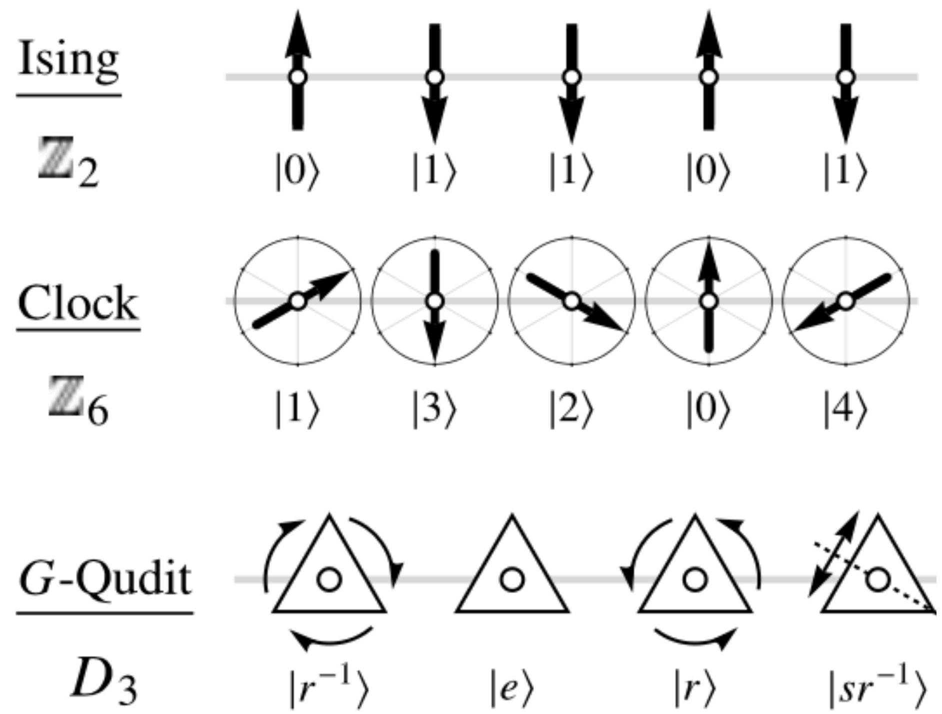
Spontaneously Broken Non-Invertible symmetries
in Transverse-Field Ising Qudit Chains



together with
Kai Chung, Umberto Borla and Andriyy Nevidomskyy

Motivation

Global generalized symmetries
of quantum gauge theories

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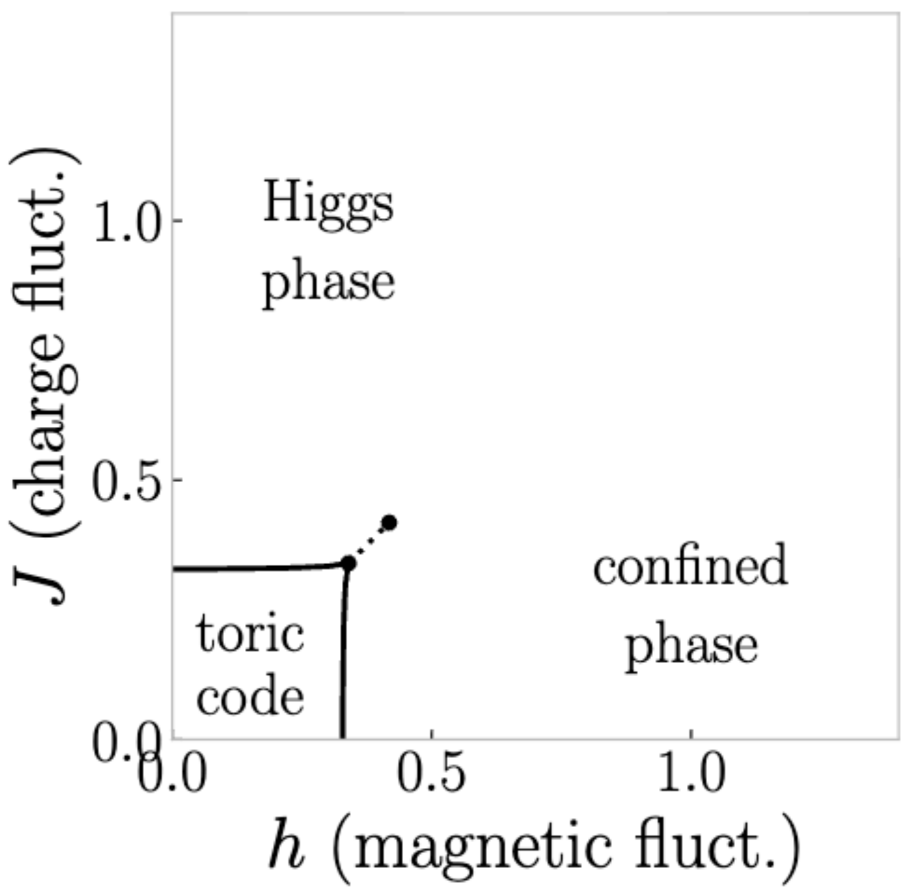

