The Simplicity of Confinement

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Work in collaboration with X. Crean and J. Giansiracusa

[arXiv:2403.07739, arXiv:2501.19320 and in preparation]

Queen Mary University of London

Workshop *Bridging Analytical and Numerical Methods for Quantum Field Theory*

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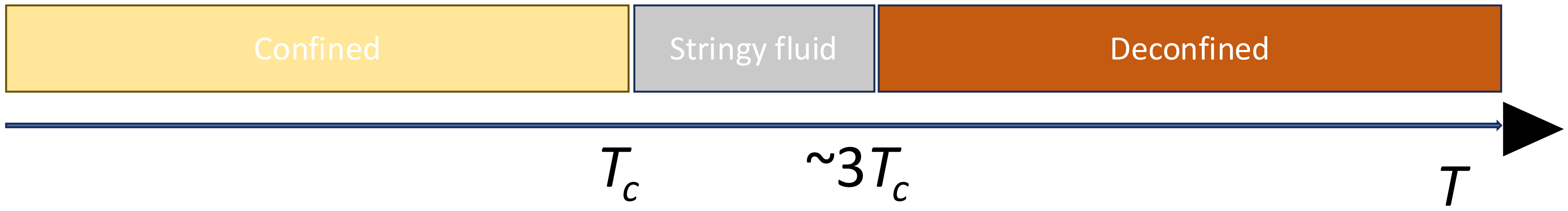
Overview

- Motivations and statement of the problem
- Lattice and Topological Data Analysis (TDA) methodology
- TDA analysis of monopoles in Compact U(1) Lattice Gauge Theory
- TDA analysis of Abelian monopoles in SU(3) Lattice Yang-Mills
- Summary and outlook

Intermediate regime of QCD above T_c?

[E.g., L. Ya. Glozman, Prog. Part. Nucl. Phys. 131 (2023) 104049, arXiv:2209.10235]



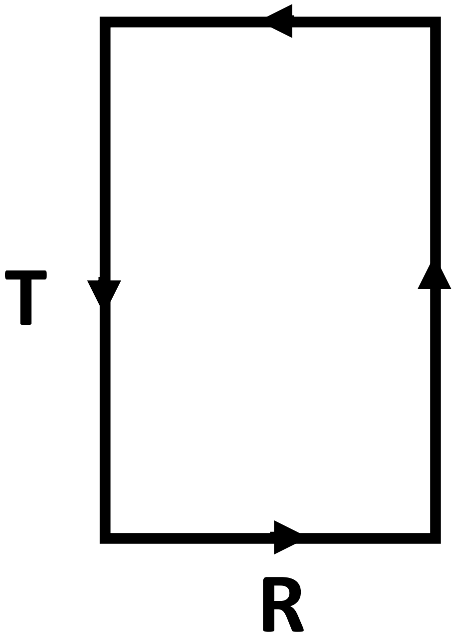
- A stringy fluid regime has been conjectured in which chiral symmetry is restored but the system still confines
- Both soft boundaries of this regime can be phase transition points at large N
- Is there a good order parameter to identify the boundaries of this regime?

Confining Potential

Confining potential V(R) = σ R between static quark and antiquark in YM theories

Confining potential derived from the area law of the Wilson loop

$$\langle W \rangle \propto e^{-\sigma RT}$$



Area law consequence of dynamics of topologically non-trivial configurations?

From the continuum to the lattice (and back)

1. Start from the Euclidean Path Integral formulation of the theory

$$\langle O \rangle = \frac{\int (\mathcal{D}\phi)\, O[\phi] e^{-S}}{\int (\mathcal{D}\phi)\, e^{-S}}$$

2. Approximate the integral on a grid of spacing a and of size $V = N_t \times N_s^3$

(Zero temperature: $N_t \simeq N_s$; finite temperature $N_t \ll N_s$)

3. Compute the integral with Monte Carlo methods

4. Extrapolate to $V \to \infty$ ($N_s \to \infty$ at finite temperature) and $a \to 0$

Fields on the lattice

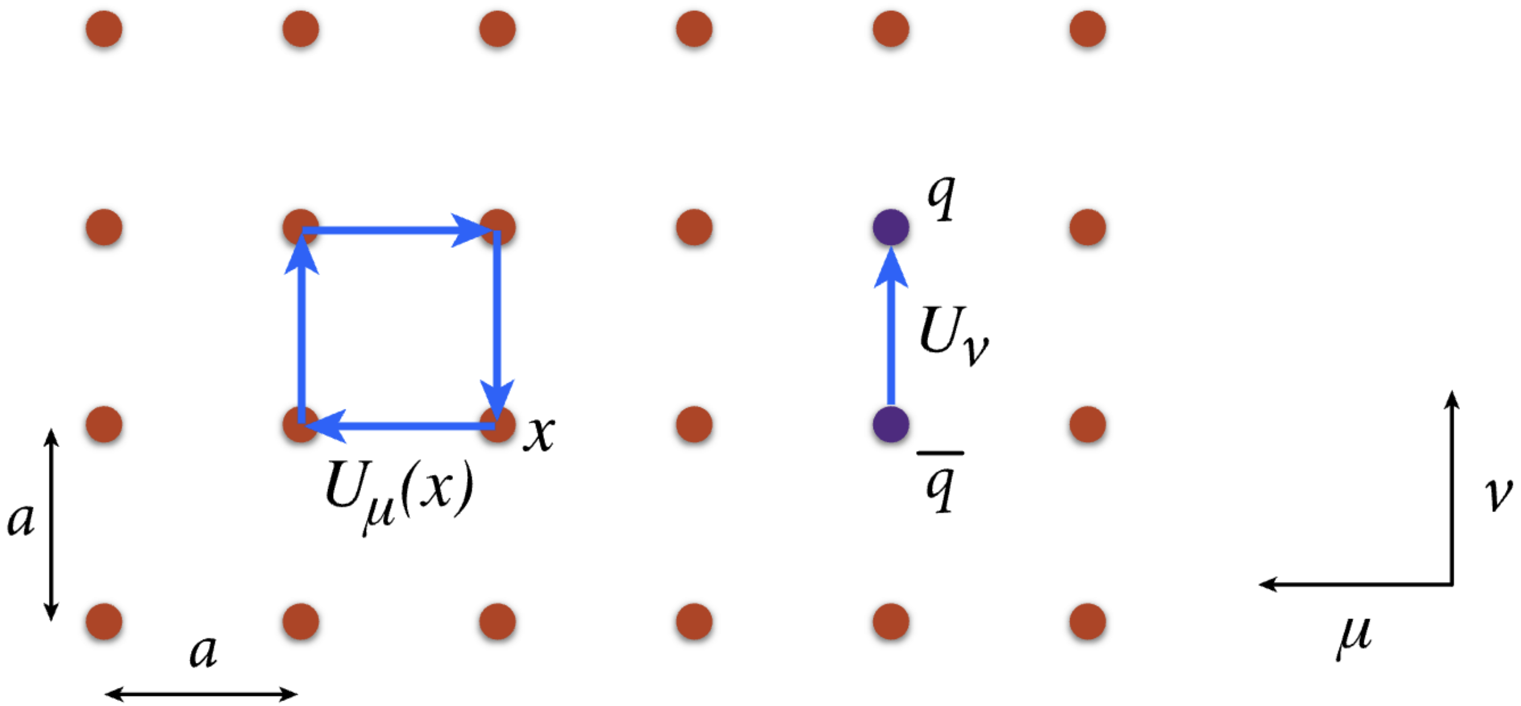
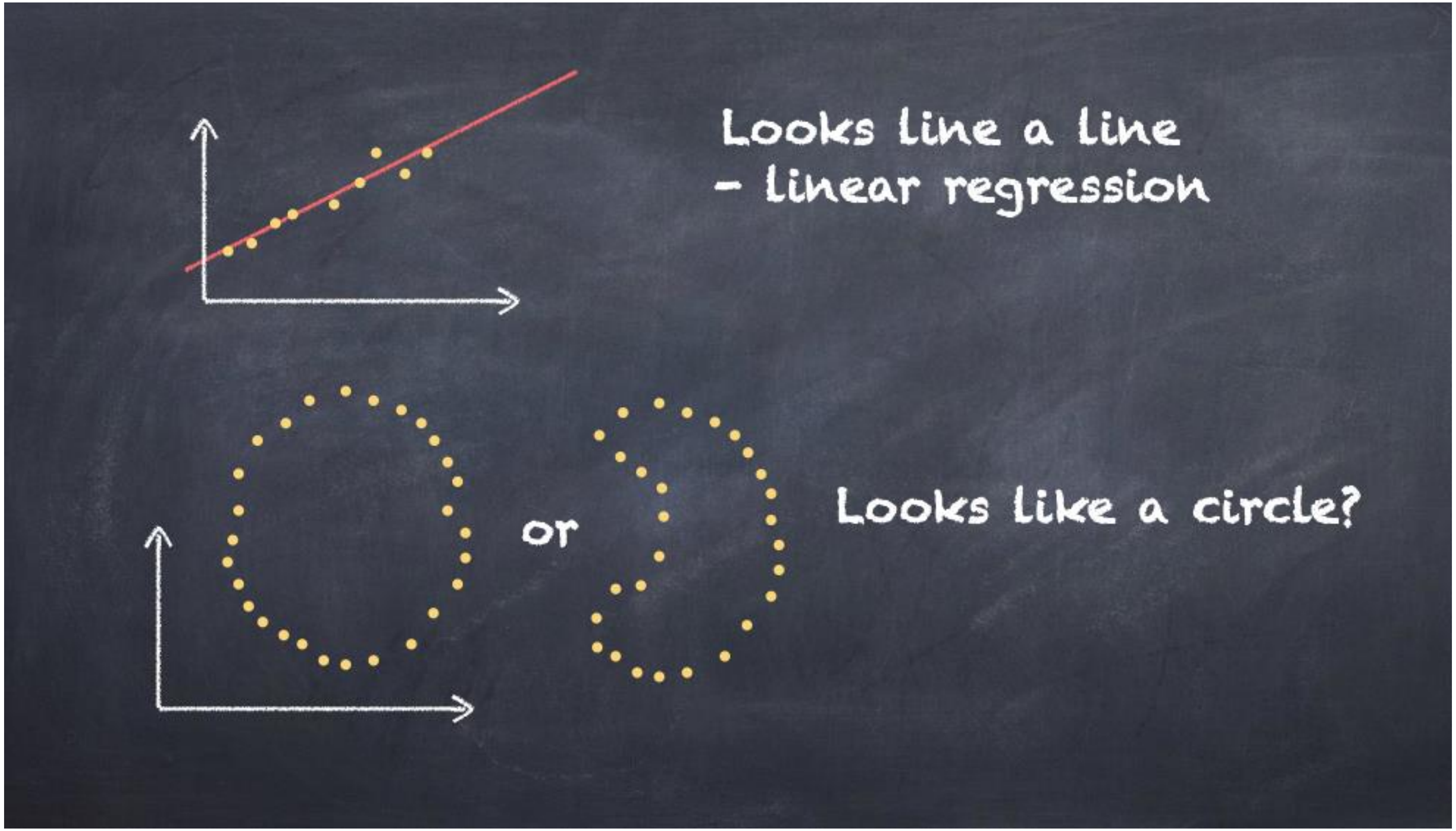


