The Nielsen-Ninomiya theorem versus bosonization

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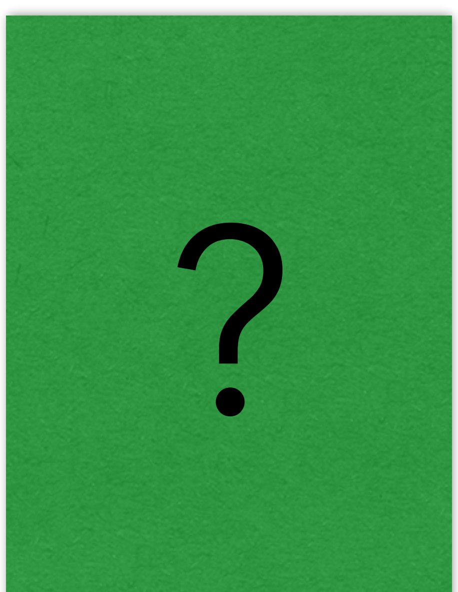
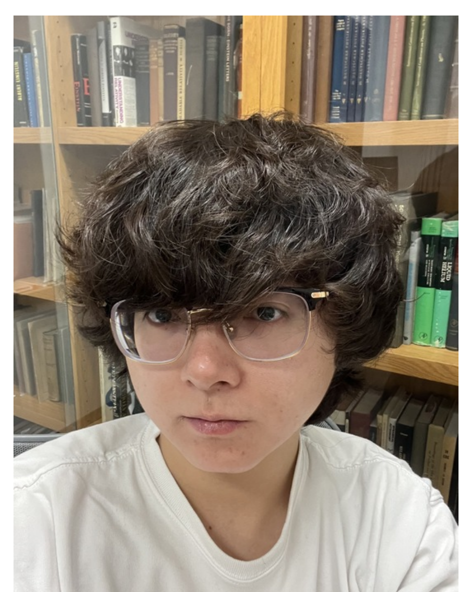
Work in progress with



Saif Ullah Baig



Shi Chen

Maria Neuzil

Symmetry of massless Dirac fermions

- In even spacetime D, a free massless Dirac fermion ψ has the symmetry

$$G_\psi = \frac{U(1)_V \times U(1)_A}{\mathbb{Z}_2}$$

$$U(1)_V : \psi \to e^{i\alpha}\psi$$
$$U(1)_A : \psi \to e^{i\Gamma\beta}\psi$$

$$\{\Gamma, \gamma^\mu\} = 0$$

- Mixed 't Hooft anomaly: turning on a $U(1)_V$ background gauge field A_μ breaks $U(1)_A$, and vice versa.
  - This is the origin of the ABJ anomaly.

Massless fermions of the lattice

- We all know that discretizing QFTs on lattices is very useful.
- It is also subtle, especially when dealing with massless fermions.
- The basic reason is that it is tricky to get anomalies right on the lattice.
- There is a famous no-go theorem, the Nielsen-Ninomiya theorem, that says it can't be done, under some assumptions.

Nielsen-Ninomiya Theorem

- Rough statement: there is no discretization of the Dirac operator $\mathscr{D} = \gamma^\mu \partial_\mu$ which enjoys all of

1. Continuity in p_μ — Locality
2. $\mathscr{D}(p) = \gamma^\mu p_\mu$ for $a|p| \ll 1$ — Free fermion as $a \to 0$
3. $\mathscr{D}$ invertible except at $|p| \to 0$ — No doublers
4. $\{\Gamma, \mathscr{D}\} = 0$ — Chiral symmetry

Naive lattice fermions

Wilson, …

- Replacing the continuum derivatives by naive lattice finite difference operators gives the naive lattice fermion discretization.
- $\mathscr{D}$ invertible except at each corner of the Brilloin zone, leading to 2^D massless fermions.

Wilson fermions

- The Wilson fermion discretization explicitly breaks chiral symmetry completely as well as the degeneracy between doubler modes.
- Must tune the bare mass to get a massless fermions in the continuum limit.

Staggered fermions

Kogut, Susskind, …

- The reduced staggered fermion discretization preserves a discrete subgroup of chiral symmetry.
- This is enough to forbid a fermion mass term from being generated, but leaves a smaller number of doubler/ `taste' modes in $D > 1$.
- Example: two `tastes' in $D = 2$ rather than $2^2 = 4$ doublers with naive fermions.

Domain wall fermions

Kaplan; Shamir; Neuberger; ...

- Overlap lattice fermions preserve a modified version of chiral symmetry, and the associated Dirac operator is not ultra-local.
- The same is true for domain-wall fermions when the extent of the extra dimensions goes to ∞.
- Hersh's talk: the resulting chiral symmetry is $\mathbb{R}$, not $U(1)$.

Symmetric mass generation

- The NN theorem has an extra assumption, which is that the lattice fermion action is bilinear.
- In the SMG approach (which is extremely closely related to older mirror fermion ideas), the idea is to give some unwanted lattice fermion excitations large masses by introducing carefully designed interactions, so that the lattice action is **not** bilinear in ψ .

Eichten, Preskill, … ; Poppitz, Giedt, …; Wang, Wen, …

- Near the continuum limit, one ends up producing an effective Dirac operator with its own non-localities associated with propagator zeros.

