Chiral anomaly and Hamiltonian Ginsparg-Wilson relations

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ECT*, Trento

2509.xxxxx | HS • 2505.20419 | HS • 2503.05900 | HS • 2309.08542 | Michael Clancy, David Kaplan, HS

The Standard Model of Particle physics is an enormously successful theory.

But it has a rather embarrassing problem.

The problem is that the Standard Model is a *chiral guage theory*.

And we do not know how to do computer simulations of a chiral gauge theory.

- This is not a “technical” problem about algorithms or hardware.
- It means that we really do not how to *define* the Standard Model in a nonperturbative way.

There is no nonperturbative lattice construction of 4d nonabelian chiral gauge theories

Dirac fermion and global chiral symmetry

- Recall that the massless Dirac fermion action in $d = 2k$,

$$S = \int d^4x \, \bar{\psi} \not{D} \psi$$

has both a vector and chiral $U(1)$ symmetry

$$\psi \xrightarrow{U(1)_V} e^{i\theta} \psi$$

$$\psi \xrightarrow{U(1)_\chi} e^{i\theta\gamma_5} \psi$$

- The chiral matrix γ^5 lets us define left and right handed Weyl fermions

$$\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi.$$

- There is a famous mixed 't Hooft anomaly between $U(1)$ and $U(1)_\chi$

Two avatars of the chiral fermion problem

- **Anomalous Global Chiral symmetry**
  - Think: QCD
  - The theory has a global chiral symmetry, with a 't Hooft anomaly
  - (Perfectly fine as a global symmetry, but cannot be gauged)
  - Physical consequences: like the $\pi \to \gamma\gamma$ cross-section
- **(Anomaly-free) Gauged chiral symmetry**
  - Think: Electroweak
  - If fermions multiplets are in the right representation, then the 't Hooft anomaly may cancel.
  - There is no obstruction to gauging, and the chiral symmetry can be gauged.
  - This is a chiral gauge theory

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On the lattice, there has been impressive progress in acheiving a global chiral symmetry with exact 't Hooft anomaly.
(the "easy" problem of chiral fermions)

However, the case of gauged chiral symmetry is open.
(the "hard" problem of chiral fermions)

In its simplest version, the problem is this:

Naive attempts to get a Weyl fermion on the lattice leads to doublers

Fermion doubling problem

What goes wrong when you try to put a Weyl fermion on the lattice?

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- Hamiltonian for a Weyl fermion in 1+1d

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- Naive discretization: Replace the derivatives with finite differences

$$\text{position space:} \qquad \partial_1 \psi(x) \to \frac{1}{2a} \left(\psi_{i+1} - \psi_{i-1}\right)$$

$$\text{momentum space:} \qquad ip\psi_p \to \frac{i}{a} \sin(ap)\psi_p$$

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- This gives the lattice Hamiltonian ($a = 1$)

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$$= \sum_p \psi^\dagger_{-p}\sin(p)\psi_p$$

- At low-energies,

$$h(q) = \sin(q) = +q + O(q^2)$$

$$h(\pi + q) = \sin(\pi + q) = -q + O(q^2)$$

Extra massless particle with opposite chirality!

Fermion doubling problem

There are infinitely many ways to discretize the free fermion Hamiltonian.

Can you be clever with the discretization and avoid this problem?

A no-go theorem

- Consider the free Dirac fermion action on the lattice

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^{2k}} \bar{\Psi}_{-p} D(p) \Psi_p$$

- In fact, the following 4 conditions cannot all hold simultaneously (Nielsen-Ninomiya '81, Karsten '81)
  1. $D(\mathbf{p})$ is a periodic, analytic function of $\mathbf{p}$ **(locality)**
  2. $D(\mathbf{p}) \propto \gamma^\mu p_\mu$ for $a|p| \ll 1$ **(continuum limit)**
  3. $D(\mathbf{p})$ is invertible everywhere except $\mathbf{p} = 0$ **(no doublers)**
  4. $\{D(\mathbf{p}), \gamma^5\} = 0$ **(chirality)**

- Take a general translationally-invariant lattice Hamiltonian for a single component fermion field. In momentum space,

$$H = \psi^\dagger_{-p} h(p) \psi_p$$

- $h(p) \sim +p$ near $p \to 0$ for the correct continuum limit (Right-moving Weyl fermion).
- Locality implies $h(p)$ is analytic, periodic function of $p \in [-\pi, \pi)$
- So if $h(p) \sim +p$ near $p \to 0$, it must cross $h(p) = 0$ again somewhere. Therefore $h(p) \sim -p$ for some $p \neq 0$. This is a left-moving Weyl fermion!
- Therefore, we end up with a Dirac fermion, instead of Weyl fermion.

Fermion doubling problem

To get a lattice theory free of doublers, we need to violate at least one of the assumptions of Nielsen-Ninomiya.

But which one?

Wilson's idea

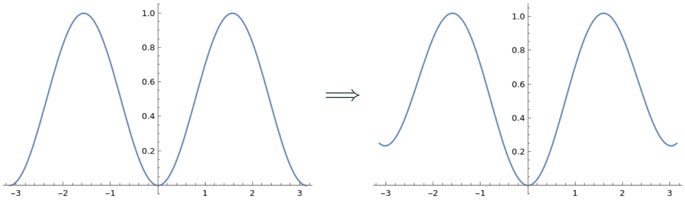
Here's an example of how to get rid of the doublers for Dirac fermions.

- Add a momentum-dependent mass term so that the doublers become heavy and decouple.

$$H = \int dp\, \bar{\psi}\Big[\gamma^i \sin(p_i) + mF(p)\Big]\psi$$

- Choose $F(p)$ such that $F(0) = 0$ but $F(p) \sim 1$ at the corners of BZ.

$$F(p) = \sum_i 1 - \cos(p_i)$$



- Doublers are gone, but so is the exact chiral symmetry
- The $U(1)$ chiral symmetry of the action is recovered only in the continuum limit

We can come up with many other such ideas to violate one of the assumptions of NN (and indeed people have).

But is there an “optimal” way to do this?

We can come up with many other such ideas to violate one of the assumptions of NN (and indeed people have).

But is there an “optimal” way to do this?

In Euclidean spacetime, the answer is yes.

Ginsparg-Wilson relation

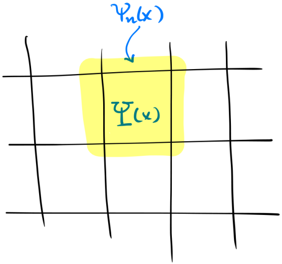
Observation:
The continuum theory has an exact chiral symmetry of the action, but the lattice theory does not.

