Casimir effect in critical $O(N)$ models with non-equilibrium methods

Andrea Bulgarelli

University of Turin and INFN

ECT* Trento, Bridging analytical and numerical methods for quantum field theory

Based on AB, Caselle, Nada, Panero; 2505.20403

UNIVERSITÀ DI TORINO

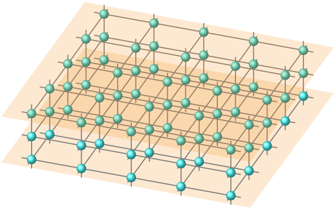
INFN Istituto Nazionale di Fisica Nucleare

Thermal CFTs

- A thermal CFT can be defined on the manifold $\mathcal{M}_D = S^1_l \times \mathbb{R}^{D-1}$
- Enhanced number of CFT data
- Critical Casimir amplitude
  $\rightarrow$ one-point function of the stress-energy tensor

$$\langle T_{tt}\rangle_{\mathcal{M}_D} = (1-D)\langle T_{ii}\rangle_{\mathcal{M}_D} = -(D-1)\Delta l^{-D}$$

Experimental results



- Attractive force between the boundaries
- Experimentally measured in $^4\mathrm{He}$ and binary liquid mixtures
  [Garcia et al.; PRL **83** (1999)]
  [Garcia et al.; PRL **88** (2002)]
  [Hertlein et al.; Nature **451** (2008)]

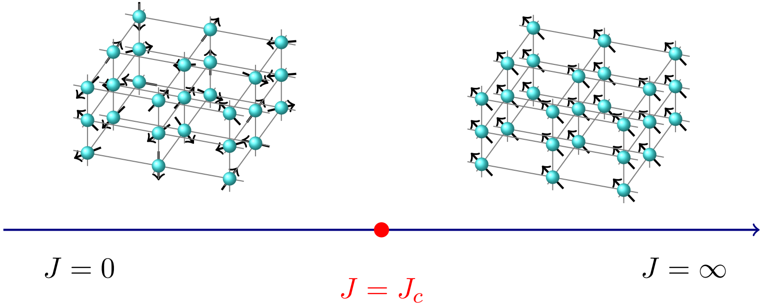
O(N) models

$$H = -J \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j$$

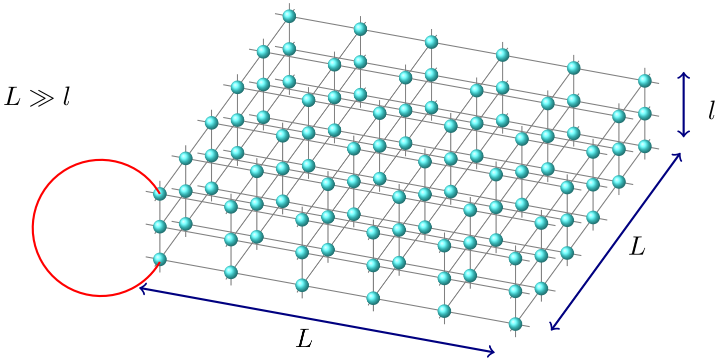
O(1): Ising O(2): XY O(3): Heisenberg

O(4) → particle physics O(∞) : spherical model → AdS/CFT

Euclidean lattice, $D = 3$, $a = 1$



Casimir effect: setup



General idea: long-range fluctuations are sensitive to the short direction l

Casimir effect: critical Casimir amplitude

[Vasilyev et al.; PRE **79** (2009)]