Golterman + Shamir; You + Xu, …

Recent developments

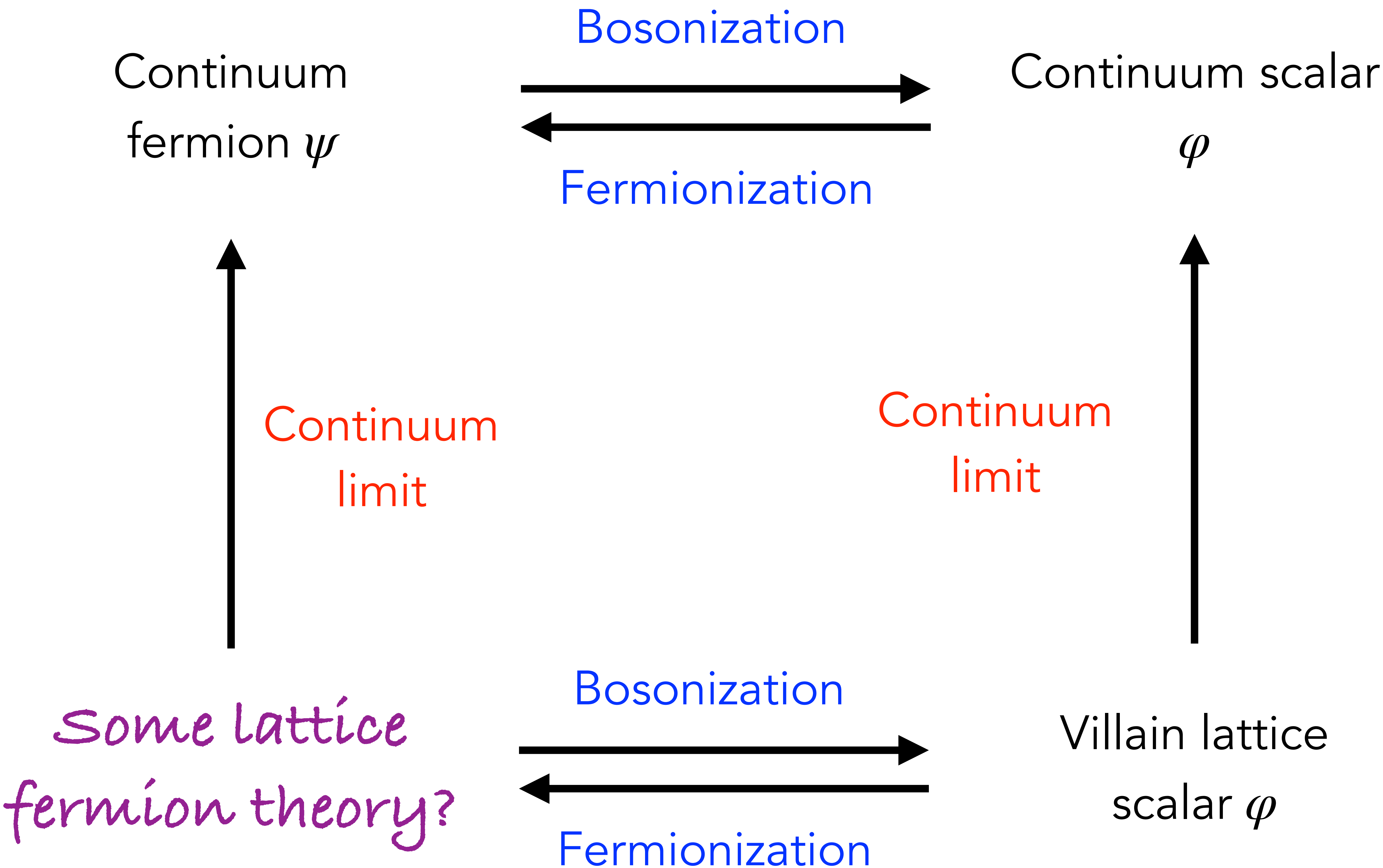
- It has become relatively widely appreciated that even scalars can have anomalies, and some anomalies can be preserved on lattice, without making the symmetry action any less local than it is in the continuum.
- People also found ways to preserve topological symmetries on lattice.

Sulejmanpasic, Shao, Seiberg, Lam, Fazzi, Gorantla, Gattringer, Cheng, Seifnashri…
2019 - now

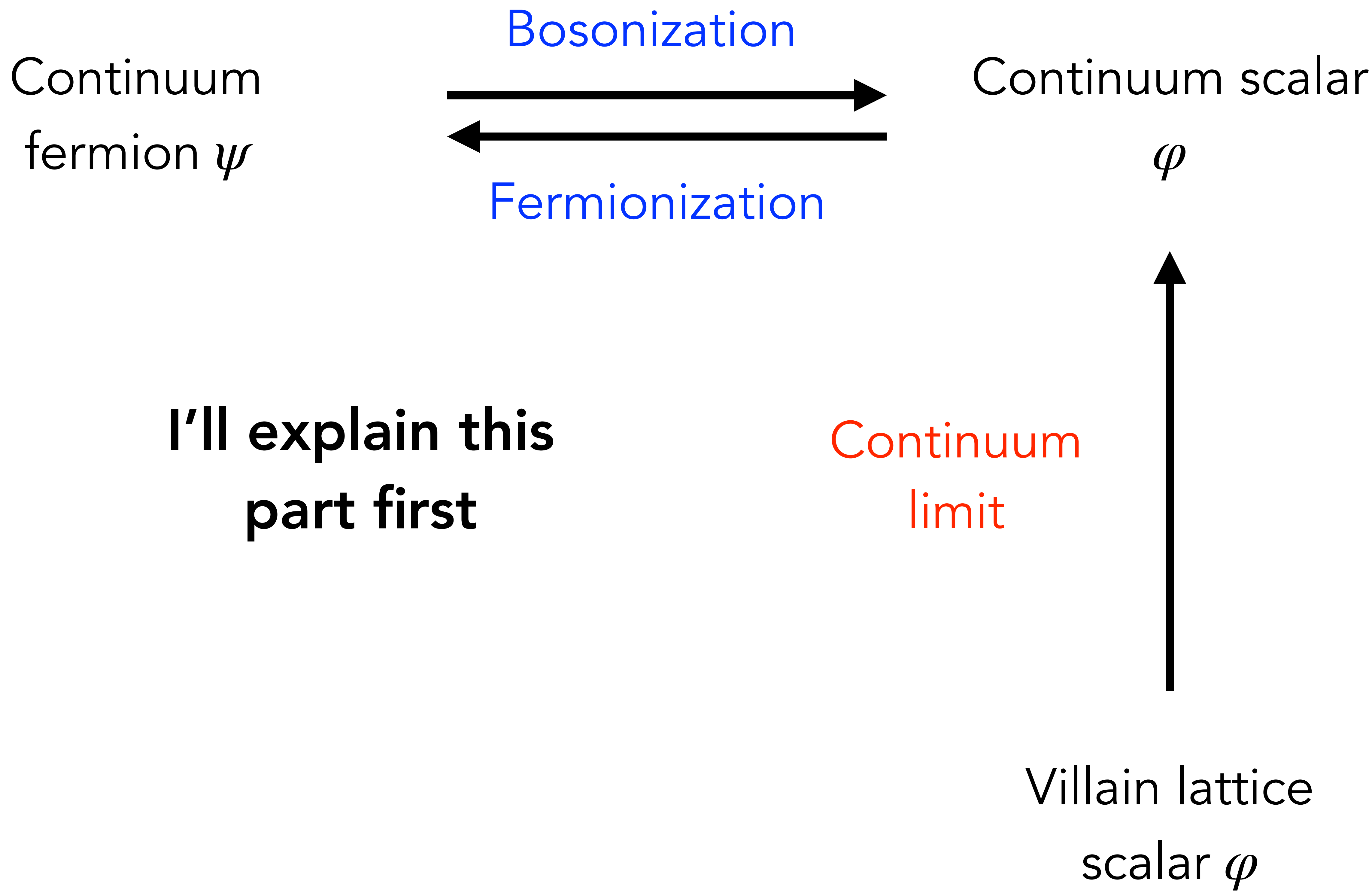
Cond-mat, hep-lat antecedents:
Catterall et al, Lieb+Shutz+Mattis, Kitaev, Kapustin+Thorngren, …

