different system sizes



Effective String Theory on the Torus the 3d Ising interface

José Matos

WIP to appear on 2509.xxxxx with David Lima, João Penedones and João Viana

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Summary

Motivation

Flux tubes, confinement a string theory

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Low energy universality and non-universal term

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Non-univers

QM Approach

TBA and partitio function

Monte Carlo

The setup: 3d Isin

Free energy for different system sizes

at u = 1Free energy as a

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D=3 ONLY!!

- EST predictions for generic interfaces $\tau=\alpha+iu$, perturbative only in the Wilson coefficients.
- Modified multicanonical algorithm for high-precision free energy data;
- Preliminary value of γ_3 ;

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Flux tubes, confinement and string theory

At leading order flux tubes are described by "string theory"

$$S_{NG} = \sigma \int d^2 \xi \sqrt{\partial_\mu X^i(\xi)} \, \partial^\mu X_i(\xi)$$

More generally, symmetry allows

$$S = \int d^2 \xi \underbrace{\sqrt{-h} \left[\sigma\right]}_{NG} + 2\gamma_3 \frac{K^4}{\sigma} + \dots ,$$
(2)

with $\gamma_3 \ge -\frac{1}{768}$ [1906.08098].

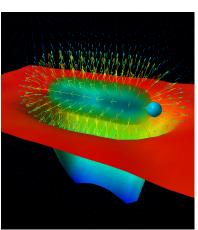


Figure: Lattice QCD simulation of a proton. Source: physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/Nobel/

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Effective string theory in the static gauge:

$$S = \mathcal{A} + S_{\mathsf{free \; boson}} + rac{S_1}{\mathcal{A}} + rac{S_2}{\mathcal{A}^2} + rac{S_3}{\mathcal{A}^3} + 2\gamma_3 rac{S_{\mathsf{NU}}}{\mathcal{A}^3} + \mathcal{O}\left(\mathcal{A}^{-4}
ight) \qquad \mathcal{A} \equiv \sigma R \mathcal{T},$$

model-independent

(3)

$$\begin{split} S_1 &= \frac{1}{8} \int d^2 \xi \left(\partial_i \pi \partial^i \pi \right)^2 \\ S_2 &= -\frac{1}{16} \int d^2 \xi \left(\partial_i \pi \partial^i \pi \right)^3 \end{split}$$

:

$$S_{\mathsf{NU}} = 2\gamma_3\sigma^2\int d^2\xi \left(\partial_i\partial_j\pi\partial^i\partial^j\pi
ight)^2.$$

Neatly packaged as

$$2\delta(s) = s/4 + \gamma_3 s^3 + \gamma_5 s^5 + \cdots \tag{4}$$

plus non-integrable corrections at s^8 in $2 \rightarrow 2$; lower orders in other channels.

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Polyakov formalism

It is equivalent to the EST description [Billò, Caselle, Ferro '06]

$$Z = \int \frac{d^2 \tau}{\tau_2} Z^{b} (q, \bar{q}) Z^{gh} (q, \bar{q}) \qquad Z^{b} (q, \bar{q}) = \operatorname{Tr} \left[q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} \right], \tag{5}$$

It is better to compute the universal partition function

$$Z_{U} = \sqrt{2\pi} \left(\frac{\sigma L_{z}^{2}}{\pi^{2} u} \right)^{1/2} \sqrt{\mathcal{A}} \sum_{k,k'} c_{k} c_{k'} \varepsilon_{k,k'} \mathcal{K}_{1} \left(\mathcal{A} \varepsilon_{k,k'} \right)$$
 (6)

where c_k are the partitions of k and

$$\varepsilon_{k,k'} = \sqrt{1 + \frac{4\pi u}{\mathcal{A}} \left(k + k' - \frac{1}{12}\right) + \left(\frac{2\pi u}{\mathcal{A}} \left(k - k'\right)\right)^2}.$$
(7)