Fradkin-Shenker model
for Z_2 gauge group

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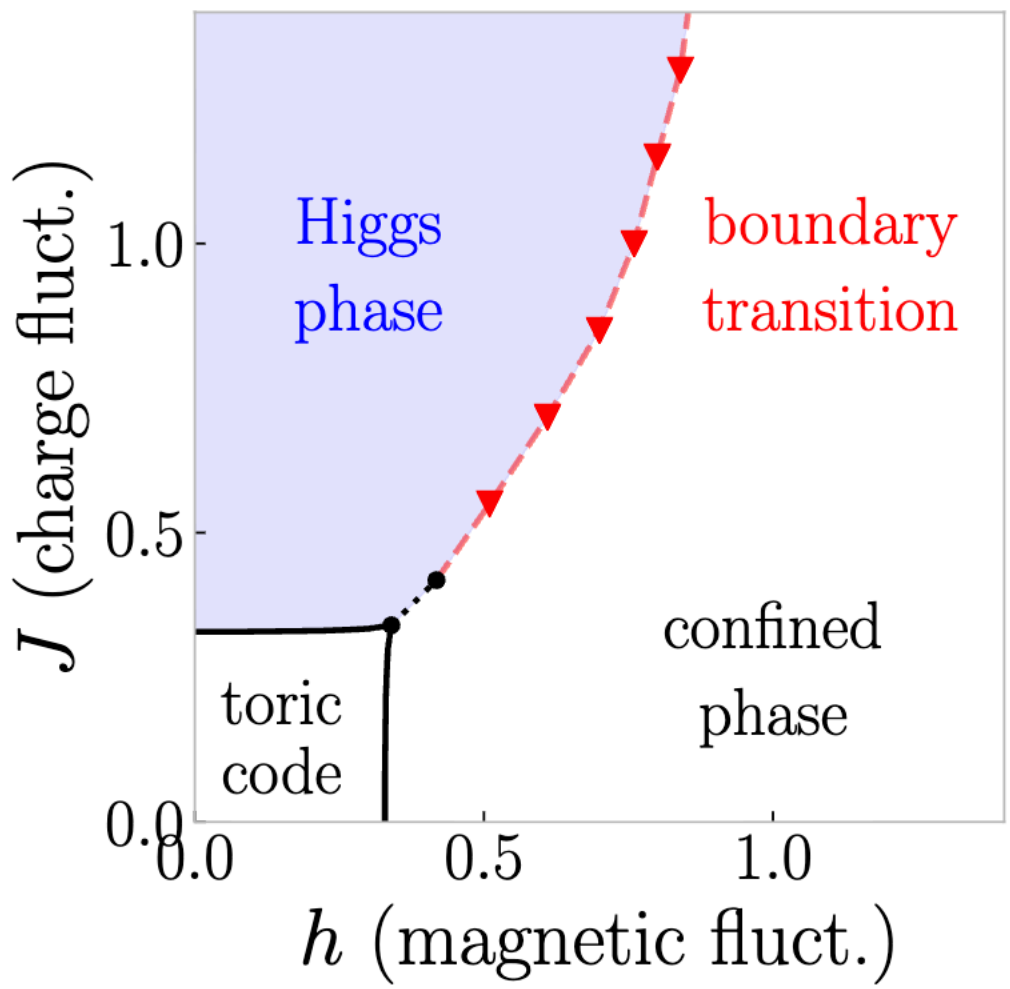

Fradkin-Shenker model
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Higgs=SPT

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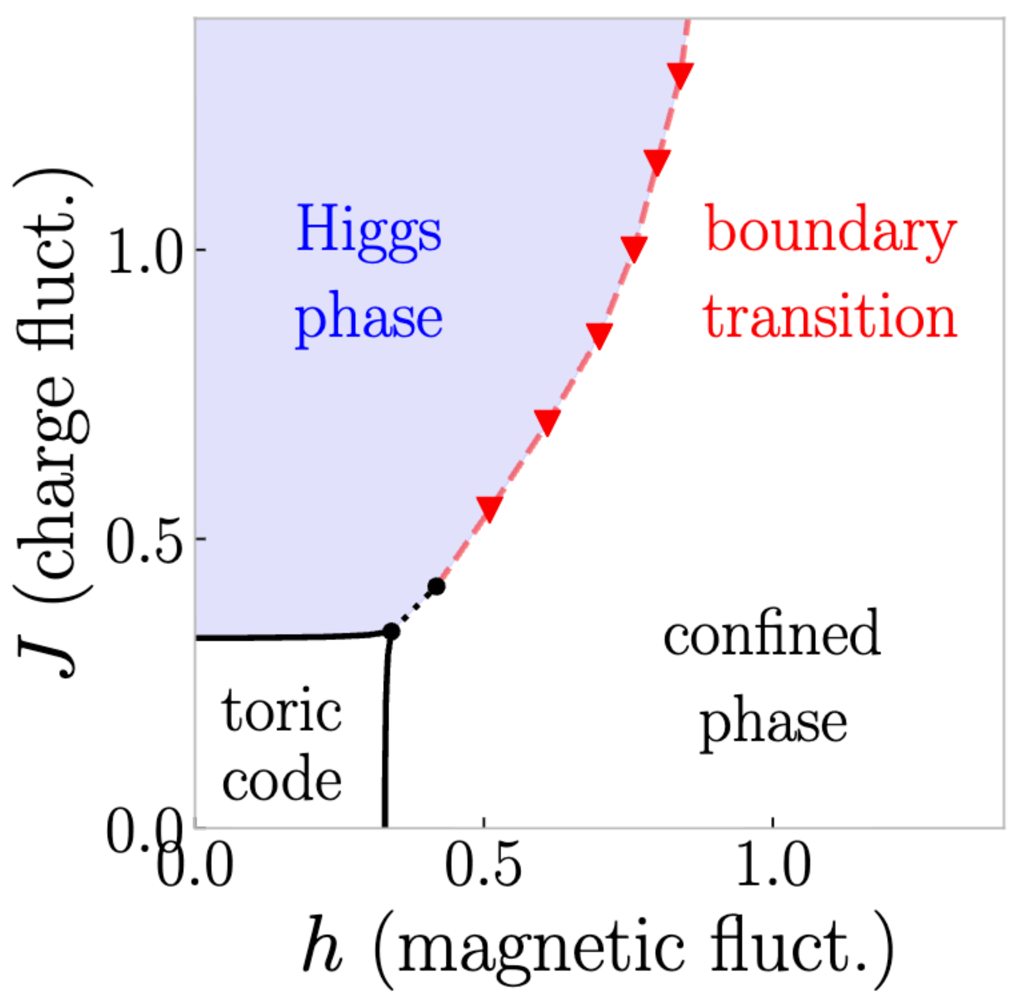


Fradkin-Shenker model
for Z_2 gauge group

Higgs=SPT

How does it work for
gauge theories with
non-Abelian groups?

