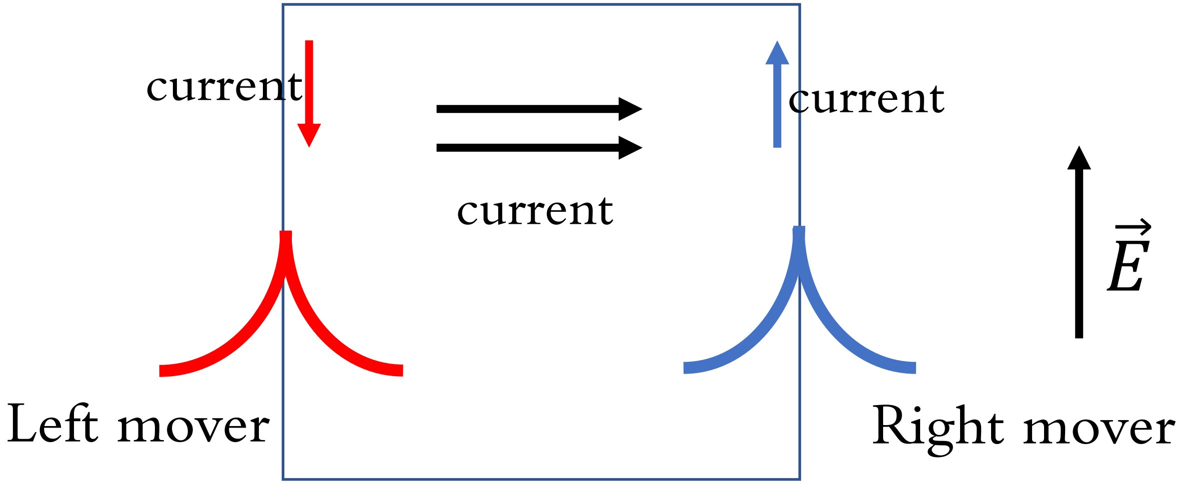
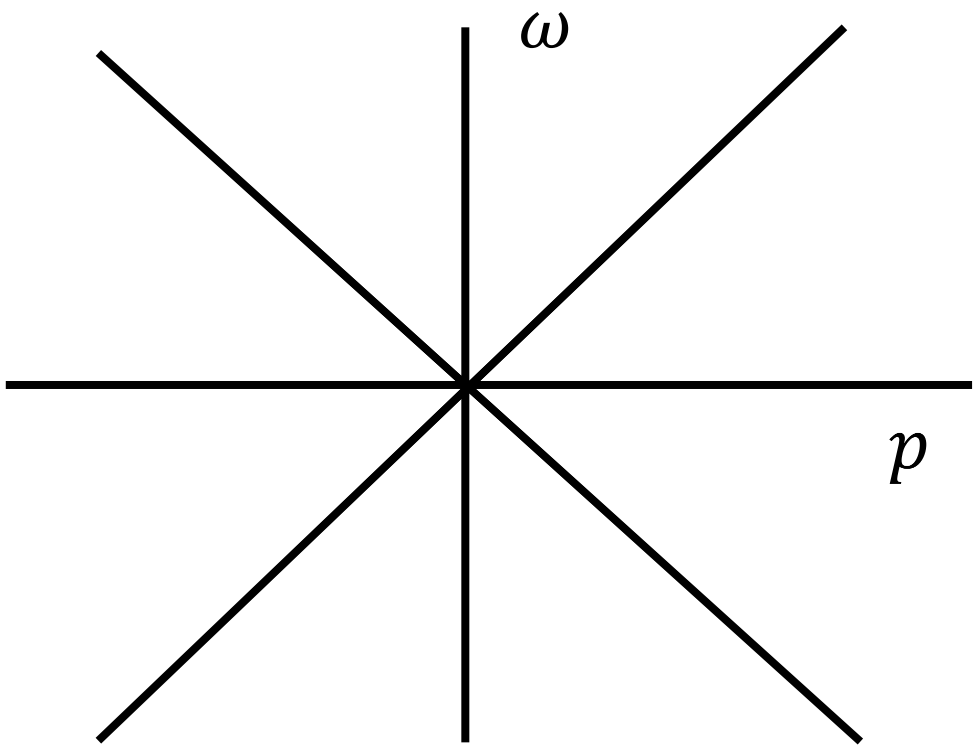