free energy density
in the limit $l \to L$

$$F = L^2 l f = L^2(\beta^{-1} f_{\text{ex}} + l f_{\text{bulk}})$$

excess of free energy due
to the compact direction

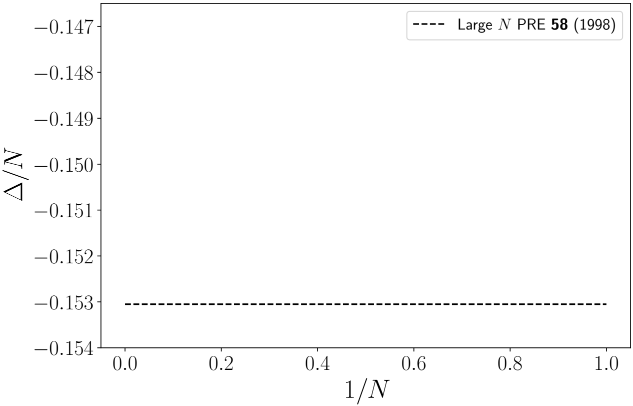
- In the scaling limit $L \gg l \gg 1$

$$f_{\text{ex}} = \Delta\, l^{-2}$$

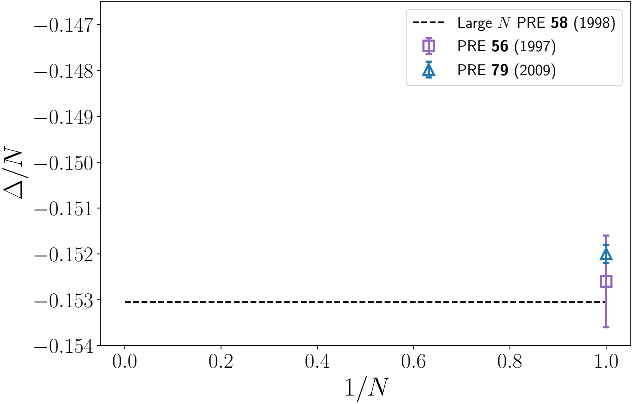
- Δ is the critical Casimir amplitude

- The Casimir amplitude is part of the **CFT data** of the theory
- Determined for different $\mathrm{O}(N)$ models with different methods

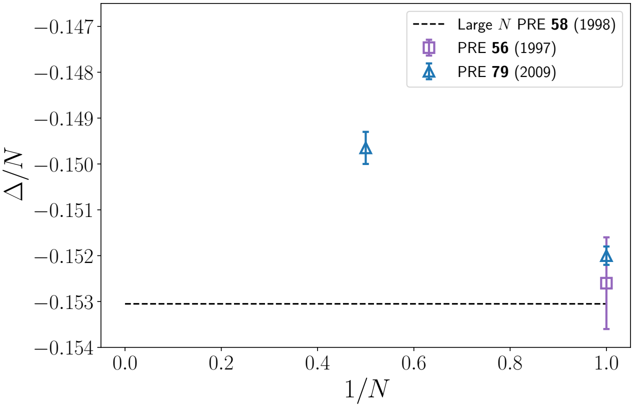
A puzzling quantity...



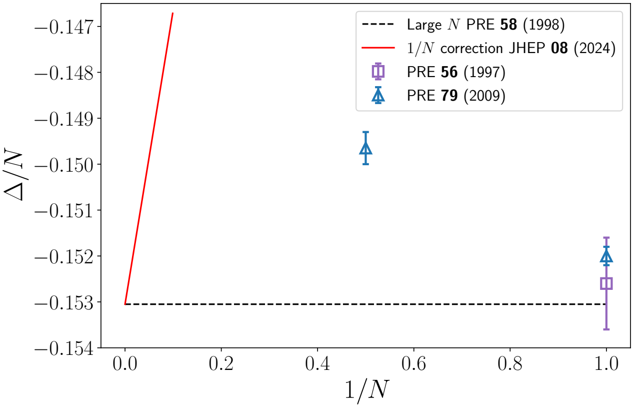
A puzzling quantity...



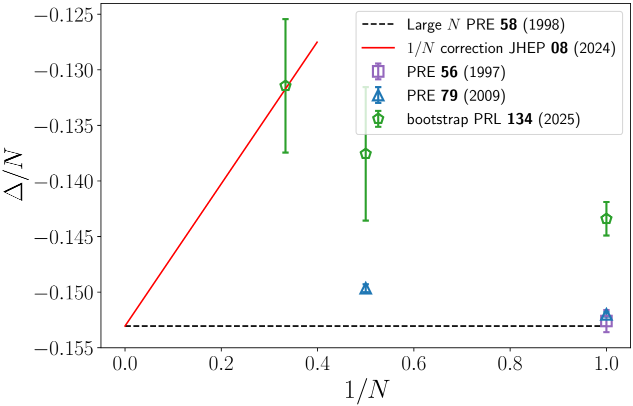
A puzzling quantity...