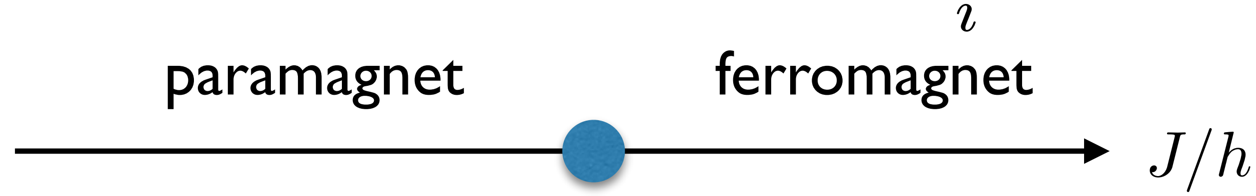
Chung et al 2024

Transverse field Ising chain

Qubit chain with NN Ising interaction and transverse field

$$H_{\mathsf{TFIM}} = -J \sum_i Z_i Z_{i+1} - h \sum_i X_i$$

Ising Z_2 charge is generated by $U = \prod_i X_i$



$$|++\cdots++\rangle$$

$$|00\ldots00\rangle$$

$$|11\ldots11\rangle$$

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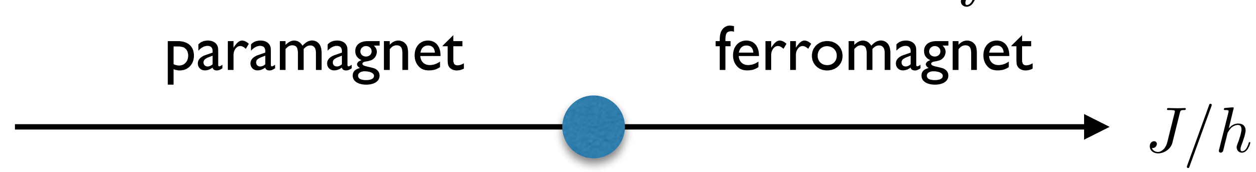

$$|++\cdots++\rangle$$

$$|00\ldots00\rangle$$
$$|11\ldots11\rangle$$

Spontaneous symmetry breaking

$$|11\ldots11\rangle = U|00\ldots00\rangle$$

order parameter Z_i charged under symmetry U

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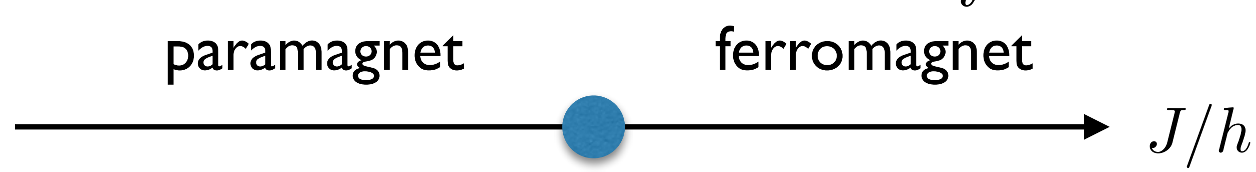

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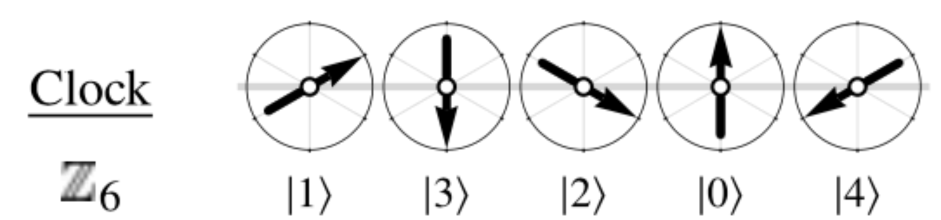
Hilbert space of qubit: $|0\rangle\,,\ |1\rangle$ Z_2 group

Z_N clock models

Two state qubit $\longrightarrow$ N-state qudit

clock operator $Z\,|n\rangle = e^{i2\pi n/N}\,|n\rangle$

shift operator $X\,|n\rangle = |n+1\rangle$



SSB states: $|\boldsymbol{n}\rangle \equiv \bigotimes_i |n\rangle_i$ $\qquad U\,|\boldsymbol{n}\rangle = |\boldsymbol{n}+\boldsymbol{1}\rangle$

order parameter
Z_N SSB

$$\langle \boldsymbol{n}|\,Z_j\,|\boldsymbol{n}\rangle = e^{2\pi i n/N}$$

$\mathbb{Z}_N$ clock models

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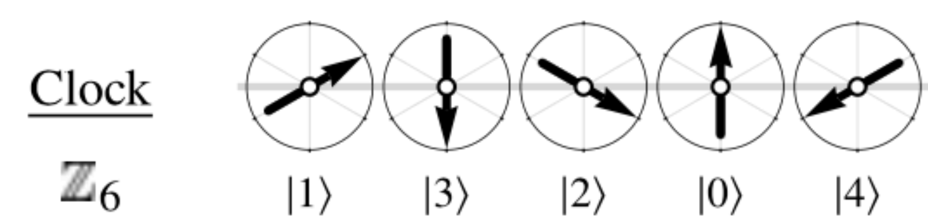

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$\mathbb{Z}_N$ SSB

$$\langle \boldsymbol{n}| Z_j |\boldsymbol{n}\rangle = e^{2\pi i n/N}$$

How to generalize to non-Abelian SSB?

$$Z^k X^m = e^{i2\pi km/N} X^m Z^k$$

G-qudit

Group-valued basis: $|g\rangle$ for each $g \in G$

shift operators $\overrightarrow{X}^h |g\rangle = |hg\rangle \,, \; \overleftarrow{X}^h |g\rangle = |gh^{-1}\rangle$

clock operators $Z^{\Gamma}_{\alpha\beta} |g\rangle = \Gamma^{g}_{\alpha\beta} |g\rangle$ Γ labels irreps