Ginsparg and Wilson (1982) asked:
If you obtain a lattice theory by “RG blocking” a continuum theory, what happens to the chiral symmetry?

Ginsparg-Wilson relation

Start with a continuum theory and construct a lattice theory by RG blocking:

$$Z = \int D\psi D\bar{\psi}\, e^{-\bar{\psi}\mathcal{D}\psi}$$



If the continuum Dirac operator $\mathcal{D}$ has an exact chiral symmetry, GW found that the lattice Dirac operator D satisfies

$$\{\mathcal{D}, \bar{\gamma}\} = 0 \implies \boxed{\{D, \bar{\gamma}\} = aD\bar{\gamma}D}$$

This is the Ginsparg-Wilson relation.

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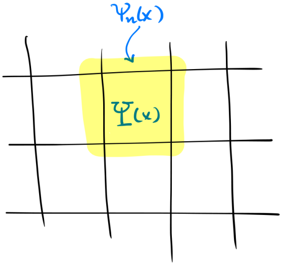

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This is the Ginsparg-Wilson relation. Why is this interesting?

An exact chiral symmetry on the lattice

- The GW relation implies an **exact** "modified" chiral symmetry of the action (Lüscher '98)

$$S = \bar{\psi} D \psi$$

- The exact symmetry is

$$\hat{\gamma}^5 = \gamma^5 (1 - aD)$$
$$\delta\psi = \hat{\gamma}^5 \psi, \qquad \delta\bar{\psi} = \bar{\psi}\gamma^5$$

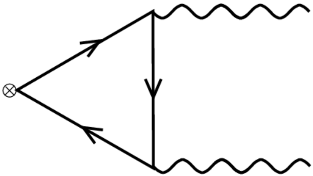
- The variation in the action is

$$\begin{aligned} \delta(\bar{\psi} D \psi) &= (\delta\bar{\psi}) D\psi + \bar{\psi} D(\delta\psi) \\ &= \bar{\psi}\left(\gamma^5 D + D\hat{\gamma}^5\right)\psi \\ &= \bar{\psi}\left(\{D, \gamma^5\} - aD\gamma^5 D\right)\psi = 0. \end{aligned}$$

- This means that any Dirac operator satisfying the GW relation automatically will not have additive mass renormalization (no fine tuning)

“No anomaly on the lattice”

- But the chiral symmetry supposed to be anomalous
- The free 4D Dirac fermion has a global chiral symmetry.
- However, the chiral symmetry is afflicted by a famous mixed $U(1)_V \times U(1)_A$ anomaly.



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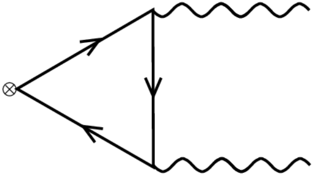

- If the lattice action is invariant, where does the anomaly come from?

Lore

There are no anomalies on the lattice.

(Fermion doubling occurs because the lattice cannot reproduce anomalies.)

"~~No~~ anomaly on the lattice"

- The exact symmetry is:

$$\psi \to U\psi \approx (1 + i\varepsilon\hat{\gamma}^5)\psi, \qquad \bar{\psi} \to \bar{\psi}\bar{U} \approx \bar{\psi}(1 + i\varepsilon\gamma^5)$$

- This implies that the measure transforms with the Jacobian

$$\begin{aligned} d\psi d\bar{\psi} \longrightarrow \quad & d\psi d\bar{\psi}\ \det(U\bar{U}) \\ &= d\psi d\bar{\psi}\ \exp(i\varepsilon\,\mathrm{tr}(\hat{\gamma}^5)) \\ &= d\psi d\bar{\psi}\ \exp(-i\varepsilon\,\mathrm{tr}(\gamma^5 D)) \\ &= d\psi d\bar{\psi}\,\exp(-i\varepsilon\ \mathrm{index}\ D) \end{aligned}$$

- This is a subtle calculation in the continuum theory! (Fujikawa '79)
- On the lattice, it is almost trivial once you identify the correct modified symmetry (Lüscher '98).

Many other examples of lattice anomalies in recent years: (Catterall et al; Nguyen, HS; Sulejmanpasic, Gattringer; Shao, Seiberg; Berkowitz, Cherman, Jacobson, ...)

Naive argument: doubling occurs because lattice cannot reproduce the anomaly.

But the GW relation implies the anomaly on the lattice.
If so, it may allow for a solution to the doubling problem.

But is there actually a Dirac operator which actually satisfies it?

A solution to the GW relation

- Consider writing

$$D = 2\frac{h}{h+1}$$

where h is some operator. The GW relation implies

$$\{\gamma^5, D\} = D\gamma^5 D \implies \{\gamma^5, h\} = 0$$

Therefore, h can be any operator which satisfies the chiral symmetry.

- So choose $h \sim \not{D}/m = \gamma^\mu D_\mu/m$, the continuum Dirac operator!
- Therefore this Dirac operator satisfies the GW relation (with $a \sim m^{-1}$)

$$D = \frac{\not{D}}{\not{D} + m}$$

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$$D = \frac{\not{D}}{\not{D} + m}$$

- But this is just a Pauli-Villars regulated Dirac fermion!

For more details, see Clancy, Kaplan, HS '24

A solution to the GW relation: the overlap operator

- Any $D = \frac{1}{2}(1 + V)$ with V unitary satisfies the GW relation
- A "continuum" solution to the GW relation:

$$V = \frac{\not{D} - m}{\not{D} + m}$$

The **overlap** operator provides a nonperturbative lattice version of this construction.

$$V = \begin{cases} \frac{\not{D} - m}{\not{D} + m} & \text{Pauli-Villars} \\ D_W / \sqrt{D_W^\dagger D_W} & \text{Overlap (Neuberger '98)} \end{cases}$$

where D_W is a Wilson-Dirac operator.