- **I'll focus on a single 2d Dirac fermion for the rest of talk.**

Idea of the talk



Idea of the talk



Idea of the talk

Then I'll focus on this part.

- Much remains unclear about the lattice fermion theory.
- I'll focus on what we understood so far, and highlight how it connects to the NN theorem.

Some lattice fermion theory?

Bosonization →

← Fermionization

Villain lattice scalar φ

2d Abelian bosonization

Free 1+1d Dirac fermion QFT with gauged fermion parity $(-1)^F$	=	2π-periodic boson QFT $\mathscr{L} = \frac{R^2}{4\pi}(d\varphi)^2, R^2 = \frac{1}{2}$

- Some entries in the dictionary:

$$\bar{\psi}_-\psi_+ \sim e^{i\varphi}$$

$$\bar{\psi}\gamma^\mu\psi \sim -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\psi$$

$$\bar{\psi}\gamma^\mu\Gamma\psi \sim \frac{i}{4\pi}\partial_\nu\psi$$

$$\psi_+\psi_- \sim e^{2i\theta} \qquad R^2 d\varphi = i\star d\theta$$

2d Abelian bosonization

Free 1+1d Dirac fermion QFT with gauged fermion parity $(-1)^F$

=

2π-periodic boson QFT
$\mathscr{L} = \frac{R^2}{4\pi}(d\varphi)^2, R^2 = \frac{1}{2}$

- What about $\psi_\pm$?

$$\psi_\pm \sim e^{i\theta \pm i\varphi/2}$$

- The left-hand side is ill-defined on its own. It must live at the end of the suitable topological line operator.
  - Natural given that $(-1)^F$ is gauged.

Compact boson review

$$\mathscr{L} = \frac{R^2}{4\pi}(d\varphi)^2, \ \varphi \simeq \varphi + 2\pi$$

- Conserved currents:

$$j_A = \frac{R^2}{2\pi} d\varphi \qquad\qquad j_V = \frac{1}{2\pi} \star d\varphi$$

- Conserved charges:

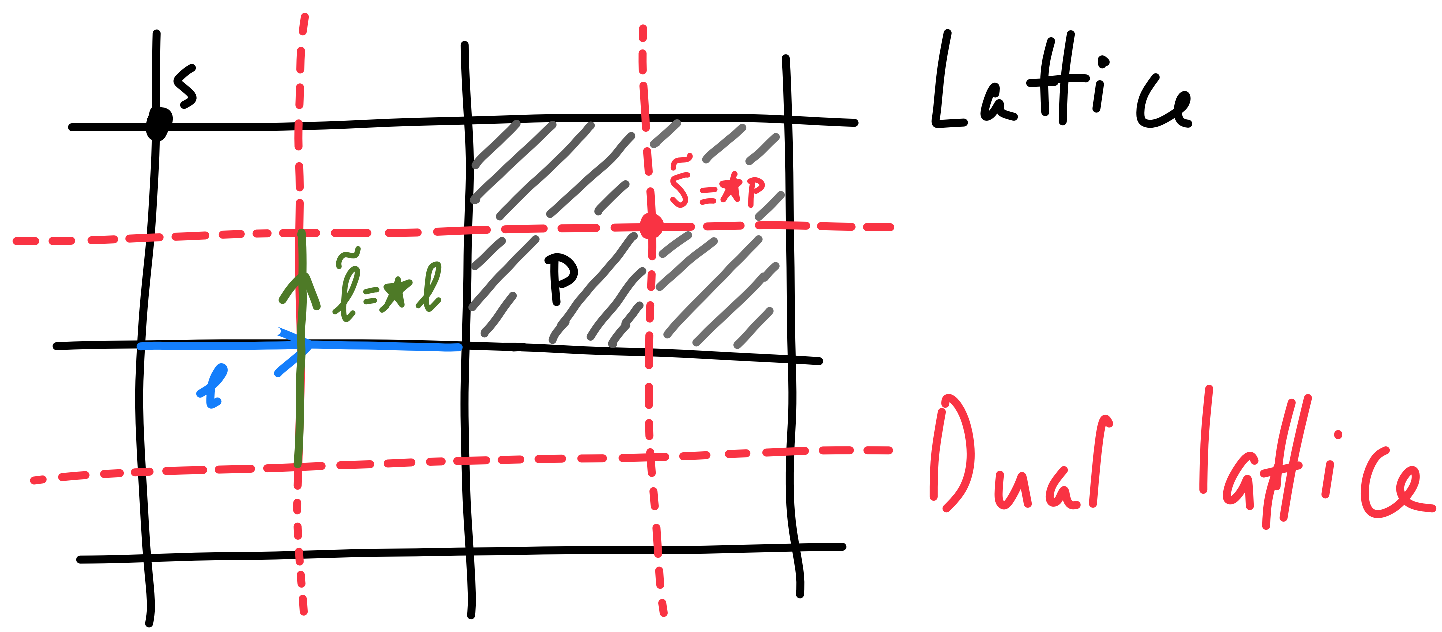
$$Q_A(C) = \int_C \star j_A = \frac{R^2}{2\pi} \int_C \star d\varphi \qquad\qquad Q_V(C) = \int_C \star j_V = \frac{-1}{2\pi} \int_C d\varphi$$

φ shift charge — Winding number charge

Conserved due to e.o.m. — Tautologically conserved

The lattice

- Work on a square lattice with sites s, links ℓ, plaquettes p, and cells on dual lattice $\tilde{s}, \tilde{\ell}, \tilde{p}$.
- "Hodge star" map from lattice to dual lattice
  $\star\, s = \tilde{p}, \star\, \ell = \tilde{\ell}, \star\, p = \tilde{s}$
- $(d\omega)_{c^{r+1}} = \sum_{c^r \in \partial c^{r+1}} \omega_{c^r}$, so that $(d\phi)_\ell = \phi_{s+a\hat{\ell}} - \phi_s$, $d^2 = 0$.