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Free energy expansion

The free energy at any order is

$$F_{U} = -\log Z_{U} = \mathcal{A} - \frac{1}{2}\log\left(\frac{\sigma}{2\pi u}L_{z}^{2}\right) + 2\log|\eta(iu)| + \sum_{n=1}^{\infty}\frac{g_{n}(iu)}{\mathcal{A}^{n}}, (8)$$

with

$$\begin{split} g_1 &= \frac{\pi^2 u^2}{72} \left| E_2 \left(i u \right) \right|^2 - \frac{\pi u}{12} E_2 + \frac{3}{8} \\ g_2 &= -\frac{\pi^4 u^4}{5184} \left(2 E_4 \left(i u \right) E_2 \left(i u \right)^2 - 2 E_4 \left(i u \right)^2 \right) \\ &+ \frac{\pi^3 u^3}{432} E_4 \left(i u \right) E_2 \left(i u \right) \\ &- \frac{\pi^2 u^2}{576} \left(6 E_2 \left(i u \right)^2 + 2 E_4 \left(i u \right) \right) + \frac{\pi u}{16} E_2 \left(i u \right) - \frac{3}{16} \\ g_3 &= \text{ommited for space} \end{split}$$

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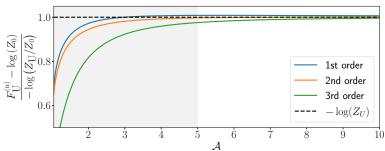
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What is the validity of the asymptotic expansion?



$$F_{\mathsf{U}}^{(n)} + \log(Z_0) = 1 + \sum_{n=1}^{n} \frac{g_n(iu)}{\mathcal{A}^n} = -\log(Z_U) + \log(Z_0)$$
 (9)

with

$$Z_{0} = e^{-A} \left(\frac{\sigma L_{z}^{2}}{2\pi u}\right)^{\frac{1}{2}} \frac{1}{|\eta(\tau)|^{2}}.$$

$$Z_{U} = \left(\frac{\sigma L_{z}^{2}}{\pi^{2} u}\right)^{1/2} \sqrt{A} \sum_{k,k'} c_{k} c_{k'} \varepsilon_{k,k'} K_{1} \left(A \varepsilon_{k,k'}\right)$$

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Non-universal corrections: Path integral

The correction is

$$V_{\text{NU}}[\pi] = -2\frac{\gamma_3}{\mathcal{A}^3} \int d^2\xi \left(\partial_i \partial_j \pi \partial^i \partial^j \pi \right)^2, \tag{10}$$

Requires regularizing series of the type [Dietz, Filk '83]

$$\sum_{(m,n)\neq(0,0)} \frac{n^4}{m^2 + u^2 n^2} \tag{11}$$

The non-universal correction for $\tau = iu$ is

$$F_{\text{NU}}(A, iu) = -\frac{32\gamma_3\pi^6}{225A^3}u^4E_4^2(iu),$$
 (12)

The ground state shift matches TBA [1906.08098]

$$\Delta E_0(R) \equiv \lim_{L \to \infty} \frac{\Delta F_{\text{NU}}(\mathcal{A}, iu)}{L} = -\frac{32\pi^6}{225} \frac{\sqrt{\sigma} \gamma_3}{(\sqrt{\sigma}R)^7}.$$
 (13)

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Considering an object with internal quantum numbers I and momentum ρ_z in the transverse direction. The partition function is

$$Z = \operatorname{Tr}\left[e^{-LH}\right] = \sum_{I,p_z} C_I e^{-\mathcal{A}\mathcal{E}_{I,p_z}} \tag{14}$$

where C_I is their degeneracy and \mathcal{E}_I its their **energy density**

$$\mathcal{E}_{I,p_z} = \sqrt{\mathcal{E}_I^2 + \frac{p_z^2}{\sigma R^2}}.$$
 (15)

The integral over p_z can be done explicitly (matches [Billò, Caselle, Ferro '06])

$$Z = \sqrt{\frac{2\mathcal{A}}{\pi}} \left(\frac{\sigma}{2\pi u}\right)^{\frac{d-2}{2}} V_{\mathcal{T}} \sum_{l} C_{l} \mathcal{E}_{l}^{\frac{d-1}{2}} K_{\frac{d-1}{2}}(\mathcal{A}\mathcal{E}_{l}). \tag{16}$$

Twist is added by inserting $e^{\alpha LP}$ into the trace.

