Summary

Motivation

Flux tubes, confinement and string theory

EST

Low energy universality and non-universal terms

Known results

Non-universal

QM Approach

TBA and partition function

Monte Carlo

The setup: 3d Ising

Multicanonical method

Free energy for different system sizes at $u = 1$

Free energy as a function of the area

Area-independent contribution

γ_3

Extrapolation to the critical point

Finite transverse volume corrections

Effective String Theory on the Torus

the 3d Ising interface

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WIP to appear on 2509.xxxxx with David Lima, João Penedones and João Viana

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Summary

Motivation

Flux tubes, confinement and string theory

EST

Low energy universality and non-universal terms

Known results

Non-universal

QM Approach

TBA and partition function

Monte Carlo

The setup: 3d Ising

Multicanonical method

Free energy for different system sizes at $u = 1$

Free energy as a function of the area

Area-independent contribution

γ_3

Extrapolation to the critical point

Finite transverse volume corrections

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D=3 ONLY!!

- EST predictions for generic interfaces $\tau = \alpha + iu$, perturbative only in the Wilson coefficients.
- Modified multicanonical algorithm for high-precision free energy data;
- Preliminary value of γ_3;

Flux tubes, confinement and string theory

At leading order flux tubes are described by "string theory"

$$S_{\rm NG} = \sigma \int d^2\xi \sqrt{\partial_\mu X^i(\xi)\, \partial^\mu X_i(\xi)} \tag{1}$$

More generally, symmetry allows

$$S = \int d^2\xi \underbrace{\sqrt{-h}\,[\sigma}_{\rm NG} + 2\gamma_3 \frac{K^4}{\sigma} + \dots\,], \tag{2}$$

with $\gamma_3 \geq -\frac{1}{768}$ [1906.08098].

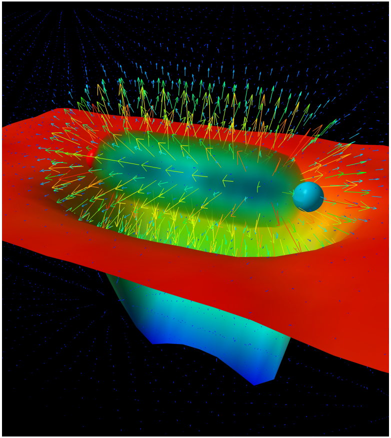


Figure: Lattice QCD simulation of a proton. Source: physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/Nobel/

EST

Effective string theory in the static gauge:

$$S = \underbrace{\mathcal{A} + S_{\text{free boson}} + \frac{S_1}{\mathcal{A}} + \frac{S_2}{\mathcal{A}^2} + \frac{S_3}{\mathcal{A}^3}}_{\text{model-independent}} + 2\gamma_3 \frac{S_{\text{NU}}}{\mathcal{A}^3} + \mathcal{O}\left(\mathcal{A}^{-4}\right) \qquad \mathcal{A} \equiv \sigma RT, \tag{3}$$

$$S_1 = \frac{1}{8} \int d^2\xi \left(\partial_i \pi \partial^i \pi\right)^2$$

$$S_2 = -\frac{1}{16} \int d^2\xi \left(\partial_i \pi \partial^i \pi\right)^3$$

$$\vdots$$

$$S_{\text{NU}} = 2\gamma_3 \sigma^2 \int d^2\xi \left(\partial_i \partial_j \pi \partial^i \partial^j \pi\right)^2 .$$

Neatly packaged as

$$2\delta\left(s\right) = s/4 + \gamma_3 s^3 + \gamma_5 s^5 + \cdots \tag{4}$$

plus non-integrable corrections at s^8 in $2 \to 2$; lower orders in other channels.

