Muonic Hydrogen: Theoretical Challenges

V. Lensky
JGU Mainz

ECT* Workshop „New perspectives in the charge radii determination for light nuclei"

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Outline

- A quick recipe how to understand theory of light muonic atoms
- Two-Photon Exchange (TPE) in muonic Hydrogen (μH)
  - Connection to other experiments
  - Pitfalls, results, and work in progress
    - Lamb shift
    - Hyperfine splitting
- Conclusion and outlook

A (Light) Muonic Atoms and Ions Cookbook

TABLE I Contributions to the $2P_{1/2} - 2S_{1/2}$ energy difference E_L in meV, with the charge radii r_C given in fm. All corrections larger than 3% of the overall uncertainty are included. Theoretical predictions for E_L are $E_L(\text{theo}) = E_{\text{QED}} + \mathcal{C}\, r_C^2 + E_{\text{NS}}$. The last two rows show the values of r_C determined from a comparison of $E_L(\text{theo})$ to $E_L(\text{exp})$.

Sec.	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
III.A	$\alpha\,(Z\alpha)^2$	eVP$^{(1)}$	205.007 38	227.634 70	1641.886 2	1665.773 1
III.A	$\alpha^2(Z\alpha)^2$	eVP$^{(2)}$	1.658 85	1.838 04	13.084 3	13.276 9
III.A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
III.B						
III.C						
III.D						
III.E						
III.F						
III.G						
III.H						
III.I						
III.J						
III.K						
III.L						
III.M						
III.N						
III.O						
III.P						
III.Q	$\alpha^2(Z\alpha)^4$	hVP with eVP$^{(1)}$	0.000 09	0.000 10	0.002 6(1)	0.002 7(1)
IV.A	$(Z\alpha)^4$	r_C^2	$-5.197\,5\, r_p^2$	$-6.073\,2\, r_d^2$	$-102.523\, r_h^2$	$-105.322\, r_\alpha^2$
IV.B	$\alpha\,(Z\alpha)^4$	eVP$^{(1)}$ with r_C^2	$-0.028\,2\, r_p^2$	$-0.034\,0\, r_d^2$	$-0.851\, r_h^2$	$-0.878\, r_\alpha^2$
IV.C	$\alpha^2(Z\alpha)^4$	eVP$^{(2)}$ with r_C^2	$-0.000\,2\, r_p^2$	$-0.000\,2\, r_d^2$	$-0.009(1)\, r_h^2$	$-0.009(1)\, r_\alpha^2$
V.A	$(Z\alpha)^5$	TPE	0.029 2(25)	1.979(20)	16.38(31)	9.76(40)
V.B	$\alpha^2(Z\alpha)^4$	Coulomb distortion	0.0	−0.261	−1.010	−0.536
V.C	$(Z\alpha)^6$	3PE	−0.001 3(3)	0.002 2(9)	−0.214(214)	−0.165(165)
V.D	$\alpha\,(Z\alpha)^5$	eVP$^{(1)}$ with TPE	0.000 6(1)	0.027 5(4)	0.266(24)	0.158(12)
V.E	$\alpha\,(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$ with TPE	0.000 4	0.002 6(3)	0.077(8)	0.059(6)
III	E_{QED}	point nucleus	206.034 4(3)	228.774 0(3)	1644.348(8)	1668.491(7)
IV	$\mathcal{C}\, r_C^2$	finite size	$-5.225\,9\, r_p^2$	$-6.107\,4\, r_d^2$	$-103.383\, r_h^2$	$-106.209\, r_\alpha^2$
V	E_{NS}	nuclear structure	0.028 9(25)	1.750 3(200)	15.499(378)	9.276(433)
	$E_L(\text{exp})$	experiment[a]	202.370 6(23)	202.878 5(34)	1258.598(48)	1378.521(48)
	r_C	this review	0.840 60(39)	2.127 58(78)	1.970 07(94)	1.678 6(12)
	r_C	previous work[a]	0.840 87(39)	2.125 62(78)	1.970 07(94)	1.678 24(83)

cookbook

noun [C]

UK /ˈkʊk.bʊk/ US /ˈkʊk.bʊk/

(UK also **cookery book**)

Add to word list

a book that explains how to prepare particular dishes:

- *She has written several award-winning cookbooks.*

Pachucki, VL, Hagelstein, Li Muli, Bacca, Pohl – theory review (2022)

[a] experiment: CREMA (2013-2023)

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III.B	$(Z, Z^2, Z^3)\,\alpha^5$	light-by-light eVP	−0.000 89(2)	−0.000 96(2)	−0.013 4(6)	−0.013 6(6)
III.C	$(Z\alpha)^4$	recoil	0.057 47	0.067 22	0.126 5	0.295 2
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III.G	$\alpha\,(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$, NLO	−0.004 43	−0.005 18	−0.174 9	−0.179 7
III.H	$\alpha^2(Z\alpha)^4$	μVP$^{(1)}$ with eVP$^{(1)}$	0.000 13	0.000 15	0.003 8	0.003 9
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III.J	$(Z\alpha)^5$	recoil	−0.044 97	−0.026 60	−0.558 1	−0.433 0
III.K	$\alpha\,(Z\alpha)^5$	recoil with eVP$^{(1)}$	0.000 14(14)	0.000 09(9)	0.004 9(49)	0.003 9(39)
III.L	$Z^2\alpha\,(Z\alpha)^4$	nSE$^{(1)}$	−0.009 92	−0.003 10	−0.084 0	−0.050 5
III.M	$\alpha^2(Z\alpha)^4$	$\mu F_1^{(2)}$, $\mu F_2^{(2)}$, μVP$^{(2)}$	−0.001 58	−0.001 84	−0.031 1	−0.031 9
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(Partial) expansion in powers of α, $Z\alpha$: light means that you can still expand

Recoil (expansion in powers of m_μ / M_A): more important than in ordinary atoms

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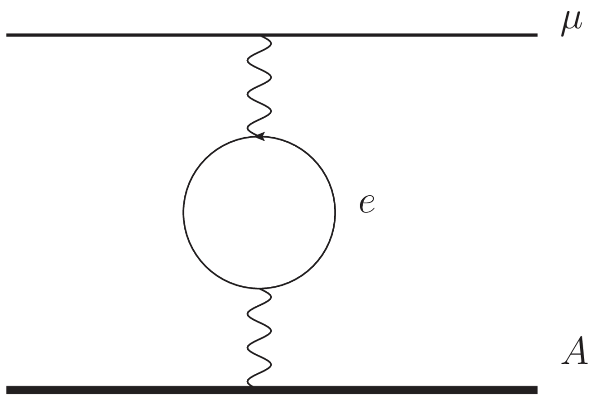
Row III.A: E_{QED}; rows IV.A–IV.C: $\mathcal{C}\, R_E^2$; rows V.A–V.E: E_{NS}.