Brell, 2015

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Brell, 2015

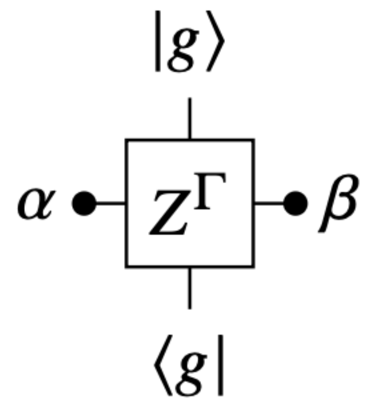
$$\sum_{\Gamma} d_{\Gamma}^{2} = |G| \longrightarrow$$ # of Z= # of X

non-Abelian generalization
of Pauli algebra

$$Z^{\Gamma}_{\alpha\beta}\overrightarrow{X}^{g} = \Gamma^{g}_{\alpha\gamma}\overrightarrow{X}^{g}Z^{\Gamma}_{\gamma\beta}$$

Tensor network representation

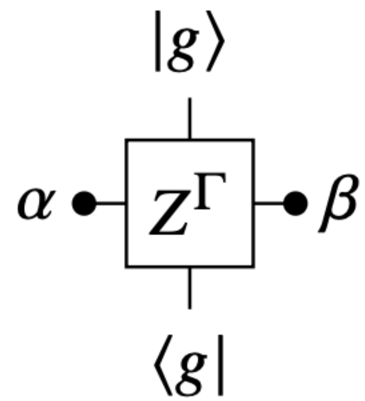
$Z^{\Gamma}_{\alpha\beta} \equiv$



Matrix product operator

$$\alpha, \beta = 1, \ldots, d_{\Gamma}$$

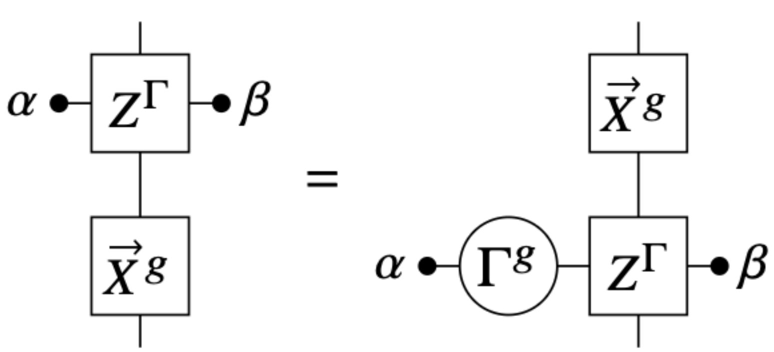
Tensor network representation

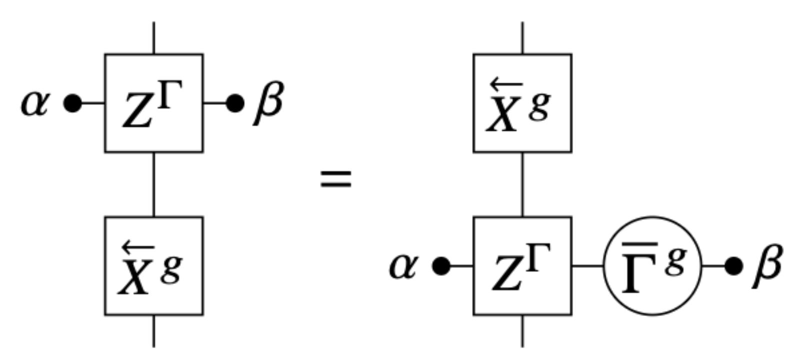
$Z^{\Gamma}_{\alpha\beta} \equiv$ 

Matrix product operator

$$\alpha, \beta = 1, \ldots, d_{\Gamma}$$

The algebra can be now written





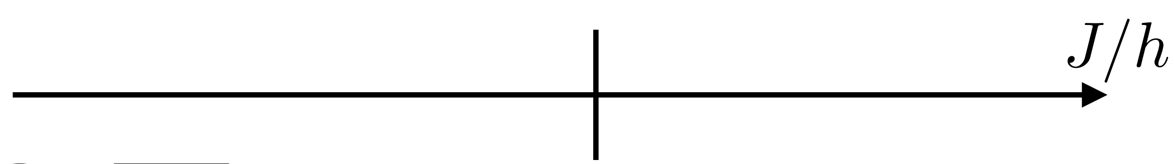
here $\overline{\Gamma}^{g} = \Gamma^{g^{-1}}$

G-symmetric TFIM

On a chain of G-qudits introduce

$$H_G = -J \sum_i \sum_\Gamma d_\Gamma \, \mathrm{Tr}[Z_i^\Gamma \cdot Z_{i+1}^{\overline{\Gamma}}] - h \sum_i \sum_g \overleftarrow{X}_i^g + \text{h.c.}$$

G symmetry generated by $$U_g = \prod_i \overrightarrow{X}_i^g$$



$$|+\rangle \equiv \bigotimes_i \sum_g |g\rangle_i$$

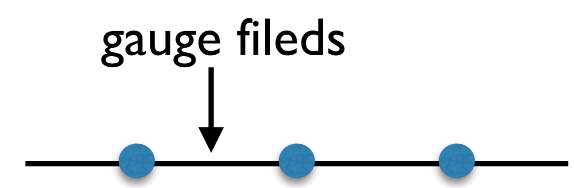
Paramagnet

$$|\boldsymbol{g}\rangle \equiv \bigotimes_i |g\rangle_i$$

SSB states

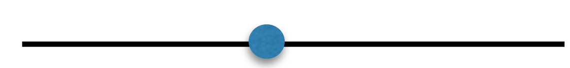
Gauging G-symmetry

Introduce G-valued gauge field on links



$$H_G \to -J \sum_i \sum_\Gamma d_\Gamma \, \mathrm{Tr}[Z_i^\Gamma \cdot \mathcal{Z}_{i+\frac{1}{2}} \cdot Z_{i+1}^{\overline{\Gamma}}] - h \sum_i \sum_g \overleftarrow{X}_i^g + \text{h.c.}$$

with Gauss law constraints: $G_i^g \equiv \overleftarrow{\mathcal{X}}^g_{i-\frac{1}{2}} \overrightarrow{X}^g_i \overrightarrow{\mathcal{X}}^g_{i+\frac{1}{2}} \stackrel{!}{=} 1$