Villain scalar

Villain 1970s;
Gross, Klebanov 1990s;
Cheng, Gattringer,
Gorantla, Fazzi, Lam,
Seiberg,
Shao,Sulejmanpasic,…
2019 - now

- A continuum scalar can be represented as

$$\varphi(x) \to \begin{cases} \varphi_s \in \mathbb{R} \\ n_\ell \in \mathbb{Z} \end{cases}$$

- Impose a discrete gauge redundancy involving $k_s \in \mathbb{Z}$

$$\varphi_s \to \varphi_s + 2\pi k_s$$
$$n_\ell \to n_\ell + (dk)_\ell$$

- Then $\{\varphi, n\}$ describe a 2π-periodic scalar.

- Discrete-gauge invariant derivatives look like

$$\partial_\mu \varphi(x) \to (d\varphi)_{\tilde{\ell}} - 2\pi n_{\tilde{\ell}}$$

Villain gauge field

Villain 1970s;

- A U(1) gauge field has a similar discretization:

$$a_\mu(x) \to \begin{cases} a_\ell \in \mathbb{R} \\ r_p \in \mathbb{Z} \end{cases}$$

- Impose a discrete gauge redundancy involving $m_\ell \in \mathbb{Z}$ and a continuous gauge redundancy involving $h_s \in \mathbb{R}$

$$a_\ell \to a_\ell + (dh)_\ell + 2\pi m_\ell$$
$$r_p \to r_p + (dm)_p$$

- Then $\{a, r\}$ describes a $U(1)$ gauge field.
- The gauge invariant field strength is

$$f_{\mu\nu}(x) \to (da)_p - 2\pi r_p$$

Topology on the lattice

- The Villain discretization ensures that winding number and instanton number are quantized at finite lattice spacing:

$$\text{winding of } \varphi = \frac{-1}{2\pi}\sum_{\tilde{\ell}\in C}\left[(d\varphi)_{\tilde{\ell}} - 2\pi n_{\tilde{\ell}}\right] = \sum_{\ell\in C} n_{\tilde{\ell}} \in \mathbb{Z}$$

$$\text{instanton number of } a_\mu = \frac{-1}{2\pi}\sum_{p\in S}\left[(da)_p - 2\pi r_p\right] = \sum_{p\in S} r_p \in \mathbb{Z}$$

- Gives us a chance to get things right!

Winding number conservation

- In the 70s Villain construct, the Cconservation equation for j_V is

$$d \star j_V = -\frac{1}{2\pi} d \left[(d\varphi)_\ell - 2\pi n_\ell \right] = (dn)_p$$

- dn can be any integer, so j_V is not conserved.
  - Physically, $(dn)_p \neq 0$ whenever there's a vortex at p. Dynamical vortices mean that winding number can jump by any integer.

- Modified Villain idea: constrain $(dn)_p$ to be zero!

Modified Villain lattice scalar

Cheng, Gattringer, Gorantla, Fazzi, Lam, Seiberg, Shao,Sulejmanpasic,… 2019 - now

- Add a Lagrange multiplier θ living on dual sites:

$$S = \frac{R^2}{4\pi} \sum_{\ell} \left[(d\varphi)_\ell - 2\pi n_\ell \right]^2 + i \sum_{p} \theta_{\star p} (dn)_p$$

- Now **both** $U(1)_V$ and $U(1)_A$ are present on the lattice, and act as $\theta_{\tilde{s}} \to \theta_{\tilde{s}} + \alpha$, $\varphi_s \to \varphi_s + \beta$ respectively.
- $\theta_{\tilde{s}}$ is the lattice version of θ, the T-dual of φ.
- 't Hooft anomaly captured exactly:

$$d \star j_A = \frac{1}{2\pi} dA_V \Rightarrow (d \star j_A)_{\tilde{p}} = \frac{1}{2\pi} \left[(dA_V)_{\tilde{p}} - 2\pi R_{\tilde{p}} \right]$$

QED and chiral gauge theory on lattice

- One can use this formalism to formulate e.g. U(1) gauge theory coupled to one or more massless Dirac fermions.
  - Preserves all Abelian symmetries at finite lattice spacing
- Can also find a discretization of `3450' chiral gauge theory with a **dynamical** gauge field that preserves all its symmetries.
  - And one can even do Monte Carlo simulations of it.

Berkowitz, AC, Jacobson 2024
+ Work in progress with Evan Berkowitz

Nielsen-Ninomiya theorem

- The NN theorem simply doesn't apply to our construction. It constrains discretization of Dirac operators.
  - We do not start with a continuum Dirac operator at all.
  - Starting point is a different path integral that doesn't even have fermions.
- But it would be nice to make contact with the NN theorem!
- Idea: if we compute the fermion-operator two-point function using the modified Villain theory, it will tell us what type of lattice fermion theory we are dealing with.

Idea of the talk

Some lattice fermion theory?