Bohr radius

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Electron loops are enhanced (matching scales)! Sotiris Pitelis's talk

Recall that in normal hydrogen eVP is a small term ~0.5% on top of the electron vertex correction

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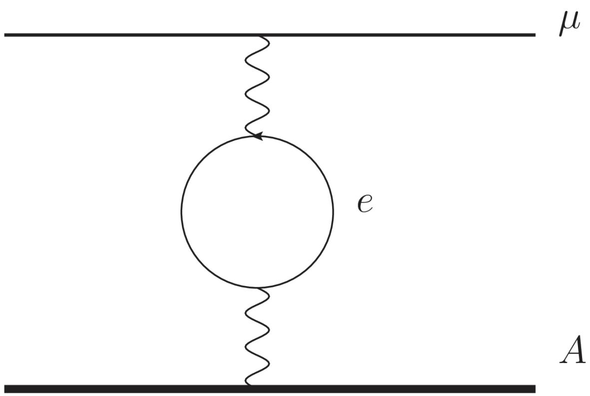

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Finite size correction is also enhanced (2nd most important term)

$\}$ $\mathcal{C}\, R_E^2$

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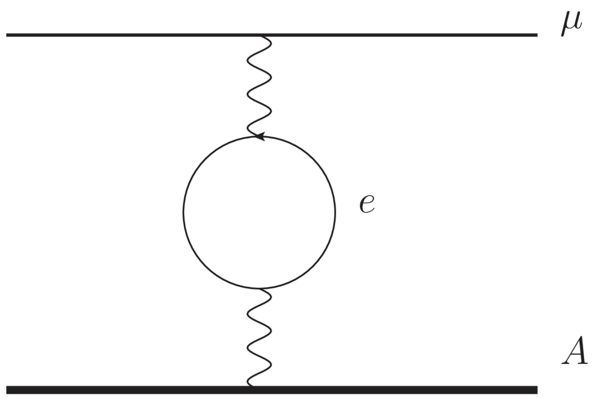
V.C	$(Z\alpha)^6$	3PE	−0.001 3(3)	0.002 2(9)	−0.214(214)	−0.165(165)
V.D	$\alpha\,(Z\alpha)^5$	eVP$^{(1)}$ with TPE	0.000 6(1)	0.027 5(4)	0.266(24)	0.158(12)
V.E	$\alpha\,(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$ with TPE	0.000 4	0.002 6(3)	0.077(8)	0.059(6)

$\}$ E_{NS}

Contributions to LS of Light Muonic Atoms/Ions

$$E_{LS}(\text{theo}) = E_{\text{QED}} + \mathcal{C}\, R_E^2 + E_{\text{NS}}$$

Sec.	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
III.A	$\alpha\,(Z\alpha)^2$	eVP$^{(1)}$	205.007 38	227.634 70	1641.886 2	1665.773 1
IV.A	$(Z\alpha)^4$	r_C^2	$-5.197\,5\, r_p^2$	$-6.073\,2\, r_d^2$	$-102.523\, r_h^2$	$-105.322\, r_\alpha^2$
V.A	$(Z\alpha)^5$	TPE	0.029 2(25)	1.979(20)	16.38(31)	9.76(40)



Bohr radius

$$a = (Z\alpha\, m_\mu)^{-1} \simeq m_e^{-1}$$

Electron loops are enhanced (matching scales)! Sotiris Pitelis's talk

Recall that in normal hydrogen eVP is a small term ~0.5% on top of the electron vertex correction

Eides, Grotch, Shelyuto 2000 (review), 2007 (book)

E_{QED}

Finite size correction is also enhanced (2nd most important term)

$\mathcal{C}\, R_E^2$

Nuclear structure corrections are enhanced, too, and, most importantly, they dominate the overall uncertainty!

E_{NS}

Finite Size and Nuclear Structure

$$E_{LS}(\text{theo}) = E_{\text{QED}} + \mathcal{C}\, R_E^2 + E_{\text{NS}}$$

- Squeezed Table

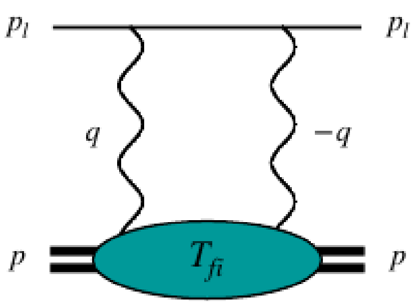
	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
E_{QED}	point nucleus	206.034 4(3)	228.774 0(3)	1644.348(8)	1668.491(7)
$\mathcal{C}\, r_C^2$	finite size	$-5.225\,9\, r_p^2$	$-6.107\,4\, r_d^2$	$-103.383\, r_h^2$	$-106.209\, r_\alpha^2$
E_{NS}	nuclear structure	0.028 9(25)	1.750 3(200)	15.499(378)	9.276(433)
$E_L(\text{exp})$	experiment[a]	202.370 6(23)	202.878 5(34)	1258.612(86)	1378.521(48)

Bohr radius

$$a = (Z\alpha m_r)^{-1}$$

- Dominant nuclear structure effect: Two-Photon Exchange (TPE)

$\propto \alpha^5$



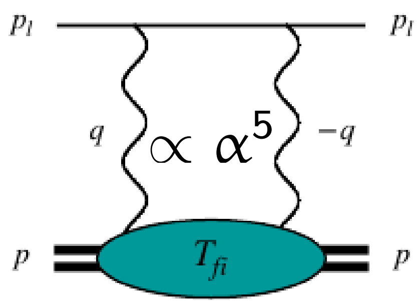
$$\Delta E_{nS} = \frac{2\pi Z\alpha}{3}\frac{1}{\pi(an)^3}\left[R_E^2 - \frac{Z\alpha m_r}{2} R_F^3\right] + \ldots$$

R_E : charge radius

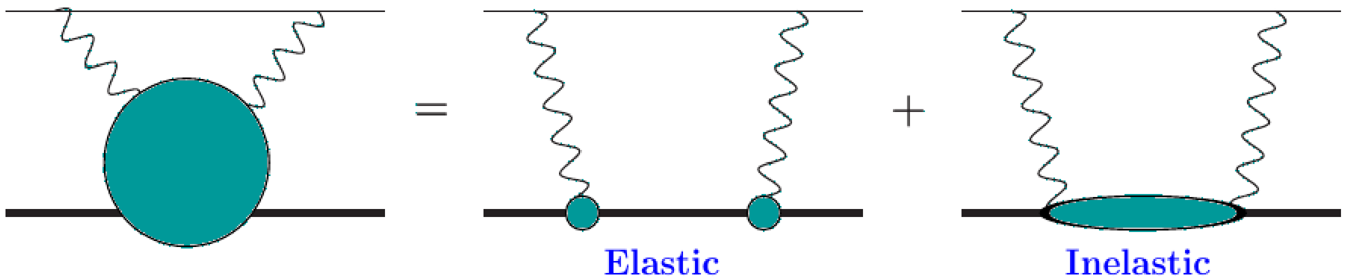
R_F : Friar radius

- TPE also dominates the uncertainty (90-95%)
- Finite size enhanced (by a factor ~10^8) – great sensitivity!
- Also greater sensitivity to subleading nuclear structure

TPE and VVCS



- TPE is naturally described in terms of (doubly virtual fwd) Compton scattering (VVCS)
- Elastic ($\nu = \pm Q^2/2M_{\text{target}}$, elastic e.m. form factors) and inelastic (~ nuclear [generalised] polarisabilities)



- Forward spin-1/2 VVCS amplitude

$$\alpha_{\text{em}}\, M^{\mu\nu}(\nu, Q^2) = -\left\{ \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) T_1(\nu, Q^2) + \frac{1}{M^2} \left(p^\mu - \frac{p\cdot q}{q^2}\, q^\mu \right) \left(p^\nu - \frac{p\cdot q}{q^2}\, q^\nu \right) T_2(\nu, Q^2) \right.$$
$$\left. + \frac{i}{M}\, \epsilon^{\nu\mu\alpha\beta}\, q_\alpha s_\beta\, S_1(\nu, Q^2) + \frac{i}{M^3}\, \epsilon^{\nu\mu\alpha\beta}\, q_\alpha (p\cdot q\ s_\beta - s\cdot q\ p_\beta)\, S_2(\nu, Q^2) \right\}$$

~HFS

Lamb Shift:

$$E^{2\gamma}_{nS} = -8 i \pi \alpha m\, [\phi_n(0)]^2 \int_s \frac{d^4 q}{(2\pi)^4} \frac{(Q^2 - 2\nu^2)\, T_1(\nu, Q^2) - (Q^2 + \nu^2)\, T_2(\nu, Q^2)}{Q^4 (Q^4 - 4 m^2 \nu^2)}$$