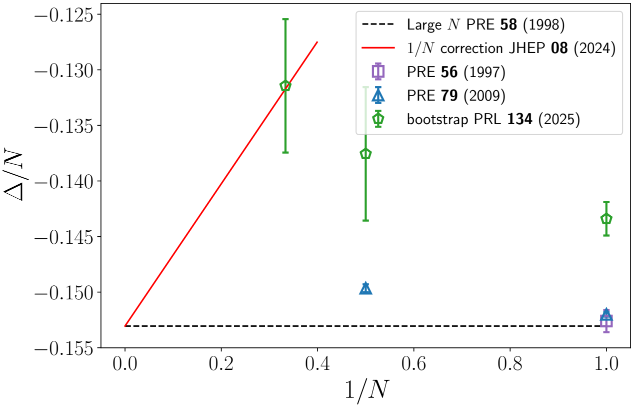


A puzzling quantity...



A puzzling quantity...





Goals

Better understanding large-N behavior

Discriminating between conflicting results

How do we compute Δ?

only depends on L

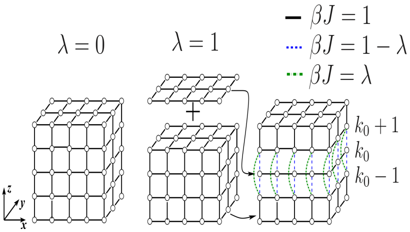
$$F = L^2 l f = L^2(\beta^{-1} f_{\mathrm{ex}} + l f_{\mathrm{bulk}})$$

$\Delta\, l^{-2}$ plus subleading dependence on L

MCMC simulations

- Markov chain Monte Carlo can be used to compute free energy differences (although it is not an ideal method...)

$$F_{\lambda=1} - F_{\lambda=0} = F(l-1) + F_{\text{slab}} - F(l) \simeq$$
$$\simeq -\frac{\partial F}{\partial l} + F_{\text{slab}}$$



From [Vasilyev et al.; PRE **79** (2009)]

unwanted term,
only depends on L

$$\frac{1}{L^2}\frac{\partial F}{\partial l} = \beta^{-1}\frac{\partial f_{\text{ex}}}{\partial l} + f_{\text{bulk}}$$

we want to isolate this
$\propto \Delta\, l^{-3}$

- An iterative procedure is employed to remove f_{bulk} and f_{slab}

Can we design a better strategy?

Non-equilibrium simulations
for efficient calculations of ΔF

Different target ΔF to automatically get rid of f_{bulk} and f_{slab}

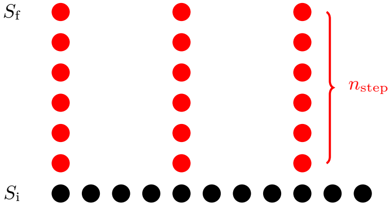
Strategy

- We want to compute $F_\mathrm{f} - F_\mathrm{i}$ defined by S_f and S_i
- Introduce a discrete protocol $c(n)$ connecting S_i and S_f
- Non-equilibrium Markov Chain following $c(n)$ in n_step

$$\phi_0 \xrightarrow{P_{c(1)}} \phi_1 \xrightarrow{P_{c(2)}} \dots \xrightarrow{P_{c(n_\mathrm{step})}} \phi_{n_\mathrm{step}}$$

Realization

[Caselle et al.; PRD **94** (2016)]