Bosonization →

← Fermionization

Villain lattice scalar φ

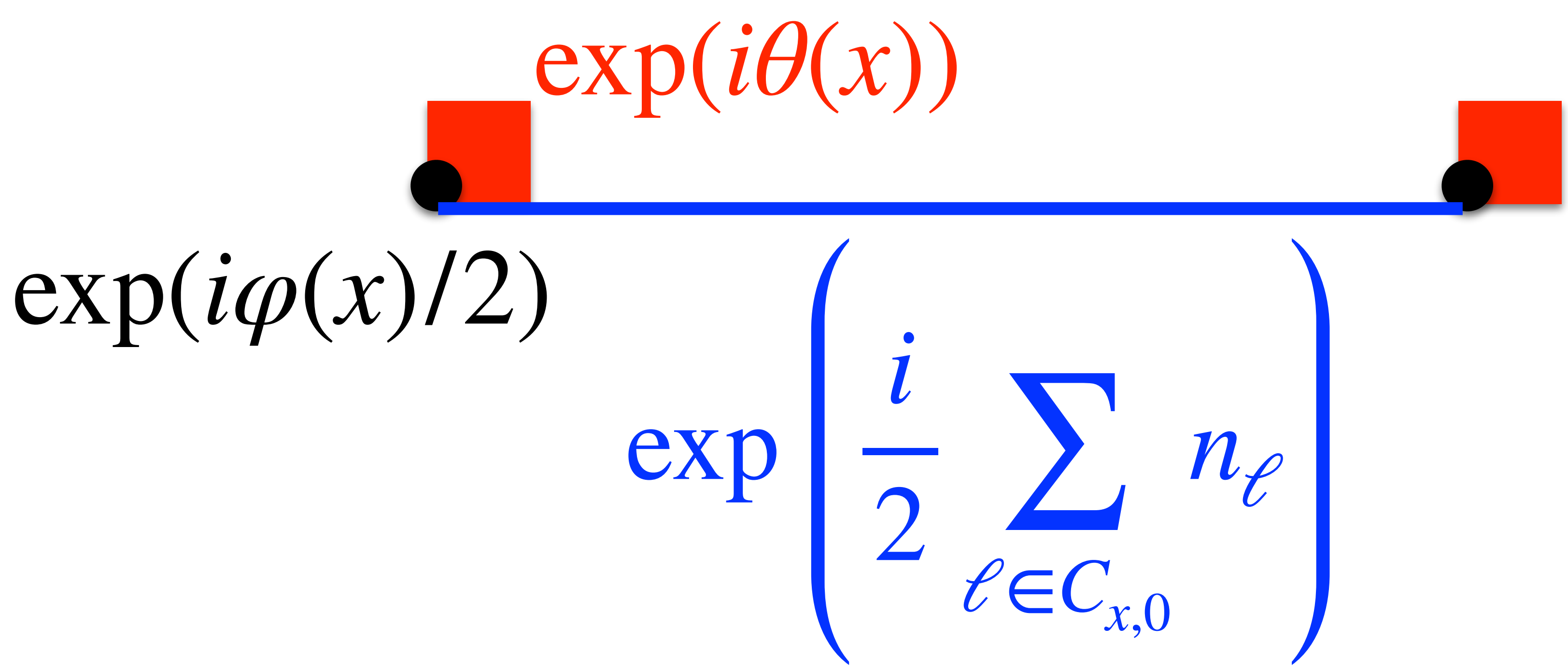
- We can compute some correlation functions of the lattice fermion theory above to try to understand what it looks like.

Fermion two-point function

- The thing to compute in the fermion theory is $\langle \psi_+(x)\bar{\psi}_+ \rangle$.
- In the bosonic variables this is

$$\left\langle e^{i\theta(x)} \exp\left(\frac{i}{2} \int_{C_{x,0}} d\varphi \right) \; e^{-i\theta(0)} \right\rangle \sim \langle e^{i\theta + i\varphi/2}(x) \; e^{-i\theta - i\varphi/2}(0) \rangle$$

- On the lattice, we should calculate a correlation function that looks like



φ correlators

- The thing we want to calculate is a bit complicated.
- Warm up:

$$\langle e^{i\varphi_s} e^{-i\varphi_{s'}} \rangle = ?$$

- Idea: if we integrate out θ, $dn = 0$. If we work on $\mathbb{R}^2$, we can fix a gauge where $n_\ell = 0$ everywhere.
- This means the action simplifies to

$$S = \frac{\kappa}{2} \sum_\ell (d\varphi)^2_\ell, \kappa = \frac{R^2}{2\pi}$$

- The Green's function of φ with this action is known!

φ Green's function

- The exact expression for the Green's function is a bit complicated:

$$G_{n_1 \geq 0, n_2 \geq 0} = \frac{1}{2\pi}\Bigg[-\log 2 - \psi(n_1+1/2) - \gamma_E - \frac{n_2+\frac{1}{2}}{n_1+\frac{1}{2}}\, {}_3F_2\left(1,1,n_2+\tfrac{3}{2};2,n_1+\tfrac{3}{2};-1\right)$$

$$+ \sum_{k=1}^{\lfloor \frac{n_1-n_2}{2} \rfloor} (-1)^k \binom{n_1-n_2}{2k} B\left(2k, n_1-k+\tfrac{1}{2}\right) {}_2F_1\left(2k, n_2+k+\tfrac{1}{2}; n_1+k+\tfrac{1}{2}; -1\right) \Bigg]$$

- G can also be defined recursively.
- For large separations, it behaves as

$$G_{r\cos\theta, r\sin\theta} \sim -\frac{1}{2\pi}\ln r - \frac{\gamma_E + \frac{3}{2}\ln 2}{2\pi} + O\left(\frac{1}{r^2}\right),$$

φ vertex operator correlator

- Let us define

$$\langle \mathcal{E}_C^{(q)} \rangle \equiv \left\langle \exp\left(iq \sum_C (d\varphi - 2\pi n) \right) \right\rangle$$

- When $q \in \mathbb{Z}$, this is just a two-point function of local operators. Otherwise it is an open line operator.
- We find

$$\left\langle \mathcal{E}_C^{(q)} \right\rangle = \xi_q^2 \exp\left(\frac{2\pi q^2}{\kappa^2} G_{x-y} \right), \; \xi_q = \exp\left[\frac{q^2}{2\kappa^2} \left(\gamma_E + \frac{3}{2} \ln 2 \right) \right]$$

- Result is independent of the contour C, as one would expect.

θ vertex operator correlator

- Next warm up step: let's compute

$$\langle \mathcal{M}^{(w)}(p, p') \rangle \equiv \langle \exp\left(iw\theta_p\right) \exp\left(-iw\theta_{p'}\right) \rangle$$

- But we integrated out θ, so we have to view this as a correlator of defect operators.
- This amounts to doing the path integral with the constraint

$$(dn)_p = w, \; (dn)_{p'} = -w$$

θ vertex operator correlator

- We find

$$\left\langle \mathcal{M}^{(w)}(p,p') \right\rangle = \zeta_w^2 \, \exp\left(2\pi\kappa^2 w^2 G_{p-p'}\right), \zeta_w = \exp\left[\frac{\kappa^2 w^2}{2}\left(\gamma_E + \frac{3}{2}\ln 2\right)\right]$$

- The techniques involved in the calculation are fairly standard, so I'm not explaining them in this talk.