TPE and Elastic Electron Scattering

- Forward spin-1/2 VVCS amplitude

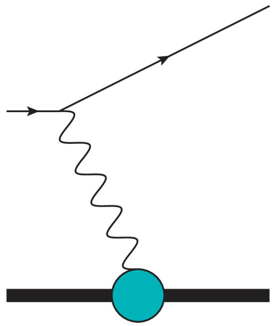
$$\alpha_{em}\, M^{\mu\nu}(\nu, Q^2) = -\Big\{\Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big)\, T_1(\nu, Q^2) + \frac{1}{M^2}\Big(p^\mu - \frac{p\cdot q}{q^2}\, q^\mu\Big)\Big(p^\nu - \frac{p\cdot q}{q^2}\, q^\nu\Big)\, T_2(\nu, Q^2)$$

$$+ \frac{i}{M}\,\epsilon^{\nu\mu\alpha\beta}\, q_\alpha s_\beta\, S_1(\nu, Q^2) + \frac{i}{M^3}\,\epsilon^{\nu\mu\alpha\beta}\, q_\alpha (p\cdot q\; s_\beta - s\cdot q\; p_\beta)\, S_2(\nu, Q^2)\Big\}$$

- Born amplitudes: proton form factors

$$T_1^{\mathrm{Born}}(\nu, Q^2) = \frac{4\pi\alpha}{M}\Big\{\frac{Q^4\big[F_1(Q^2) + F_2(Q^2)\big]^2}{Q^4 - 4M^2\nu^2} - F_1^2(Q^2)\Big\}$$

$$T_2^{\mathrm{Born}}(\nu, Q^2) = \frac{16\pi\alpha M Q^2}{Q^4 - 4M^2\nu^2}\Big\{F_1^2(Q^2) + \frac{Q^2}{4M^2} F_2^2(Q^2)\Big\}$$



- Elastic TPE contribution to the Lamb shift: Friar radius

$$\Delta E_{nS} = \frac{2\pi Z\alpha}{3}\frac{1}{\pi(an)^3}\Big[R_E^2 - \frac{Z\alpha m_r}{2} R_F^3\Big] + \dots$$

$$R_{\mathrm{F}}^3 = \frac{48}{\pi}\int_0^\infty \frac{dQ}{Q^4}\big[G_E^2(Q^2) - 1 - 2G_E'(0)\, Q^2\big]$$

Friar and Charge Radii Correlation

- Friar and charge radii are moments of the same form factor/charge distribution, so they are correlated
- The value of R_E intrinsic to the form factor influences the value of R_F (elastic TPE) → and the extraction from μH
- The form factor used to calculate TPE has to be consistent with the extracted value of R_E
- A solution: split the integral

Karshenboim (2014)

$$R_\mathrm{F}^3 = \frac{48}{\pi} \int_0^\infty \frac{dQ}{Q^4} \left[G_E^2(Q^2) - 1 - 2G_E'(0)\,Q^2\right] = \frac{48}{\pi} \left[\int_0^{q_0} + \int_{q_0}^\infty\right] \frac{dQ}{Q^4} \left[G_E^2(Q^2) - 1 - 2G_E'(0)\,Q^2\right]$$

- In the first integral, use a simple polynomial model to cancel the divergence: $G_E(Q^2) = 1 - R_E^2 Q^2/3 - C\,Q^4$; the last term can be varied
- The second integral converges and can even be taken using data points, and the value of R_E can be adjusted
- Alternatively, fit the form factors constraining the charge radius and see if the resulting elastic TPE is consistent with the extraction from μH

Friar and Charge Radii Correlation: Fits of the FFs

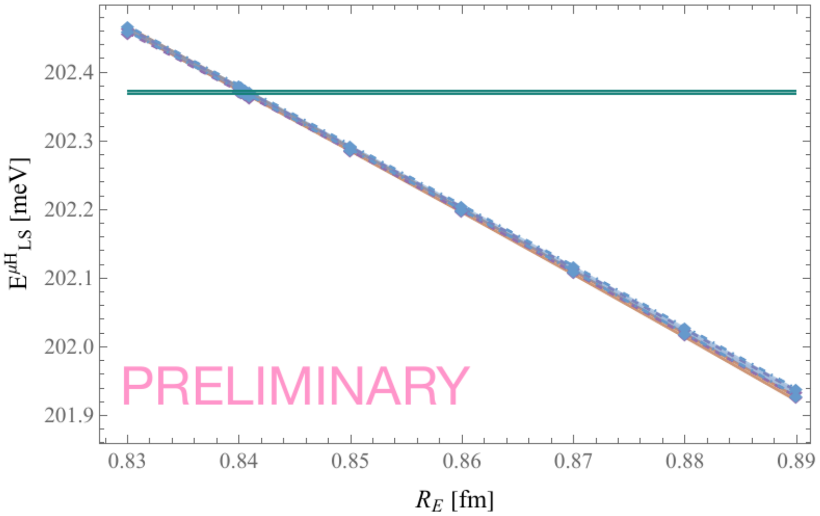
- Fit *ep* scattering world data

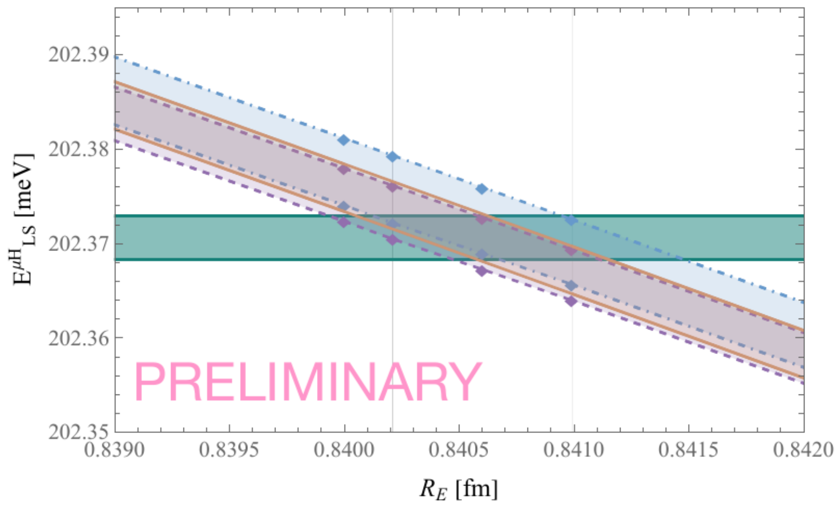
F. Hagelstein, VL, V. Sharkovska, in preparation (2025)

  - Unpolarized cross sections: 33 experiments, 2055 data points
  - Polarization transfer: 14 experiments, 69 data points
  - Initial state radiation extraction: 1 experiment, 25 points
  - 10 values of $R_E = 0.83 \dots 0.89$ fm set as a constraint
  - 40 different fit Ansätze

- TPE = the resulting Friar term + recoil (small) + the inelastic TPE compare with **experiment** ($E_{LS} = 202.3076(23)$ meV) and **theory review** ($E_{\mathrm{LS}}^{\mathrm{th}} = \left[206.0344(3) - 5.2259(R_E/\mathrm{fm})^2 + 0.0289(25)\right]$ meV)

- Consistent with the review: $R_E = 0.84060(39)$fm





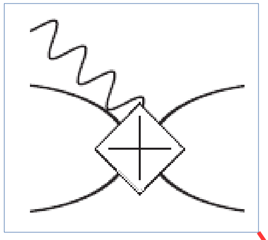
Deuteron Charge Form Factor and TPE in μD

- Self-consistency isn't trivial: An example from μD
- Pionless EFT at N3LO
- Correlation between R_F and R_E – analytic result, generated by an N3LO LEC