Equilibrium MCMC

NEMC for free energy calculations II

Jarzynski's theorem

- How to compute equilibrium quantities?
- Jarzynski's theorem [Jarzynski; PRL **78** (1997)]

$$\Delta F = -\frac{1}{\beta} \log \langle e^{-\beta W} \rangle_{\text{non-eq}}$$

$$W = \sum_{n=0}^{n_{\text{step}}-1} S_{c(n+1)}(\phi_n) - S_{c(n)}(\phi_n)$$

- Mathematically exact result, also for expectation values

$$\langle \mathcal{O} \rangle_{\text{eq}} = \frac{\langle \mathcal{O} e^{-W} \rangle_{\text{non-eq}}}{\langle e^{-W} \rangle_{\text{non-eq}}}$$

Advantages

✓ Systematics well under control

✓ Linear scaling with the number of d.o.f.
[Bonanno at al.; JHEP **04** (2024)]
[AB et al.; PRD **111** (2025)]

✓ Systematically improvable by increasing n_{step} or using **Stochastic Normalizing Flows**

Second derivatives

✗ $\frac{\beta}{L^2}\frac{\partial F}{\partial l} = \frac{\partial f_{\text{ex}}}{\partial l} + \beta f_{\text{bulk}}$ ✓ $\frac{\beta}{L^2}\frac{\partial^2 F}{\partial l^2} = \frac{\partial^2 f_{\text{ex}}}{\partial l^2} \propto \Delta l^{-4}$

On the lattice second derivatives are computed as differences of first derivatives

$$\frac{\partial^2 F}{\partial l^2} \simeq [F(l+1) - F(l)] - [F(l) - F(l-1)]$$

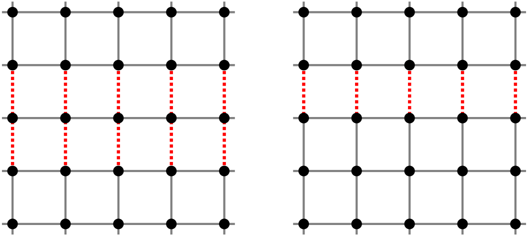
Forward derivative — Backward derivative

Drawback: not a primary observable, the two derivatives are uncorrelated.
We propose an alternative strategy

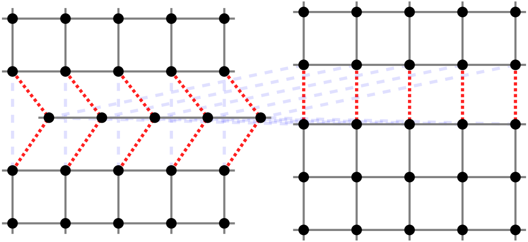
$$\frac{\partial^2 F}{\partial l^2} \simeq F(l+1) + F(l-1) - 2F(l)$$

