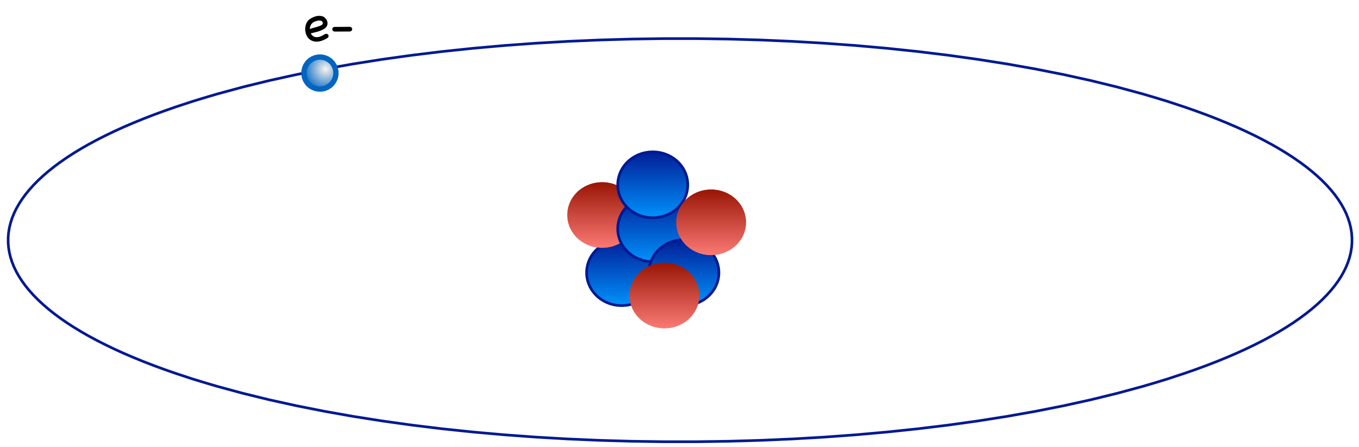
Nuclear structure corrections in light muonic atoms from ab-initio nuclear theory

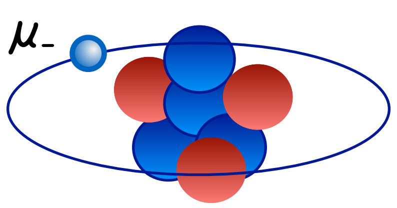
Sonia Bacca

Hydrogen-like muonic atoms

Ordinary atoms



Muonic atoms



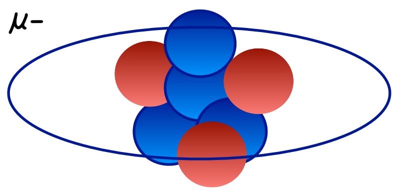
muon more sensitive to the nucleus

Can be used as a precision probe for the nucleus

- To extract charge radii from Lamb shift or 2P-1S data
- To extract Zemach radii from hyperfine splitting data
  → Chen Ji's talk

Extracting the charge radius

Strong experimental program by the CREMA collaboration to extract the nuclear charge radius from the Lamb shift measurement in **muonic atoms**



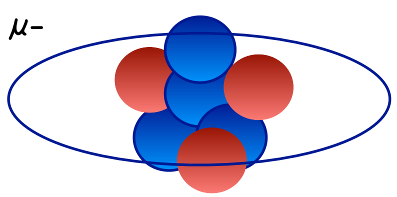
$$\Delta E_{2S-2P} = \delta_{\rm QED} + \frac{m_r^4 (Z\alpha)^4}{12} \langle r_c^2 \rangle + \delta_{\rm TPE}$$

what is measured

what you want to extract

Extracting the charge radius

Strong experimental program by the CREMA collaboration to extract the nuclear charge radius from the Lamb shift measurement in **muonic atoms**