Figure credit: physicstoday



The edge world: 1+1 D massless Dirac fermion spectrum



Dispersion for a
Dirac Hamiltonian

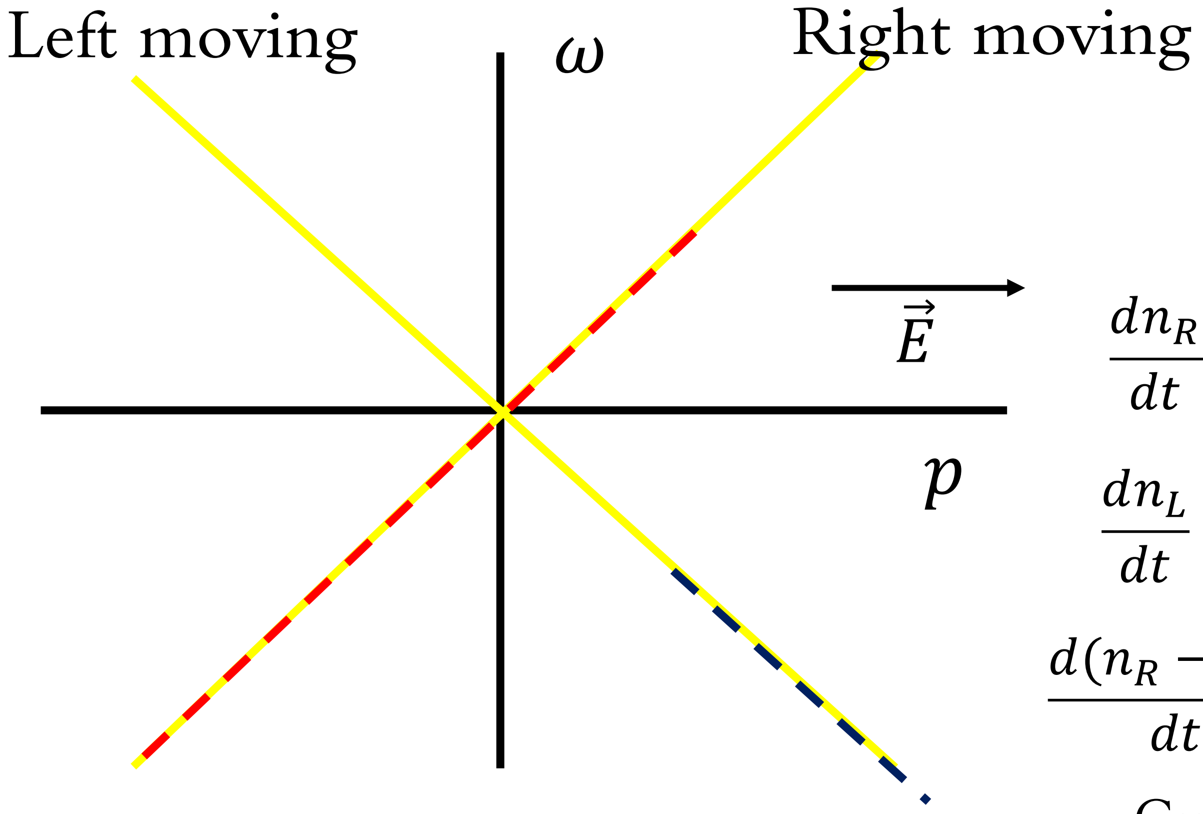
Minkowski space-time.

Relativistic dispersion:

$$\omega^2 - p^2 = 0$$

$$\Rightarrow \omega = \pm p$$

The edge world: 1+1 D massless Dirac fermion spectrum



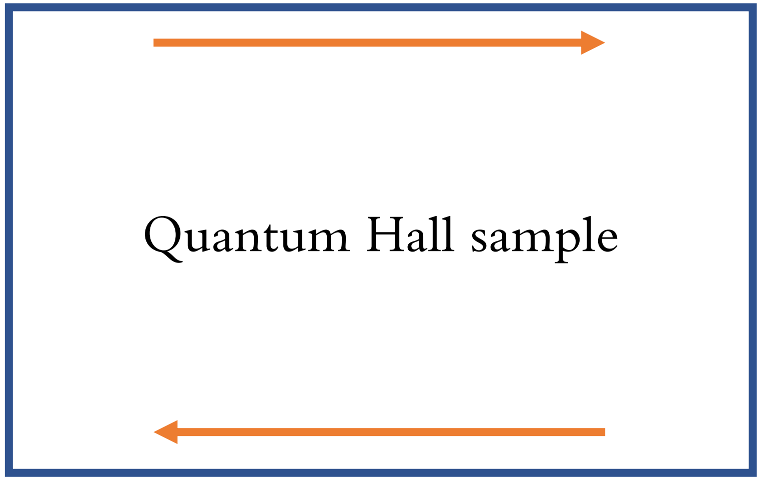
$$\frac{dn_R}{dt} = \frac{eE}{2\pi}$$

$$\frac{dn_L}{dt} = -\frac{eE}{2\pi}$$

$$\frac{d(n_R - n_L)}{dt} = \frac{eE}{\pi}$$

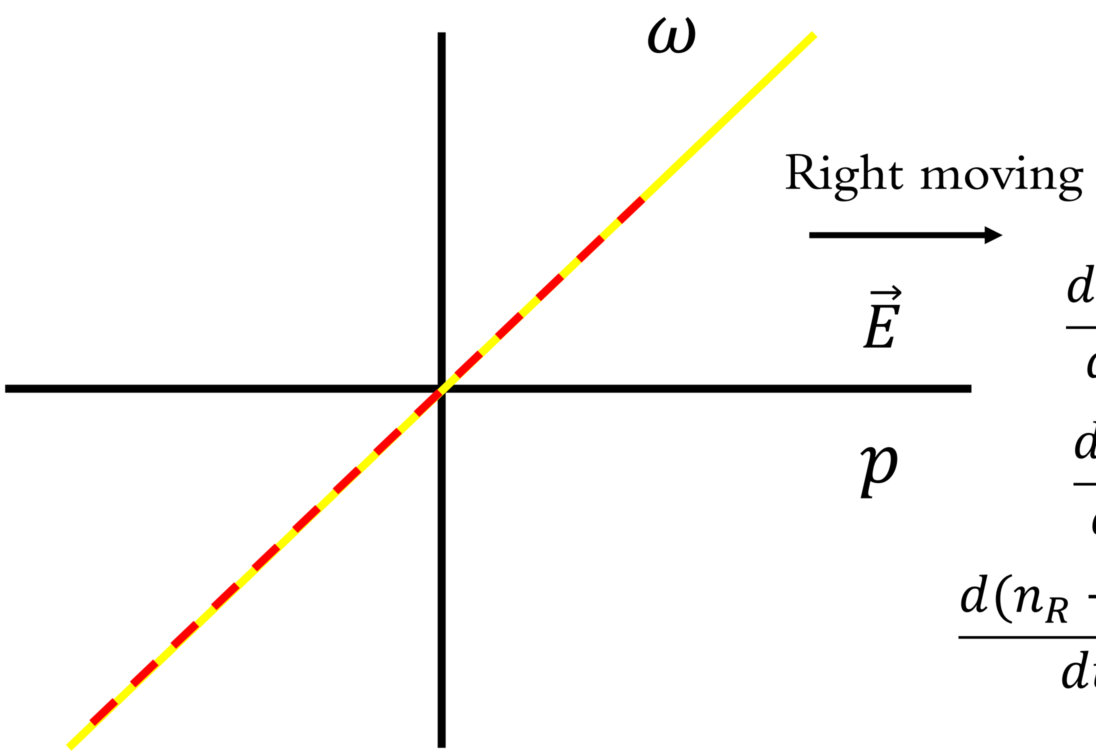
Current $\propto E$

Anomalous transport on the edges



Vector current or charge $n_R + n_L$ conserved, axial not so.