Fermion correlator from Villain scalar

- The fermion two-point function can be written as

$$\left\langle \psi_+(s)\bar{\psi}(s')\right\rangle = \left\langle \mathcal{M}^{(1)}(p,p')\,\mathcal{E}^{(1/2)}_{C_{s,s'}}\right\rangle$$

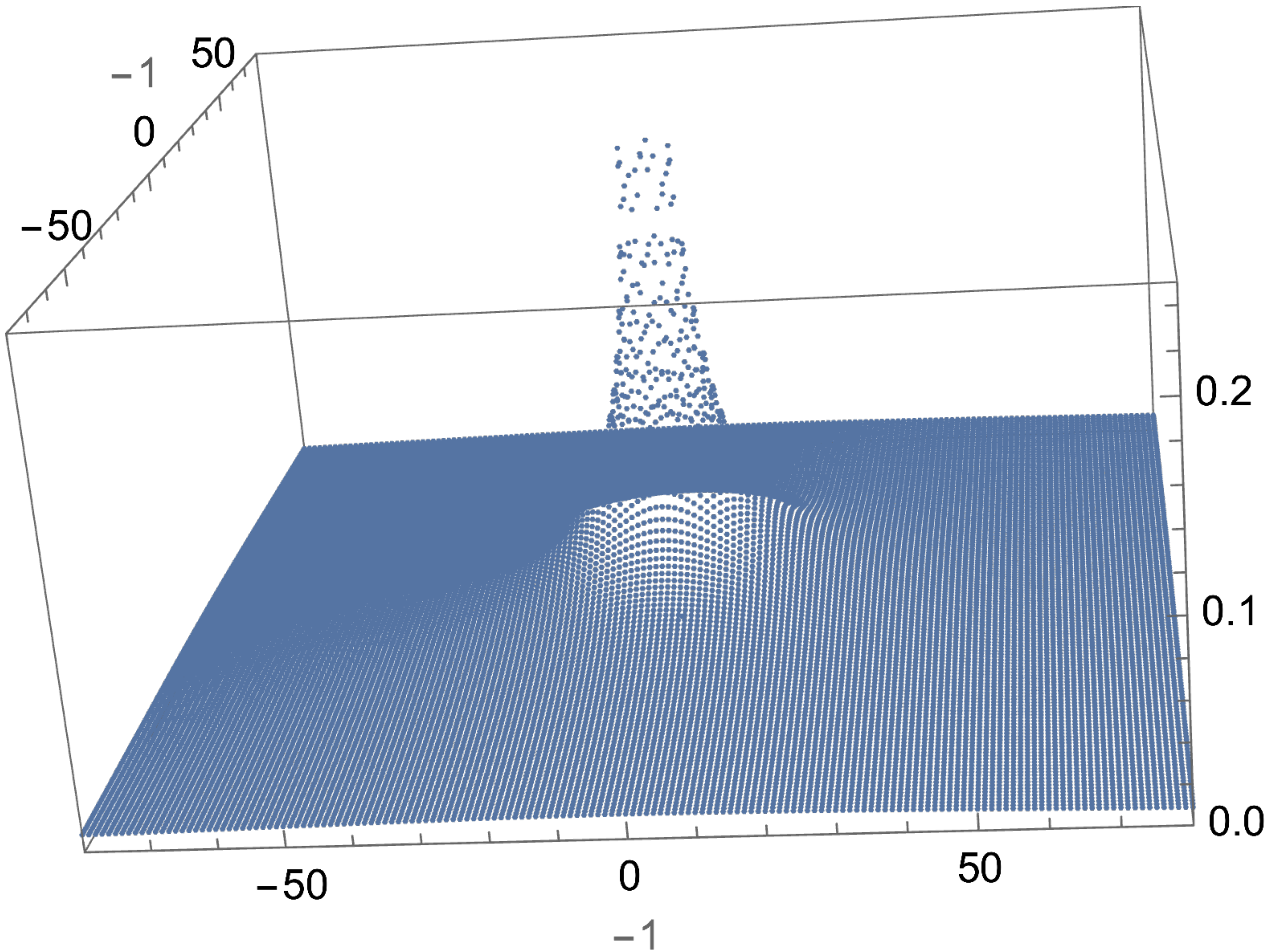
- We find that

$$\left\langle \mathcal{M}^{(1)}(p,p')\,\mathcal{E}^{(1/2)}_{C}\right\rangle = \left\langle \mathcal{M}^{(1)}(p,p')\right\rangle \left\langle \mathcal{E}^{(1/2)}_{C}\right\rangle \,\exp\left[-i\pi \sum_{\ell\in C}\left(\delta G_p - \delta G_{p'}\right)_\ell\right]$$