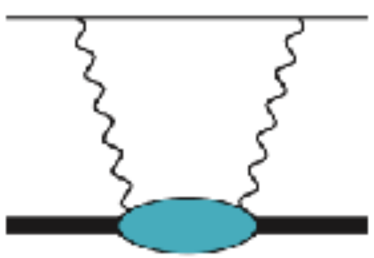
$$\Delta E_{2S-2P} = \delta_{\rm QED} + \frac{m_r^4 (Z\alpha)^4}{12} \langle r_c^2 \rangle + \delta_{\rm TPE}$$

well known — not well known

- μD $\longrightarrow$ results released in 2016
- μ^4He$^+$ $\longrightarrow$ results released in 2021
- μ^3He$^+$ $\longrightarrow$ results released in 2023
- μ^3H $\longrightarrow$ not feasible
- μ^6Li^{2+} $\longrightarrow$ future plan for QUARTET
- μ^7Li^{2+} $\longrightarrow$ future plan for QUARTET
- …

see Mehdi Drissi's talk

Two-photon exchange (TPE)

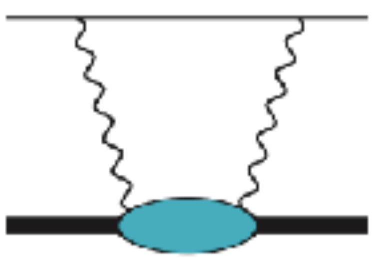


$$\delta_{\rm TPE} = \underbrace{\delta^A_{\rm Zem} + \delta^N_{\rm Zem}}_{\text{Elastic terms}} + \underbrace{\delta^A_{\rm pol} + \delta^N_{\rm pol}}_{\text{Inelastic terms}}$$

Elastic terms

Inelastic terms

Two-photon exchange (TPE)