Ginsparg-Wilson for chiral symmetry: summary

- Continuum Dirac fermions in $d = 2k$ dimensions.

$$S = \int \bar{\psi}(D + m)\psi$$

- Chiral symmetry for $m = 0$:

$$\psi \to e^{i\varepsilon\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{i\varepsilon\gamma^5}$$

- GW relation:

$$\{\gamma^5, D\} = aD\gamma^5 D$$

- Lüscher symmetry: $\psi \to \psi + \varepsilon\delta\psi$

$$\delta\psi = \hat{\gamma}^5\psi, \quad \delta\bar{\psi} = \bar{\psi}\gamma^5$$

- Exact anomaly from the noninvariance of the measure

$$d\psi d\bar{\psi} \to d\psi d\bar{\psi}\, e^{-i\varepsilon\ \mathrm{tr}\,\gamma_5 D} = d\psi d\bar{\psi}\, e^{-i\varepsilon(n_+ - n_-)}$$

- GW relation implies
  - ✓ An exact symmetry of the action
  - ✓ Exact anomaly on the lattice (noninvariance of the measure)
  - ✓ No doublers
  - ✓ No additive mass renormlization

So far, everything I said was strictly in a **Euclidean** spacetime setting.

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But many important questions require us to go beyond Euclidean lattice Monte Carlo methods, such as nonequilibrium phenomenon, realtime evolution, or formulations with sign problems θ vacua,

Quantum computing and tensor network methods may help us go beyond lattice MC. But we need a Hamiltonian formulation.

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Quantum computing and tensor network methods may help us go beyond lattice MC. But we need a Hamiltonian formulation.

Is there a Hamiltonian formulation of Ginsparg-Wilson fermions?

A Hamiltonian overlap operator

- There is actually a very natural guess for a Hamiltonian overlap operator (Cruetz-Horvath-Neuberger (2002))
- Start with the continuum Dirac Hamiltonian

$$H = \int d^d x \, \psi^\dagger \gamma^0 (\gamma^i \partial_i) \psi.$$

- Replace the continuum **spatial** Dirac operator with a spatial overlap operator

$$\psi^i \partial_i \to D$$

which satisfies the usual GW relation

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

- Also impose γ_5 hermiticity $\gamma_5 D \gamma_5 = D^\dagger$ which gives a GW relation

$$D + D^\dagger = 2aD^\dagger D$$

- Does this work?

A Hamiltonian overlap operator (CHN)

- CHN overlap Hamiltonian

$$H = \psi^\dagger h \psi = \psi^\dagger \gamma^0 D \psi$$

with $h = \gamma^0 D$, and where D is now a **spatial** Dirac operator which satisfies the GW relation.

Is there a conserved chiral charge in this model?

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Is there a conserved chiral charge in this model?

- An exactly conserved charge can again be defined with a "modified" γ_5

$$\hat{Q}_5 = \psi^\dagger \hat{\gamma}_5 \psi$$

with

$$\hat{\gamma}_5 = \gamma_5(1 - aD)$$

- The GW relation implies that this charge is conserved!

$$[\hat{Q}_5, H] = 0$$

Chiral charge

- Since both h and $\hat{\gamma}_5$ are defined in terms of D,

$$h = \gamma^0 D$$
$$\hat{\gamma}_5 = \gamma_5(1 - aD)$$

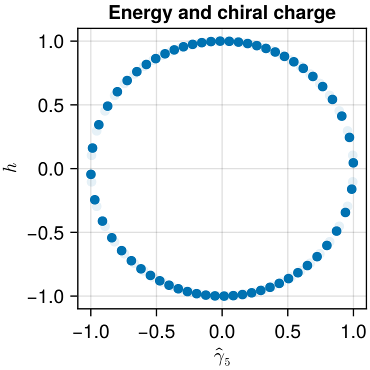
  there is actually an interesting relationship between the two

$$(ah)^2 + \hat{\gamma}_5^2 = 1$$

- What does this mean?

Chiral charge

$$(ah)^2 + \hat{\gamma}_5^2 = 1$$



- Charge varies with energy
  - Low-energy modes $|ah| \sim 0 \implies \hat{\gamma}_5 \sim \pm 1$,
  - High-energy modes $|ah| \sim 1 \implies \hat{\gamma}_5 \sim 0$

Chiral anomaly

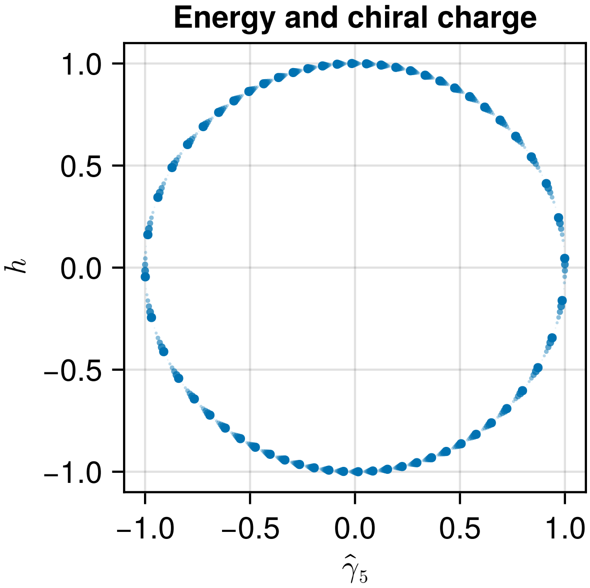
- The chiral anomaly equation in 1+1d says

$$\partial_\mu j_5^\mu = \frac{1}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}$$

- On a spatial circle, with temporal gauge $A_0 = 0$, and spatially constant $A_1 = A$

$$\partial_0 \int dx\, j_5^0 + \int dx\, \partial_1 j_5^1 = \frac{1}{\pi} \int dx\, \partial_0 A_1$$
$$\implies \partial_0 \hat{Q}_5 = \frac{L}{\pi}\, \partial_0 A$$
$$\implies \hat{Q}_5(A) = \frac{AL}{\pi} + \text{const}$$

- The chiral charge must increase *linearly* with the gauge field


Energy and chiral charge
1.0
0.5
0.0
−0.5
−1.0
h
−1.0
−0.5
0.0
0.5
1.0
$\hat{\gamma}_5$

Charge quantization

Note that the chiral charge is not quantized.

That is a bit strange from a continuum perspective.

Charge quantization

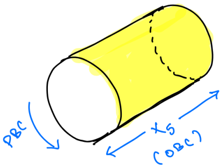
Recently, there has been some discussion about the **quantization** of the axial charge

- Recent (Shao, Chatterjee, Pace '25) and earlier work Horvath, Thacker '98:
    - a definition of a quantized charge in 1+1D but it does not commute with the vector charge $Q = \psi^\dagger \psi$
    - Has been used to propose a construction of the 3-4-5-0 model in 1+1 dimensions (Xu, 2025)
- (Gioai, Thorgren '25)
    - Construct local Hamiltonians and unquantized charges in 3+1d
- No-go theorem (Xu, Fidkowski '23): impossible to have a quantized charge for local Hamiltonian for a Weyl fermion
- (Haegeman, Lootens, Mortier, Stottmeister, Ueda, Verstraete '24)
    - Tensor network methods which use a quantized charge by construction run into trouble when trying to reproduce the anomaly

Can we gain insight into this using a GW approach?
Is there a way to improve the charge quantization?