Figure from Particle Data Group

Topological Data Analysis for Lattice Field Theory

- Topological excitations determine non-perturbative features of non-Abelian quantum field theories
- The lattice is a crucial tool to understand non-perturbative features of non-Abelian quantum field theories
- Can we characterise topological properties of Yang-Mills theories after discretization using a general methodology that is rigorous also on a discrete spacetime?

Topology of a discrete set of points

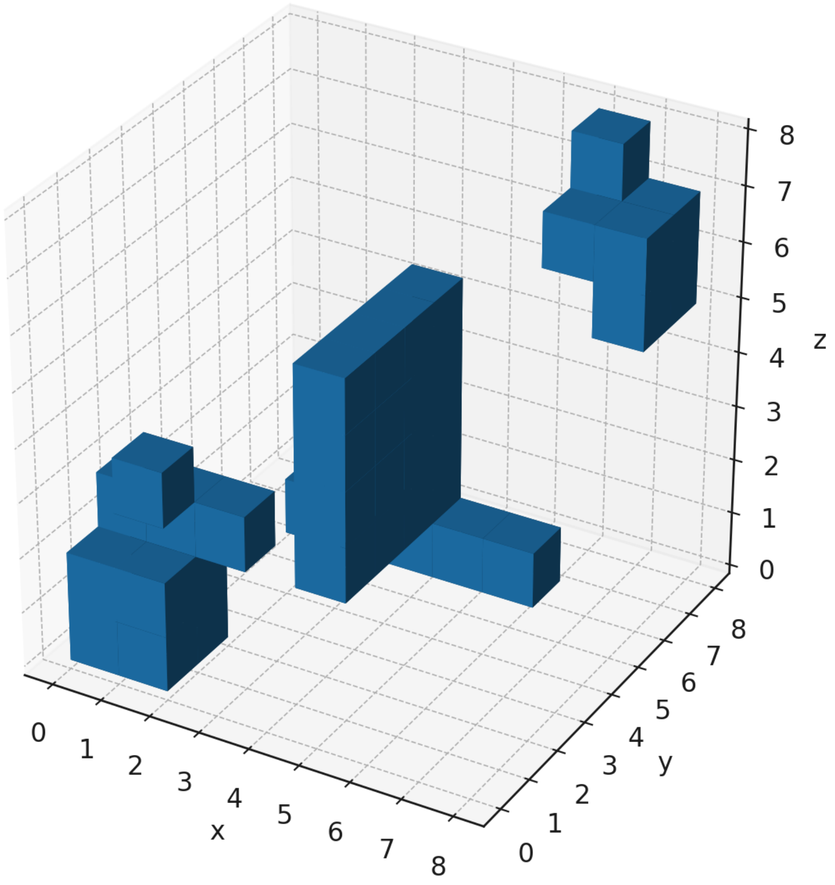


Homology

- Input: a cubic complex X determined from configurations
  - ➢ List of vertices
  - ➢ List of which of those elements span k-dimensional cubes

- Output:

H_0	connected components
H_1	holes (loops)
H_2	voids
H_3	higher voids



Programme

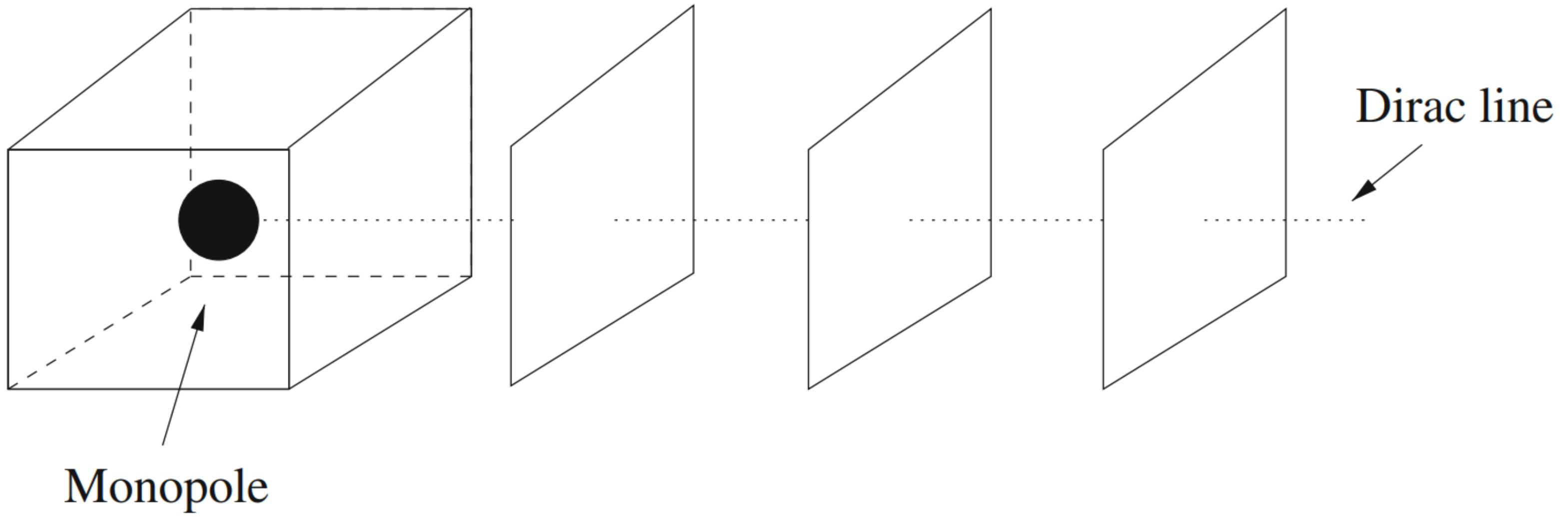
- Build a cubical complex using topological objects present in a given configuration of a discrete quantum field theory
- Determine the homology of that cubical complex
- Build observables based on ensemble averages of homological content of importance-sampled configurations
- Investigate the behaviour of those observables as a function of the parameters of the relevant theories

Monopoles in Lattice Compact U(1)

- Action $S = \beta \sum_{i,\mu<\nu} (1 - \cos\theta_{\mu\nu}(i))$, $\theta_{\mu\nu}(i) = \theta_\mu(i) + \theta_\nu(i+\hat{\mu}) - \theta_\mu(i+\hat{\nu}) - \theta_\nu(i))$
- Invariance under gauge transformations $\theta_\mu(i) \to \theta_\mu(i) + \lambda(i+\hat{\mu}) - \lambda(i)$
- Cosine only sensitive to angles in [-π, π], but plaquette varies in [-4π, 4π]
- Potential excess flux $2\pi n$, with n = -2,-1, 1, 0 interpreted as a Dirac string
- Monopoles sitting at cubes where an imbalanced number of fluxes is present
- Confinement phase at strong coupling due to condensation of monopoles

[T. DeGrand, D. Toussaint, Phys. Rev. D22, 2478 (1980)]

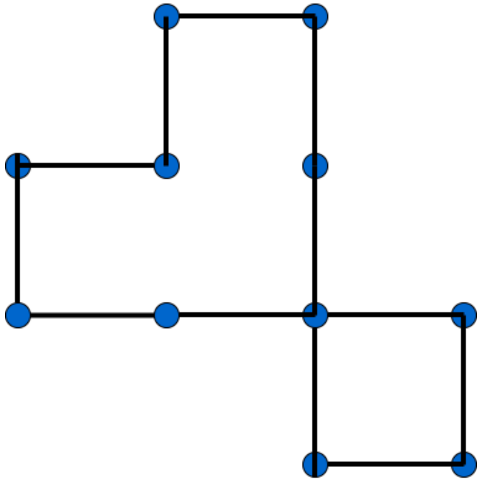
Monopole currents

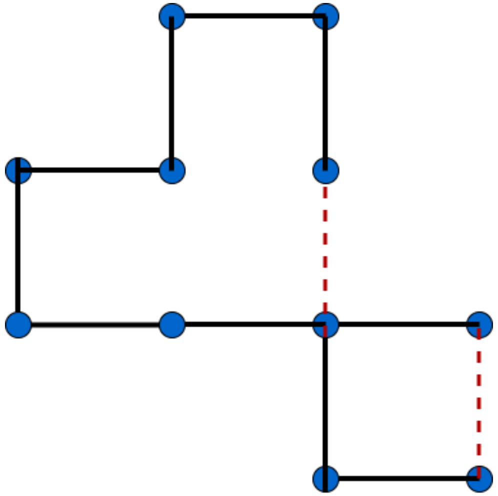


In D=4 monopoles live on dual links and form closed loops that can wrap around a toroidal lattice

In the confined phase, a small number of multiply looping connected components are present, while in the deconfined phase many connected components forming small loops arise