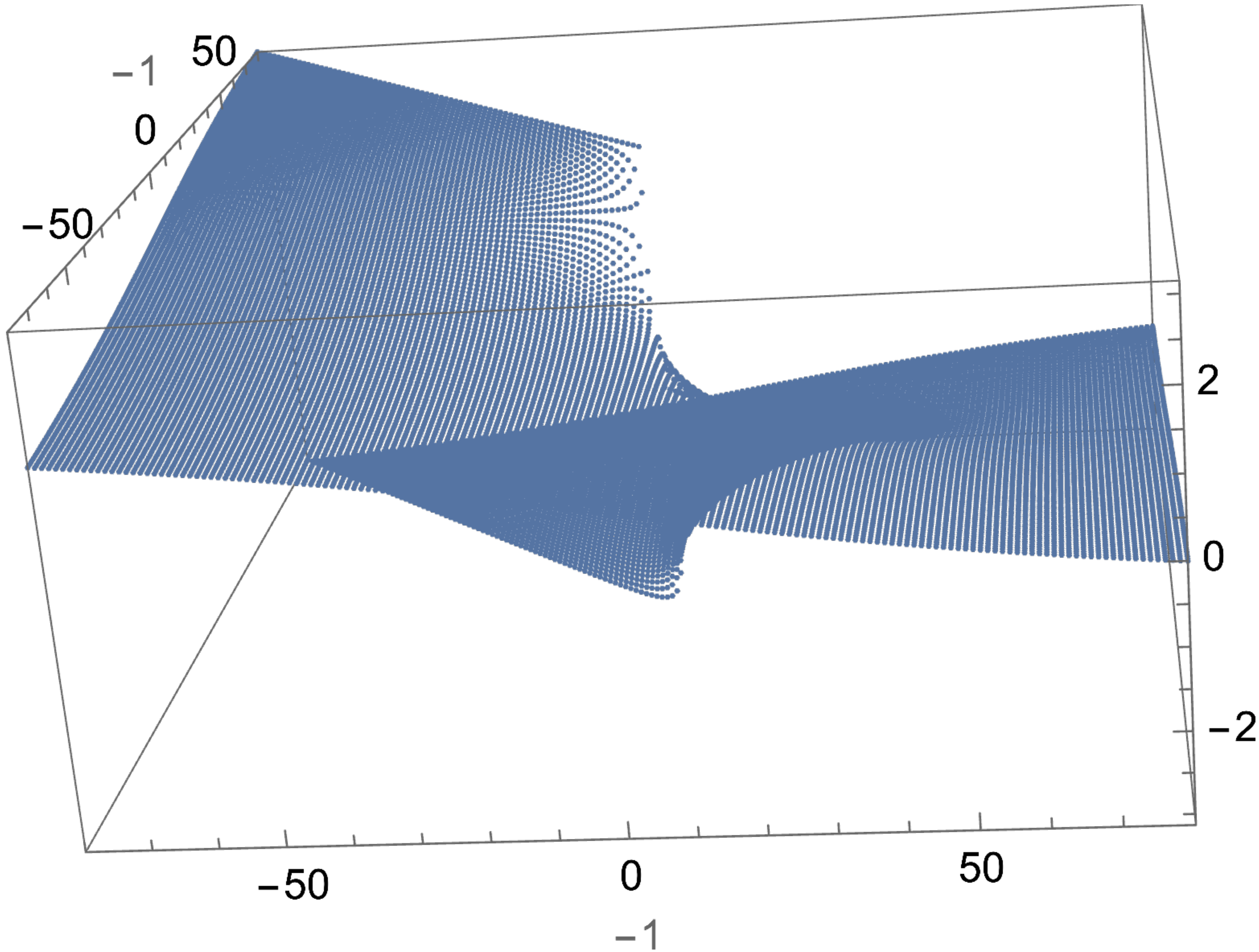
where $\delta = \star\, d\, \star$.

- The first two factors are positive, while the last factor is a non-vanishing phase. **This is the same structure one sees in a continuum CFT calculation.**

Magnitude of lattice Weyl correlator



Phase of lattice Weyl correlator



Fermion correlator from Villain scalar

- For large separations, we find

$$\langle \psi_\pm(s_1)\bar{\psi}_\pm(s_2)\rangle = \frac{1}{s_\uparrow \pm i s_\downarrow} + O\left(\frac{1}{|s_2 - s_1|^3}\right)\ , \ |x| \to \infty$$

where $s_\uparrow = \frac{1}{2}(s_1 + s_2), s_\downarrow = \frac{1}{2}(s_1 - s_2)$.

- This is exactly what we expect in the continuum, where

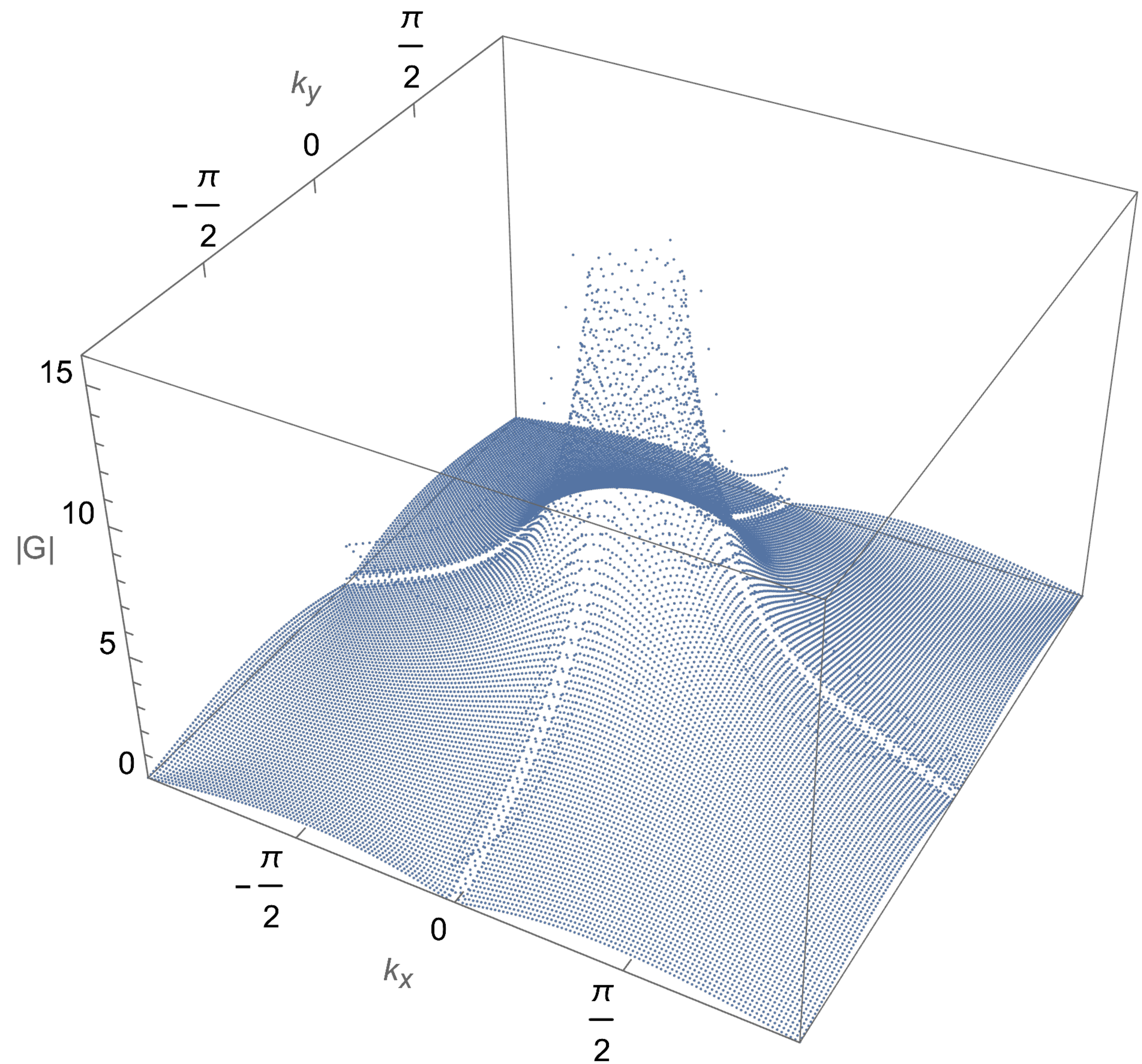
$$\langle \psi_\pm(x_1)\bar{\psi}_\pm(x_2)\rangle = \frac{1}{x_1 \pm i x_2}$$

- So the lattice Weyl correlators reduce to the continuum correlators in terms of lattice `light-cone' coordinates.