Domain-wall fermions

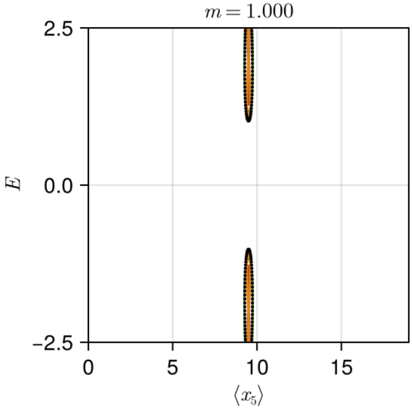
- Domain-wall fermions provide a very physically transparent setting for understanding how the anomaly arises
- Consider a *massive* Wilson-Dirac Hamiltonian h in 2+1 dimensions
- Choose PBC along the boundary, and OBC in the bulk.



- Tune the mass m through a topological phase transition.
- Plot the energy h vs the x_5 coordinate along the bulk direction of the eigenmodes of h

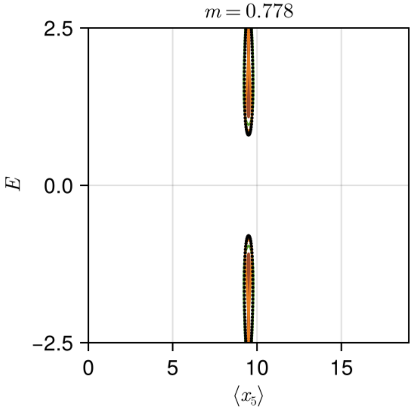
$$\gamma^0 h(\mathbf{p}) = i\gamma_5 \hat{\delta}_5 - \frac{r}{2}\Delta_5 + m + \sum_{a=1}^{d}\left[\gamma^a \sin(p_a) + r(1-\cos p_a)\right].$$

Callan-Harvey ’85, Kaplan ’92, Creutz-Horvath ’94



$m = 1.000$
2.5
0.0
−2.5
E
0
5
10
15
$\langle x_5 \rangle$

Domain-wall fermions and the anomaly



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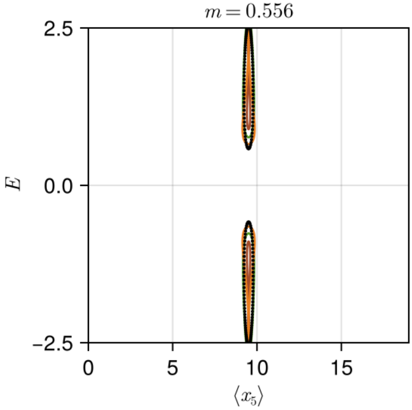

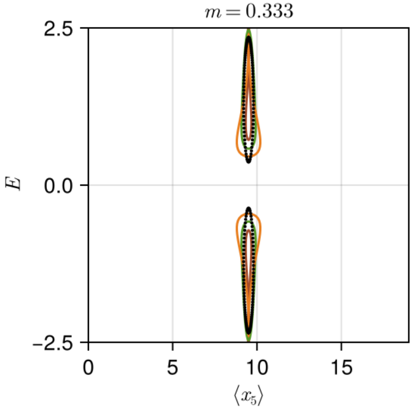
$m = 0.333$
2.5
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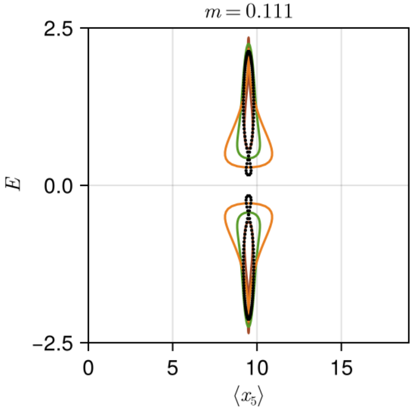
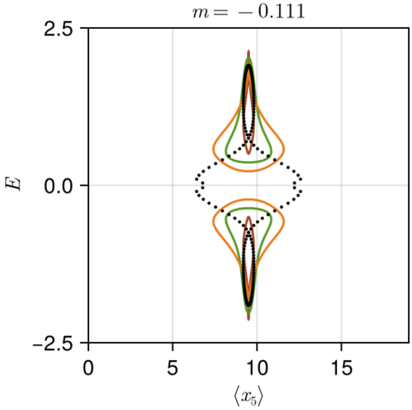
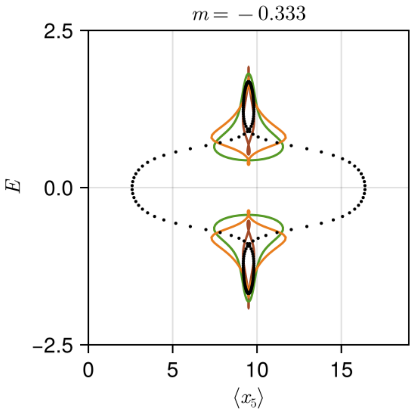
$\langle x_5 \rangle$

Domain-wall fermions and the anomaly



Domain-wall fermions and the anomaly





$m = -0.333$
2.5
0.0
−2.5
E
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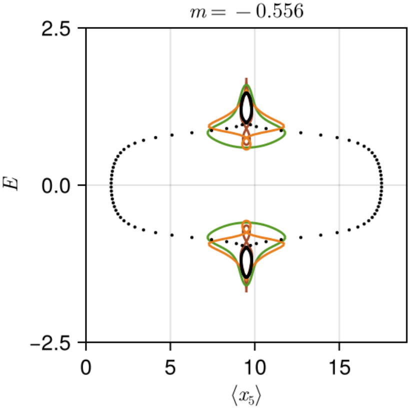
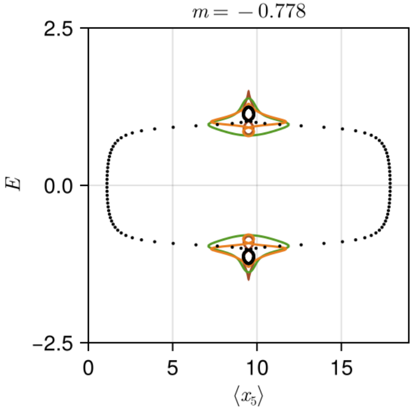
10
15
$\langle x_5 \rangle$