Polyakov formalism

It is equivalent to the EST description [Billò, Caselle, Ferro '06]

$$Z = \int \frac{d^2\tau}{\tau_2} Z^{\mathrm{b}}(q,\bar{q}) Z^{\mathrm{gh}}(q,\bar{q}) \qquad Z^{\mathrm{b}}(q,\bar{q}) = \mathrm{Tr}\left[q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}\right], \tag{5}$$

It is better to compute the universal partition function

$$Z_U = \sqrt{2\pi}\left(\frac{\sigma L_z^2}{\pi^2 u}\right)^{1/2} \sqrt{\mathcal{A}} \sum_{k,k'} c_k c_{k'} \varepsilon_{k,k'} K_1\left(\mathcal{A}\varepsilon_{k,k'}\right) \tag{6}$$

where c_k are the partitions of k and

$$\varepsilon_{k,k'} = \sqrt{1 + \frac{4\pi u}{\mathcal{A}}\left(k + k' - \frac{1}{12}\right) + \left(\frac{2\pi u}{\mathcal{A}}(k-k')\right)^2}. \tag{7}$$

Free energy expansion

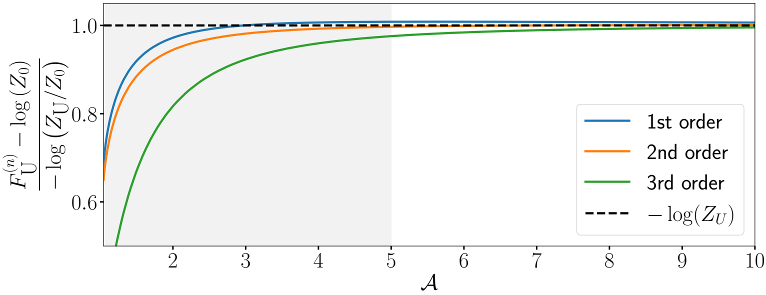
The free energy at any order is

$$F_{\mathrm{U}} = -\log Z_{\mathrm{U}} = \mathcal{A} - \frac{1}{2}\log\left(\frac{\sigma}{2\pi u}L_z^2\right) + 2\log|\eta(iu)| + \sum_{n=1}^{\infty}\frac{g_n(iu)}{\mathcal{A}^n}, \quad (8)$$

with

$$\begin{aligned} g_1 &= \frac{\pi^2u^2}{72}|E_2(iu)|^2 - \frac{\pi u}{12}E_2 + \frac{3}{8} \\ g_2 &= -\frac{\pi^4u^4}{5184}\left(2E_4(iu)E_2(iu)^2 - 2E_4(iu)^2\right) \\ &+ \frac{\pi^3u^3}{432}E_4(iu)E_2(iu) \\ &- \frac{\pi^2u^2}{576}\left(6E_2(iu)^2 + 2E_4(iu)\right) + \frac{\pi u}{16}E_2(iu) - \frac{3}{16} \\ g_3 &= \textbf{ommited for space} \end{aligned}$$

What is the validity of the asymptotic expansion?



$$F_{\mathrm{U}}^{(n)} + \log(Z_0) = 1 + \sum_{n=1}^{n} \frac{g_n(iu)}{\mathcal{A}^n} \text{,,} = \text{''} - \log(Z_U) + \log(Z_0) \tag{9}$$

with

$$Z_0 = e^{-\mathcal{A}} \left(\frac{\sigma L_z^2}{2\pi u}\right)^{\frac{1}{2}} \frac{1}{|\eta(\tau)|^2}.$$

$$Z_U = \left(\frac{\sigma L_z^2}{\pi^2 u}\right)^{1/2} \sqrt{\mathcal{A}} \sum_{k,k'} c_k c_{k'} \varepsilon_{k,k'} K_1(\mathcal{A}\varepsilon_{k,k'})$$

Non-universal corrections: Path integral

The correction is

$$V_{\mathrm{NU}}[\pi] = -2\frac{\gamma_3}{\mathcal{A}^3}\int d^2\xi \left(\partial_i\partial_j\pi\partial^i\partial^j\pi\right)^2, \tag{10}$$

Requires regularizing series of the type [Dietz, Filk '83]

$$\sum_{(m,n)\neq(0,0)} \frac{n^4}{m^2+u^2n^2} \tag{11}$$

The non-universal correction for $\tau = iu$ is

$$F_{\mathrm{NU}}(\mathcal{A}, iu) = -\frac{32\gamma_3\pi^6}{225\mathcal{A}^3}u^4E_4^2(iu), \tag{12}$$