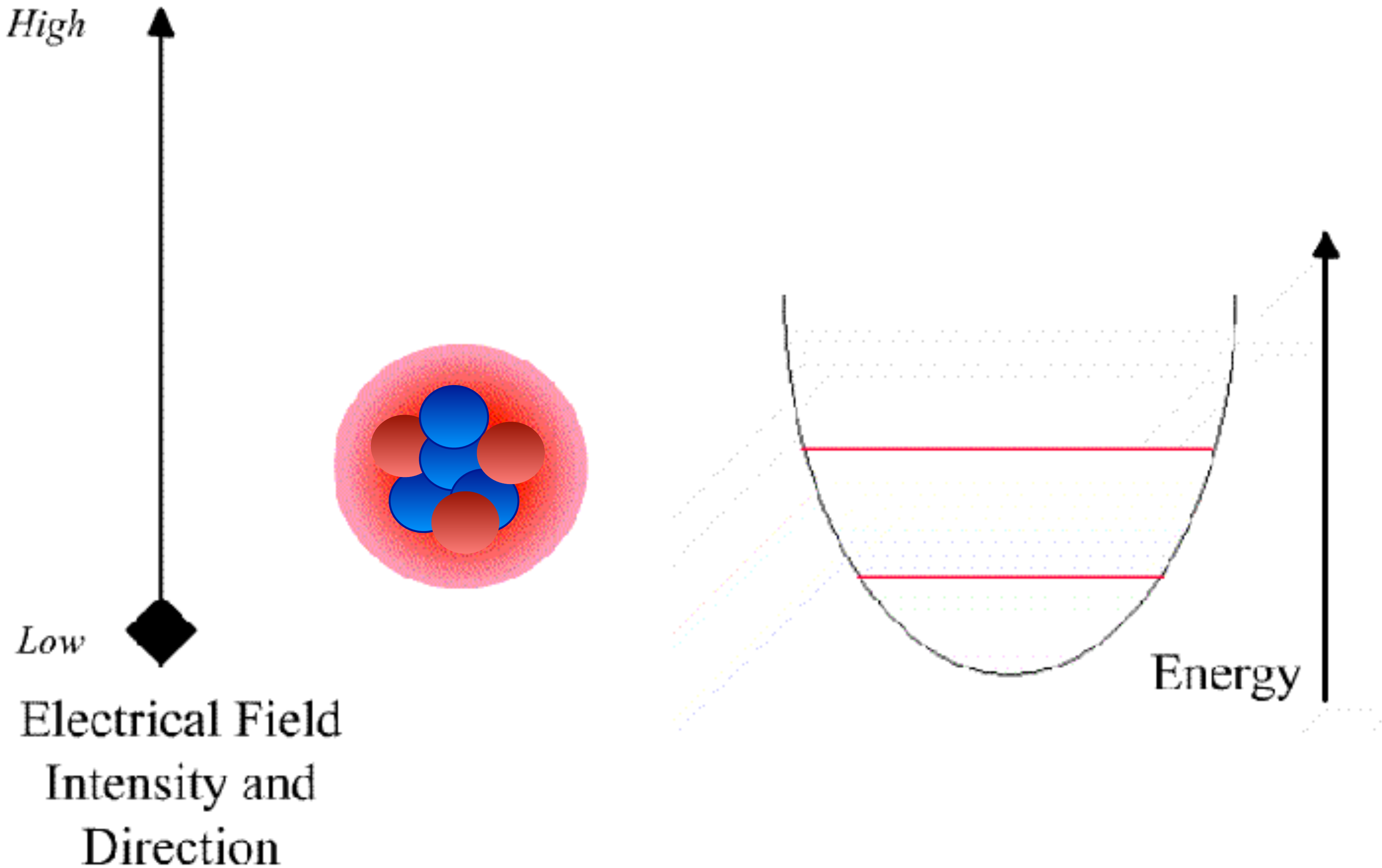


$$\delta_{\mathrm{TPE}} = \delta^{A}_{\mathrm{Zem}} + \delta^{N}_{\mathrm{Zem}} + \delta^{A}_{\mathrm{pol}} + \delta^{N}_{\mathrm{pol}}$$

A: Few-nucleon terms

N: Single-nucleon terms

Inelastic effects, aka polarizability



$$\mathbf{P} = \alpha_{\mathbf{E1}} \mathbf{E}$$

Curtesy of V. Pascalutsa

A matter of precision

The uncertainty of the extracted radius depends on the precision of the TPE

$$\Delta E_{2S-2P} = \delta_{\rm QED} + \mathcal{A}_{\rm OPE}\langle r_c^2 \rangle + \delta_{\rm TPE}$$

Even though, roughly: 95% 4% 1%

The uncertainty on TPE exceeds the experimental precision, hence reducing uncertainties is important

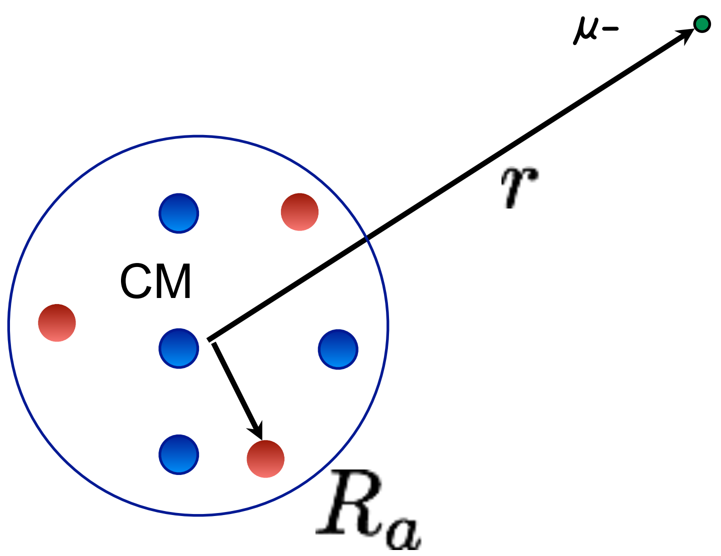
Uncertainties comparison (as of 2013)