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resolving this constraint we end up with

$$H_G \to -J \sum_i \sum_\Gamma d_\Gamma \,\mathrm{Tr}[\mathcal{Z}_{i+\frac{1}{2}}] - h \sum_i \sum_g \overleftarrow{\mathcal{X}}_{i-\frac{1}{2}}^g \overrightarrow{\mathcal{X}}_{i+\frac{1}{2}}^g + \text{h.c.}$$

KW duals are different for non-Abelian G

Non-invertible symmetry

$$H_{\widetilde{G}} = -J \sum_i \sum_g \overleftarrow{X}_i^g \overrightarrow{X}_{i+1}^g - h \sum_i \sum_\Gamma d_\Gamma \, \mathrm{Tr}[Z_i^\Gamma] + \text{h.c.}$$

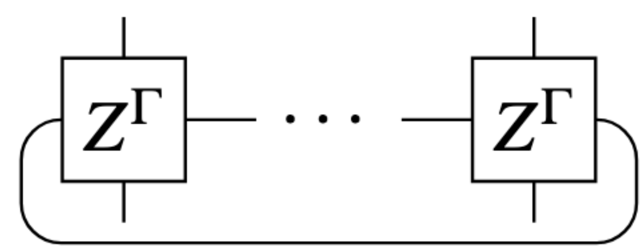
Symmetry genertors

$$R_\Gamma = \mathrm{Tr} \prod_i Z_i^\Gamma$$

labeled by irreps

MPO form:

R_Γ =



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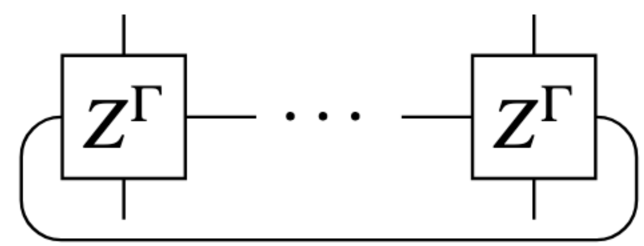


Symmetries multiply according to irrep Rep(G) algebra

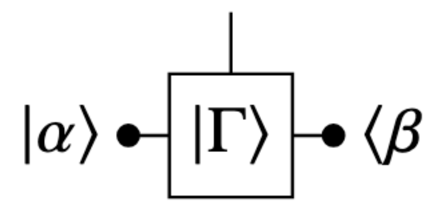
$$R_{\Gamma_a} R_{\Gamma_b} = \sum_c N_{ab}^c R_{\Gamma_c} \quad \Leftrightarrow \quad \Gamma_a \otimes \Gamma_b = \bigoplus_c N_{ab}^c \Gamma_c$$

For non-Abelian groups G this algebra is <u>non-invertible</u>

Spontaneous symmetry breaking

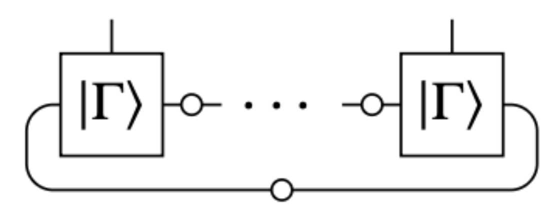
To find SSB states, introduce a dual basis of G-qudit

$$|\Gamma_{\alpha\beta}\rangle = \sqrt{\frac{d_\Gamma}{|G|}}\sum_{g\in G}\Gamma^{g}_{\alpha\beta}\,|g\rangle \equiv$$



which block-diagonalizes X operators $\vec{X}^{\,g}\,|\Gamma_{\alpha\beta}\rangle = \Gamma^{g^{-1}}_{\alpha\gamma}\,|\Gamma_{\gamma\beta}\rangle$

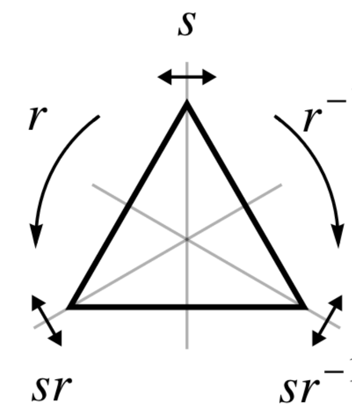
SSB states for h=0: $$|\boldsymbol{\Gamma}\rangle = \sum_{\{\alpha_i\}}\bigotimes_i \frac{1}{\sqrt{d_\Gamma}}\,|\Gamma_{\alpha_i\alpha_{i+1}}\rangle =$$



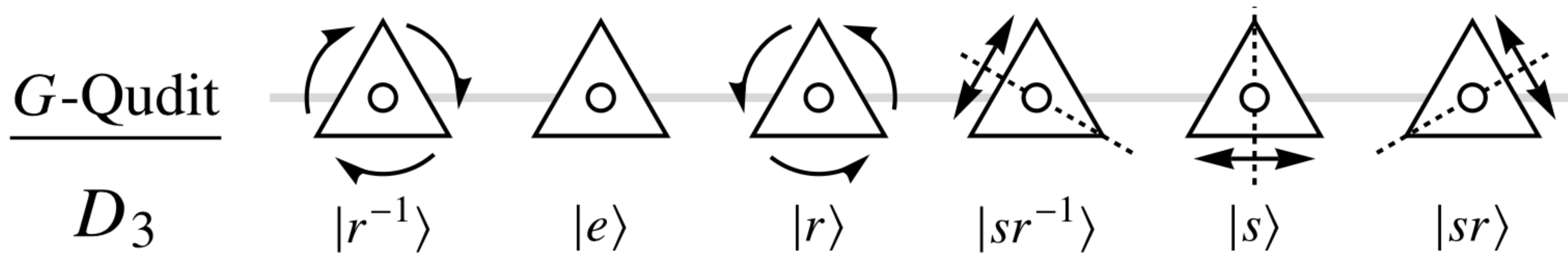
local
order parameter

$$\langle\boldsymbol{\Gamma}|\vec{X}^{\,g}\,|\boldsymbol{\Gamma}\rangle = \mathrm{Tr}[\Gamma^{g}]/d_\Gamma$$