What about the NN theorem?

- To see how this can be reconciled with the NN theorem, we can study the Fourier transform of the lattice Weyl correlator.
- The result is easiest to explain via pictures.
- Sublety: numerically we do the Fourier transform in a spacetime box. This leads to some artifacts in the plot that should be ignored.

Magnitude of momentum-space lattice Weyl correlator



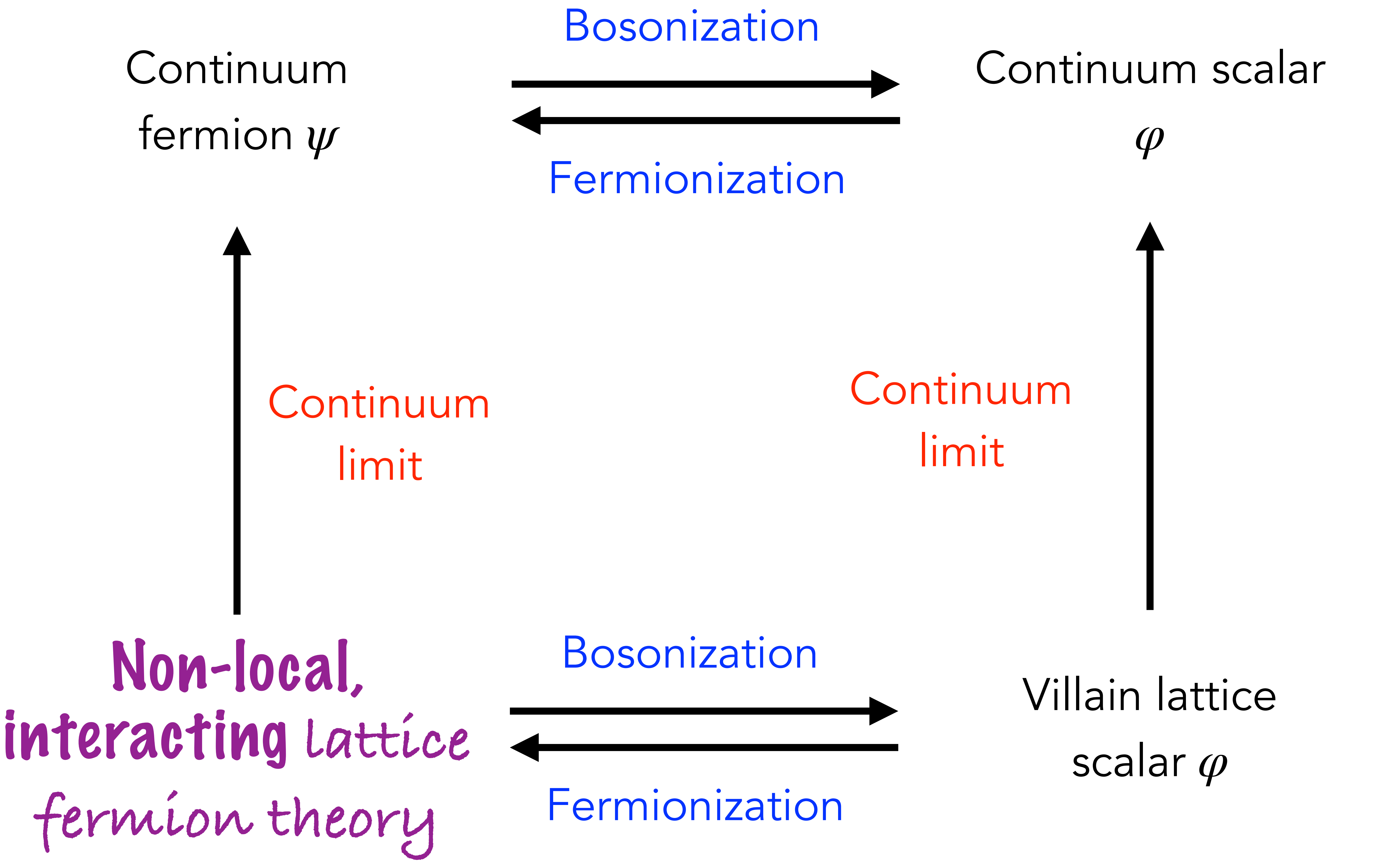
What about the NN theorem?

- The momentum-space correlator has a pole at the origin.
- It also has three zeros in the Brilloin zone at $(\pi, \pi), (\pi, 0), (0, \pi)$.
- This means that the associated Dirac operator is not local!
- Is this a disaster? Yes and no.

Yes, it's a disaster

- The non-locality means that is not at all obvious how to couple the `fermionic' lattice theory to gauge fields.
- Moreover, we have calculated the fermion four-point function using our lattice theory, and compared to what it would be if the theory was quadratic.
  - The results are not the same.
- So the lattice fermion theory associated to the modified Villain scalar is both non-local and interacting.
- At least we can see how the NN theorem is obeyed.
  - The NN theorem certainly does not apply to lattice fermion models of this sort.

Idea of the talk



No, it’s not a disaster

- We have the bosonic version of the theory, which is very nice for concrete calculation, and does not suffer from any of these problems!
- We can just use it.
- Also, despite the weird features of the related lattice fermion model, we suspect it is worth exploring further.

Conclusions

- The modified Villain approach allows one to describe the physics of massless Dirac fermions while preserving all their symmetries and anomalies, thanks to bosonization.
- The symmetries act locally on the lattice scalar fields, enabling clean discussions of lattice gauge theories.
- The lattice fermion theory associated to the modified Villain scalar turns out to be neither local nor free, so Nielen-Ninomiya theorem does not apply to it.
  - Yet it has the desired continuum limit anyway, by construction.
  - It would be nice to describe this exotic lattice fermion theory more explicitly!

Thanks for listening!