(18)

$$2\delta\left(s\right) = s/4 + \gamma_3 s^3 \tag{17}$$

_

TBA spits out

$$\frac{\Delta E_{k,k',s,s'}}{R} = -\frac{32\pi^6 u^4 \gamma_3}{225\mathcal{A}^4} \frac{(240s+1)(240s'+1)}{\mathcal{E}_{k,k'} \left(\left(\mathcal{E}_{k,k'} + 1 \right)^2 - \frac{\pi^2 u^2}{\mathcal{A}^2} \left(k - k' \right)^2 \right)^3} + \mathcal{O}\left(\gamma_3^2 \right)^{\frac{1}{2}}$$

with $k = \sum_i n_i$, $s = \sum_i n_i^3$ and

$$\varepsilon_{k,k'} = \sqrt{1 + \frac{4\pi}{A} \left(k + k' - \frac{1}{12} \right) + \left(\frac{2\pi u}{A} \left(k - k' \right) \right)^2}. \tag{19}$$

Ground state shift

$$\begin{split} \Delta E_0(R) &= -\frac{32\pi^6\gamma_3}{225R^7} \frac{1}{\sqrt{1 - \frac{\pi}{3R^2}} \left(\sqrt{1 - \frac{\pi}{3R^2}} + 1\right)^6} + \mathcal{O}\left(\gamma_3^2\right) \\ &= -\frac{32\pi^6\gamma_3}{225R^7} - \frac{64\pi^7\gamma_3}{675R^9} - \frac{2\pi^8\gamma_3}{45R^{11}} - \frac{22\pi^9\gamma_3}{1215R^{13}} + \mathcal{O}\left(R^{-15}\right) + \mathcal{O}\left(\gamma_3^2\right) \end{split}$$

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Non-universal partition function: TBA

Then

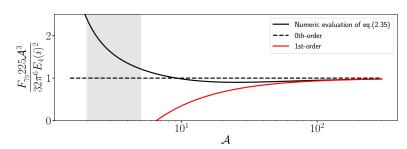
$$Z = Z_U + \gamma_3 Z_{\gamma_3} + \mathcal{O}\left(\gamma_3^2\right) \Longrightarrow F = -\log\left(Z_U\right) - \gamma_3 F_{\gamma_3} + \mathcal{O}\left(\gamma_3^2\right) \ \ (20)$$

with

$$F_{\gamma_3} = A \frac{\sum_{k,k',s,s'} p(k,s) p(k',s') \Delta \mathcal{E}_{k,k',s,s'} \mathcal{E}_{k,k'} K_0 \left(A \mathcal{E}_{k,k'} \right)}{\sum_{k,k'} p(k) p(k') \mathcal{E}_{k,k'} K_1 \left(A \mathcal{E}_{k,k'} \right)}, \quad (21)$$

whose large area expansion for square interfaces $(\tau = i)$ is

$$F_{\gamma_3} = \frac{32\pi^6}{225\,\mathcal{A}^3} E_4(i)^2 \left(1 - \frac{13}{2\mathcal{A}} + \frac{14013 + 280\pi^4 E_4(i)^2}{432\mathcal{A}^2} + \mathcal{O}(\mathcal{A}^{-2})\right)$$



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Higher-order Wilson Coefficients

The spectrum at first order in γ_n is

$$\Delta E_{k,k',s,s'} = -2^{6n-1} \pi^{2n} \zeta \left(-n\right)^2 \frac{\left(1 + \frac{2}{\zeta(-n)}s\right) \left(1 + \frac{2}{\zeta(-n)}s'\right)}{R^{1+2n} \mathcal{E}_{k,k'} \left(\left(\mathcal{E}_{k,k'} + 1\right)^2 - \frac{4\pi^2 \left(k - k'\right)}{R^4}\right)^n},$$
(22)

with $k=\sum_i n_i$ and $s=\sum_i n_i^n$. The free energy correction is

$$F_{\gamma_{n}} = A \frac{\sum_{k,k',s,s'} p(k,s) p(k',s') \Delta \mathcal{E}_{k,k',s,s'} \mathcal{E}_{k,k'} K_{0} (A \mathcal{E}_{k,k'})}{\sum_{k,k'} p(k) p(k') \mathcal{E}_{k,k'} K_{1} (A \mathcal{E}_{k,k'})}$$

$$= -\frac{1}{2} (4^{n} \pi^{n} \zeta (-n))^{2} \frac{u^{n+1}}{A^{n}} (|E_{n+1} (\tau)|^{2} + \mathcal{O}(A^{-n-1}))$$

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How can we realize an interface? Ising... It is always the Ising.