Domain-wall fermions and the anomaly

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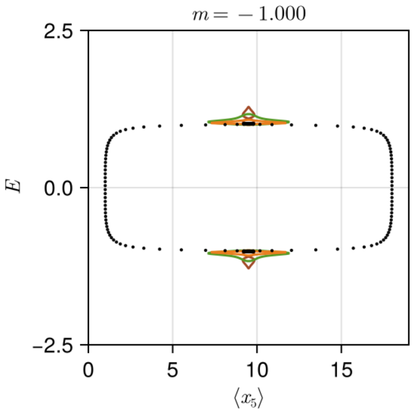
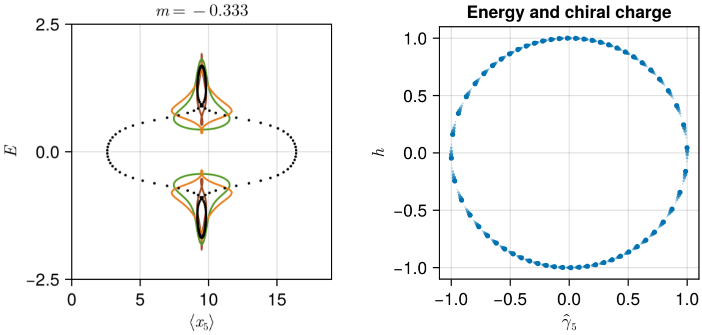
Domain-wall fermions and the anomaly



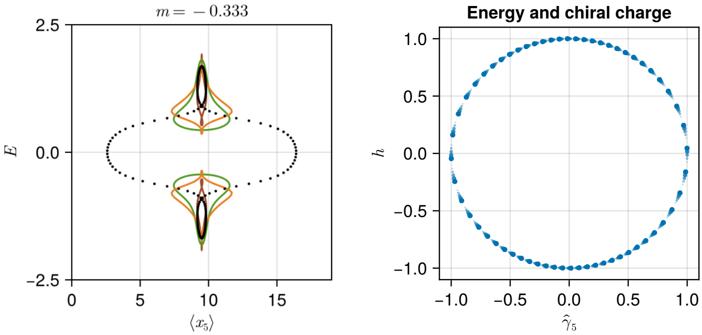


- In the domain-wall picture a reasonable definition of the boundary chiral charge simply the distance from the center. So we identify (Creutz, Horvath '94)

$$\langle x_5 \rangle \sim \langle \hat{\gamma}_5 \rangle$$

- When going from the domain-wall to the overlap operator, the purely bulk bands are discarded.
- The Hamiltonian overlap captures the essential "bulk" modes – those which participate in anomaly inflow

What does it mean to “improve” the chiral symmetry?



- In the domain-wall picture, it seems clear that the violations to the quantization of the chiral charge are coming from the bulk modes.
- These bulk modes are necessary for anomaly inflow
- To improve the quantization, the best we can do is imagine “pushing the bulk modes closer to wall.”

Can we implement this idea directly in the overlap operator?

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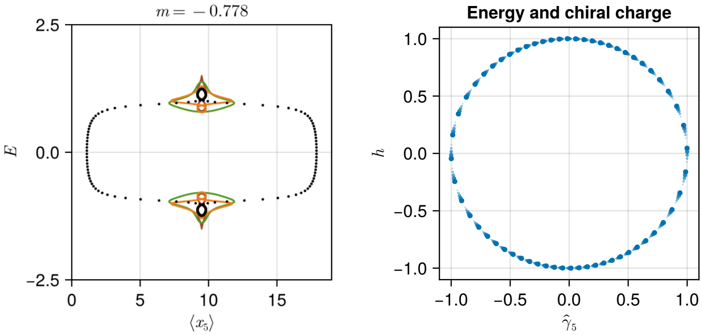

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Can we implement this idea directly in the overlap operator?

Fujikawa's generalization of the GW relation

- Interestingly, Fujikawa (2000) proposed an algebraic generalization of the (Euclidean) GW relation

$$D^\dagger + D = 2aD^\dagger D$$
$$\downarrow$$
$$D^\dagger + D = 2a^{2k+1}(D^\dagger D)^{k+1}$$

  to improve the chiral properties of the overlap operator. For $k = 0$, this reduces to the standard GW relation.
- What happens if we replace the GW relation in the Hamiltonian overlap with Fujikawa's generalization of the GW relation?

CHN Overlap + Fujikawa's generalized GW

- The construction of the CHN Hamiltonian overlap goes through:

$$H = \psi^\dagger \gamma^0 D \psi$$

where D now satisfied the order $(2k+1)$-GW relation

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- Again, a modified chiral charge can be defined

$$\hat{\gamma}_5 = \gamma_5[1 - D(\gamma_5 D)^{2k}]$$

- **Relation between the modified chiral charge $\hat{\gamma}_5$ and energies h?**

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$$\hat{\gamma}_5^2 + (ah)^2 = 1$$

$$\Big\downarrow \quad \text{Order-}(2k+1)\text{ GW relation}$$

$$\hat{\gamma}_5^2 + (ah)^{4k+2} = 1$$

Chirality-improved Hamiltonian overlap

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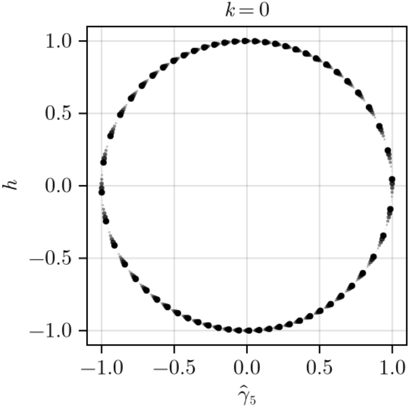

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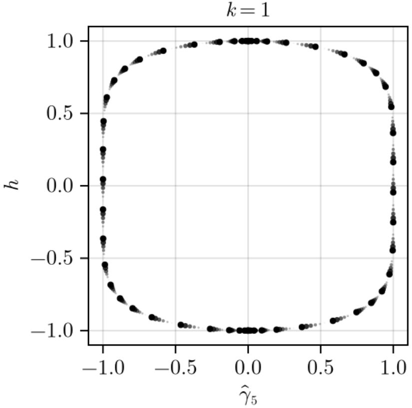