D$_3$ group



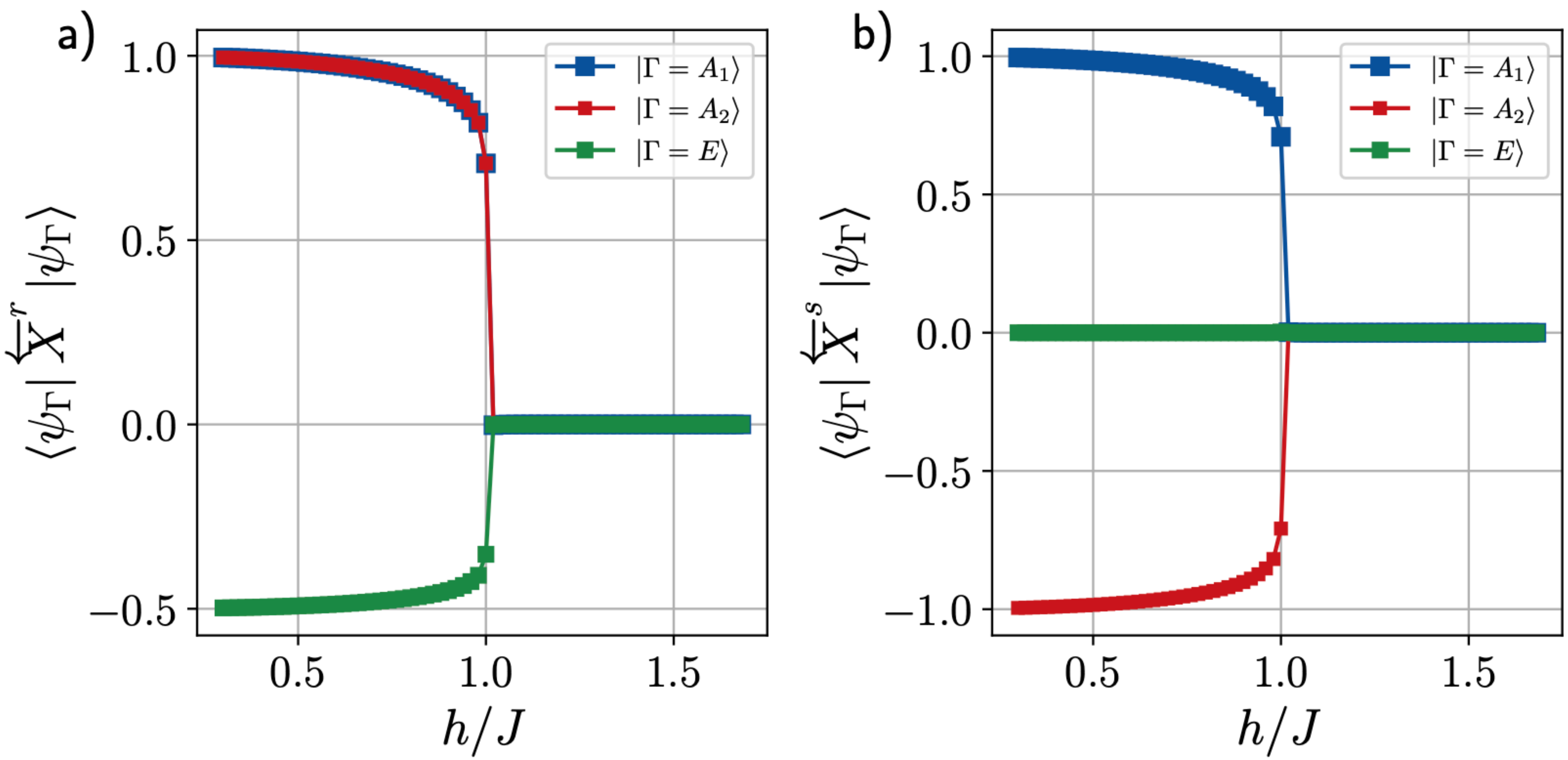
Symmetries of equilateral triangle: non-Abelian $sr = r^{-1}s$



Three irreps: one-dimensional— A_1, A_2 two-dimensional— E

$\Gamma_a \otimes \Gamma_b$	A_1	A_2	E
A_1	A_1	A_2	E
A_2	A_2	A_1	E
E	E	E	$\boldsymbol{A_1 \oplus A_2 \oplus E}$

Rep(D_3) SSB



iDMRG: local order parameter, transition at h=J

Rep(G) SSB: new features

$$R_{\Gamma_a} |\mathbf{\Gamma}_b\rangle = \sum_c N_{ab}^c |\mathbf{\Gamma}_c\rangle \longleftarrow$$ cat states for non-Abelian G

SSB ground states have different entanglement structure

D_3 group: product states $|\boldsymbol{A_1}\rangle, |\boldsymbol{A_2}\rangle$

entangled MPS $|\boldsymbol{E}\rangle$ with bond dim=2

Rep(G) SSB: new features

$$R_{\Gamma_a} |\boldsymbol{\Gamma}_b\rangle = \sum_c N_{ab}^c |\boldsymbol{\Gamma}_c\rangle$$ $\longleftarrow$ cat states for non-Abelian G