Target F_{f} — Prior F_{i}

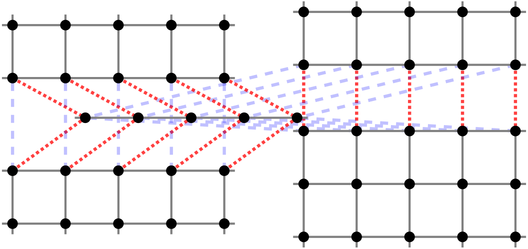
The slab-exchange protocol



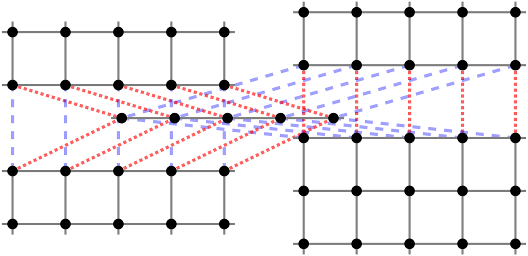
The slab-exchange protocol



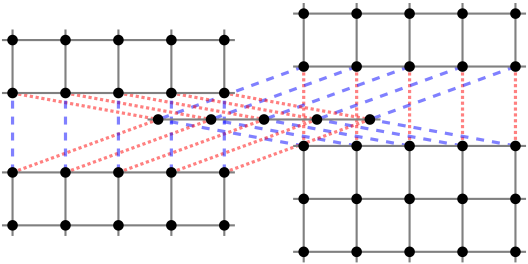
The slab-exchange protocol



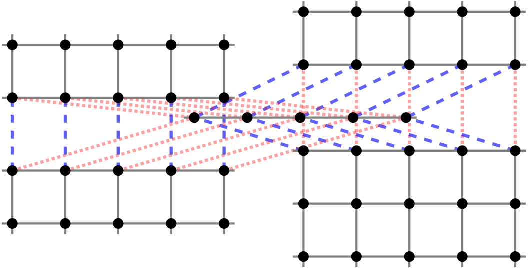
The slab-exchange protocol



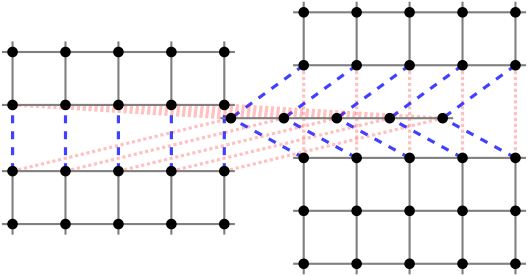
The slab-exchange protocol



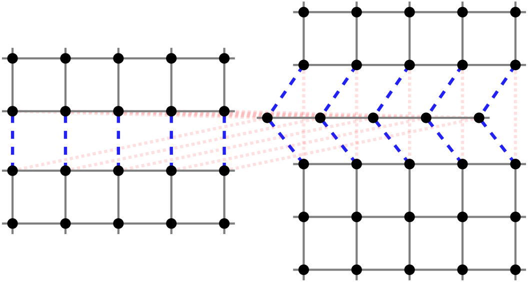
The slab-exchange protocol



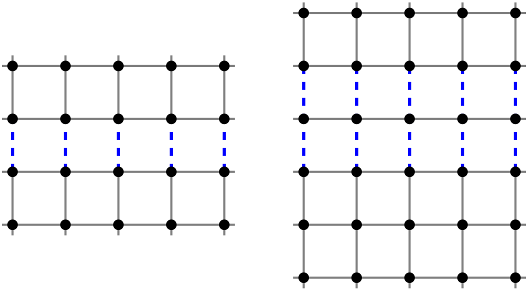
The slab-exchange protocol



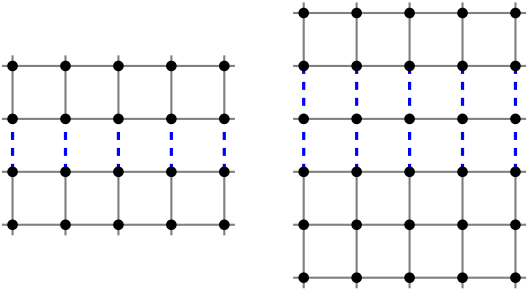
The slab-exchange protocol



The slab-exchange protocol

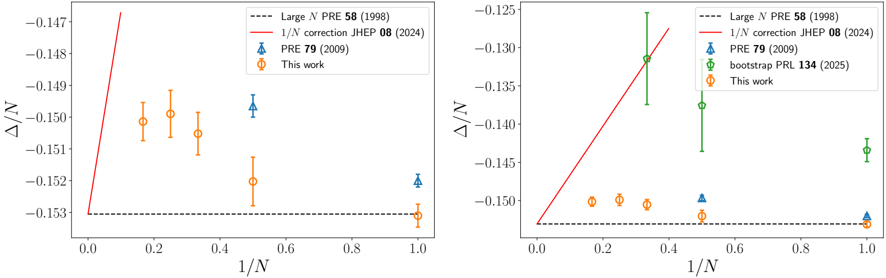


The slab-exchange protocol



$$\frac{\partial^2 F}{\partial l^2} \simeq F(l+1) + F(l-1) - 2F(l) = -\frac{1}{\beta}\log \langle e^{-\beta W} \rangle_{\text{non-eq}}$$

Our results



Large N PRE **58** (1998)
$1/N$ correction JHEP **08** (2024)
PRE **79** (2009)
This work
bootstrap PRL **134** (2025)
Δ/N
$1/N$