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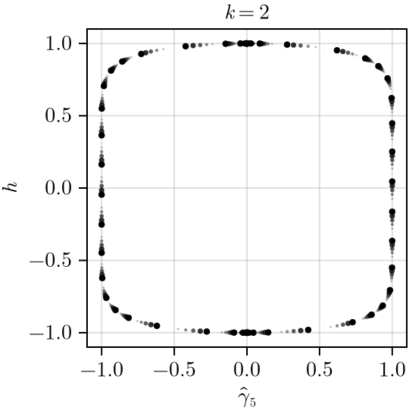

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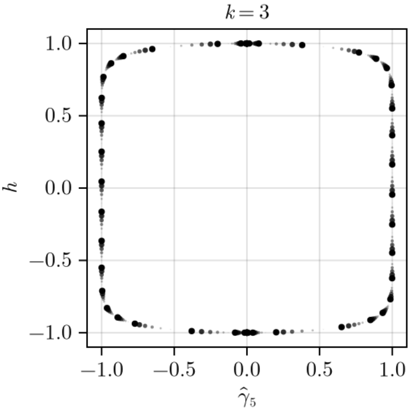

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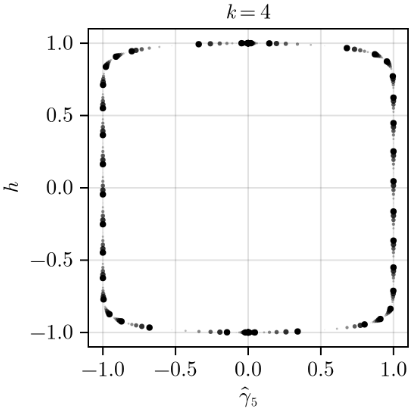

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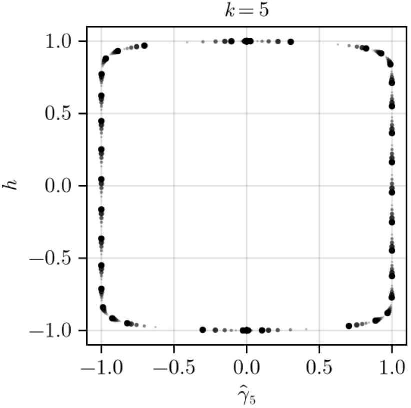

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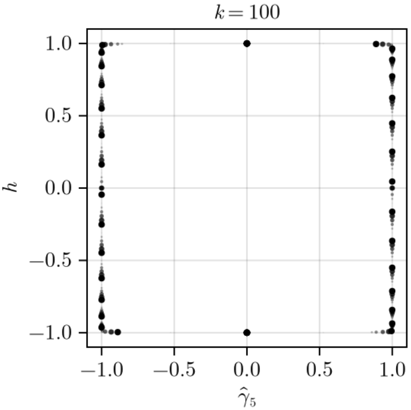

This is doing exactly what we wanted: "pushing all the low-energy modes closer to the wall," improving the charge quantization at low-energies

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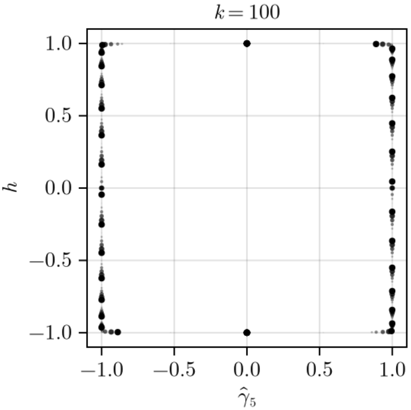

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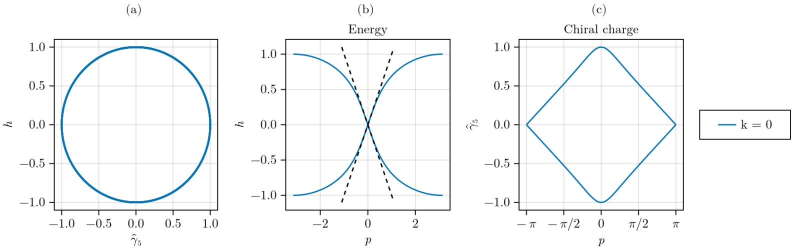
The charge-quantization for modes $|h| < 1$ becomes exact at $k \to \infty$.

Locality vs Quantization

- An interesting tension seems to appear as we change k
- As we make k larger, the Hamiltonian and charge both become more nonlocal.

Locality vs Quantization



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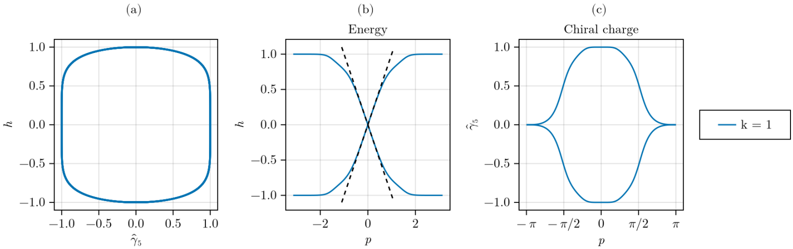

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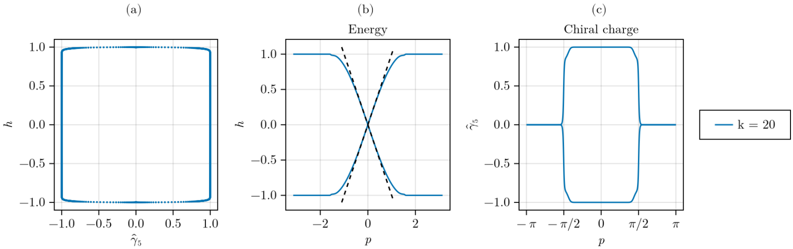
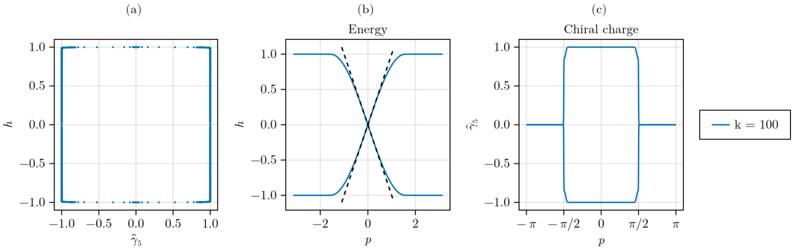
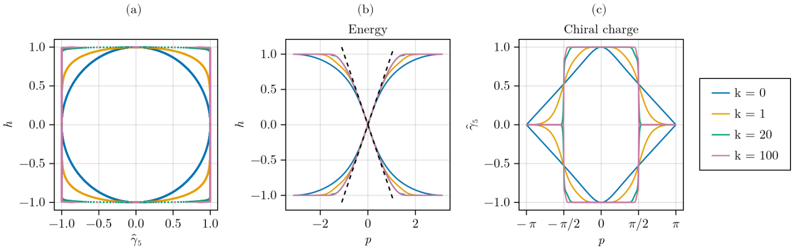