System	ΔE_{2S-2P}	$\Delta\delta_{TPE}$
μ^2H	0.003 meV	0.03 meV
μ^3He^+	0.08 meV	1 meV
μ^4He^+	0.06 meV	0.6 meV
$\mu\,^{6,7}Li^{++}$	0.7 meV	4 meV

Theoretical derivation

$$H = H_N + H_\mu + \Delta V$$

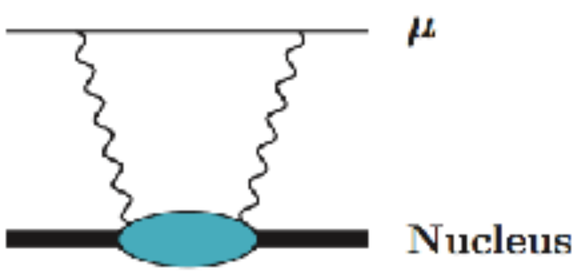
$$H_\mu = \frac{p^2}{2m_r} - \frac{Z\alpha}{r}$$



Perturbative potential: correction to the bulk Coulomb

$$\Delta V = \sum_a^Z \alpha \left(\frac{1}{r} - \frac{1}{|\vec{r} - \vec{R}_a|} \right)$$

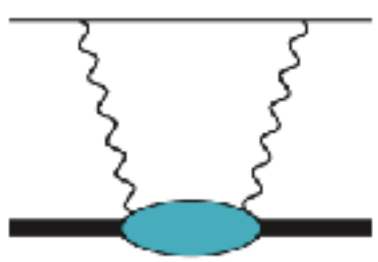
Using perturbation theory at second order
one obtains the expression for TPE
up to order $(Z\alpha)^5$



Inelastic terms (cfr. Pachucki)

- Non relativistic term

Take non-relativistic kinetic energy in muon propagator neglecting the Coulomb force in the intermediate state



$$P \simeq \frac{m_r^3 (Z\alpha)^5}{12} \sqrt{\frac{2m_r}{\omega}} \left[|\boldsymbol{R} - \boldsymbol{R}'|^2 - \frac{\sqrt{2m_r\omega}}{4} |\boldsymbol{R} - \boldsymbol{R}'|^3 + \frac{m_r\omega}{10} |\boldsymbol{R} - \boldsymbol{R}'|^4 \right]$$

η expansion $\delta^{(0)}$ $\delta^{(1)}$ $\delta^{(2)}$

★ $|\boldsymbol{R} - \boldsymbol{R}'|$ "virtual" distance traveled by the proton between the two-photon exchange

★ Uncertainty principle $|\boldsymbol{R} - \boldsymbol{R}'| \sim \frac{1}{\sqrt{2m_N\omega}}$

★ $\eta = \sqrt{2m_r\omega} |\boldsymbol{R} - \boldsymbol{R}'| \sim \sqrt{\frac{m_r}{m_N}}$

expand the muon matrix elements in powers of η up to the second order

Inelastic terms (cfr. Pachucki)

- Non relativistic term

★ $\delta^{(0)} \propto |\boldsymbol{R} - \boldsymbol{R}'|^2$

dominant term, related to the energy-weighted integral

$$\delta^{(0)}_{D1} = -\frac{2\pi m_r^3}{9}(Z\alpha)^5 \int_{\omega_{\rm th}}^{\infty} d\omega \sqrt{\frac{2m_r}{\omega}} S_{D_1}(\omega)$$

of the dipole response function

$$S_{D1}(\omega) = \frac{1}{2J_0+1} \sum\!\!\!\!\!\!\!\int_{N\neq N_0} |\langle NJ||\hat{D}_1||N_0J_0\rangle|^2 \delta(\omega - \omega_N)$$

Inelastic terms (cfr. Pachucki)

- Non relativistic term

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dominant term, related to the energy-weighted integral

$$\delta^{(0)}_{D1} = -\frac{2\pi m_r^3}{9}(Z\alpha)^5 \int_{\omega_{\rm th}}^{\infty} d\omega \sqrt{\frac{2m_r}{\omega}} S_{D_1}(\omega)$$