Different SSB ground state have different entanglement

D_3 example: product states $|\boldsymbol{A_1}\rangle, |\boldsymbol{A_2}\rangle$

entangled MPS $|\boldsymbol{E}\rangle$

<u>SPT-like features:</u>

edge modes

entanglement spectrum degeneracy

string order parameter

Rep(G) SSB: new features

on open chain |G| SSB states

<u>Edge modes:</u>

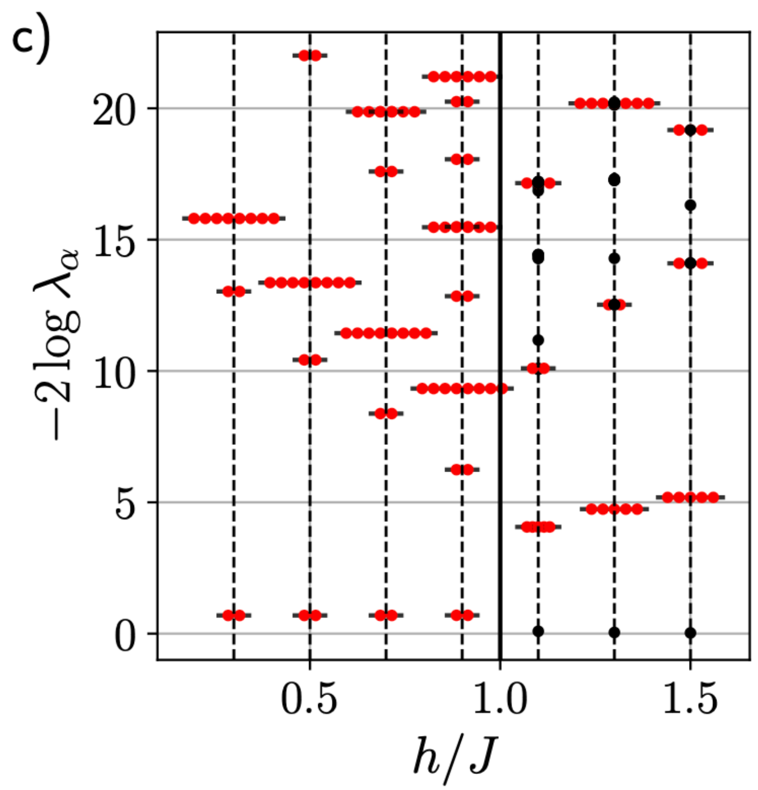
$$|\mathbf{\Gamma}_{\alpha\beta}\rangle \;=\; |\alpha\rangle \bullet\!\!-\!\boxed{|\Gamma\rangle}\!-\!\circ- \cdots -\circ\!-\!\boxed{|\Gamma\rangle}\!-\!\!\bullet \langle\beta|$$

transitions between different SSB states
with local edge operators

partial <u>fractionalization</u> of G-qudit between two edges

Rep(G) SSB: new features

<u>Entanglement spectrum:</u>



Schmidt decomposition

$$|\psi\rangle = \sum_\alpha \lambda_\alpha |\alpha L\rangle |\alpha R\rangle$$

in the SSB entangled state $|\boldsymbol{E}\rangle$

all Schmidt eigenvalues are two-fold degenerate

Pollmann et al
Perez-Garcia et al

Rep(G) SSB: new features

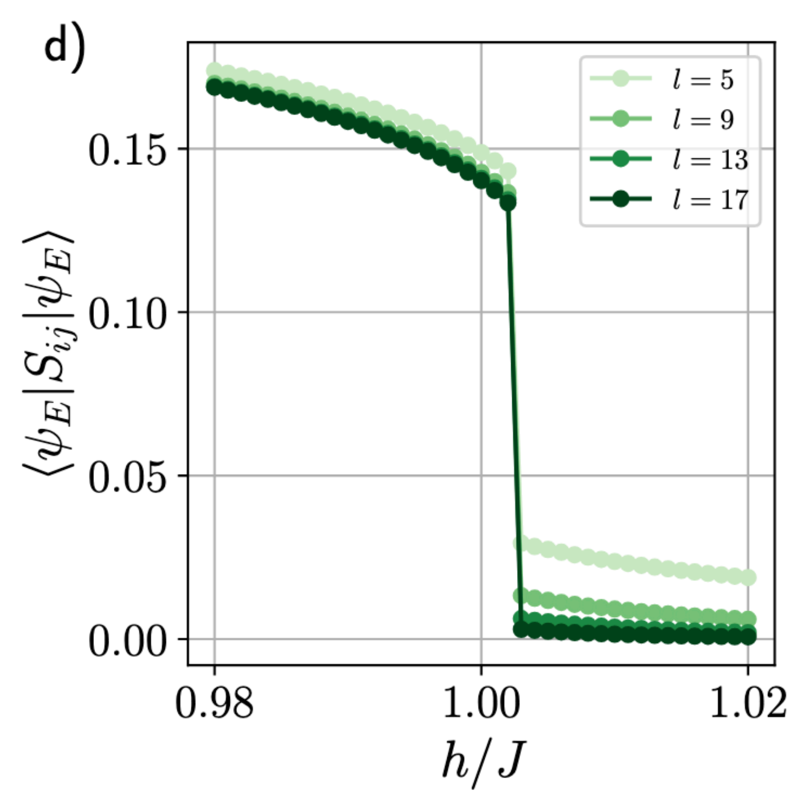
Ordinary symmetry multiplets: $U_g \mathcal{O}^\alpha_\Gamma = \left(\Gamma^g_{\alpha\beta}\mathcal{O}^\beta_\Gamma\right) U_g$

multiplets of non-invertible symmetry contain local and non-local operators

Bhardwaj et al

$$S_{ij} = \overleftarrow{X}^r_i \prod_{i<k<j} Z^{A_2}_k \vec{X}^r_j$$

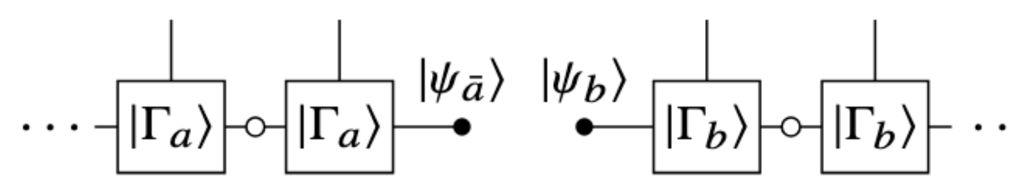
SSB state $|\boldsymbol{E}\rangle$ supports both local and non-local order parameters