The ground state shift matches TBA [1906.08098]

$$\Delta E_0(R) \equiv \lim_{L\to\infty}\frac{\Delta F_{\mathrm{NU}}(\mathcal{A}, iu)}{L} = -\frac{32\pi^6}{225}\frac{\sqrt{\sigma}\gamma_3}{(\sqrt{\sigma}R)^7}. \tag{13}$$

Summary

Motivation

Flux tubes, confinement and string theory

EST

Low energy universality and non-universal terms

Known results

Non-universal

QM Approach

TBA and partition function

Monte Carlo

The setup: 3d Ising

Multicanonical method

Free energy for different system sizes at $u = 1$

Free energy as a function of the area

Area-independent contribution

γ_3

Extrapolation to the critical point

Finite transverse volume corrections

QM approach

Considering an object with internal quantum numbers I and momentum p_z in the transverse direction. The partition function is

$$Z = \mathrm{Tr}\left[e^{-LH}\right] = \sum_{I,p_z} C_I e^{-\mathcal{A}\mathcal{E}_{I,p_z}} \tag{14}$$

where C_I is their degeneracy and $\mathcal{E}_I$ its their **energy density**

$$\mathcal{E}_{I,p_z} = \sqrt{\mathcal{E}_I^2 + \frac{p_z^2}{\sigma R^2}}. \tag{15}$$

The integral over p_z can be done explicitly (matches [Billò, Caselle, Ferro '06])

$$Z = \sqrt{\frac{2\mathcal{A}}{\pi}} \left(\frac{\sigma}{2\pi u}\right)^{\frac{d-2}{2}} V_T \sum_I C_I \mathcal{E}_I^{\frac{d-1}{2}} K_{\frac{d-1}{2}}(\mathcal{A}\mathcal{E}_I)\,. \tag{16}$$

Twist is added by inserting $e^{\alpha LP}$ into the trace.

TBA

Using

$$2\delta(s) = s/4 + \gamma_3 s^3 \tag{17}$$

TBA spits out

$$\frac{\Delta E_{k,k',s,s'}}{R} = -\frac{32\pi^6 u^4 \gamma_3}{225\mathcal{A}^4} \frac{(240s+1)(240s'+1)}{\mathcal{E}_{k,k'}\left((\mathcal{E}_{k,k'}+1)^2 - \frac{\pi^2 u^2}{\mathcal{A}^2}(k-k')^2\right)^3} + \mathcal{O}\left(\gamma_3^2\right) \tag{18}$$

with $k = \sum_i n_i$, $s = \sum_i n_i^3$ and

$$\varepsilon_{k,k'} = \sqrt{1 + \frac{4\pi}{\mathcal{A}}\left(k + k' - \frac{1}{12}\right) + \left(\frac{2\pi u}{\mathcal{A}}(k-k')\right)^2}. \tag{19}$$

Ground state shift

$$\Delta E_0(R) = -\frac{32\pi^6\gamma_3}{225R^7} \frac{1}{\sqrt{1-\frac{\pi}{3R^2}}\left(\sqrt{1-\frac{\pi}{3R^2}}+1\right)^6} + \mathcal{O}\left(\gamma_3^2\right)$$

$$= -\frac{32\pi^6\gamma_3}{225R^7} - \frac{64\pi^7\gamma_3}{675R^9} - \frac{2\pi^8\gamma_3}{45R^{11}} - \frac{22\pi^9\gamma_3}{1215R^{13}} + \mathcal{O}\left(R^{-15}\right) + \mathcal{O}\left(\gamma_3^2\right).$$

Non-universal partition function: TBA

Then

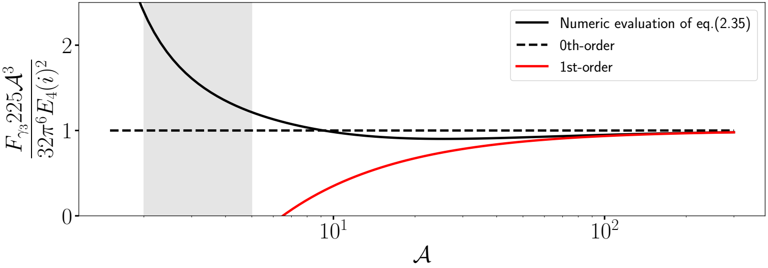
$$Z = Z_U + \gamma_3 Z_{\gamma_3} + \mathcal{O}\left(\gamma_3^2\right) \Longrightarrow F = -\log\left(Z_U\right) - \gamma_3 F_{\gamma_3} + \mathcal{O}\left(\gamma_3^2\right) \quad (20)$$

