

Non-perturbative physics of $(1+1)$ d gauge theories using symmetric matrix product states

Benjamin T. Søgaaard
Princeton University

Based on work with Ross Dempsey, Anna-Maria Glück, Igor Klebanov and Silviu Pufu
[2508.16363], [2409.19164], [2311.09334]

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Introduction

- ▶ $(1 + 1)$ -dimensional gauge theory as a platform:
 - ▶ Toy model for formal theoretical developments
 - ▶ Stepping stone for Hamiltonian lattice simulation of more realistic models
 - ▶ Physical realization in experiment or on digital quantum computer
- ▶ Many works on LGT using MPS: How to deal with **gauge field**?
 - ▶ Integrate out: non-local charge-charge interaction $\Rightarrow$ finite lattices
[Byrnes, Sriganesh, Bursill, Hamer 2002; Bañuls, Cichy, Jansen, Cirac 2013; ...]
 - ▶ Same footing as matter: extra constraints $\Rightarrow$ local, but mostly $U(1)$
[Buyens, Haegeman, Van Acoleyen, Verschelde, Verstraete 2013; Fujikura, Kikukawa, Okuda, Pedersen 2024; ...]
- ▶ **This talk:** Matrix product operators construction for generic LGT
 - ▶ Allows for **infinite lattices** and **non-abelian** gauge groups
 - ▶ Application: massive Schwinger model & adjoint QCD_2

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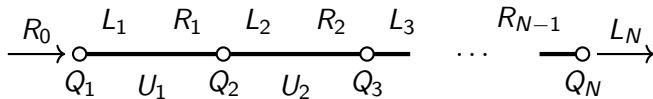
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Hamiltonian lattice gauge theories

- Gauge group G : Parallel propagators $U_n \in G$, electric fields $L_n, R_n \in \mathfrak{g}$ on links and matter fields/charges ψ_n/Q_n on sites



- Dynamics + constraint:

Hamiltonian : $H = \frac{ag^2}{2} \sum_{n=1}^{N-1} L_n^2 + H_{\text{matter}},$ Gauss law: $L_n - R_{n-1} = Q_n,$

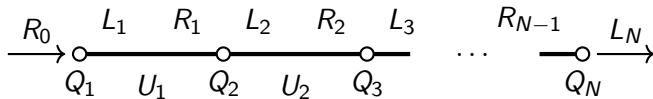
BCs for electric field set by irreps $\mathbf{r}_0/\mathbf{r}_N$ acted on by R_0/L_N

- Hilbert spaces:

$$\mathcal{H}_{\text{site},n} = \mathbf{R}_n, \quad \mathcal{H}_{\text{gauge},n} = L^2(G) = \bigoplus_{\mathbf{r} \in \text{irreps}} \mathbf{r}_L \otimes \overline{\mathbf{r}}_R \quad (= \text{span}\{|q\rangle\} \quad \text{for } \text{U}(1)).$$

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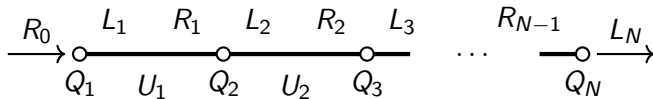
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Gauge fixing & the Gauss law

- ▶ No dynamical gluon in $(1+1)d \Rightarrow$ resolve gauge redundancy
- ▶ Use gauge invariance to set $U_n \mapsto \mathbb{1}$ and solve Gauss law for electric field:

$$L_n = R_n = R_0 + \sum_{k=1}^n Q_k, \quad L_N - R_0 = \sum_{n=1}^N Q_n.$$

Breaks **translational invariance**, the Hilbert space $\mathcal{H}_{\text{gauge}}$ has “disappeared”

- ▶ Solve quantum chain w/ global symm. $G = R_0 - L_N + \sum_{n=1}^N Q_n$

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$\Rightarrow$ Solve using MPS + global constraint $\Rightarrow$ restores locality