Anyonic domain walls

Domain walls can carry internal dofs- open legs of MPS

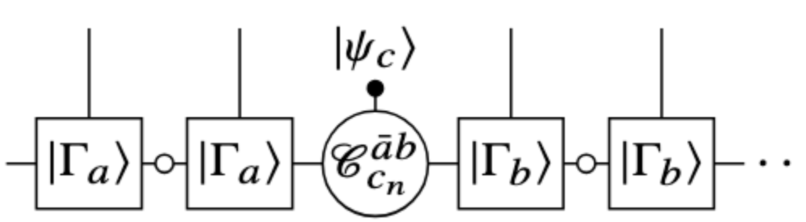
They fuse as non-abelian anyons with multiple possibilities

Two domain walls nearby: $|\psi_{ab}\rangle = \cdots$ 

Using Clebsch-Gordon decomposition

$$|\Gamma_{c_n}, \gamma\rangle = \sum_{\alpha,\beta} [C^{c_n}_{\bar{a}b}]^{\gamma}_{\bar{\alpha}\beta} \, |\Gamma_{\bar{a}}, \bar{\alpha}\rangle \otimes |\Gamma_b, \beta\rangle$$

$$|\psi_{ab}\rangle = \sum_c \sum_{n=1}^{N^c_{ab}} \cdots$$



different fusion channels

Outlook

- Realization with quantum hardware
- Higgs=SPT with non-Abelian groups
- Phase transition
- extension to 2d:
  non-Abelian anyons and quantum computing

arXiv: 2508.11003

Extra slides

Proof of non-invertability

Consider the action of symmetry on the basis state

$$R_\Gamma \left| g_1, \cdots, g_L \right\rangle = \mathrm{Tr}[\Gamma^{g_1 \cdots g_L}] \left| g_1, \cdots, g_L \right\rangle$$

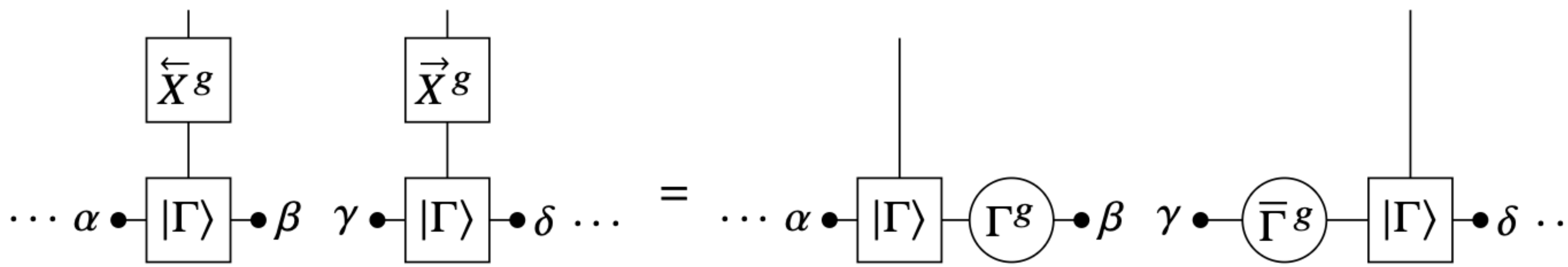
character

for irreps with $d_\Gamma > 1$ at least one group element must have zero character

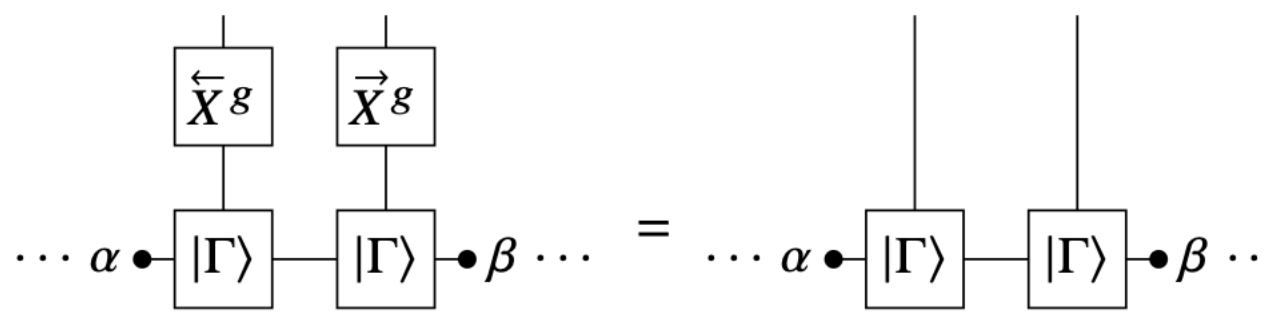
R_Γ must have finite kernel $\longrightarrow$ symmetry cannot be inverted

SSB details

Using

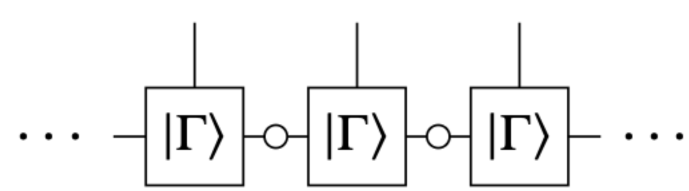


we find



eigenvalue
+1

therefore

$|\mathbf{\Gamma}\rangle =$  $\equiv \sum_{\{\alpha_i\}} \bigotimes_i \frac{1}{\sqrt{d_\Gamma}} |\Gamma_{\alpha_i \alpha_{i+1}}\rangle_i$

Multiplet of Rep(D_3)

$$R_E \vec{X}_j^r = \left[\mathrm{Re}(\omega) \vec{X}_j^r + i\, \mathrm{Im}(\omega) \left(\prod_{k<j} Z_k^{A_2} \right) \vec{X}_j^r \right] R_E$$

local local non-local string

Bhardwaj et al

with $\omega = \exp(i2\pi/3)$