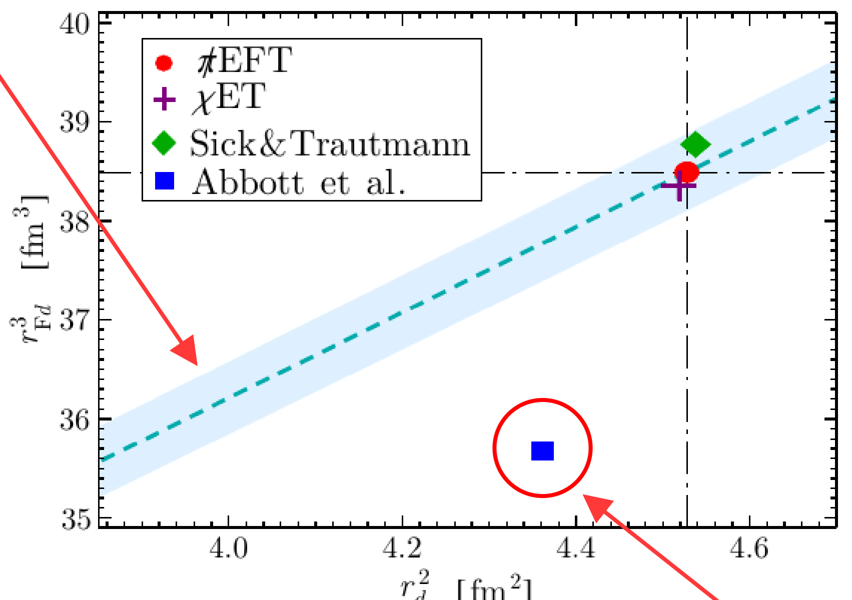
$$R_F^3 = \frac{48}{\pi}\int_0^\infty \frac{dQ}{Q^4}\left[G_C^2(Q^2) - 1 - 2G_C'(0)\,Q^2\right]$$

$$= \frac{3}{80\gamma^3}\Big\{\mathcal{Z}\left[5 - 2\mathcal{Z}(1-2\ln 2)\right] - 320/9\; r_0^2\gamma^2\left[\mathcal{Z}(1-4\ln 2) - 2 + 2\ln 2\right] + 80(\mathcal{Z}-1)^3\,\ell_1^{C0_S}\Big\}$$

$$R_E^2 = \frac{1}{8\gamma^2} + \frac{\mathcal{Z}-1}{8\gamma^2} + 2r_0^2 + \frac{3(\mathcal{Z}-1)^3}{\gamma^2}\,\ell_1^{C0_S}$$



VL, Hagelstein, Pascalutsa (2022)



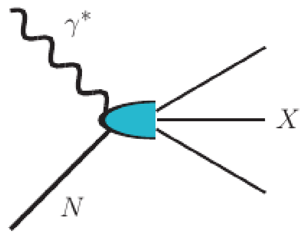
- Some form factor parametrizations do not reproduce the correlation!

TPE and Structure Functions

- Forward spin-1/2 VVCS amplitude

$$\alpha_{em} M^{\mu\nu}(\nu, Q^2) = -\left\{\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu, Q^2) + \frac{1}{M^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) T_2(\nu, Q^2) + \frac{i}{M}\epsilon^{\nu\mu\alpha\beta} q_\alpha s_\beta S_1(\nu, Q^2) + \frac{i}{M^3}\epsilon^{\nu\mu\alpha\beta} q_\alpha (p\cdot q\, s_\beta - s\cdot q\, p_\beta) S_2(\nu, Q^2)\right\}$$

- Unitarity and analyticity, data-driven: dispersive relations



Structure functions $F_1(x, Q^2)$, $F_2(x, Q^2)$, $g_1(x, Q^2)$, $g_2(x, Q^2)$

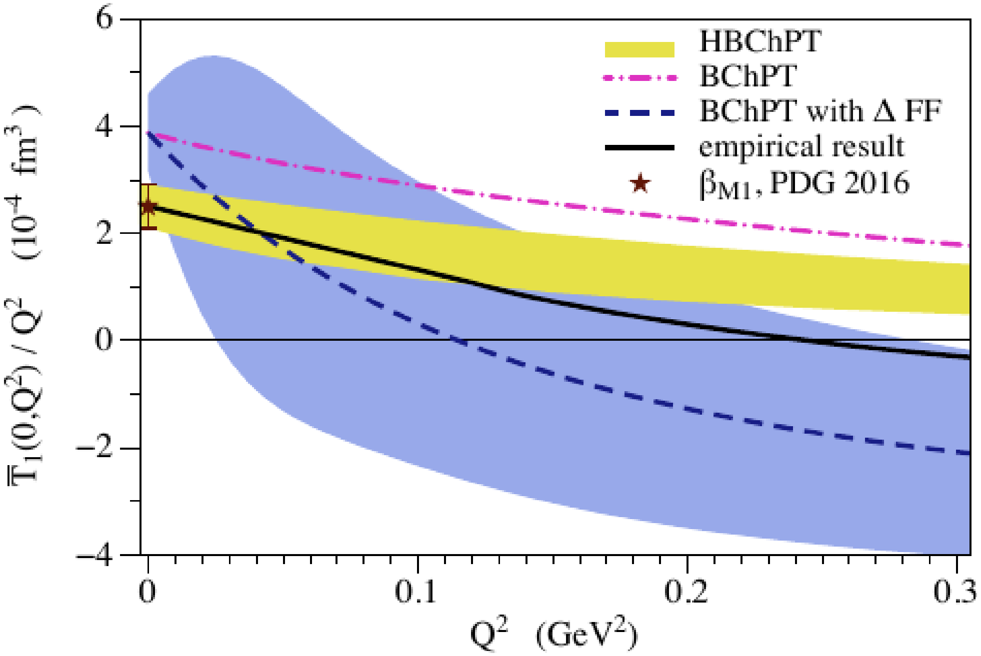
$$T_1(\nu, Q^2) = T_1(0, Q^2) + \frac{32\pi M\nu^2}{Q^4}\int_0^1 dx \frac{x f_1(x, Q^2)}{1 - x^2(\nu/\nu_{el})^2 - i0^+},$$

$$T_2(\nu, Q^2) = \frac{16\pi M}{Q^2}\int_0^1 dx \frac{f_2(x, Q^2)}{1 - x^2(\nu/\nu_{el})^2 - i0^+}$$

- In practice, one deals with the non-Born subtraction function (everything minus Born)
- The subtraction function is not directly accessible in experiment

Various Subtraction Functions

- The diversity of the results for the proton subtraction function $T_1(0, Q^2)$
  - HBChPT: dipole FF, matches β_{M1}[PDG] and the slope at 0
    modification of Birse, McGovern (2012)
  - BChPT: transition FFs change the subtraction function
  - Empirical: Regge asymptotic at high energy subtracted
    Tomalak, Vanderhaeghen (2015)



VL, Hagelstein, Pascalutsa, Vanderhaeghen (2017)

- Zero crossing at low Q^2 – emerges in BChPT with FFs; established in the empirical derivation, but the position not well known (0.1..0.4 GeV^2)
- Big cancellations between different mechanisms (πN and πΔ loops vs. Δ pole), also cancellations in the LS integral because of the sign change
- Empirical derivation has sizeable errors towards $Q^2 = 0$ (not shown) attributed to mismatch between structure function fit in the resonance region (Christy-Bosted) and at high energies (Donnachie-Landshoff) => needs a better (combined) structure function parametrization