★ $\delta^{(1)} \propto |\boldsymbol{R} - \boldsymbol{R}'|^3$ Related to Zemach moment elastic contribution

$$\delta^{(1)}_{Z3} = \frac{\pi}{3} m_r (Z\alpha)^2 \phi^2(0) \iint d^3R d^3R' |\boldsymbol{R} - \boldsymbol{R}'|^3 \rho_0^p(\boldsymbol{R}) \rho_0^p(\boldsymbol{R}')$$

★ $\delta^{(2)} \propto |\boldsymbol{R} - \boldsymbol{R}'|^4$

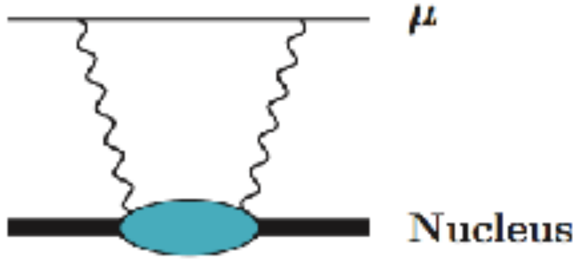
leads to energy-weighted integrals of three different response functions

$S_{R^2}(\omega), S_Q(\omega), S_{D1D3}(\omega)$

Inelastic terms (cfr. Pachucki)

Coulomb term

Consider the Coulomb force in the intermediate states
Naively $\delta_C^{(0)} \sim (Z\alpha)^6$, actually logarithmically enhanced
$\delta_C^{(0)} \sim (Z\alpha)^5 \log(Z\alpha)$ Friar (1977), Pachucki (2011)
Related to the dipole response function



Relativistic terms

Take the relativistic kinetic energy in muon propagator
Related to the dipole response function

$$\delta_{L(T)}^{(0)} = \frac{2m_r^3}{9}(Z\alpha)^5 \int_{\omega_{\rm th}}^{\infty} d\omega\, K_{L(T)}\left(\frac{\omega}{m_r}\right) S_{D_1}(\omega)$$

Finite nucleon-size corrections

Consider finite nucleon-size by including their charge distributions and obtain terms, e.g.,

$$\delta_{R1}^{(1)} = -8\pi m_r (Z\alpha)^2 \phi^2(0) \int\int d^3R d^3R' |\boldsymbol{R} - \boldsymbol{R}'| \left[\frac{2}{\beta^2}\rho_0^{pp}(\boldsymbol{R}, \boldsymbol{R}') - \lambda \rho_0^{np}(\boldsymbol{R}, \boldsymbol{R}')\right]$$

Elastic + Inelastic

$$\delta_{\mathrm{TPE}} = \delta^{A}_{\mathrm{Zem}} + \delta^{n}_{\mathrm{Zem}} + \delta^{A}_{\mathrm{pol}} + \delta^{n}_{\mathrm{pol}}$$

$$\delta^{A}_{\mathrm{pol}} = \delta^{(0)}_{D1} + \delta^{(1)}_{R3} + \cancel{\delta^{(1)}_{Z3}} + \delta^{(2)}_{R^2} + \delta^{(2)}_{Q} + \delta^{(2)}_{D1D3} + \delta^{(0)}_{C}$$
$$+\delta^{(0)}_{L} + \delta^{(0)}_{T} + \delta^{(0)}_{M} + \delta^{(1)}_{R1} + \cancel{\delta^{(1)}_{Z1}} + \delta^{(2)}_{NS}$$

$$\delta^{A}_{\mathrm{Zem}} = -\cancel{\delta^{(1)}_{Z3}} - \cancel{\delta^{(1)}_{Z1}}$$

Friar an Payne ('97)