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Matrix product states

- ▶ Gapped ground states in $(1 + 1)d$ are well-approximated by matrix product states
[Verstraete, Cirac 2006]

$$|A_1, \dots, A_N\rangle = \sum_{\{i_n\}} A_1^{i_1} A_2^{i_2} \dots A_N^{i_N} |i_1\rangle |i_2\rangle \dots |i_N\rangle = \begin{array}{c} \cdots \text{---} A_{n-1} \text{---} A_n \text{---} A_{n+1} \text{---} \cdots \\ | \quad \quad | \quad \quad | \\ \cdots \quad |i_{n-1}\rangle \quad |i_n\rangle \quad |i_{n+1}\rangle \quad \cdots \end{array} .$$

$A_n^{s_n}$ has dimension $\chi \times \chi \times d$, χ = bond dim, d = local Hilbert space dim
MPS has $N\chi^2 d$ parameters vs a generic state d^N

- ▶ Overlaps can be computed efficiently by horizontal contraction because we store individual MPS tensors A_n

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Symmetric matrix product states

- ▶ Global **on-site** symmetry $U = \prod_n U_n \Rightarrow$ local constraints on MPS tensors
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$$\left[(\mathcal{Q}_{n-1}^a)_{\alpha_{n-1}\alpha'_{n-1}} + (Q_n)^{i_n i'_n} - (\mathcal{Q}_n)_{\alpha_n \alpha'_n} \right] (A_n)^{i'_n}_{\alpha'_{n-1}\alpha'_n} = 0.$$

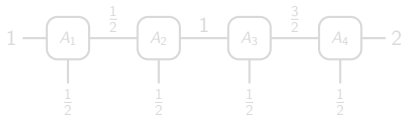
- ▶ Keep track of **virtual irreps** e.g. for U(1) symmetry

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$i_n \uparrow \quad \uparrow q(i_n)$

In general, A_n can be non-zero when $\mathbf{r}_n \subset \mathbf{R}_n \otimes \mathbf{r}_{n-1}$

- ▶ SU(2) example of symmetric (matrix) product state:



A symmetric MPS includes sum over **fusion channels** and **multiplicities**

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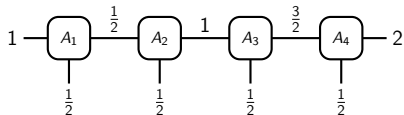
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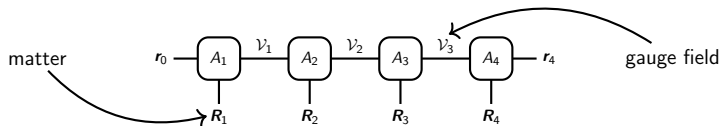
- ▶ SU(2) example of symmetric (matrix) product state:



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Symmetric MPS as gauge theory states

- Symmetric MPS tensors inherently **solve a Gauss law** and **virtual spaces include information about $\mathcal{H}_{\text{gauge}}$**



- The gauge field data is accessible because we keep track of individual MPS tensors A_n

Link operators

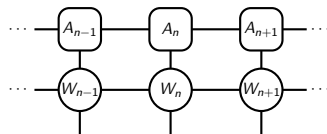
- ▶ We can extract the state of the electric field by acting on **virtual** indices
- ▶ L_n^2 acts as new type of operators acting on links [Dempsey, Glück, Pufu, BTS 2025]
 $\Rightarrow$ restores **locality** and **translational invariance**

$$L_n^2 |A_1, \dots, A_N\rangle = \begin{array}{c} \cdots - \boxed{A_{n-1}} - \boxed{A_n} - \boxed{A_{n+1}} - \cdots \\ \hline \cdots \quad \underbrace{\hspace{10em}}_{(R_0 + \sum_{k=1}^n Q_k)^2} \quad \cdots \\ \hline \end{array} = \cdots - \boxed{A_{n-1}} - \boxed{A_n} - \bigcirc Q_n^2 - \boxed{A_{n+1}} - \cdots ,$$