with

$$F_{\gamma_3} = \mathcal{A}\frac{\sum_{k,k',s,s'} p(k,s)p(k',s')\Delta\mathcal{E}_{k,k',s,s'}\mathcal{E}_{k,k'}K_0\left(\mathcal{A}\mathcal{E}_{k,k'}\right)}{\sum_{k,k'} p(k)p(k')\mathcal{E}_{k,k'}K_1\left(\mathcal{A}\mathcal{E}_{k,k'}\right)}, \quad (21)$$

whose large area expansion for square interfaces ($\tau = i$) is

$$F_{\gamma_3} = \frac{32\pi^6}{225\,\mathcal{A}^3}E_4(i)^2\left(1 - \frac{13}{2\mathcal{A}} + \frac{14013 + 280\pi^4 E_4(i)^2}{432\mathcal{A}^2} + \mathcal{O}(\mathcal{A}^{-2})\right)$$



Higher-order Wilson Coefficients

The spectrum at first order in γ_n is

$$\Delta E_{k,k',s,s'} = -2^{6n-1}\pi^{2n}\zeta(-n)^2 \frac{\left(1+\frac{2}{\zeta(-n)}s\right)\left(1+\frac{2}{\zeta(-n)}s'\right)}{R^{1+2n}\mathcal{E}_{k,k'}\left((\mathcal{E}_{k,k'}+1)^2-\frac{4\pi^2(k-k')}{R^4}\right)^n}, \tag{22}$$

with $k = \sum_i n_i$ and $s = \sum_i n_i^n$. The free energy correction is

$$\begin{aligned} F_{\gamma_n} &= \mathcal{A}\frac{\sum_{k,k',s,s'} p(k,s)p(k',s')\Delta\mathcal{E}_{k,k',s,s'}\mathcal{E}_{k,k'}K_0(\mathcal{A}\mathcal{E}_{k,k'})}{\sum_{k,k'} p(k)p(k')\mathcal{E}_{k,k'}K_1(\mathcal{A}\mathcal{E}_{k,k'})} \\ &= -\frac{1}{2}\left(4^n\pi^n\zeta(-n)\right)^2\frac{u^{n+1}}{\mathcal{A}^n}\left(\left|E_{n+1}(\tau)\right|^2+\mathcal{O}(\mathcal{A}^{-n-1})\right) \end{aligned}$$

Summary

Motivation

Flux tubes, confinement and string theory

EST

Low energy universality and non-universal terms

Known results

Non-universal

QM Approach

TBA and partition function

Monte Carlo

The setup: 3d Ising

Multicanonical method

Free energy for different system sizes at $u = 1$

Free energy as a function of the area

Area-independent contribution

γ_3

Extrapolation to the critical point

Finite transverse volume corrections

How can we realize an interface? Ising... It is always the Ising.

Summary

Motivation

Flux tubes, confinement and string theory

EST

Low energy universality and non-universal terms

Known results

Non-universal

QM Approach

TBA and partition function

Monte Carlo

The setup: 3d Ising

Multicanonical method

Free energy for different system sizes at $u = 1$

Free energy as a function of the area

Area-independent contribution

γ_3

Extrapolation to the critical point

Finite transverse volume corrections

The 3d Ising

Domain wall in the 3d Ising

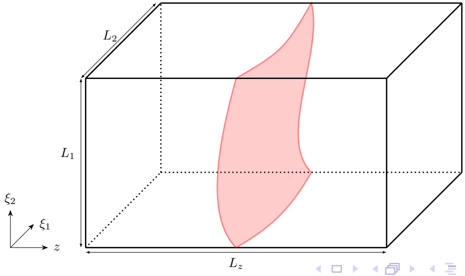
$$H[\{s\}; J] = \sum_{\langle i,j \rangle} J_{ij} s_i s_j \tag{23}$$

and

$$Z[\beta, J] = \int dJ e^{-F(J)} \qquad e^{-F(J)} \equiv \sum_{\{s_i\}} e^{-\beta H[\{s_i\};J]}. \tag{24}$$