Edge world: Anomaly, chiral fermion



$$\frac{dn_R}{dt} = \frac{eE}{2\pi}$$

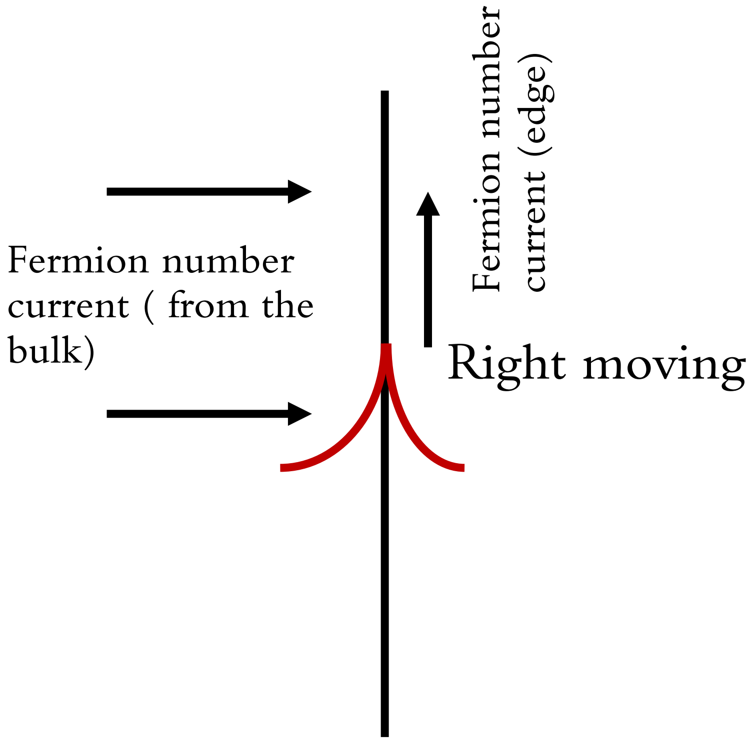
$$\frac{dn_L}{dt} = 0$$

$$\frac{d(n_R + n_L)}{dt} = \frac{eE}{\pi}$$

Current $\propto E$

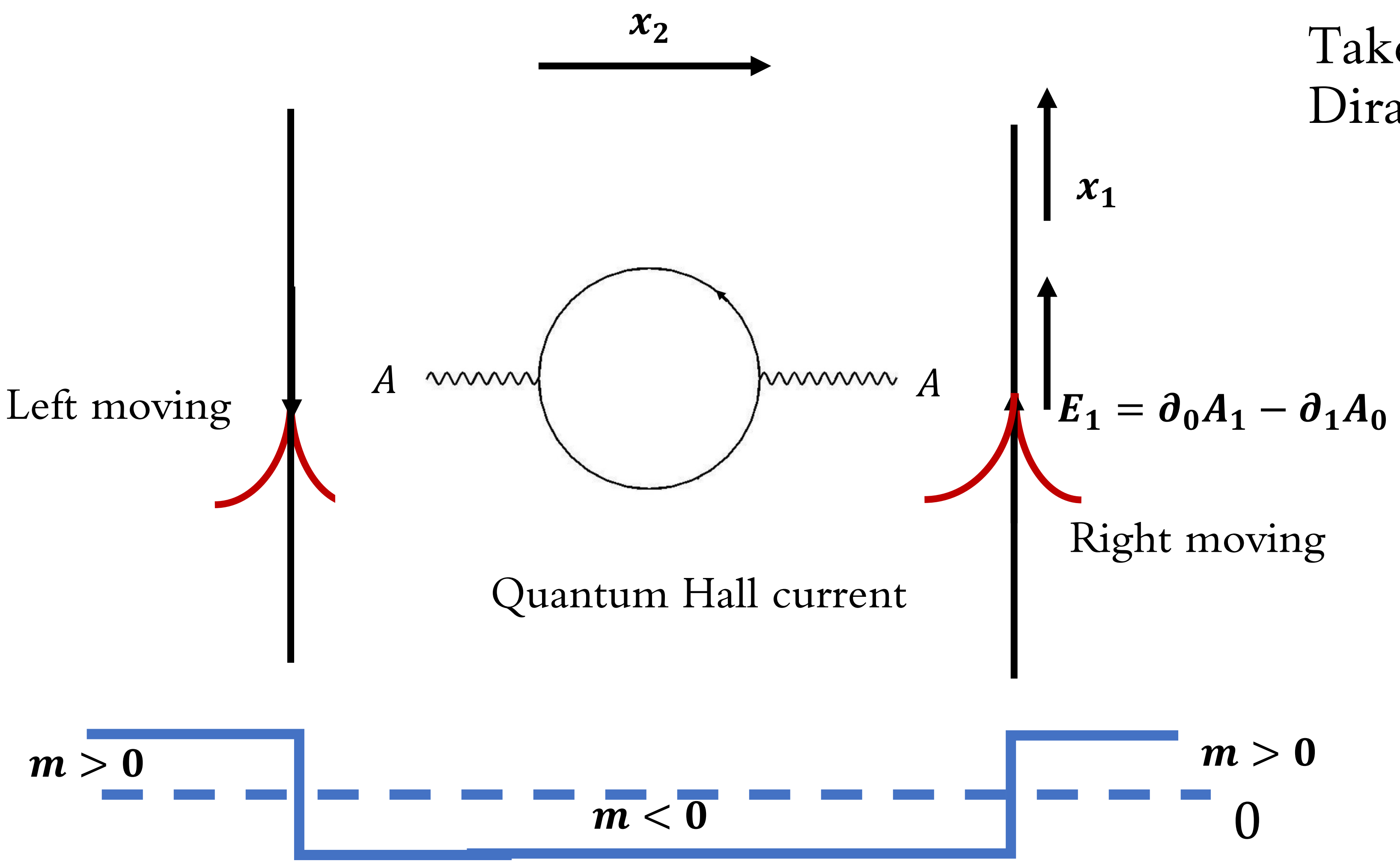
Edge of QHE sample