Counting loops in graphs





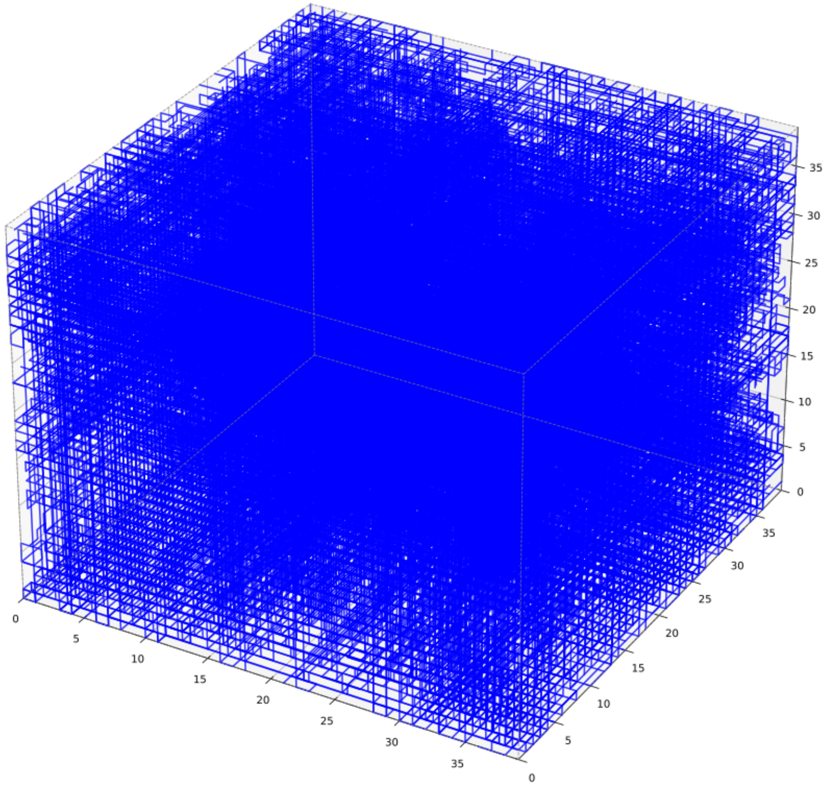
Relevant quantities: $H_0=b_0$ and $H_1=b_1$

We extract a graph from a (connected) monopole current loop by removing orientation and then we build a spanning tree of that graph by removing edges so that all cycles disappear

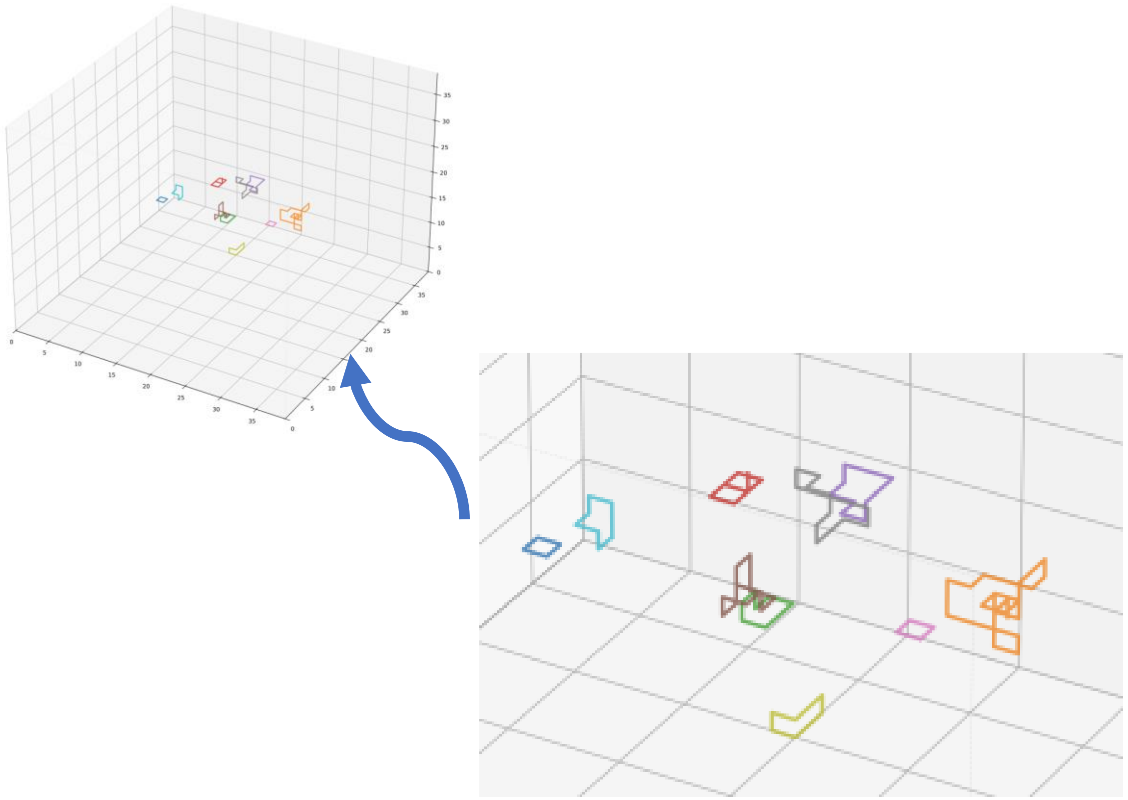
The number of removed edges is the number of loops

Expected properties of currents

At strong coupling very few complex loops



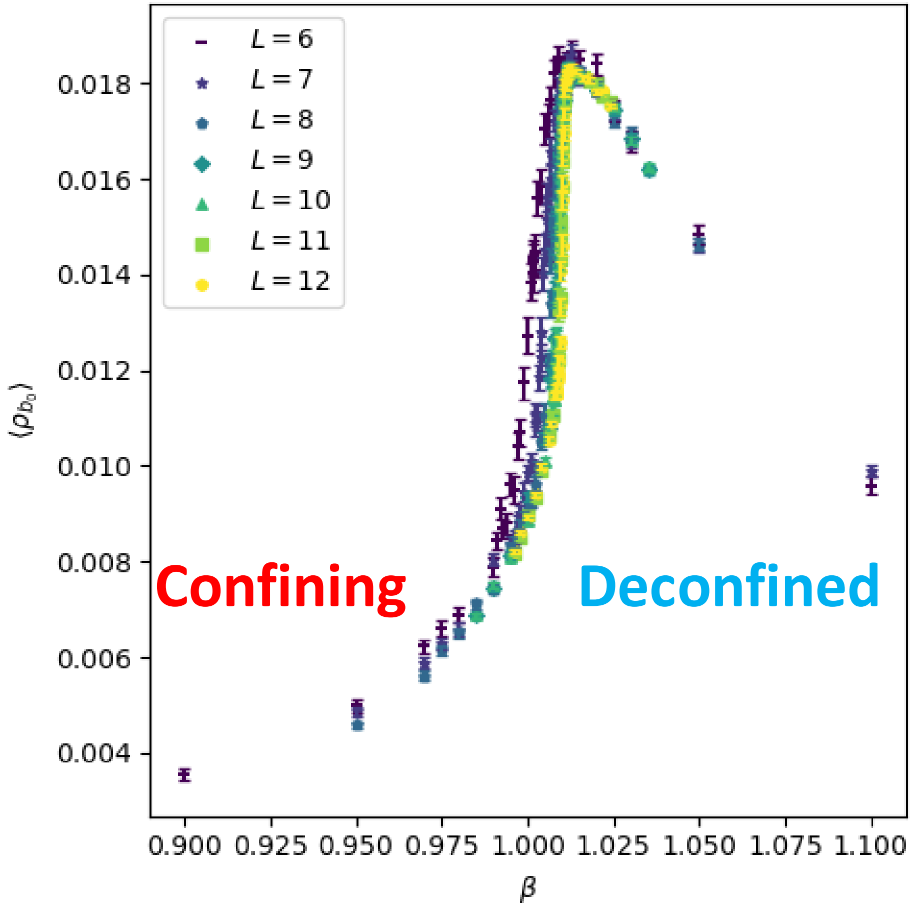
At weak coupling few simple loops

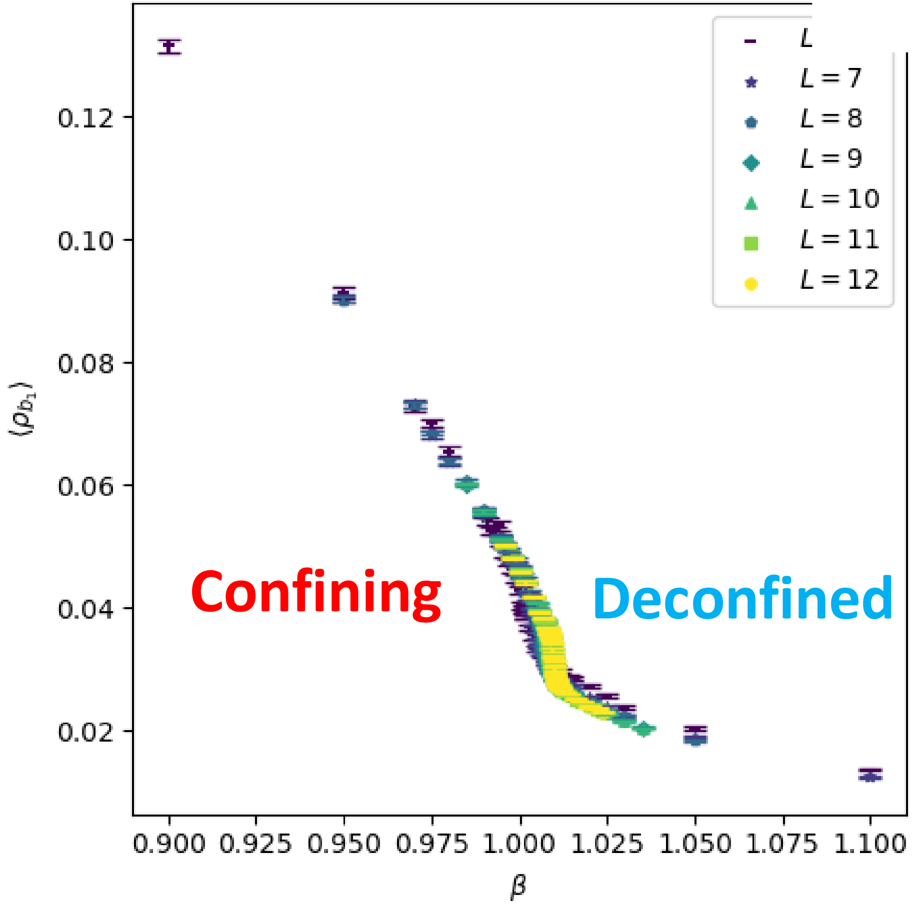


Measurements

- For each configuration, extract the current loop graph on the dual lattice and compute its Betti numbers b_0 and b_1
- Compute the averages $\rho_0 = \frac{\langle b_0 \rangle}{V}$ and $\rho_1 = \frac{\langle b_1 \rangle}{V}$
- Compute the associated susceptibilities χ_0 and χ_1
- Reweight those observables and locate their peaks
- Define $\beta_c(N_s)$ as the position of those peaks
- Extrapolate using the ansatz $\beta_c(N_s) = \beta_c + \frac{a}{N_s^3}$