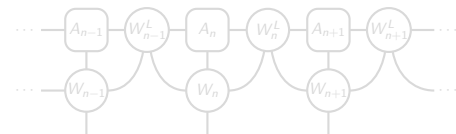
with Q_n^2 block diagonal computing the Casimir of irreps

Link-enhanced matrix product operator

- ▶ Gauge-fixed matter Hamiltonian as an MPO [Verstraete, García-Ripoll, Cirac 2004; McCulloch 2007]

$$H_{\text{matter}} |A_1, \dots, A_N\rangle =$$


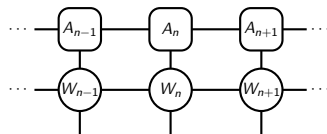
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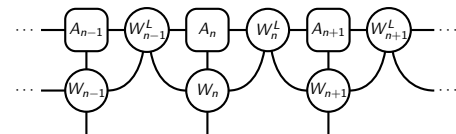
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What can LEMPOs do for you?

- ▶ **Translational invariance:** Works directly in the thermodynamic limit
- ▶ **Abstracts the group theory:** Gauss law, fusion of irreps etc. are handled by well-known symmetric MPSs $\Rightarrow$ Existing codes e.g. TeNPy, ITensors.jl, QSpace, or MPSKit.jl
- ▶ **Physical virtual bonds:** MPS ansatz for non-trivial flux tube sector
- ▶ **No fixed electric cut-off:** The virtual space can be dynamically adjust (e.g. DMRG)
- ▶ **Works with MPS algorithms:** DMRG, VUMPS, TDVP etc.
- ▶ Rest of this talk: **Infinite system size** methods for the massive Schwinger model and adjoint QCD₂

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Uniform MPS

- ▶ Translational invariance allows for methods that work directly with an infinite system using uMPSs

$$|\psi\rangle = \cdots \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \cdots .$$

- ▶ VUMPS algorithm, dynamically adapt reps on virtual links [Zauner-Stauber, Vanderstraeten, Fishman, Verstraete, Haegeman 2017]
- ▶ Quasiparticle ansatz: Bound states as local excitations of the ground state [Haegeman, Osborne, Verstraete 2013]

$$|B, k\rangle = \sum_n e^{ikn} \cdots \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{B} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \cdots$$

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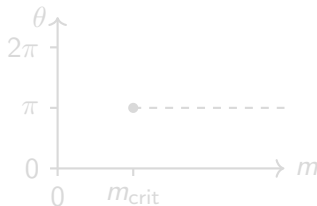
Massive Schwinger model: Continuum theory

- ▶ U(1) gauge theory coupled to massive Dirac fermion

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi,$$

by $(\mathbb{Z}_2)_\chi : \psi \rightarrow -\psi$, we take $m \geq 0$

- ▶ Solvable at $m = 0$: Free boson $M_S = \frac{g}{\sqrt{\pi}}$ [Schwinger 1962]
- ▶ The θ -angle is periodic $\theta \sim \theta + 2\pi$ and is interpreted as a background electric field
- ▶ Ising transition at $\theta = \pi$ [Coleman 1976], $m_{\text{crit}}/g \approx 0.33$, degenerate C-broken vacua for $m > m_{\text{crit}}$



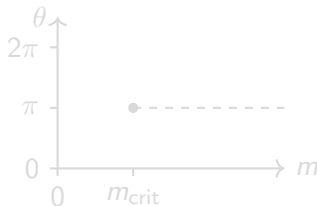
Massive Schwinger model: Continuum theory

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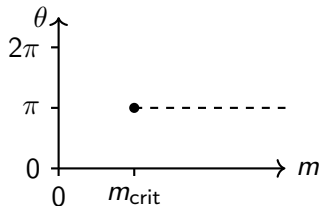
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Massive Schwinger model: Lattice Hamiltonian