Vector current not conserved, edge by itself is sick in an electric field.



Vector current is conserved, when Hall current from bulk is taken into account.

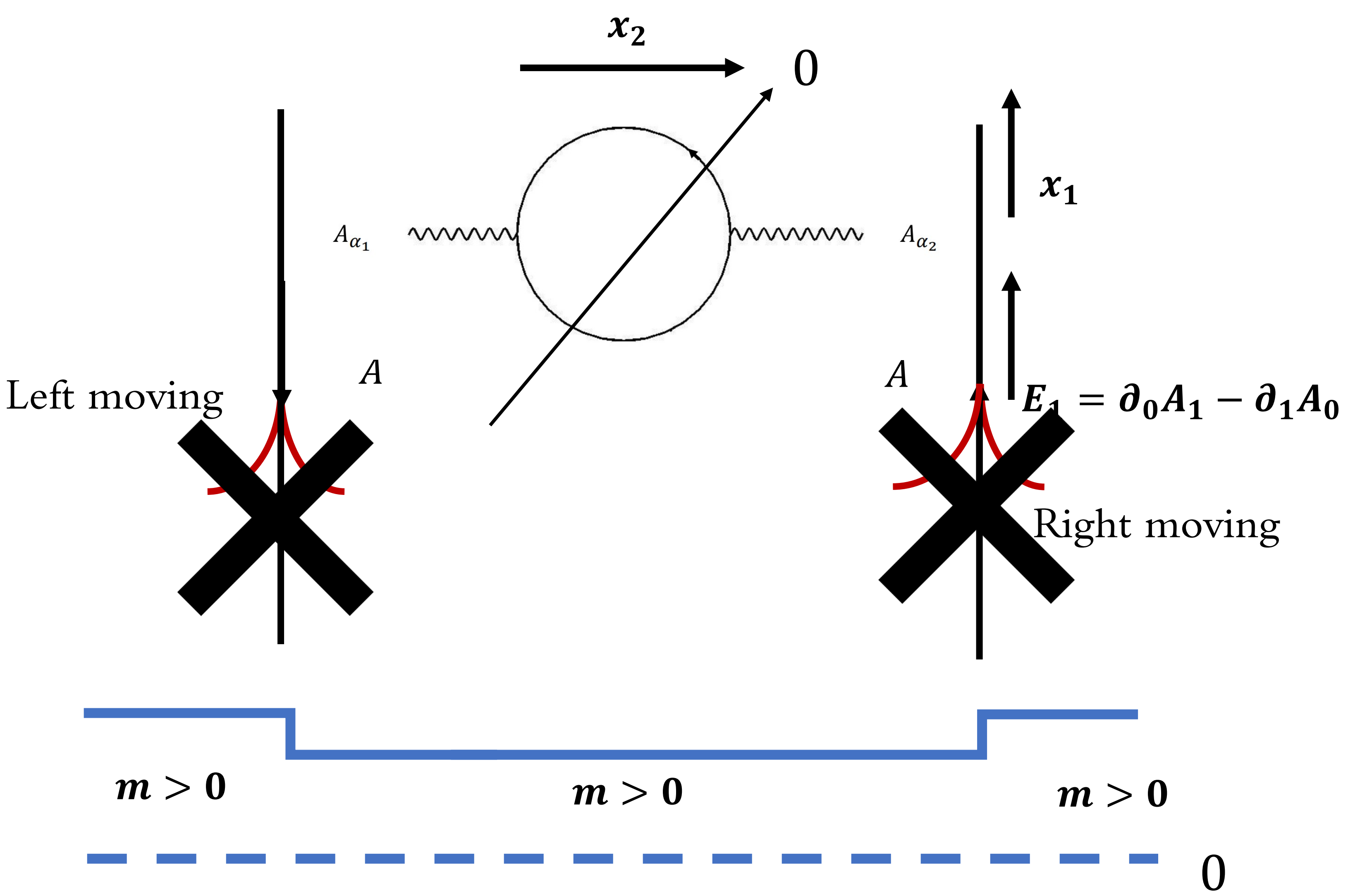
Relativistic fermion: Quantum Hall Effect(QHE)(Callan-Harvey 1984)