Conclusions and outlook

Conclusions

- New methods and range of parameters never explored before
- **Careful analysis** of extrapolations and systematic effects
- $N = 6$ is not large-N (differently from $\mathrm{SU}(N)$ gauge theories)
- Our analysis aligns with other MC studies over bootstrap

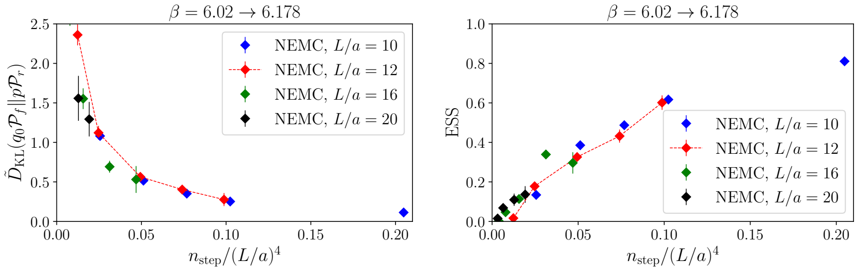
Outlook

- For better understanding N dependence:
  - simulations at larger N (~ 10)
  - calculations at higher orders ($1/N^2$)
- Understanding the discrepancy with bootstrap
- Why is $N = 1$ so close to $N = \infty$?

Backup slides

[AB, Cellini, Nada; PRD **111** (2025)]

- Theory: $\mathrm{SU}(3)$ pure gauge theory in $4D$
- Protocol: linear interpolation from $\beta = 6.02$ (mild autocorrelation) to $\beta = 6.178$ (larger autocorrelation)

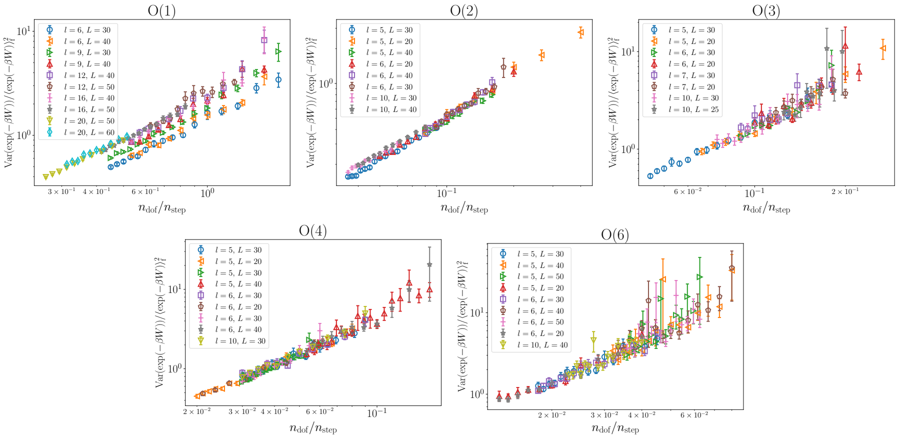


Calibration of the algorithm I

$$\mathrm{ESS} = \frac{\langle e^{-\beta W} \rangle^2_{\mathrm{non-eq}}}{\langle e^{-2\beta W} \rangle_{\mathrm{non-eq}}} \qquad \frac{\mathrm{Var}(e^{-\beta W})}{\langle e^{-\beta W} \rangle^2_{\mathrm{non-eq}}} = \frac{1}{\mathrm{ESS}} - 1$$

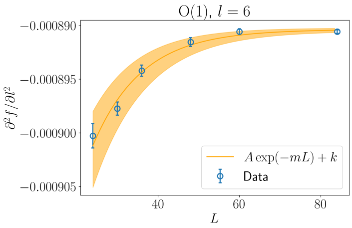
$$\frac{\mathrm{Var}(e^{-\beta W})}{\langle e^{-\beta W} \rangle^2_{\mathrm{non-eq}}} = \sum_{k=1}^{k_{\mathrm{max}}} v_k \left(\frac{n_{\mathrm{dof}}}{n_{\mathrm{step}}} \right)^k$$