- ▶ Hamiltonian lattice theory using staggered fermions $\{c_n, c_m^\dagger\} = \delta_{nm}$ [Banks, Susskind, Kogut 1976]

$$H = \sum_n \left[\frac{g^2 a}{2} \left(L_n + \frac{\theta}{2\pi} \right)^2 - \frac{i}{2a} \left(c_n^\dagger U_n c_{n+1} - c_{n+1}^\dagger U_n^\dagger c_n \right) + m_{\text{lat}} (-1)^n c_n^\dagger c_n \right] .$$

- ▶ Matter charge $Q_n = c_n^\dagger c_n - \delta_{n,\text{odd}}$

$$R_n = \begin{cases} 0 \oplus +1, & n \text{ even}, \\ -1 \oplus 0, & n \text{ odd}. \end{cases}$$

- ▶ Mass shift preserves part of the chiral anomaly at $m = 0$ [Dempsey, Klebanov, Pufu, Zan 2022]

$\Rightarrow$ improves convergence to the continuum

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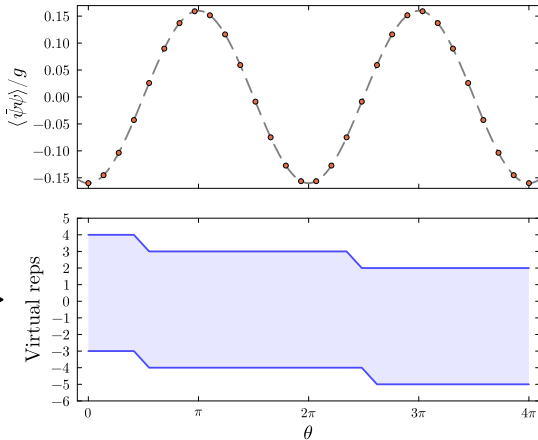
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Dynamical electric field cutoff

- For the effective electric field $L_{\text{eff}} = L + \frac{\theta}{2\pi}$ to be periodic in θ , the virtual reps have to compensate. This is done **dynamically** in VUMPS e.g. $\langle \bar{\psi}\psi \rangle$ is periodic



Range of U(1) irrep used to
encode the electric field

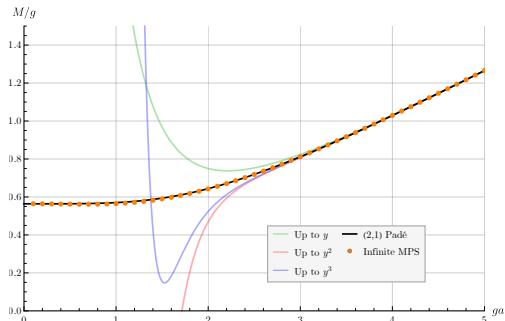
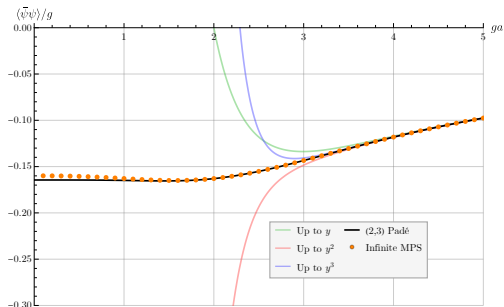


Exact result @ $m = 0$: $\langle \bar{\psi}\psi \rangle / g = -\frac{e^\gamma}{2\pi^{3/2}} \cos \theta$

Strong coupling expansion and Padé approximants

- Compute observables perturbatively in $y \equiv \frac{1}{(ag)^4} \ll 1$ and extrapolate to $a \rightarrow 0$ using Padé approximants [Banks, Susskind, Kogut 1976; Hamer, Weihong, Oitmaa 1997]
 $\Rightarrow$ Improve using mass shift [Dempsey, Klebanov, Pufu, Zan 2022]

$$M_S/g = \frac{ga}{4} (1 + 8y - 144y^2 + 448y^3 + \dots) \approx \frac{ga}{4} \left(\frac{3 + 190y + 2432y^2}{3 + 94y} \right)^{1/4}.$$



[Dempsey, Glück, Pufu, BTS 2025]