The interface free energy is

$$F^{\text{Interface}} = -\log\left(\frac{Z[\beta,-1]}{Z[\beta,1]}\right) \tag{25}$$



Multicanonical method

Deform the partition function with a function ω

$$Z[J;\omega] = \int dJ e^{-F(J)+\omega(J)} \tag{26}$$

such that

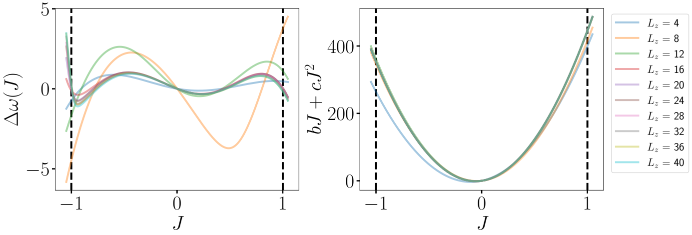
$$e^{-F(J)+\omega(J)} = \text{constant}. \tag{27}$$

Moreover, we split ω as

$$\omega(J) = bJ + cJ^2 + \Delta\omega(J). \tag{28}$$

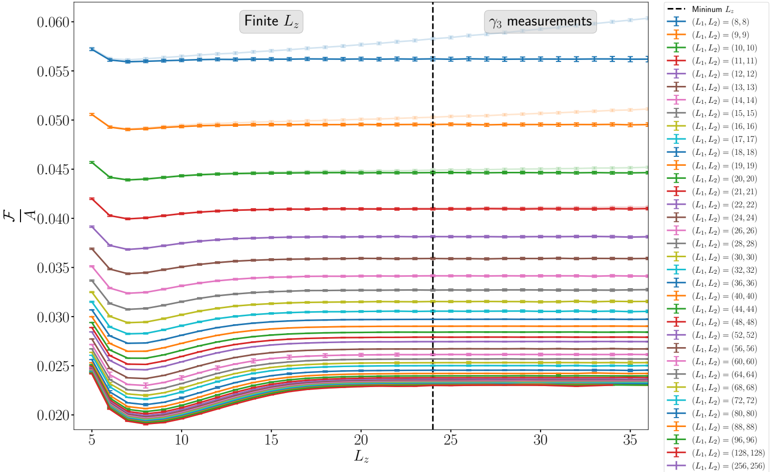
The EST free energy becomes

$$F^{\text{Interface}} = 2b + \Delta\omega(1) - \Delta\omega(-1). \tag{29}$$

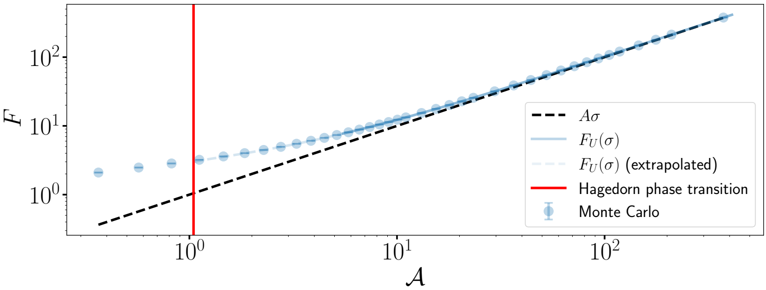


Summary
Motivation
Flux tubes, confinement and string theory
EST
Low energy universality and non-universal terms
Known results
Non-universal
QM Approach
TBA and partition function
Monte Carlo
The setup: 3d Ising
Multicanonical method
Free energy for different system sizes at $u = 1$
Free energy as a function of the area
Area-independent contribution
γ_3
Extrapolation to the critical point
Finite transverse volume corrections