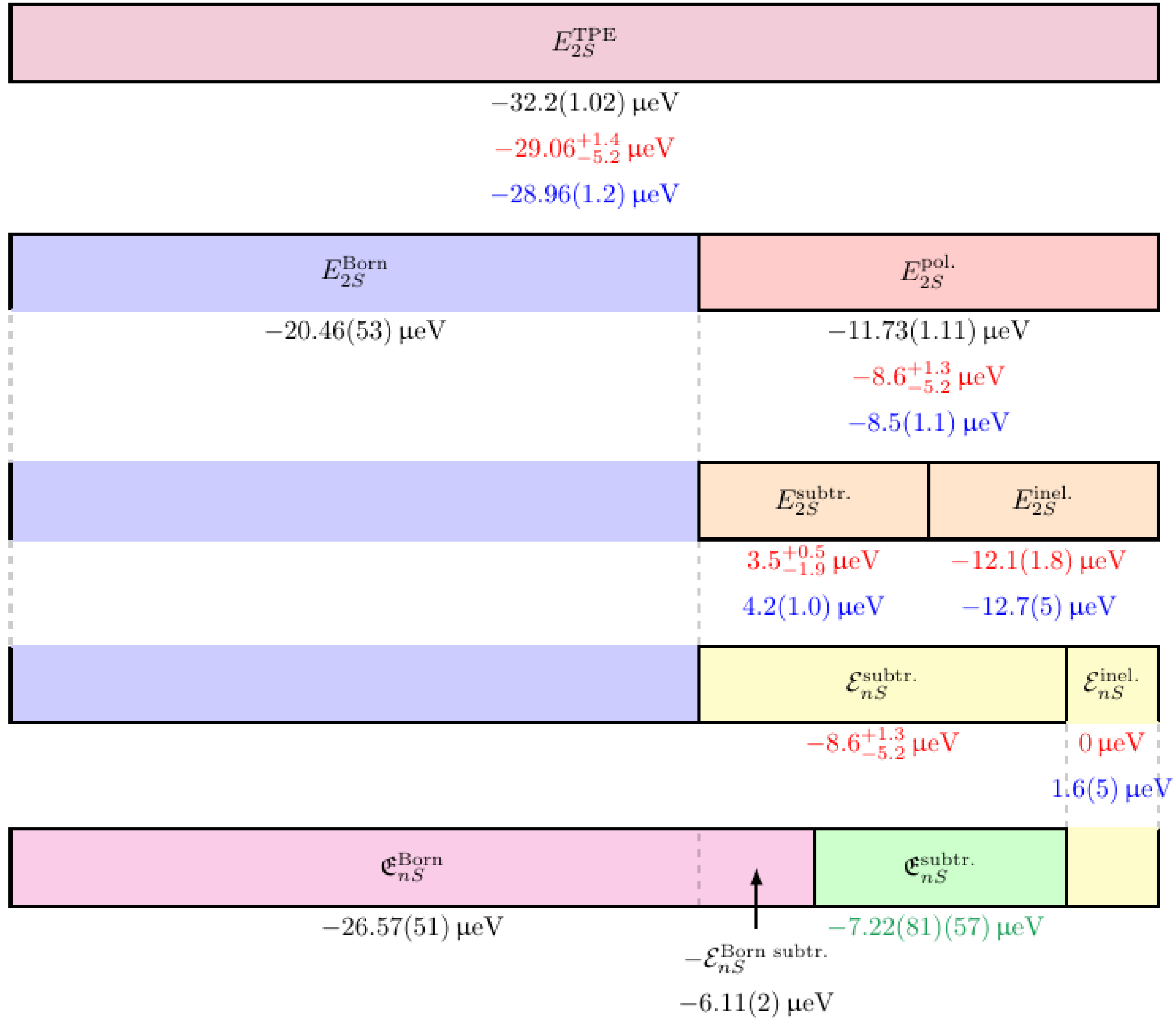
Yet Another Subtraction Function

- One doesn't have to subtract at $\nu = 0$, one can use any value $\nu = \nu_0$

$$T_1(\nu, Q^2) = T_1(\nu_0, Q^2) + \frac{32\pi M}{Q^4} \int_0^1 \mathrm{d}x\, x\, f_1(x, Q^2)\, \frac{\nu^2 - \nu_0^2}{\left[1 - x^2 \left(\frac{\nu}{\nu_{\mathrm{el}}}\right)^2\right]\left[1 - x^2 \left(\frac{\nu_0}{\nu_{\mathrm{el}}}\right)^2\right]}$$

- In particular, using $\nu = iQ$ is advantageous:
  - Shrinks the inelastic contribution — Hagelstein, Pascalutsa (2020)
  - Simplifies a Lattice QCD calculation — Fu et al. (2025)
  - The (non-Born) limit at $Q \to 0$ is proportional to the relatively bigger α_{E1}
  - Potentially, $T_1(iQ, Q^2)$ is less affected by cancellations than $T_1(0, Q^2)$ – more reliable results in an empirical fit
- Results for $T_1(iQ, Q^2)$ have recently been obtained in LQCD — Fu et al. (2025)
- It's slightly cumbersome to compare the different results because of different separation of the TPE… but we have a recipe for that, too!

TPE: Results



F. Hagelstein, VL, S. Pitelis, V. Sharkovska, in preparation (2025)

FIG. 1. LQCD in green. This work (+ LQCD) in black. Data-driven in blue. BChPT in red.

- Results across different approaches are in agreement, also on the sizes of separate contributions
- Subtraction function still carries the biggest uncertainty and needs to be further constrained, especially in view of a more precise experiment

LS in μH

- Elastic form factors seem to be under control, although shrinking the uncertainty is desirable
- Subtraction function: new promising results from LQCD; a more reliable empirical extraction can be possible (especially if a new structure function parametrization becomes available); this can be supplemented at low virtualities by the BChPT results

Hyperfine Splitting (HFS) of μH

$$E_{\text{hfs}}(nS) = \frac{8}{3}\frac{Z\alpha}{(an)^3}\frac{1+\kappa}{mM}\left(1+\Delta_{\text{QED}}+\Delta_{\text{weak}}+\Delta_{\text{strong}}\right)$$

$$\Delta_{\text{strong}} = \Delta_{\text{Z}} + \Delta_{\text{recoil}} + \Delta_{\text{pol}}$$

$$\Delta_{\text{pol.}} = \Delta_1 + \Delta_2 = \frac{Zm}{2\pi(1+\kappa)M}(\delta_1+\delta_2),$$

$$\delta_1 = 2\int_0^\infty \frac{\mathrm{d}Q}{Q}\left\{\frac{5+4v_l}{(v_l+1)^2}\left[4I_1(Q^2)/Z^2+F_2^2(Q^2)\right]-\frac{32M^4}{Q^4}\int_0^{x_0}\mathrm{d}x\,x^2 g_1(x,Q^2)\right.$$
$$\left.\times\frac{1}{(v_l+v_x)(1+v_x)(1+v_l)}\left(4+\frac{1}{1+v_x}+\frac{1}{v_l+1}\right)\right\},$$

$$\delta_2 = 96M^2\int_0^\infty\frac{\mathrm{d}Q}{Q^3}\int_0^{x_0}\mathrm{d}x\,g_2(x,Q^2)\left(\frac{1}{v_l+v_x}-\frac{1}{v_l+1}\right)$$

$$I_1(Q^2) = \frac{2M^2Z^2}{Q^2}\int_0^{x_0}\mathrm{d}x\,g_1(x,Q^2)$$

The generalised GDH integral

$$v_l = \sqrt{1+1/\tau_l},\;\; v_x = \sqrt{1+x^2\tau^{-1}},\;\; \tau_l = Q^2/4m^2,\;\; \tau = Q^2/4M^2$$

Kinematic functions

- No need for subtraction
- Cancellations between the elastic and the inelastic contribution

HFS of μH

$$E_{\rm hfs}(nS) = \frac{8}{3}\frac{Z\alpha}{(an)^3}\frac{1+\kappa}{mM}\left(1+\Delta_{\rm QED}+\Delta_{\rm weak}+\Delta_{\rm strong}\right)$$

$$\Delta_{\rm strong} = \Delta_{\rm Z}+\Delta_{\rm recoil}+\Delta_{\rm pol}$$

$$\Delta_{\rm pol.} = \Delta_{LT}+\Delta_{TT}+\Delta_{F_2} = \frac{m}{2\pi(1+\kappa)M}\left(\delta_{LT}+\delta_{TT}+\delta_{F_2}\right),$$

$$\delta_{LT} = \frac{4M}{\alpha\pi^2}\int_0^\infty \mathrm{d}Q\int_0^{x_0} dx\,\frac{1}{v_l+v_x}\frac{1}{x^2+\tau}\left[1-\frac{1}{(1+v_l)(1+v_x)}\right]\sigma_{LT}(x,Q^2),$$

$$\delta_{TT} = \frac{4M^2}{\alpha\pi^2}\int_0^\infty \frac{\mathrm{d}Q}{Q}\int_0^{x_0}\frac{\mathrm{d}x}{x}\frac{1}{1+v_l}\left[\frac{2\tau}{x^2+\tau}+\frac{1}{(v_l+v_x)(1+v_x)}\right]\sigma_{TT}(x,Q^2),$$

$$\delta_{F_2} = 2\int_0^\infty \frac{Q}{Q}\frac{5+4v_l}{(v_l+1)^2}F_2^2(Q^2)$$

$v_l = \sqrt{1+1/\tau_l},\ v_x = \sqrt{1+x^2\tau^{-1}},\ \tau_l = Q^2/4m^2,\ \tau = Q^2/4M^2$ Kinematic functions