Calibration of the algorithm II



Computing Δ

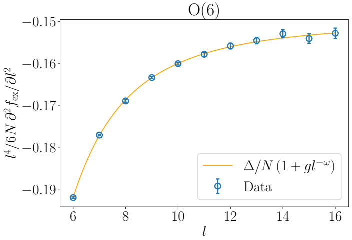
$L \to \infty$ limit



$$\frac{\partial^2 f_{\rm ex}}{\partial l^2}(L; A, k, m) = A \exp(-mL) + k$$

- Removing fine size effects in the transverse direction
- m global parameter for fixed N

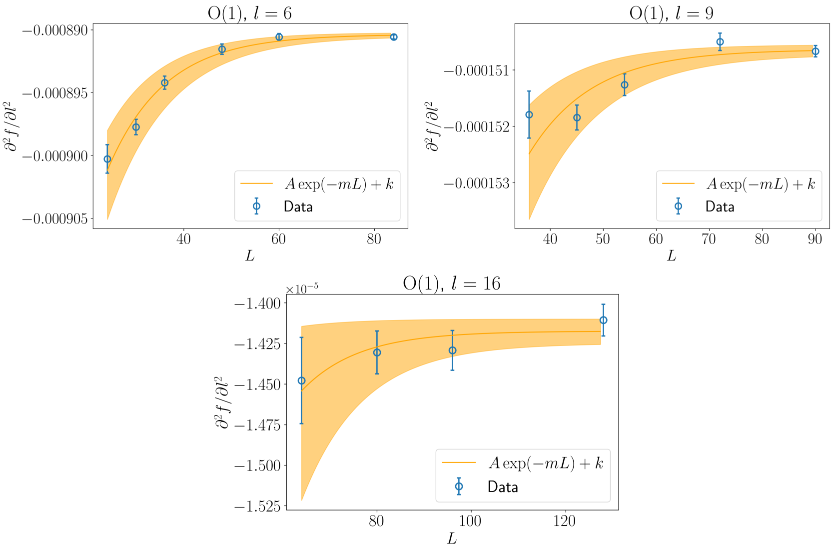
Scaling in l



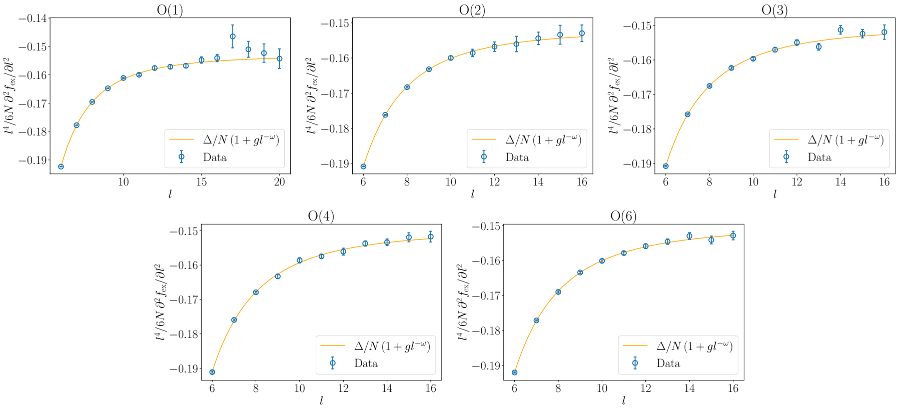
$$\frac{\partial^2 f_{\rm ex}}{\partial l^2}(l; \Delta, g, \omega) = 6\,\Delta\, l^{-4}(1 + gl^{-\omega})$$

- Fit function motivated by **finite size scaling** [Vasilyev et al.; PRE **79** (2009)]
- Good quality of the fit for all Ns

$L \to \infty$ limit



Scaling in l



	O(1)	O(2)	O(3)	O(4)	O(6)
$\chi^2_{\rm red}[\#_{\rm dof}]$	1.18 [12]	0.21 [8]	1.33 [8]	0.78 [8]	0.46 [8]
g	57(4)	63(10)	59(9)	49(9)	44(6)
ω	3.02(5)	3.08(10)	3.01(10)	2.90(11)	2.83(8)