Zeroth and first Betti numbers

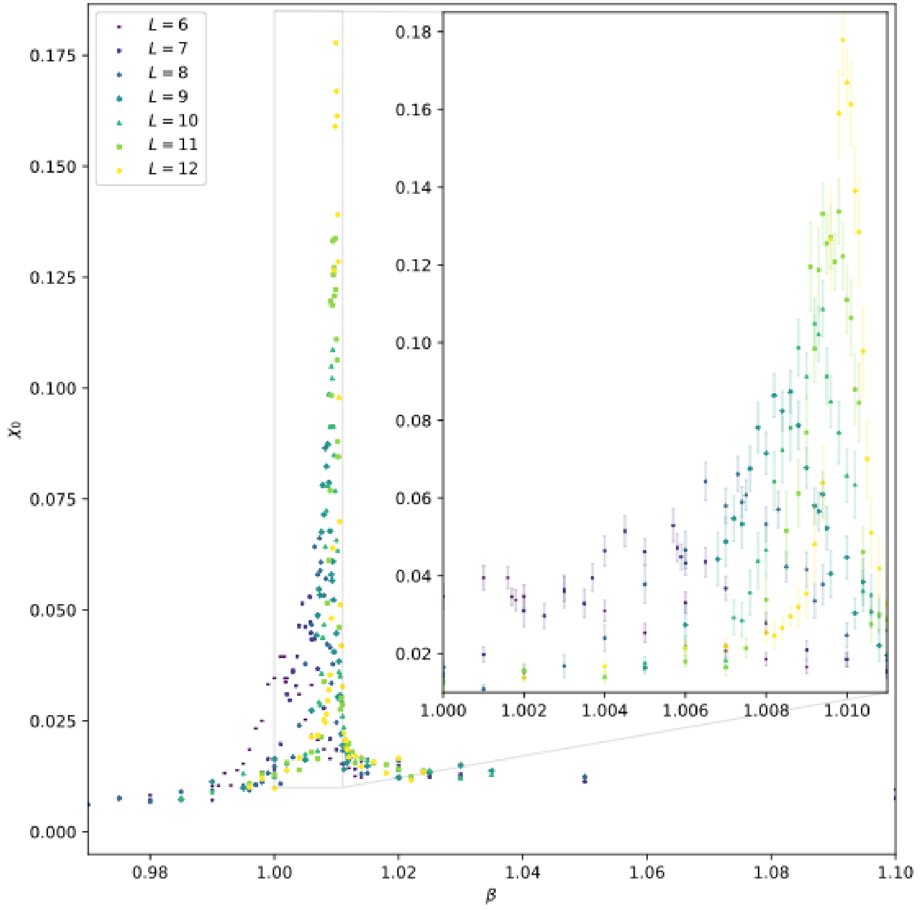


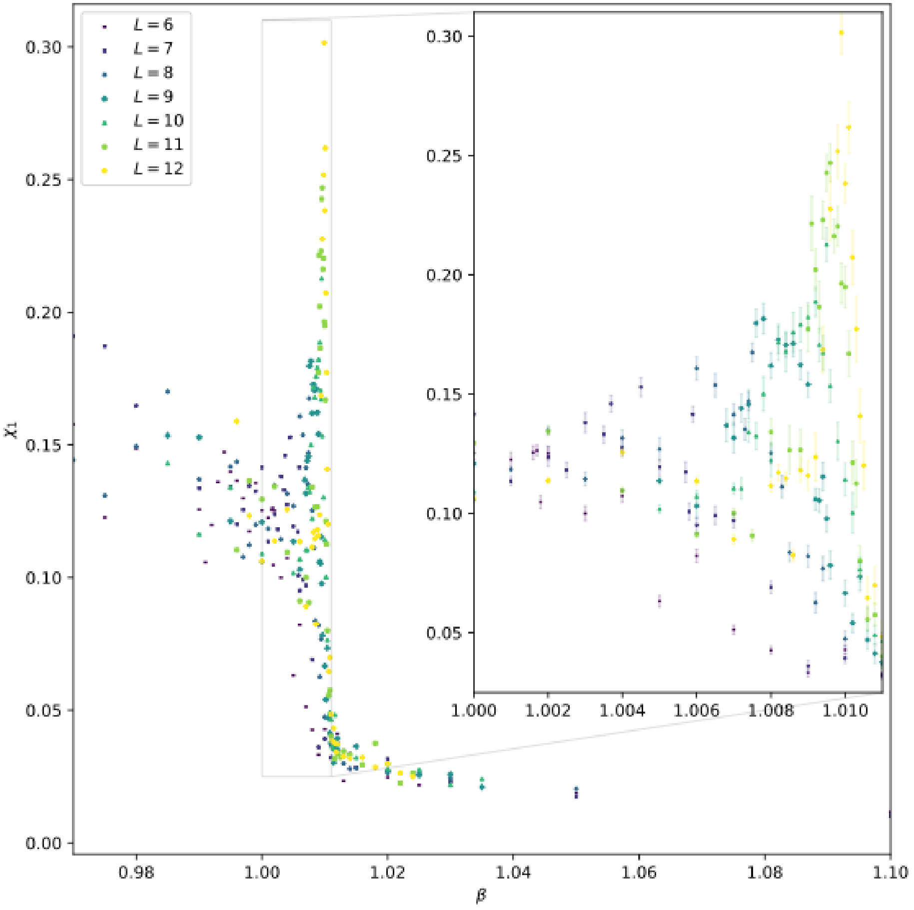


Singularity developing at the the phase transition as the volume increases

[X. Crean, J. Giansiracusa and B. Lucini, SciPost Phys. 17 (2024) 4, 100, arXiv:2403.07739]

Susceptibilities of Betti numbers





The peak is becoming sharper as the lattice size increases

Finite size scaling analysis

Critical coupling β_c

E	ρ_{b_0}	ρ_{b_1}
1.01071(3)	1.01076(6)	1.01076(6)

Literature value: $\beta_c = 1.011127(3)$

[E.g., B. Lucini *et al.* , Eur. Phys. J. C 76 (2016) 6, 306 [arXiv:1509.08391]]

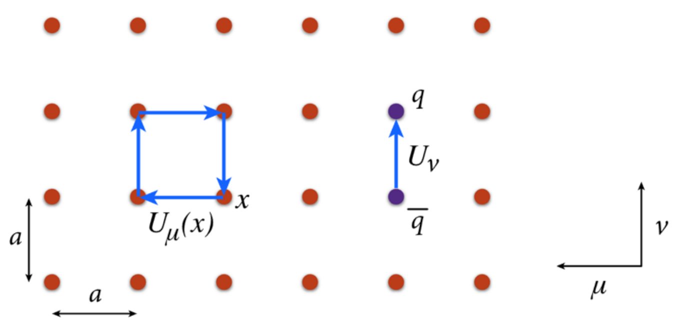
Lattice action for SU(N) Yang-Mills

- Plaquette variable

$$U_P(i) = U_{\mu\nu}(i) = U_\mu(i)U_\nu(i+\hat{\mu})\left(U_\mu(i+\hat{\nu})\right)^\dagger\left(U_\nu(i)\right)^\dagger \ , \qquad U_\mu(i) \in \mathrm{SU(N)}$$

- Action

$$S = \beta \sum_{x,\mu<\nu}\left(1 - \frac{1}{N}\mathcal{R}e\mathrm{Tr}U_{\mu\nu}(i)\right) \ , \qquad \beta = 2N/g^2$$



- Invariance under gauge transformations

$$U_\mu(i) \to G^\dagger(i)U_\mu(i)G(i+\hat{\mu}) \ , \qquad G(i) \in \mathrm{SU(N)}$$

Abelian monopoles in SU(3) Yang-Mills

- Classically, the existence of monopoles in gauge theories requires the presence of a self-interacting bosonic field transforming in the adjoint representation (e.g., the ‘t Hooft-Polyakov monopole in the Georgi-Glashow model)
- However, monopoles can arise if an effective dynamics develops in which an adjoint operator plays the role of a Higgs field
- Abelian monopoles are located at points in which two eigenvalues of this adjoint operator are degenerate

Maximal Abelian Gauge (MAG)

- Gauge fixing corresponding to the diagonalization of the adjoint operator

$$\tilde{X}(n) = \sum_{\mu} \left[U_{\mu}(n)\tilde{\lambda}U_{\mu}^{\dagger}(n) + U_{\mu}^{\dagger}(n-\hat{\mu})\tilde{\lambda}U_{\mu}(n-\hat{\mu}) \right] \quad , \qquad \tilde{\lambda} = \mathrm{diag}(1,0,-1)$$