- Rewritten in terms of scattering cross sections

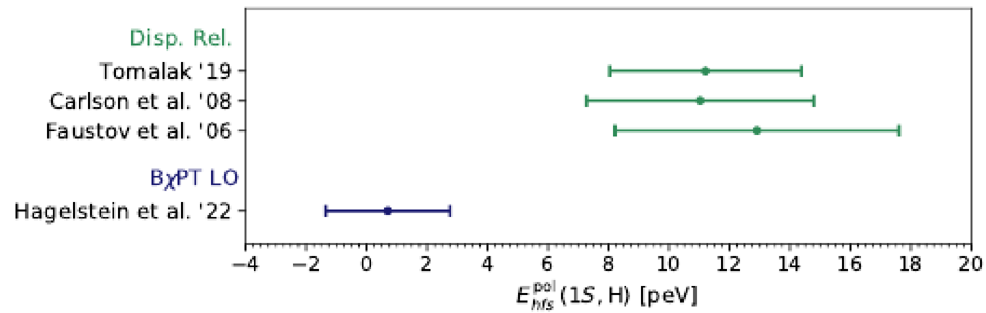
HFS of μH in Covariant BχPT: Cancellations

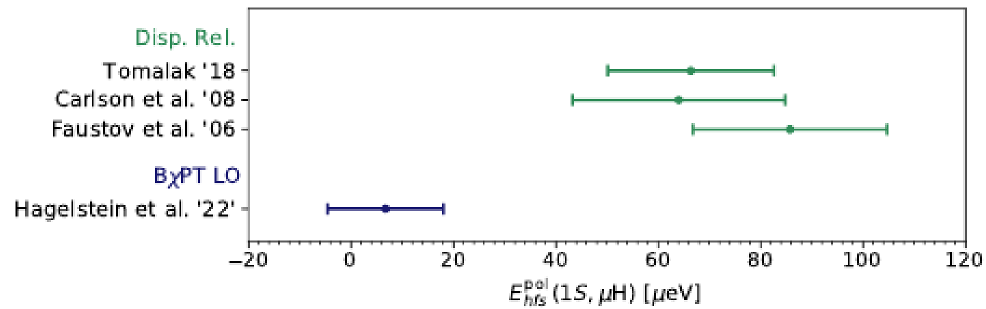
- LO BχPT result

$$E_{\text{hfs}}^{\langle\text{LO}\rangle\,\text{pol.}}(1S,\text{H}) = 0.69(2.03)\,\text{peV}$$

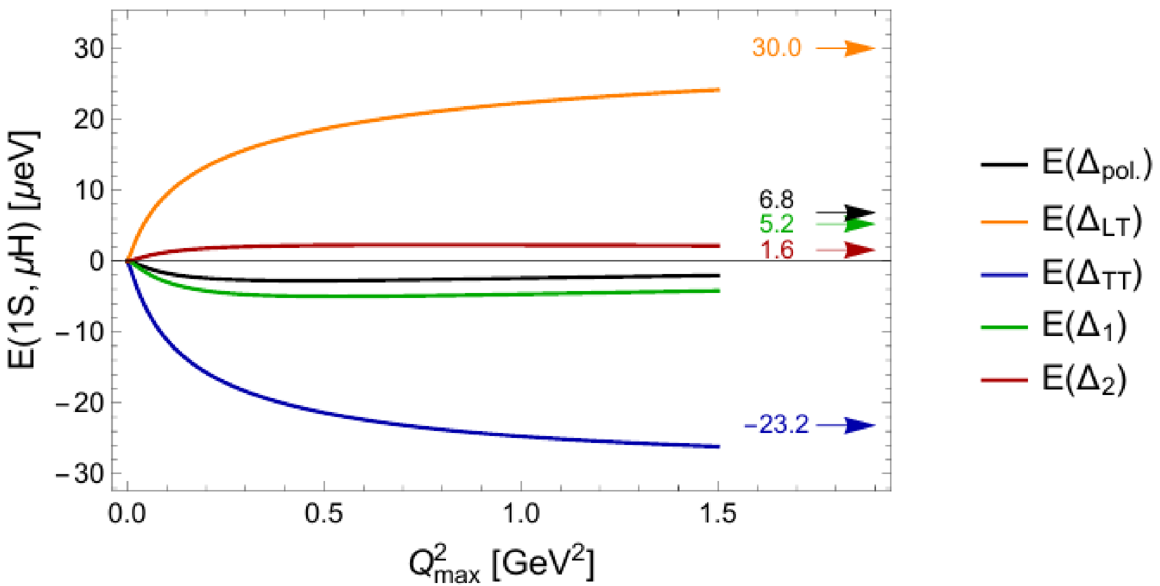
$$E_{\text{hfs}}^{\langle\text{LO}\rangle\,\text{pol.}}(1S,\mu\text{H}) = 6.8(11.4)\,\mu\text{eV}$$

- Consistent with zero
- Cancellations!





Hagelstein, VL, Pascalutsa (2023)

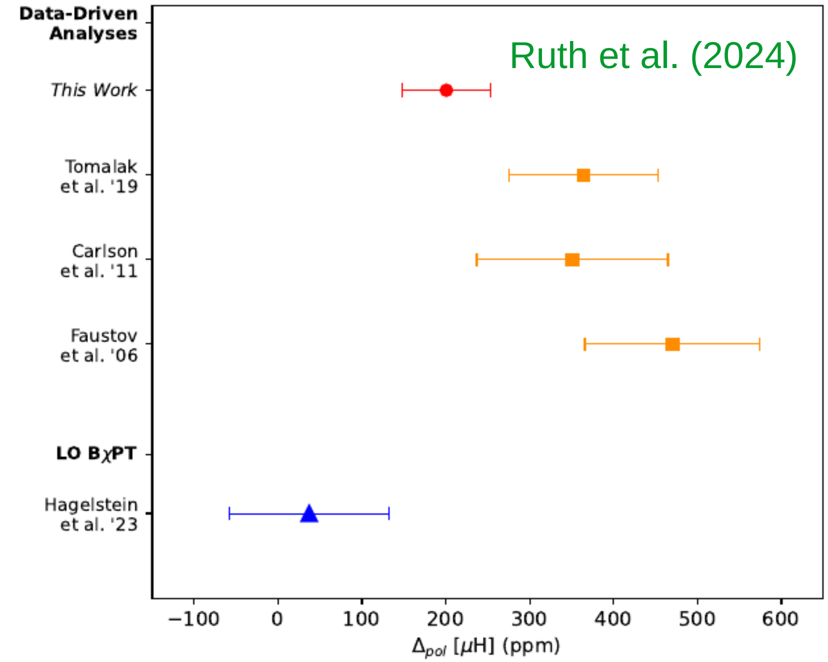


- The LT and TT contributions are large and almost cancel each other
- The LO BχPT result is nearly zero
- Sizeable uncertainty