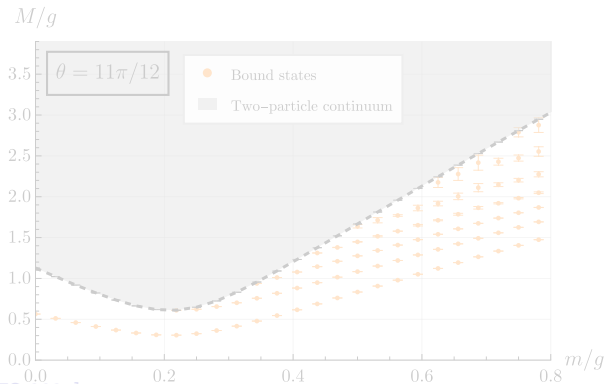
Precision study of observables and bound states

- Benchmark at $m = 0$ vs. exact solution

$$\langle \bar{\psi}\psi \rangle / g = -0.159926(3) \quad (\text{lattice}) \quad = -0.1599288\dots \quad (\text{exact}),$$

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- Bound state spectrum



[Dempsey, Glück, Pufu, BTS 2025]

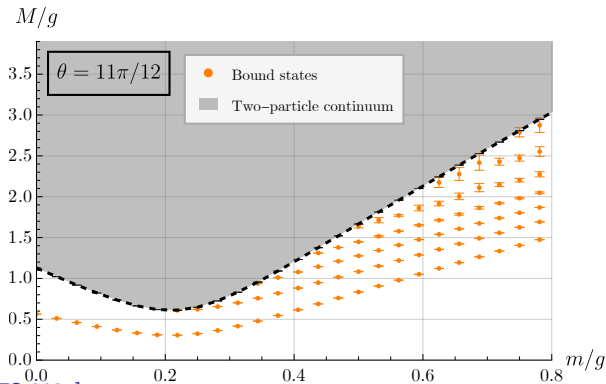
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Adjoint QCD₂: Continuum theory

- ▶ Two dimensional $SU(N_c)$ gauge theory with adjoint Majorana fermion of mass m
[Dalley, Klebanov 1993; . . . ; Katz, Tavares, Xu 2014; Cherman, Jacobson, Tanizaki, Ünsal 2020; . . .]

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- ▶ One parameter m/g , by $(\mathbb{Z}_2)_\chi : \psi \rightarrow -\psi$, we take $m \geq 0$
 - ▶ $m = 0$, mass gap + vanishing string tension [Gross, Klebanov, Matytsin, Smilga 1996]
 - ▶ Supersymmetry at $m_{\text{SUSY}} = g\sqrt{\frac{N_c}{2\pi}}$ [Kutasov 1994; Popov 2022; Klebanov, Pufu, BTS, Witten 2025]
 - ▶ Generic m , mass gap + confinement
- ▶ Center symmetry: N_c different flux tube sectors $\Leftarrow$ probe of confinement
- ▶ At $m = 0$, there are 2^{N_c-1} degenerate vacua due to non-invertible symmetries [Komargodski, Ohmori, Roumpedakis, Seifnashri 2021]

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[Dempsey, Klebanov, Pufu, BTS 2023, 2024]

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$$R_n = \begin{cases} \sqrt{2} \times \mathbf{2}, & N_c = 2, \\ \mathbf{2} \times \mathbf{8}, & N_c = 3. \end{cases}$$

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$p = 0$:



$p = 1$:



⇒ Construct MPS ansatz with virtual spaces in p th sector

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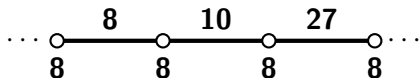
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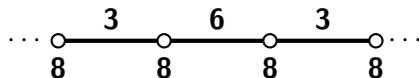
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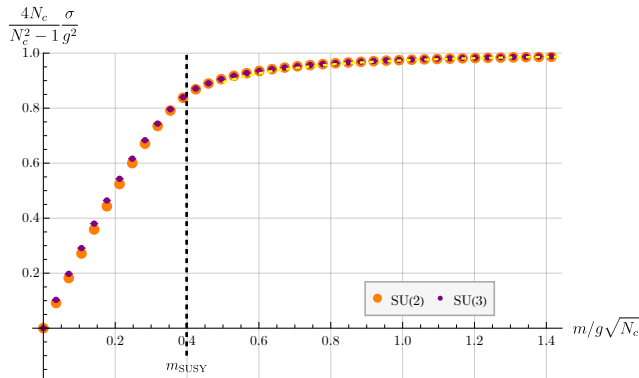
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String tension $N_c = 2, 3$