- Equivalent gauge fixing condition

$$\{\tilde{g}\} = \arg \max_{\{g\}} \tilde{F}_{\mathrm{MAG}}(U,g)$$

$$\tilde{F}_{\mathrm{MAG}}(U,g) = \sum_{\mu,n} \mathrm{tr}\left(g(n)U_{\mu}(n)g^{\dagger}(n+\hat{\mu})\tilde{\lambda}g(n+\hat{\mu})U_{\mu}^{\dagger}(n)g^{\dagger}(n)\,\tilde{\lambda} \right)$$

[C. Bonati, M. D'Elia, Nucl. Phys. B877, 233 (2013) [arXiv:1308.0302]]

Abelian fields

- Gauge fixed configuration $\tilde{U}_\mu(n)$
- Diagonal elements $\tilde{U}_{ii} = r_i e^{i\varphi_i}$, $\sum_i \varphi_i = 2\pi n + \delta\varphi$
- Define $\phi_i = \varphi_i - \delta\varphi \frac{\left|\tilde{U}_{ii}\right|^{-1}}{\sum_j \left|\tilde{U}_{jj}\right|^{-1}}$
- Set $\theta_1 = \phi_1$ and $\theta_2 = -\phi_3$, and use the DeGrand and Toussaint prescription for identifying the monopoles associated to each Abelian field

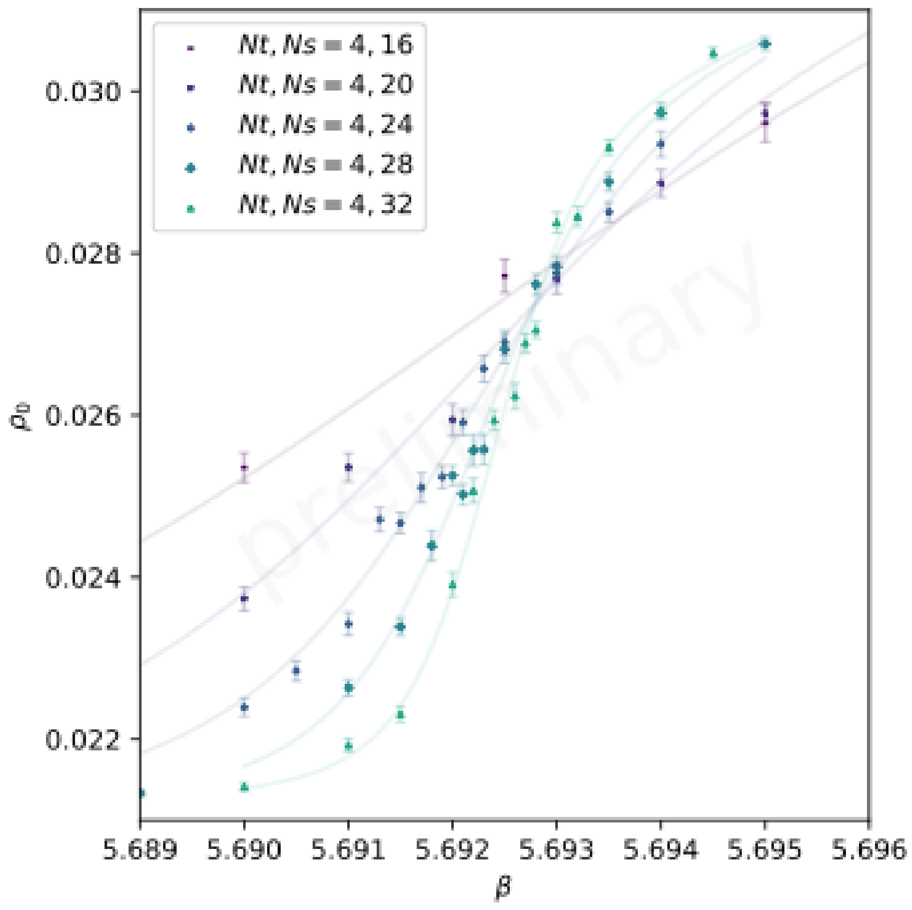
[C. Bonati, M. D'Elia, Nucl. Phys. B877, 233 (2013) [arXiv:1308.0302]]

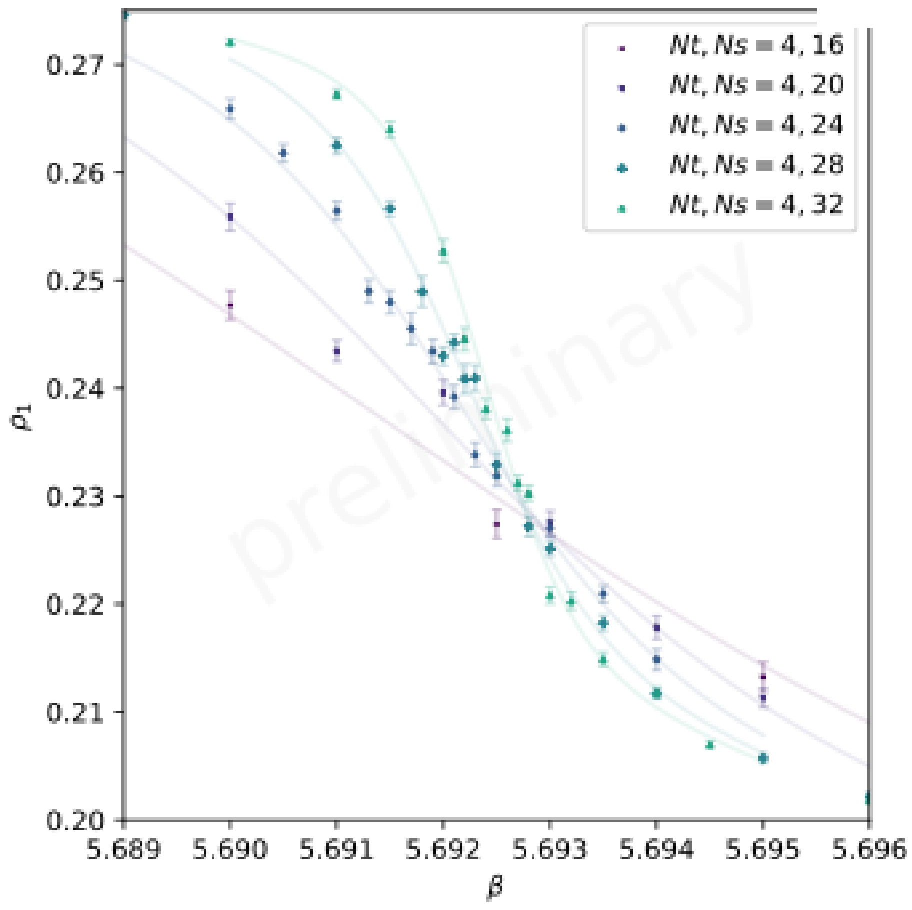
Numerical setup

- Asymmetric lattices of size $N_t \times N_s^3 = N_t \times V$, $N_t = 4{,}6{,}8$ and at various sizes V respecting the condition $N_t \ll N_s$
- Choose a set of β near the expected critical point $\beta_c(N_t)$
- Generate 400-600 thermalized configurations separated by 2000 composite sweeps (1 composite sweep = 1 heat bath + 4 over relaxation sweeps)
- Perform projection to the MAG and measure ρ_0, ρ_1, χ_0 and χ_1 as in the U(1) case

(*) E.g., B. Lucini, M. Teper and U. Wenger, JHEP 01 (2004) 061, arXiv: hep-lat/0307017