Free energy for different system sizes at $u = 1$

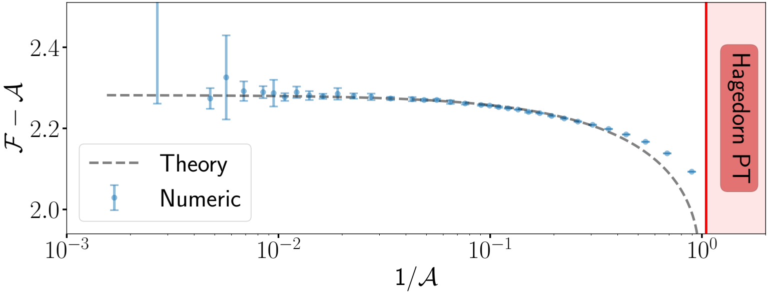


Free energy as a function of the area



$$F_U = -\log(Z_U) = -\log\left[\left(\frac{\sigma L_z^2}{\pi^2 u}\right)^{1/2}\sqrt{\mathcal{A}}\sum_{k,k'} c_k c_{k'}\varepsilon_{k,k'}K_1\left(\mathcal{A}\varepsilon_{k,k'}\right)\right] \quad (30)$$

Area-independent contribution



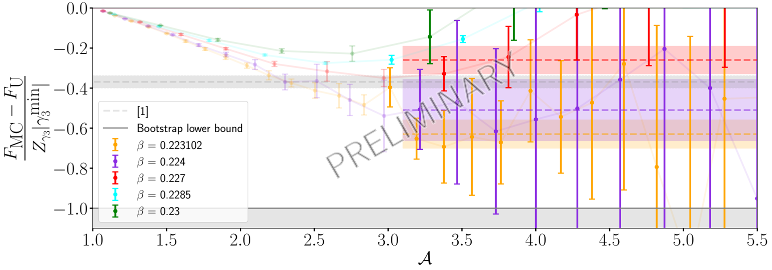
$$c_0 = -\frac{1}{2}\log\left(\frac{\sigma}{2\pi u}\right) + 2\log|\eta(iu)| \tag{31}$$

Summary
Motivation
Flux tubes, confinement and string theory
EST
Low energy universality and non-universal terms
Known results
Non-universal
QM Approach
TBA and partition function
Monte Carlo
The setup: 3d Ising
Multicanonical method
Free energy for different system sizes at $u = 1$
Free energy as a function of the area
Area-independent contribution
γ_3
Extrapolation to the critical point
Finite transverse volume corrections

γ_3

For $\tau = i$ the EST predicts

$$\gamma_3^{\mathrm{MC}} \equiv \frac{F_{\mathrm{MC}} - F_{\mathrm{U}}}{F_{\gamma_3}} + \mathcal{O}(\gamma_3) \qquad \gamma_3^{\min} \equiv -\frac{1}{768} \tag{32}$$



Summary

Motivation

Flux tubes, confinement and string theory

EST

Low energy universality and non-universal terms

Known results

Non-universal

QM Approach

TBA and partition function

Monte Carlo

The setup: 3d Ising

Multicanonical method

Free energy for different system sizes at $u = 1$

Free energy as a function of the area

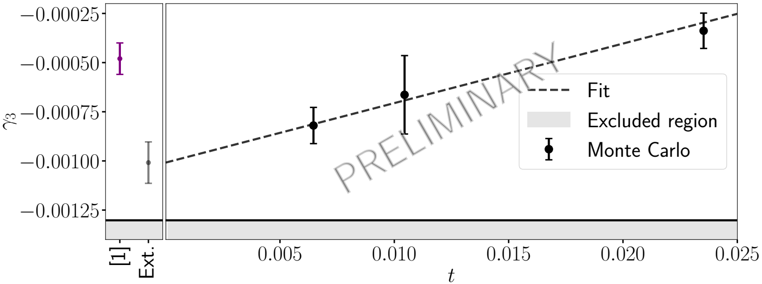
Area-independent contribution

γ_3

Extrapolation to the critical point

Finite transverse volume corrections

Extrapolation to the critical temperature



Finite transverse volume corrections: to appear 251x.xxxxx

We define an effective string tension as

$$\sigma_{\text{eff}}(L_z) \equiv \lim_{A\to\infty} \frac{F(\mathcal{A}, u, L_z; \sigma)}{A}. \quad (33)$$

The EST prediction is

$$\sigma_{\text{eff}}(L_z) = \sigma_{\text{eff}}(\infty) \equiv \sigma_\infty \quad (34)$$