- Fundamental string tension from vacuum energy density (extrapolated to $a = 0$)

$$\sigma = \lim_{a \rightarrow 0} [\varepsilon_{p=1}(a) - \varepsilon_{p=0}(a)].$$

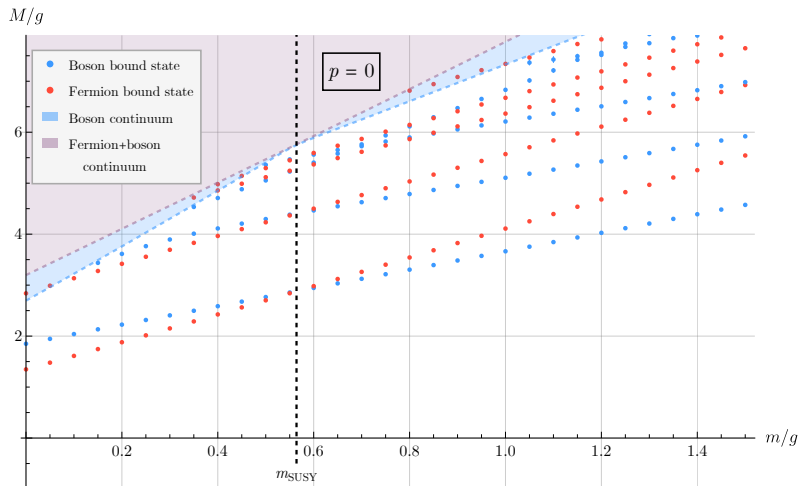
- Consistent with $m = 0$ prediction and 1-loop estimate for $m \gg g$



[Dempsey, Glück, Pufu, BTS 2025]

Bound states $N_c = 2$

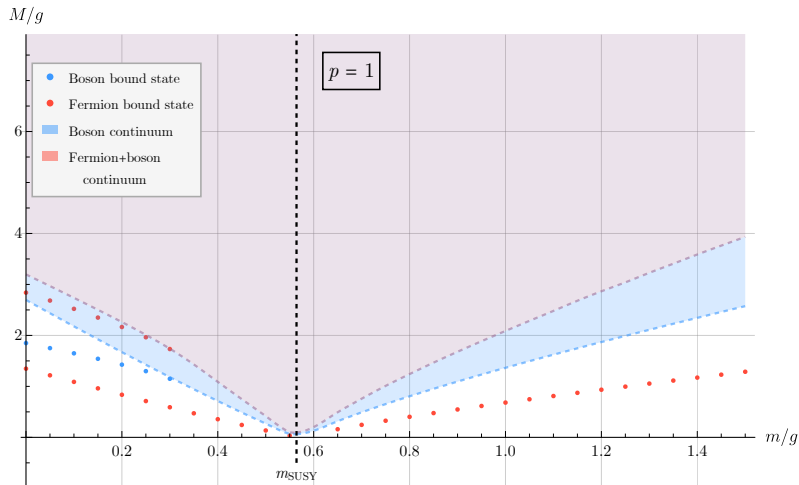
- ▶ Gapped @ $m = 0$, fermion/boson degeneracy @ $m = m_{\text{SUSY}}$
- ▶ Agrees with DLCQ at finite N_c [Dempsey, Klebanov, Lin, Pufu 2023]



[Dempsey, Glück, Pufu, BTS 2025]