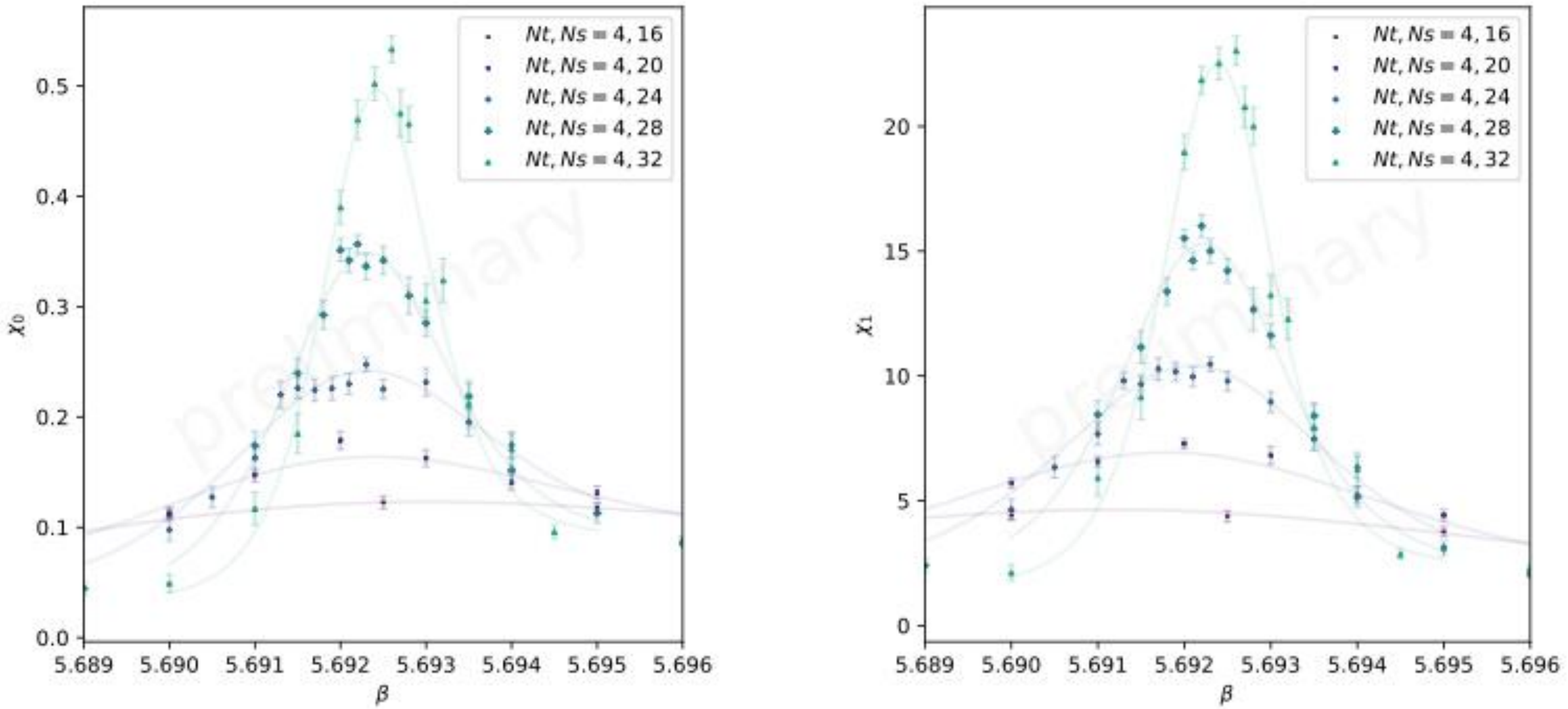
Betti numbers – SU(3) Yang-Mills





Singularity developing at the the phase transition as the volume increases

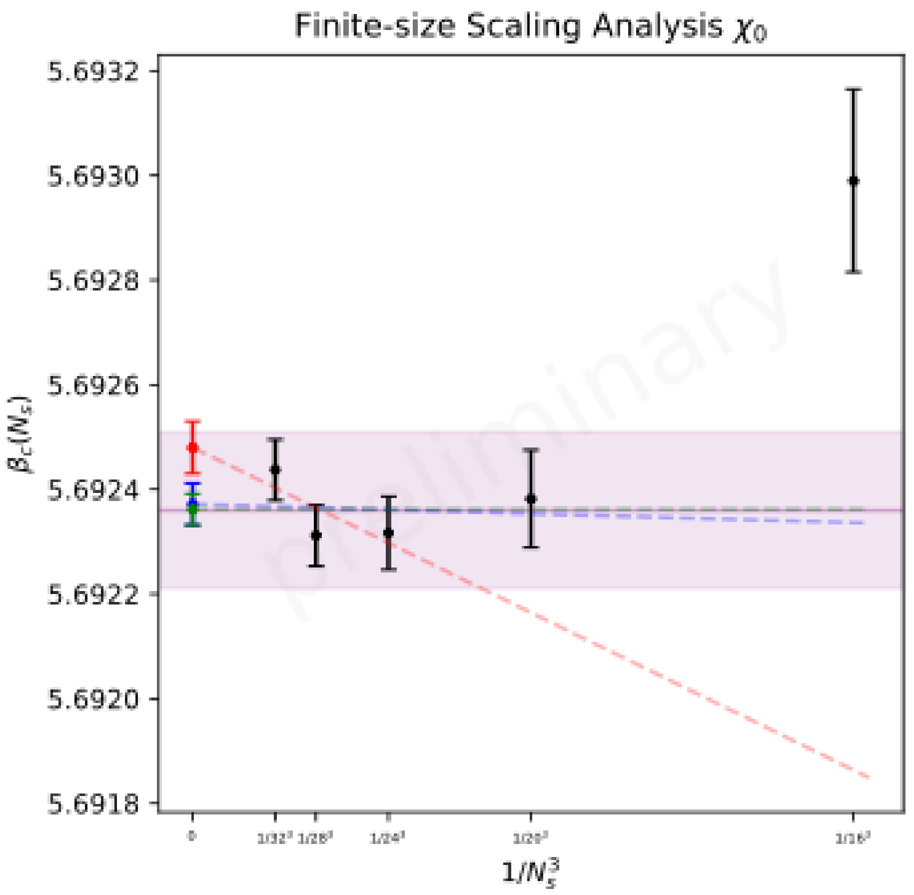
Susceptibilities of Betti numbers

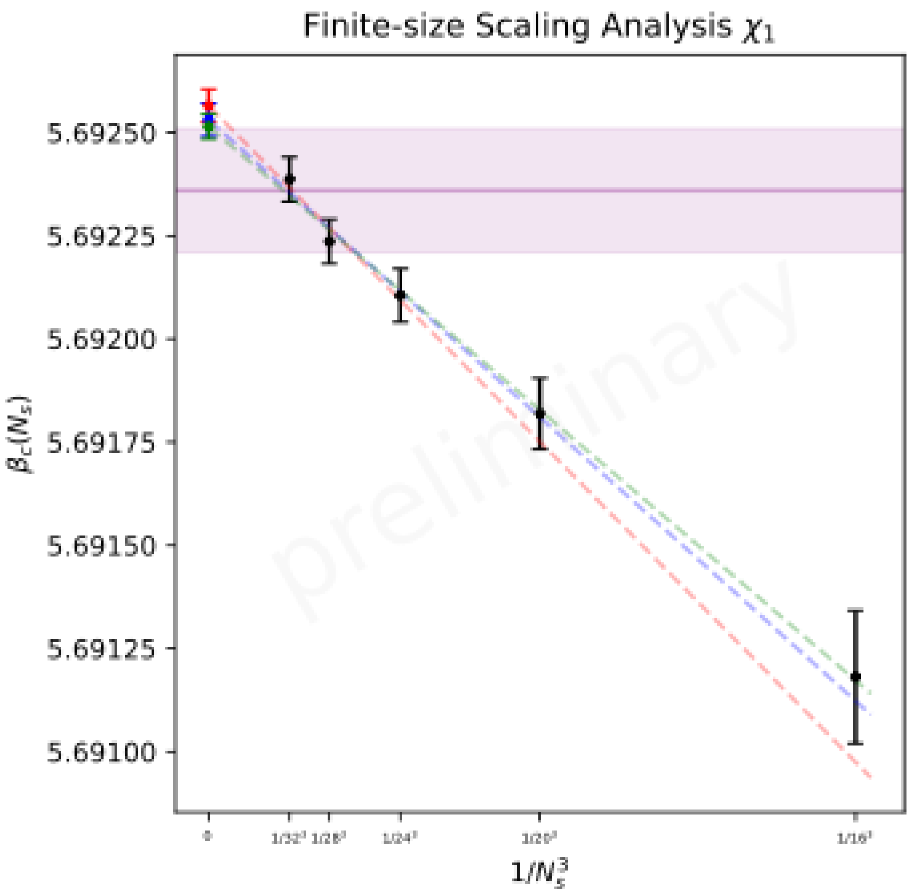


The peak is becoming sharper as the lattice size increases

[Horizontal band from B. Lucini, M. Teper and U. Wenger, JHEP 01 (2004) 061, arXiv: hep-lat/0307017]

Scaling of position of peaks





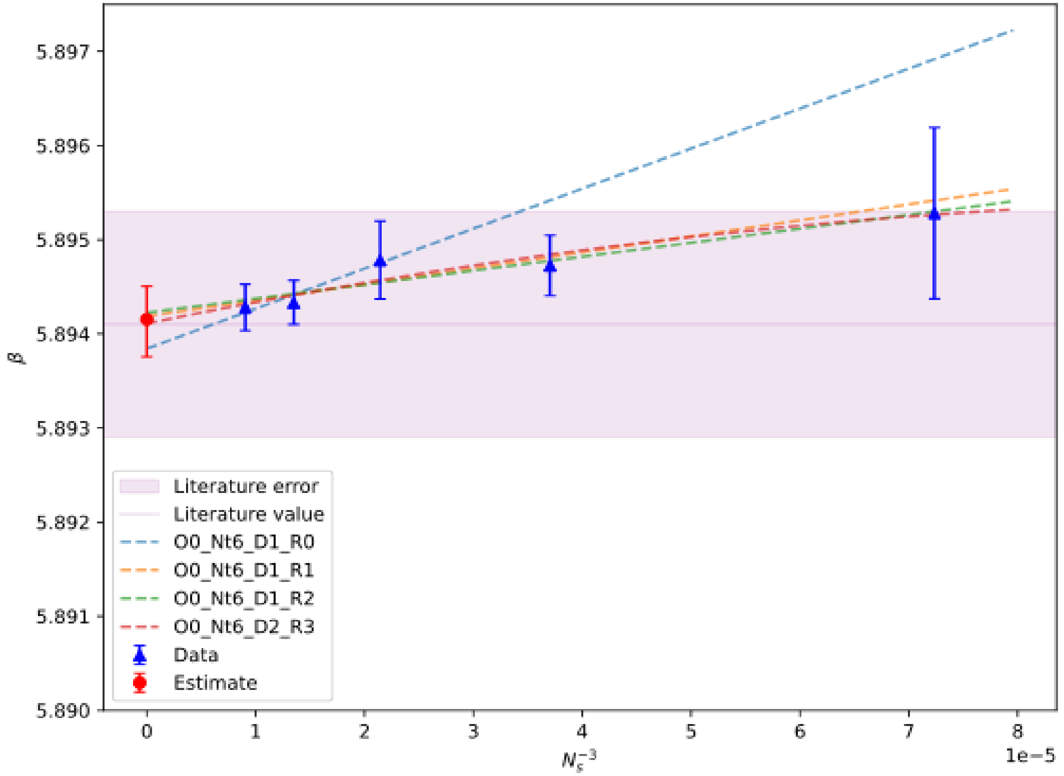
Results compatible with standard calculations, hints of better precision