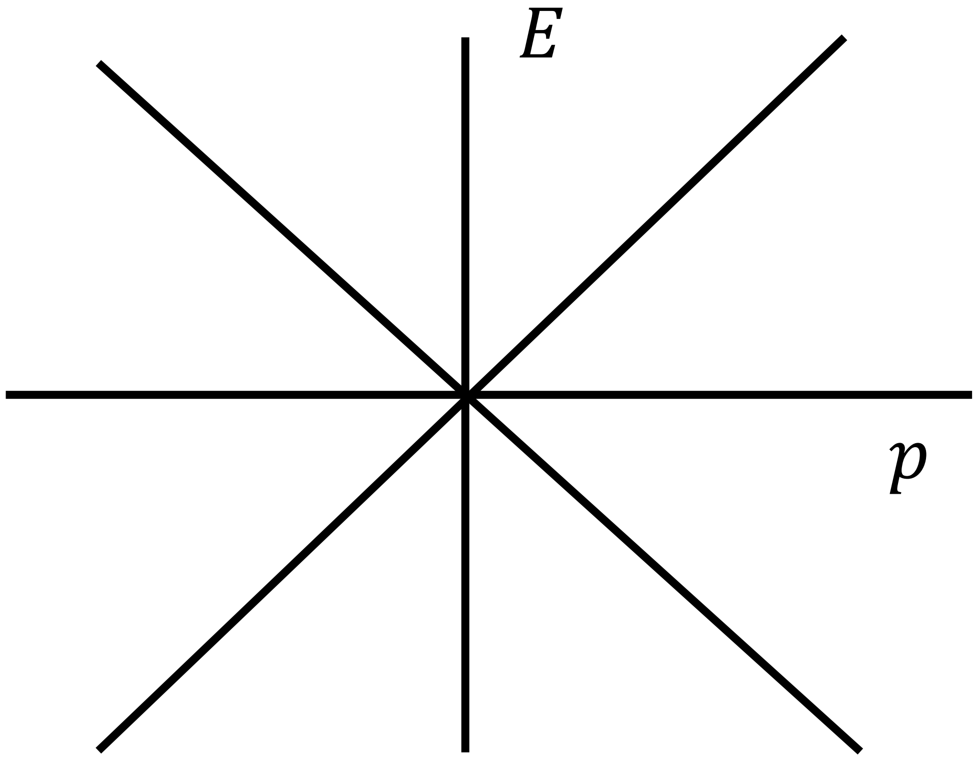
Take a $2+1$ dimensional Dirac fermion

Step function in Dirac mass m in x_2 (domain walls)

Transition to boring (non-topological)



Massless Dirac Dispersion (1 spatial dimension)