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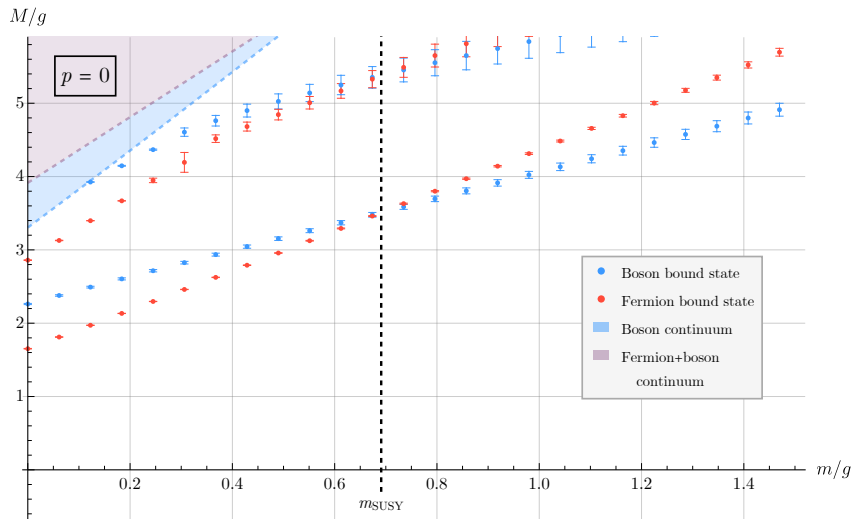
- ▶ m_{SUSY} : Non-zero string tension $\Rightarrow$ spontaneous breaking of SUSY $\Rightarrow$ goldstino
- ▶ Non-trivial flux tube sector not observable in DLCQ



[Dempsey, Glück, Pufu, BTS 2025]

Bound states $N_c = 3$

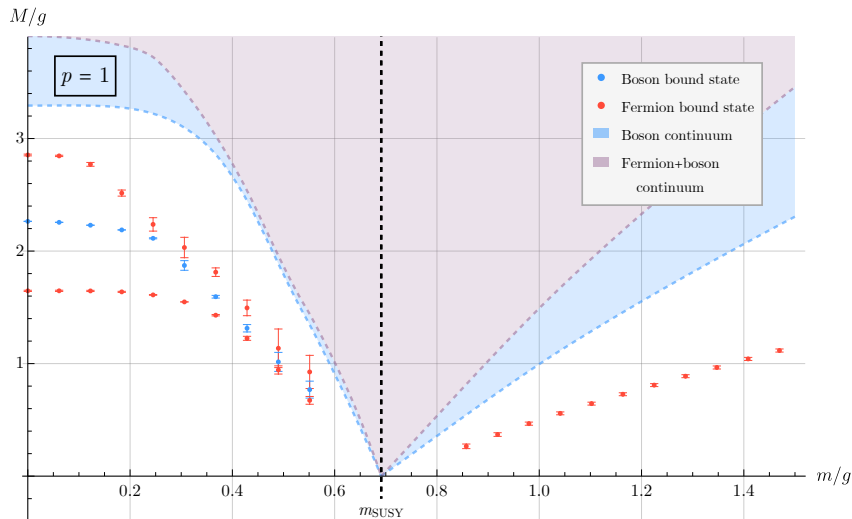
- Numerics are more challenging for $N_c = 3$, predicted behavior at $m = 0, m_{\text{SUSY}}$



[Dempsey, Glück, Pufu, BTS 2025]

Bound states $N_c = 3$

- Small mass gaps are computationally expensive: still trend towards SUSY mass



[Dempsey, Glück, Pufu, BTS 2025]

Conclusion

- ▶ Symmetric MPSs furnish a natural setting for Hamiltonian LGT states
- ▶ We introduced LEMPOs as a MPO formulation of general gauge theories compatible with standard MPS algorithms
- ▶ LEMPOs allow for infinite system size methods, which give high precision results for ground state properties and bound state masses
- ▶ Analyzed lattice realization of the massive Schwinger model and $N_c = 2, 3$ adjoint QCD₂ and found agreement with theoretical predictions

Future directions:

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Extra Slides

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