Determining β_c from fits

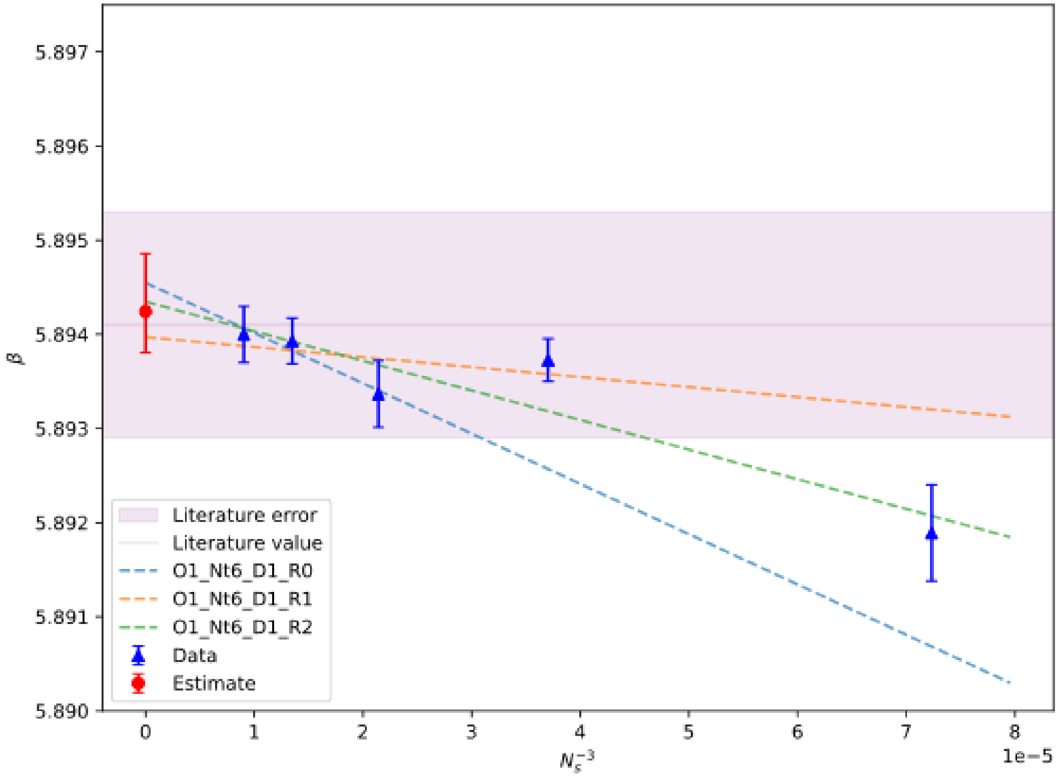
- Methodology: extract central value and error by weighting all (reasonable) fits in a polynomial in $\frac{1}{V}$
- For each fit, take the central value μ_i and the error σ_i as the parameters of a normal distribution $\mathcal{N}(\beta; \mu_i, \sigma_i)$
- Define the weights $w_i = \exp(-\frac{1}{2}(\chi^2 + 2\ \mathrm{npar} - \mathrm{ndata}))$
- Construct $P(\beta) = \Sigma_i w_i \frac{\mathcal{N}(\beta;\mu_i,\sigma_i)}{N}$
- Extract the best value, the lower bound and the upper bound respectively from the 50%, 16% and 84% confidence level

[S. Borsanyi et al., Nature 593, 51 (2021), arXiv:2002.12347]

Results for β_c at $N_t = 6$



χ_0



χ_1

Summary of results

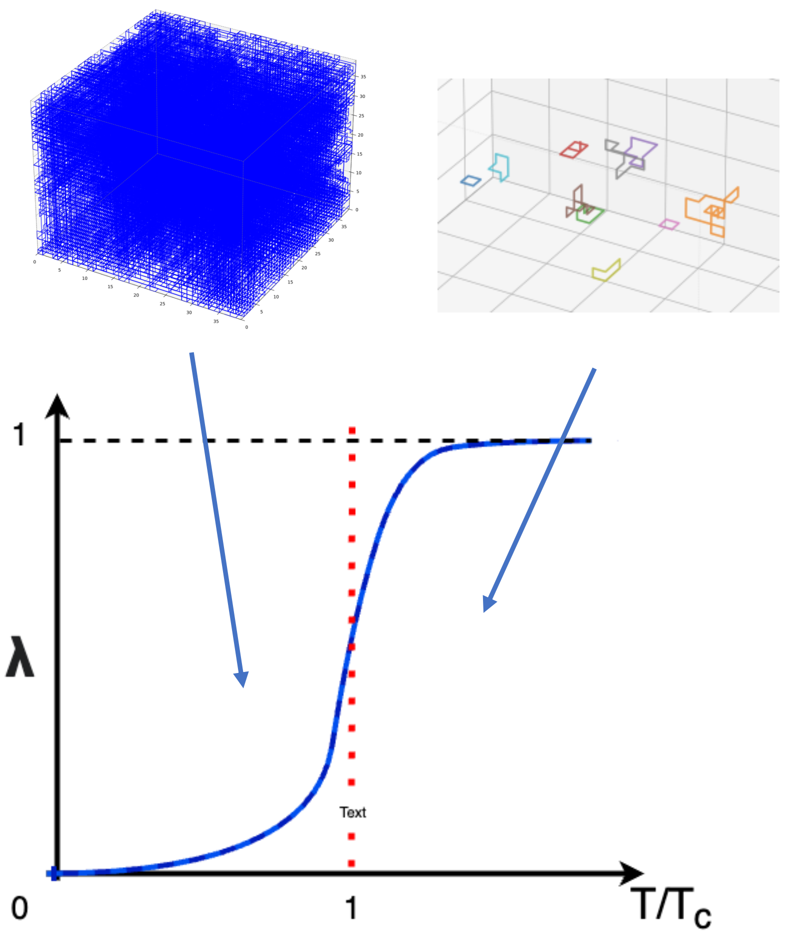
Observable	$N_t = 4$	$N_t = 6$	$N_t = 8$
ρ_0	$5.69247^{+0.00005}_{-0.00006}$	$5.8942^{+0.0004}_{-0.0004}$	$6.0645^{+0.0014}_{-0.0017}$
ρ_1	$5.69257^{+0.00010}_{-0.00008}$	$5.8942^{+0.0006}_{-0.0004}$	$6.0625^{+0.0006}_{-0.0009}$
Literature value	$5.69236^{+0.00015}_{-0.00015}$	$5.8941^{+0.0012}_{-0.0012}$	$6.0625^{+0.0018}_{-0.0018}$

Complexity and simplicity in topology

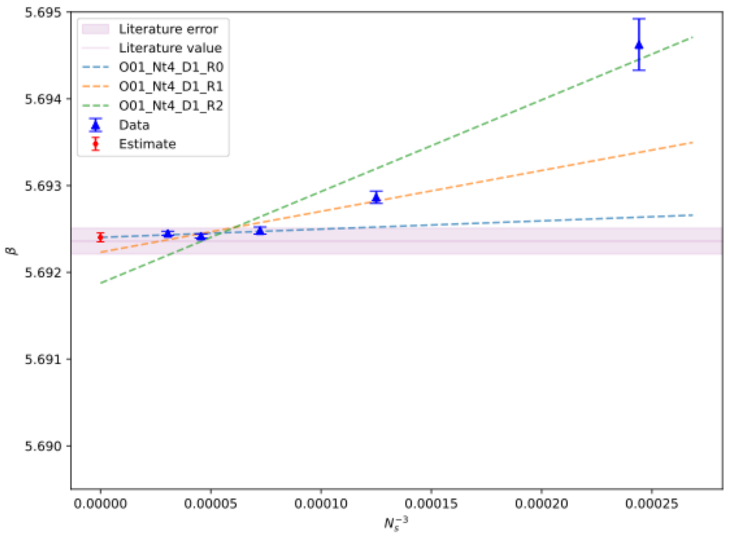
- Complexity is the number of loops per connected component: the higher this number, the more complex the graph is
- We define the simplicity as the inverse of the complexity

$$\lambda = \langle b_0 / b_1 \rangle$$

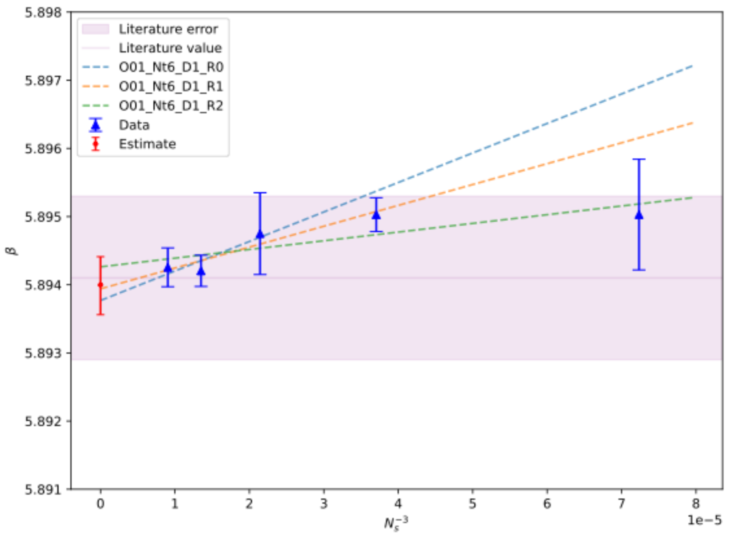
- Simplicity can be used as a phase indicator: at low temperature, simplicity approaches zero in the high volume limit, while at high temperature it approaches one