(a)
1.0
0.5
0.0
−0.5
−1.0
h
−1.0
−0.5
0.0
0.5
1.0
$\hat{\gamma}_5$
(b)
Energy
h
−2
0
2
p
(c)
Chiral charge
$\hat{\gamma}_5$
$-\pi$
$-\pi/2$
0
$\pi/2$
π
p
k = 100



- As quantization improves, locality becomes worse
- The $k \to \infty$ limit shows how you get a compact $U(1)$ symmetry with power-law locality,
  - (this may not be a problem for certain quantum hardware such as those based on ion traps)

You cannot get an exponentially local theory without the “bulk” modes necessary for reproducing anomaly

- Consistent with the no-go theorem of Fidkowski, Xu ’23

But there is more to this story...

Let's look at the question of quantization again.

A quantized charge

- Recall the *Staggered Fermion* in 1+1D

$$H = \frac{i}{2} \sum_j c_j^\dagger c_{j+1} + \text{h.c.}$$

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- Curiously, there is also a **quantized** chiral charge (Chatterjee-Pace-Shao ’25, Horvath-Thacker ’98)

$$Q_A^1 = \frac{1}{2}\sum_j (c_j + c_j^\dagger)(c_{j+1} - c_{j+1}^\dagger)$$

But $[Q_A^1, Q] \neq 0$.

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Does the quantized charge fit into the Ginsparg-Wilson picture?

HS '25, in preparation

A GW perspective on quantization

- A symmetry between the CHN overlap Hamiltonian and Chiral charge if you exchanged $\gamma^0 \leftrightarrow \gamma^5$ and $V \to -V$:

$$h = \gamma^0(1 + V)$$
$$\hat{\gamma}_5 = \frac{1}{2}\gamma^5(1 - V)$$

- The chiral charge itself can be thought of as a (doubler-free) Hamiltonian with gapless modes at $k = \pi$.

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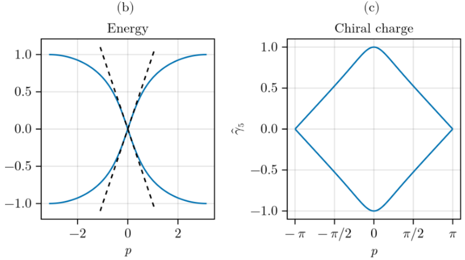


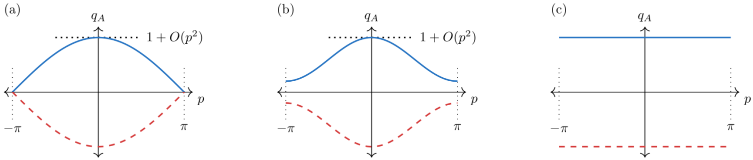
- To get a quantized charge, we need to give mass to the $k = \pi$ modes

$$\hat{\gamma}_5' = \frac{1}{2}\gamma^5(1 - V) + M$$

such that $[\hat{\gamma}_5', h] = 0$

Quantization = Gapping charge operator

- To make the charge quantized, we need to add "mass" term to the charge which would gap the massless mode at $k = \pi$.



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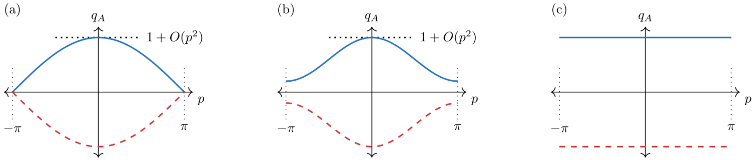

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- In the traditional (CHN) GW formulation, this is impossible! (Any mass term for the Hamiltonian breaks chiral symmetry, and vice versa)

But what about Majorana mass terms?

- If we sacrifice $U(1)$ vector symmetry, then maybe we can add Majorana mass terms
- A natural way to treat this is to treat the Dirac fermions as a Majorana fermion by writing $\psi, \psi^\dagger$ as a doubled Majorana field

$$\chi = \begin{pmatrix} \psi + \psi^\dagger \\ i(\psi - \psi^\dagger) \end{pmatrix}$$

- This is the BdG formalism

Ginsparg-Wilson Hamiltonian in a BdG formalism

- In the BdG formalism with

$$\chi = \begin{pmatrix} \psi + \psi^\dagger \\ i(\psi - \psi^\dagger) \end{pmatrix}$$

  the overlap Hamiltonian and chiral charges:

$$H = \chi^T h_{\text{BdG}} \chi$$
$$Q = \chi^T q_{\text{BdG}}^{\text{CHN}} \chi$$

- We find

$$h_{\text{BdG}} = \mathbb{I} \otimes \gamma^0 (1 + V)$$
$$q_{\text{BdG}}^{\text{CHN}} = \tau_y \otimes \gamma^5 (1 - V)$$

- Symmetry between the Hamiltonian h and the chiral charge q:

$$h \leftrightarrow q$$
$$\gamma^0 \leftrightarrow \gamma^5$$
$$V \leftrightarrow -V$$
$$k = 0 \qquad \text{massless} \leftrightarrow \text{gapped}$$
$$k = \pi \qquad \text{gapped} \leftrightarrow \text{massless}$$

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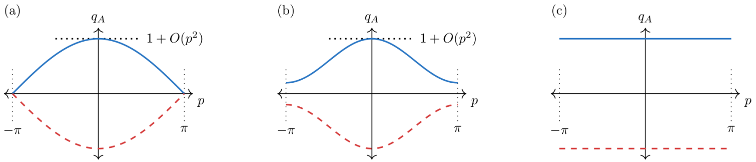

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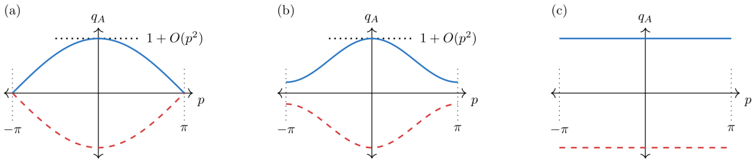

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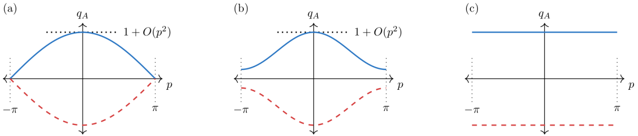
- In the traditional (CHN) GW formulation, there is no such mass term.
- But a BdG-Ginsparg-Wilson formulation naturally allows for Majorana mass terms.