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The 3d Ising

Domain wall in the 3d Ising

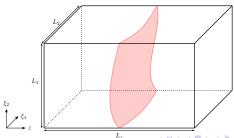
$$H[\{s\};J] = \sum_{\langle i,j\rangle} J_{ij} s_i s_j \tag{23}$$

and

$$Z[\beta, J] = \int dJ e^{-F(J)} \qquad e^{-F(J)} \equiv \sum_{\{s_i\}} e^{-\beta H[\{s_i\}; J]}.$$
 (24)

The interface free energy is

$$F^{\text{Interface}} = -\log\left(\frac{Z\left[\beta, -1\right]}{Z\left[\beta, 1\right]}\right) \tag{25}$$



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Multicanonical method

Deform the partition function with a function ω

$$Z[J;\omega] = \int dJ e^{-F(J)+\omega(J)}$$
 (26)

such that

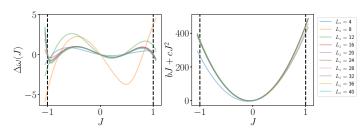
$$e^{-F(J)+\omega(J)} = \text{constant}.$$
 (27)

Moreover, we split ω as

$$\omega(J) = bJ + cJ^2 + \Delta\omega(J). \tag{28}$$

The EST free energy becomes

$$F^{\text{Interface}} = 2b + \Delta\omega (1) - \Delta\omega (-1). \tag{29}$$



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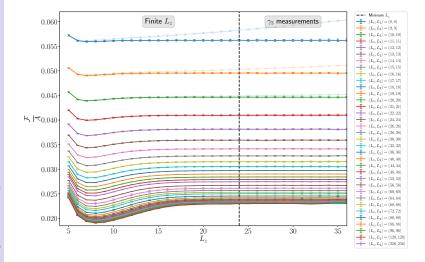
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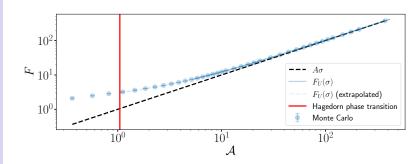
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Free energy as a function of the area



$$F_{U} = -\log(Z_{U}) = -\log\left[\left(\frac{\sigma L_{z}^{2}}{\pi^{2}u}\right)^{1/2} \sqrt{A} \sum_{k,k'} c_{k} c_{k'} \varepsilon_{k,k'} K_{1}\left(A \varepsilon_{k,k'}\right)\right]$$
(30)

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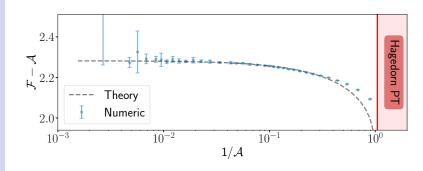
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Area-independent contribution



$$c_0 = -\frac{1}{2}\log\left(\frac{\sigma}{2\pi u}\right) + 2\log|\eta(iu)| \tag{31}$$

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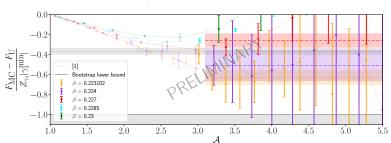
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For $\tau = i$ the EST predicts

$$\gamma_3^{\mathsf{MC}} \equiv \frac{F_{\mathsf{MC}} - F_{\mathsf{U}}}{F_{\gamma_3}} + \mathcal{O}\left(\gamma_3\right) \qquad \gamma_3^{\mathsf{min}} \equiv -\frac{1}{768} \tag{32}$$



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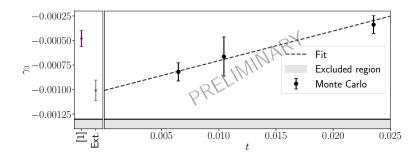
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Finite transverse volume corrections

Finite transverse volume corrections: to appear 251x.xxxxx

We define an effective string tension as

$$\sigma_{\text{eff}}(L_z) \equiv \lim_{A \to \infty} \frac{F(A, u, L_z; \sigma)}{A}.$$
 (33)

The EST prediction is

$$\sigma_{\rm eff}(L_z) = \sigma_{\rm eff}(\infty) \equiv \sigma_{\infty}$$
 (34)