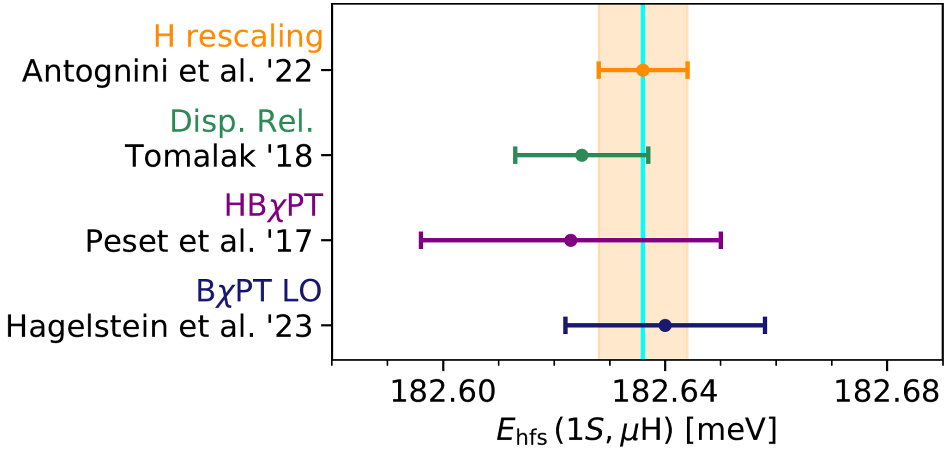
HFS of μH

- New results from g2p@JLab shrink discrepancy between data and BχPT

$$0.02\ \text{GeV}^2 < Q^2 < 0.12\ \text{GeV}^2$$



- Compare with expected experimental precision (cyan line)
- Theory needs to do better than that
- Rescaling non-recoil contributions from the HFS in H



$$E^{\text{Z+pol}}_{nS-\text{hfs}}(\mu\text{H}) = \frac{E_{\text{F}}(\mu\text{H})\, m_r(\mu\text{H})\, b_{nS}(\mu\text{H})}{n^3 E_{\text{F}}(\text{H})\, m_r(\text{H})\, b_{1S}(\text{H})} E^{\text{Z+pol}}_{1S-\text{hfs}}(\text{H}) - \frac{E_{\text{F}}(\mu\text{H})}{n^3} \Delta_{\text{pol}}(\mu\text{H}) \underbrace{\left[c_{1S}(\text{H}) \frac{b_{nS}(\mu\text{H})}{b_{1S}(\text{H})} - c_{nS}(\mu\text{H}) \right]}_{\simeq 10^{-5}}$$

Antognini, Hagelstein, Pascalutsa (2022)

- Potentially can allow one to disentangle Zemach and polarizability terms

HFS in μH

- One needs to increase theory precision to constrain the frequency scan window

see Ahmed Ouf's talk

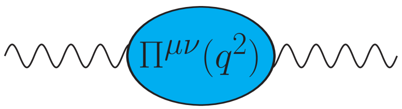
- The rescaling needs to be further investigated
- Possible „missing“ contributions

see Sotiris Pitelis's talk

- Revisit other contributions
  - Hadronic Vacuum Polarization (HVP)

Hadronic Vacuum Polarization (HVP)

- Hadronic Vacuum Polarization



$$\Pi^{\mu\nu}(q^2) = -(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(Q^2) \qquad Q^2 = -q^2$$

- Dispersion relation

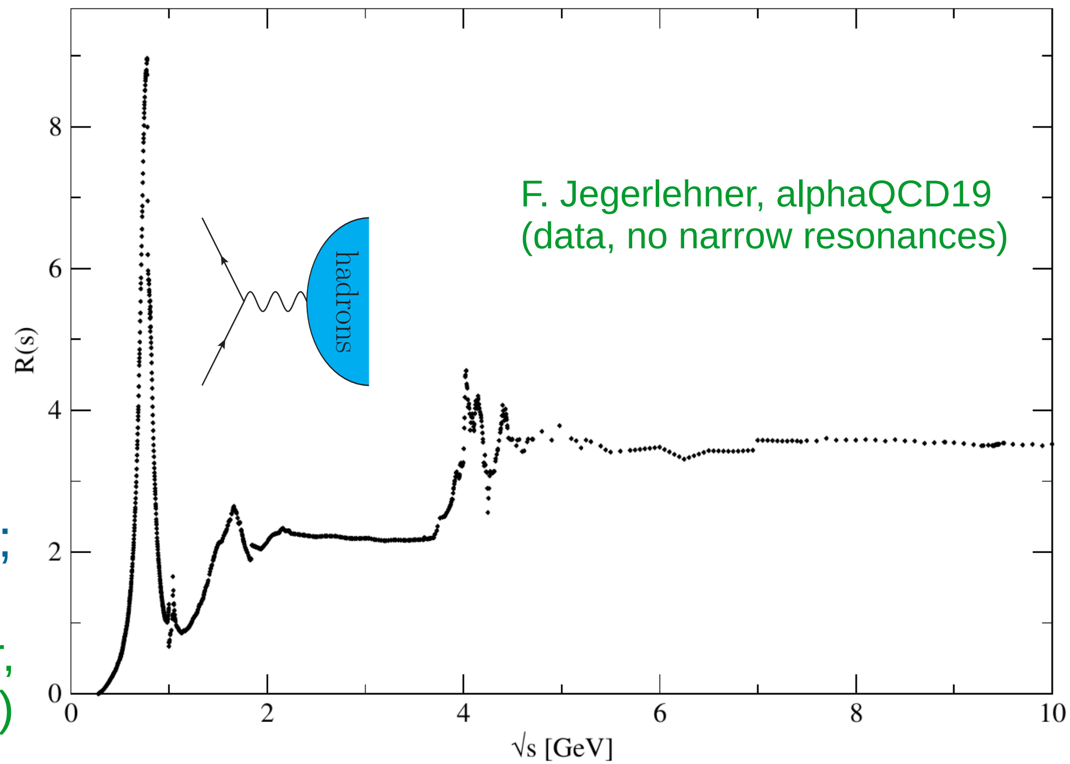
$$\overline{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0) = -\frac{Q^2}{\pi}\int_{t_0}^{\infty} \mathrm{d}t\, \frac{\mathrm{Im}\Pi(t)}{t(t+Q^2-i0)},$$

- (Semi-)empirical evaluation

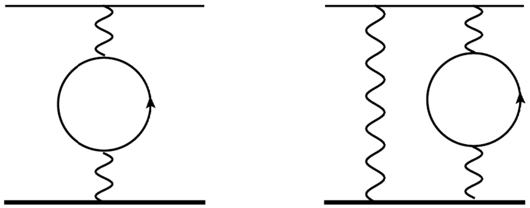
$$\mathrm{Im}\Pi(t) = -\frac{\alpha}{3}R(t)$$

$$R(t) = \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$

- A few compilations on the market; we use those of F. Jegerlehner (αQCD19) and M. Davier, A. Hoecker, B. Malaescu, Z. Zhang (DHMZ, 2019)



HVP for Point-Like Nucleus



- In the point-like limit

$$E_{nS}^{\mathrm{VP}} = -\frac{4Z\alpha}{n^3 a^3}\Pi'(0) + O(Z^5\alpha^6) \quad \text{Lamb shift contribution}$$

$$\Pi'(0) = -\frac{1}{\pi}\int_{t_0}^{\infty} \mathrm{d}t\,\frac{\mathrm{Im}\Pi(t)}{t^2}$$

$$a = (Z\alpha m_r)^{-1}$$

$$E_{\mathrm{F}} = \frac{8Z}{3a^3}\frac{1+\kappa_N}{mM}$$

$$b = t/4M^2$$

$$\frac{E_{nS-\mathrm{HFS}}^{\mathrm{VP}}}{E_{\mathrm{F}}} = -\frac{2Z\alpha}{\pi^2 n^3(1+\kappa_N)}\frac{mM}{M^2-m^2}\int_{t_0}^{\infty}\frac{\mathrm{d}t}{t}\mathrm{Im}\Pi(t)W(t) \quad \text{HFS contribution}$$

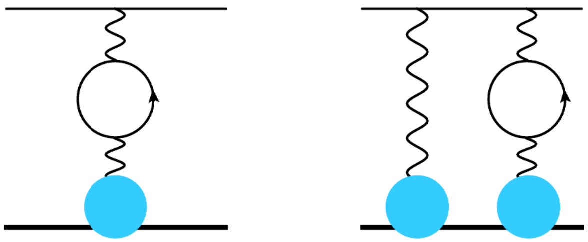
$$W(t) = -8\pi(M-m)t^{-1/2} \quad \text{in the non-recoil limit}$$

- HFS contains recoil corrections (two-photon exchange)
- HVP: typical momenta are of the order of a few m_π, comparable with muon or nuclear masses $\Longrightarrow$ one cannot use the non-recoil limit!