Quantization = Gapping charge operator



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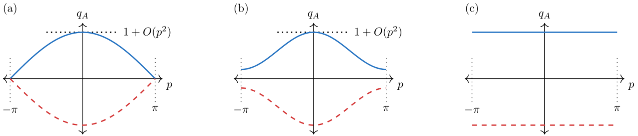
Quantization = Gapping charge operator



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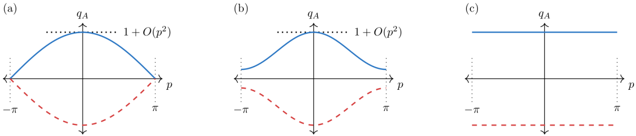

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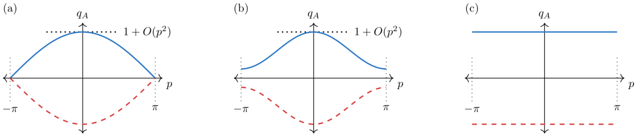

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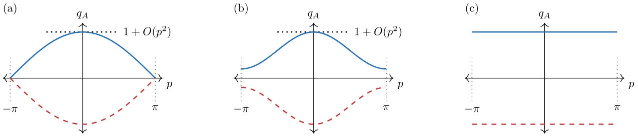

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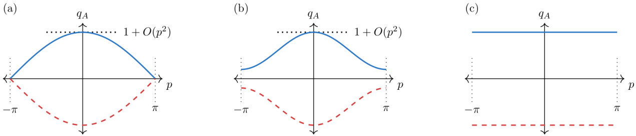

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- Only one allowed choice!

$$M = \tau_x\sigma_y.$$

Quantization with a BdG-GW formalism

- We finally obtain

$$h_{\mathrm{BdG}} = \tau_0 \gamma^0 (1 - V)$$
$$q_{\mathrm{BdG}}^{\mathrm{CHN}} = \tau_y \gamma^5 (1 + V)$$
$$q_{\mathrm{BdG}}^{0} = q_{\mathrm{BdG}}^{\mathrm{CHN}} + \tau_x \gamma^0 (1 - V)$$

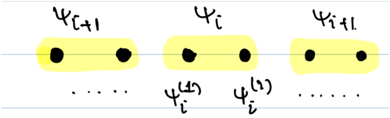
- The quantized charge has the structure

$$q_{\mathrm{BdG}}^{0} = \tau_y \gamma_5 \left[(1 + V) + M' (1 - V) \right]$$

which is nothing but a **massive Majorana overlap** operator! (Clancy, Kaplan, Singh '24)

Quantized charge

- Actually, if you unpack the two components of a GW fermion in 1+1d, you get exactly 1+1d staggered fermions



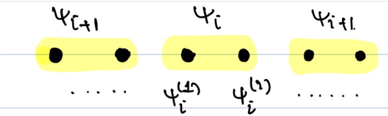
- It has to be an ultralocal Hamiltonian with a Dirac fermion at each site, with no doublers.

Staggered = CHN Overlap

- The standard (unquantized) chiral charge is exactly the CHN chiral charge we have been discussing all this time, and..

Quantized charge

- Actually, if you unpack the two components of a GW fermion in 1+1d, you get exactly 1+1d staggered fermions



- It has to be an ultralocal Hamiltonian with a Dirac fermion at each site, with no doublers.

Staggered = CHN Overlap

- The standard (unquantized) chiral charge is exactly the CHN chiral charge we have been discussing all this time, and..
- the quantized charge we found is exactly the one discussed by Chatterjee-Shao-Pace '25

The BdG-Ginsparg-Wilson formalism provides a powerful unifying framework for to understand the quantization issue.

What about locality?

GW + Ultralocality

- The usual GW/overlap are not ultralocal, except in 1+1d.

$$V = \frac{D_W}{\sqrt{{D_W}^\dagger D_W}}$$

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$$V = \frac{D_W}{\sqrt{{D_W}^\dagger D_W}}$$

- What if we try to enforce ultralocality in higher dimensions?
- Impossible with $N = 1$ flavors, but becomes possible with $N > 1$
- What is the mininum number of flavors for which GW becomes ultralocal? $N = 2^{\frac{d}{2}}$, whihch is exactly the number of flavors in staggered fermions
- Indeed,

Ultralocality + GW = Staggered!

- This also puts recently discussed charges of Gioao, Thorngren '25 in a GW framework

HS '25, in preparation

It's been over 40 years since the original work of GW.

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A remnant of chiral symmetry on the lattice

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The GW relation keeps on giving.

The many avatars of Ginsparg-Wilson structures

- Euclidean Generalized GW relations = **lattice bulk-boundary correspondence** (Clancy, Kaplan, Singh 24)
  - The bulk-boundary correspondence is used almost all recent attempts for nonabelian chiral lattice gauge theories (symmetric mass generation, single-wall)
  - Global anomalies, Majorana fermions, arbitrary dimensions

- Constructing improved overlap Hamiltonians (HS '25)
  - **Tension** between locality, realtime, unitarity and compactness.
  - Physical picture of how the compactness can be improved at the cost of locality.
  - Quantum algorithms (for certain hardware) might benefit from this improved chirality

- Quantized chiral charge from a BdG-Ginsparg-Wilson approach (HS '25)
  - A straightforward Hamiltonian formulation of the GW leads to an unquantized charge.
  - A "BdG-Ginsparg-Wilson" formulation provides new perspective on the recently discussed quantized charge, allows further generalizations.
- Ultralocality + Ginsparg Wilson (HS '25)

Generalized GW relations offer a unified approach to capturing fermionic anomalies, both in Euclidean and Hamiltonian formulations.

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Are all "good" chiral fermions just GW in disguise?

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Are all “good” chiral fermions just GW in disguise?

Is a complete picture of fermionic anomalies on the lattice needed to solve the problem of nonabelian chiral gauge theories?

Thank you.