Continuum dispersion for a Dirac Hamiltonian

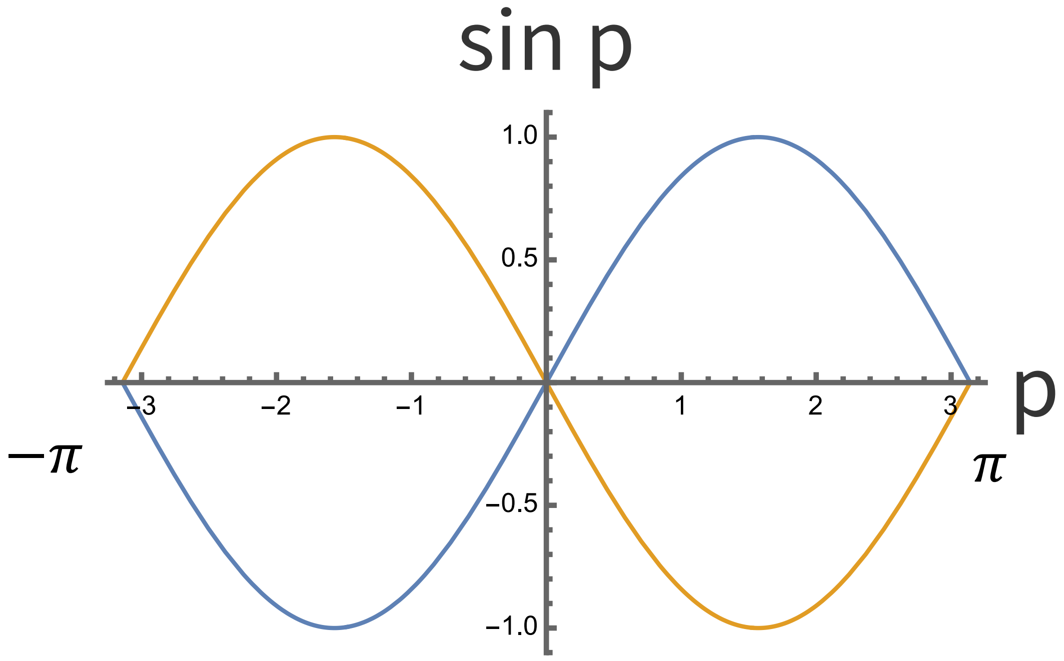
Minkowski space-time.

Relativistic dispersion:

$$E^2 - p^2 = 0$$

$$\Rightarrow \; E = \pm p$$

Brilliuoin zones (Dirac)



Two Dirac fermions

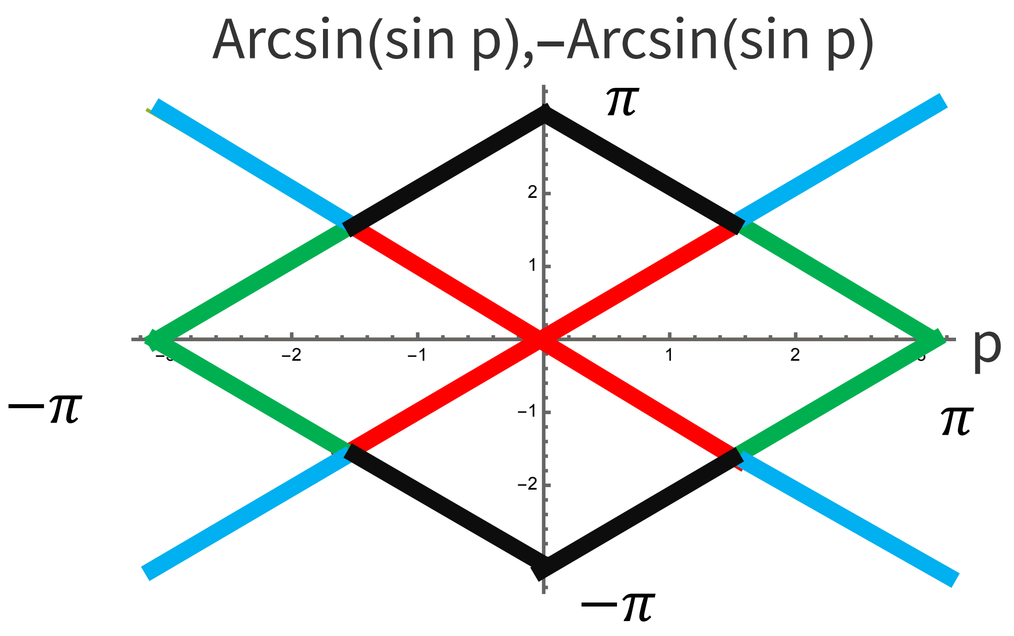
Lattice in space.

Time not discretized.

Solving the naively discretized Dirac Hamiltonian with eigenvalues $\pm\sin p$

$$E = \pm \sin p$$

Brilliuoin zones discrete time (Dirac)



Differently colored Dirac flavors

Solving the discrete time Dirac equation with naïve discretization

$$\sin p_0 = \sin p$$

Solutions:

$$p_0 = \mathrm{ArcSin}(\pm \sin p) = \pm p, \pi \mp p$$

Is there an explicit way to connect Floquet insulators to discrete time systems?

Can the Floquet spectrum be reinterpreted as a time lattice theory of some undriven Hamiltonian (with time lattice spacing T)?

Even if this was the case, what kind of undriven Hamiltonian would those be?

Floquet and lattice

Answer: We don’t know if this is generally true.

But possible for certain models for certain parameters.

Simple example: driven SSH (Dirac) Hamiltonian.