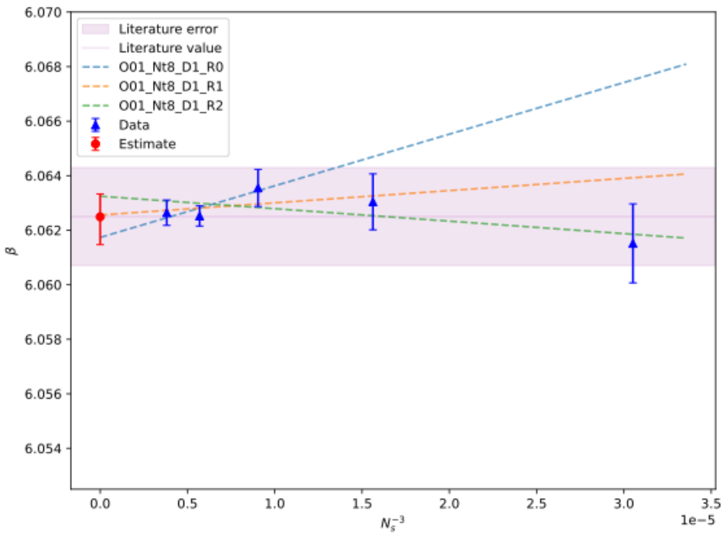
β_c from simplicity



$N_t = 4$

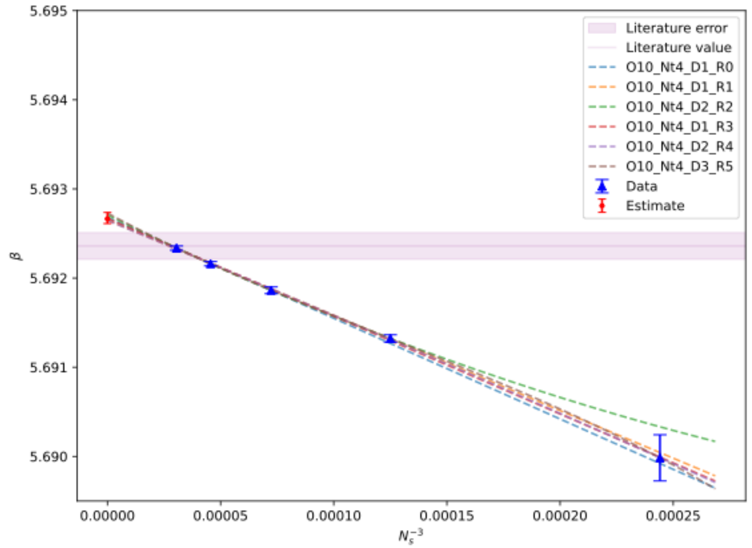


$N_t = 6$

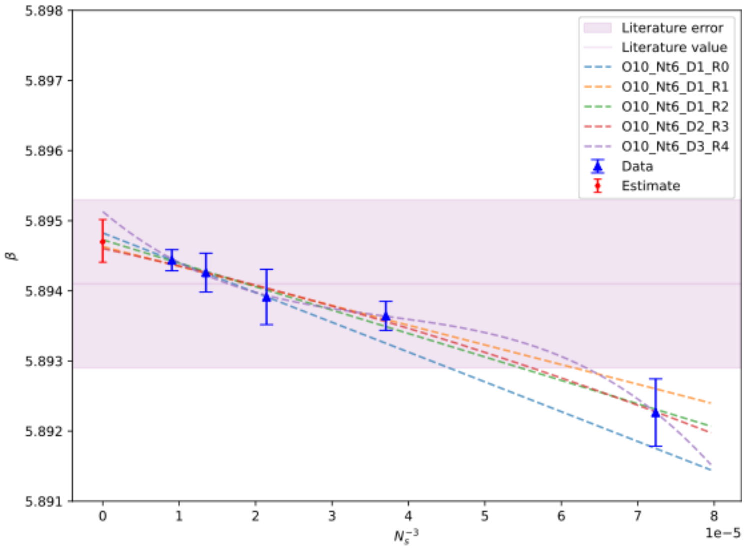


$N_t = 8$

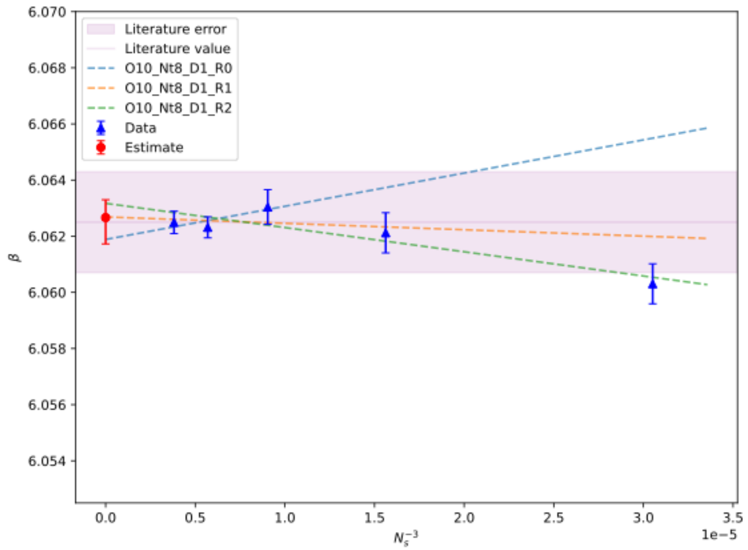
β_c from complexity



$N_t = 4$



$N_t = 6$



$N_t = 8$

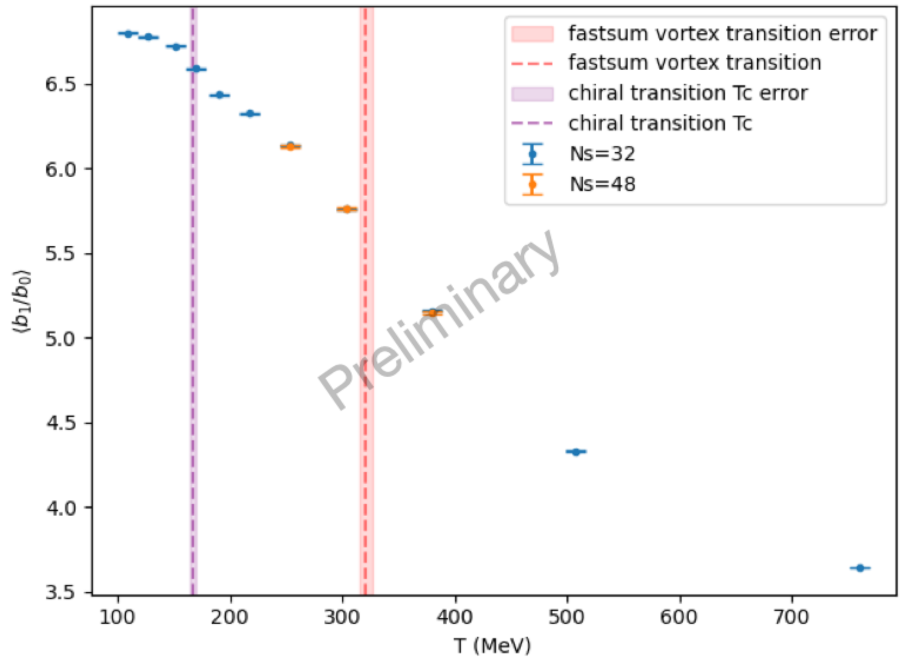
Conclusions

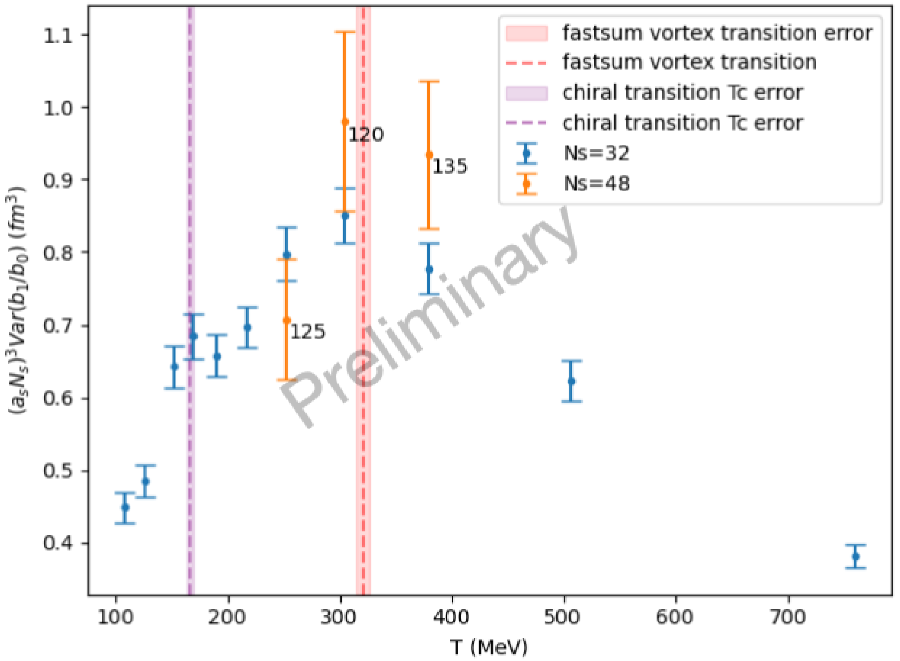
- Topological data analysis provides a robust way to understand the non-trivial topological content of lattice configurations
- Based on this information, phase indicators can be constructed that provide a precise quantitative characterization of the deconfinement phase transition in gauge theories
- Explicit observables constructed and tested for both Abelian and non-Abelian Lattice Gauge Theories
- In progress: persistent homology analysis of U(1) and SU(3)
- Also in progress: extension of the approach to full QCD

Towards QCD

(G. Aarts, C. Allton, R. Bignell, X. Crean, G. Giansiracusa, B.L., in progress)

First preliminary calculation on 2 + 1 flavours of O(a)-improved Wilson fermions on anisotropic lattices (FASTSUM Gen 2L ensembles, G. Aarts et al., arXiv:2209.14681)





Visible peak in the susceptibility, possibly small finite size effects

(For the FASTSUM vortex transition, see J. Mickley, C. Allton, R. Bignell, D. Leinweber , arXiv: 2504.08131)

Other selected works

- T. Hirakida, K. Kashiwa, J. Sugano, J. Takahashi, H. Kouno, and M. Yahiro, Int. J. Mod. Phys. A 35 (2020) 10, 2050049 [arXiv:1810.07635]
- A. Cole, G.J. Loges and G. Shiu, Phys. Rev. B 104 (2021) 10, 104426 [arXiv:2009.14231]
- N. Sale, G. Giansiracusa and B. Lucini, Phys. Rev. E 105 (2022) 2, 024121 [arXiv:2109.10960]
- N. Sale, G. Giansiracusa and B. Lucini, Phys. Rev. D 107 (2023) 3, 034501 [arXiv:2207.13392]
- D. Spitz, J. Urban, J.M. Pawlowski, Phys. Rev. D 107 (2023) 3, 034506 [arXiv:2208.03955]
- D. Spitz, J. Urban, J.M. Pawlowski, Phys. Rev. D 111 (2025) 11, 114519 [arXiv:2412.09112]