$$W(t) = (b+2)\sqrt{1-1/b}\,\ln\frac{1-\sqrt{1-1/b}}{1+\sqrt{1-1/b}} + (b+3/2)\ln 4b - 1/2 - (M\to m)$$

Sapirstein, Terray, Yennie (1984), Karshenboim, Shelyuto (2021)

(H)VP and Finite Size Effects



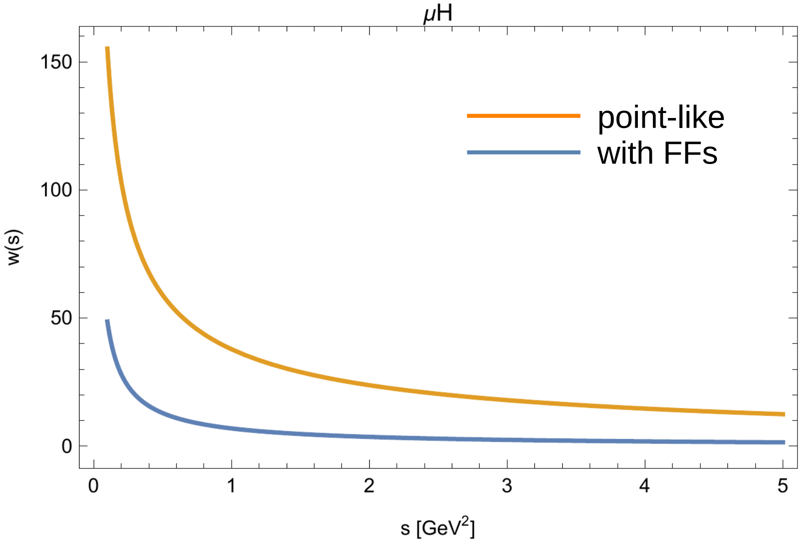
- One needs to account for the effects of the nuclear form factors
- Since FFs depend on Q^2 only, it is very convenient to treat VP in a dispersive approach
- This boils down to having a different weighting function

$$\frac{E_{\rm HFS}^{\rm Born}(nS)}{E_{\rm F}(nS)} = \frac{2Z}{1+\kappa_N}\left(\frac{\alpha}{\pi}\right)^2 \frac{mM}{M^2-m^2}\int_{s_0}^{\infty}\frac{dt}{3t}R(t)W(t)$$

$$v=\sqrt{1+4M^2/Q^2}$$
$$v_l=\sqrt{1+4m^2/Q^2}$$

$$W(t)=\int_0^{\infty}\frac{dQ}{Q}\left\{2(v-v_l)G_M(Q^2)\left(2F_1(Q^2)+\frac{F_1(Q^2)+3F_2(Q^2)}{(v_l+1)(v+1)}\right)-\left(1-\frac{m^2}{M^2}\right)\frac{5+4v_l}{(1+v_l)^2}F_2^2(Q^2)\right\}\frac{Q^2}{t+Q^2}$$

- F_1 and F_2 Dirac and Pauli, G_M magnetic FFs
- Point-like limit correspoinds to F_1=1, F_2=0
- The weighting function is suppressed!
- We implicitly assume that HVP radiative corrections are subtracted from the FFs



Results: 1S HFS

Previous work	Mu [kHz]	H [kHz]	μH [μeV]	μ^3He^+ [μeV]
Sapirstein et al. (1984)	0.22(3)			
Faustov et al. (1999)	0.2397(70)			
Czarnecki et al. (2002)	0.233(3)			
Nomura, Teubner (2013)	0.23268(144)			
Keshavarzi et al. (2020)	0.23204(82)			
Karshenboim, Shelyuto (2021)	0.236(5)			
Karshenboim (1997) point-like		0.19(8)		
finite-size		0.14(3)		
Faustov, Martynenko (1998)			3.5610	
Borie (2012)			4.8(8)	-72.7
This work				
— DHMZ R function				
non-recoil	1.236(6)*	0.395(2)	9.44(5)	-152(1)
— Jegerlehner R function (αQCD19)				
non-recoil	1.226(8)	0.392(2)	9.37(6)	-151.0(9)
with recoil	0.230(1)	0.2097(13)	5.42(3)	-110.4(7)
finite-size**	--- ---	0.0827(5)	2.07(1)	-15.49(8)

* uncertainties shown are solely due to the R function(s)
** dipole form factors with $r_E = r_M = 0.8406$ fm [H] or 1.9643 fm [^{3}He]

Results: 1S HFS

Previous work	Mu [kHz]	H [kHz]	μH [μeV]	μ^3He^+ [μeV]
Sapirstein et al. (1984)				
Faustov et al. (1999)				
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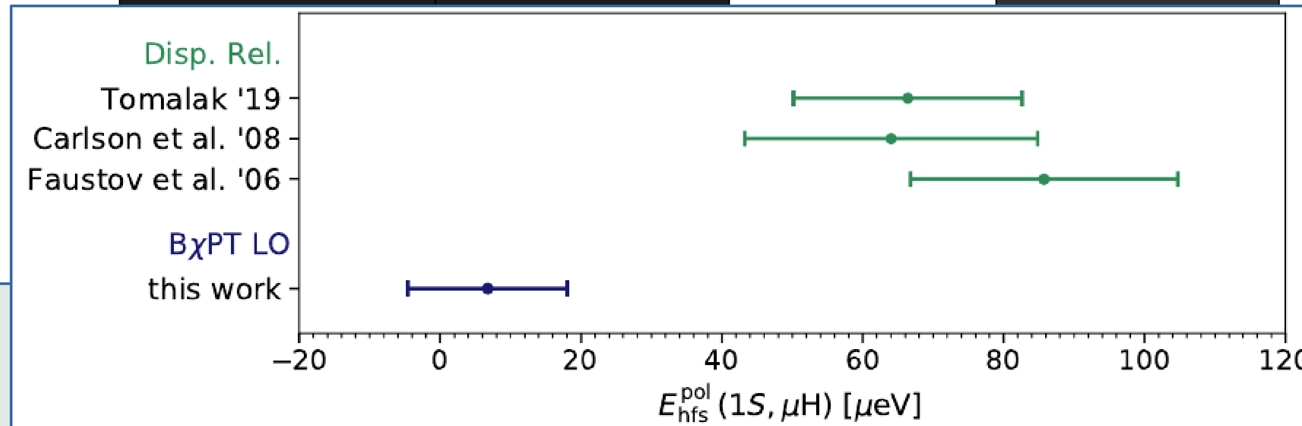
The hyperfine splitting of μH (theory update):

Antognini et al. (2022)

$$E_{1S\text{-hfs}} = \Big[\underbrace{182.443}_{E_F} \underbrace{+1.350(7)}_{\text{QED+weak}} \underbrace{+0.004}_{\text{hVP}} \underbrace{-1.30653(17)\left(\frac{r_{Zp}}{\text{fm}}\right) + E_F\left(1.01656(4)\,\Delta_{\text{recoil}} + 1.00402\,\Delta_{\text{pol}}\right)}_{2\gamma\text{ incl. radiative corr.}}\Big]\,\text{meV}$$



HVP contribution is of the order of the polarizability contribution

It is crucial to know it well!

Conclusions and Outlook

- To evaluate TPE in μH: an intricate interplay of different methods and inputs coming from many sources [ep elastic scattering, ep inclusive scattering, proton CS, …]
- Consistency checks are very important at this level of precision (e.g., verify that proton form factors give consistent R_E and R_F)
- Work is in progress in many directions
  - Form factor moments: use data directly, investigate Zemach radius
  - LS subtraction function: various approaches are consistent, work is going on, trying to achieve a better precision [Lattice? Empirical? NLO BchPT?]
  - HFS: different approaches agree, various bits and pieces are being revisited – important for a prospective improvement of precision
  - Investigating form factors and revisiting various contributions (HVP, $\pi^0\gamma$, ...)
- Stay tuned!

Thank you!

- To my collaborators for the great work
- And to you for listening!