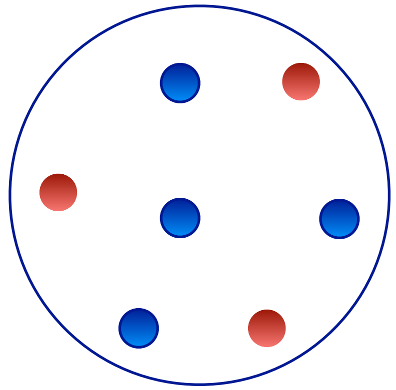
Some terms depend on the ground state of the nucleus, some (most) on inelastic response functions

Ab Initio Nuclear Theory

- Solve the Schrödinger equation for few-nucleons

$$H_N|\Psi_{NJ}\rangle = E_{NJ}|\Psi_{NJ}\rangle$$

using numerical methods that allow to assign uncertainties



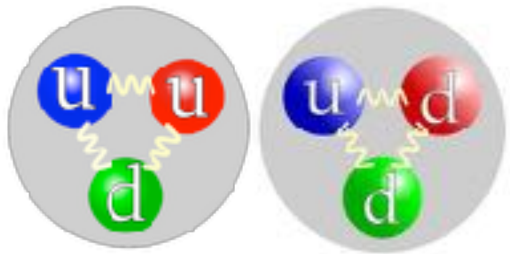
- Starting from a nuclear Hamiltonian

$$H_N = T + V$$

V Phenomenology or Chiral Effective Field Theory

There will also be an uncertainty due to the modelling of the nuclear Hamiltonian

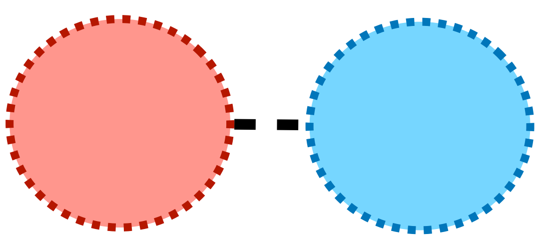
Chiral Effective Field Theory



Fundamental: Quarks/gluons

Explicit and spontaneously broken **chiral symmetry**

Weinberg



Effective: Nucleons/pions

Construct the most general theory compatible with explicit and spontaneous **chiral symmetry breaking.** **Low-energy constants** encapsulate the non-resolved high energy physics

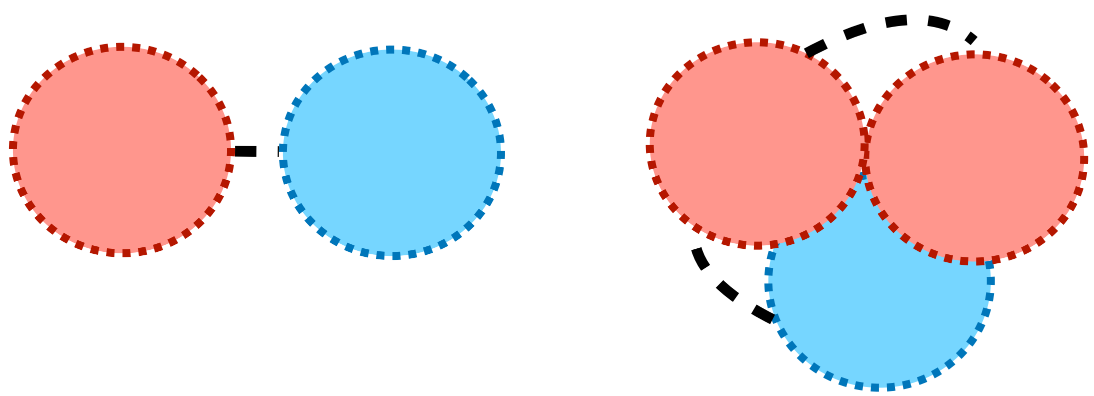
Chiral Effective Field Theory

Systematic expansion in powers of Q/Λ

$$V = V_{\rm LO} + V_{\rm NLO} + V_{\rm NNLO} \cdots$$

Three-nucleon forces appear naturally and consistently with two-nucleon forces

$$V = V_{NN} + V_{3N} + \ldots$$



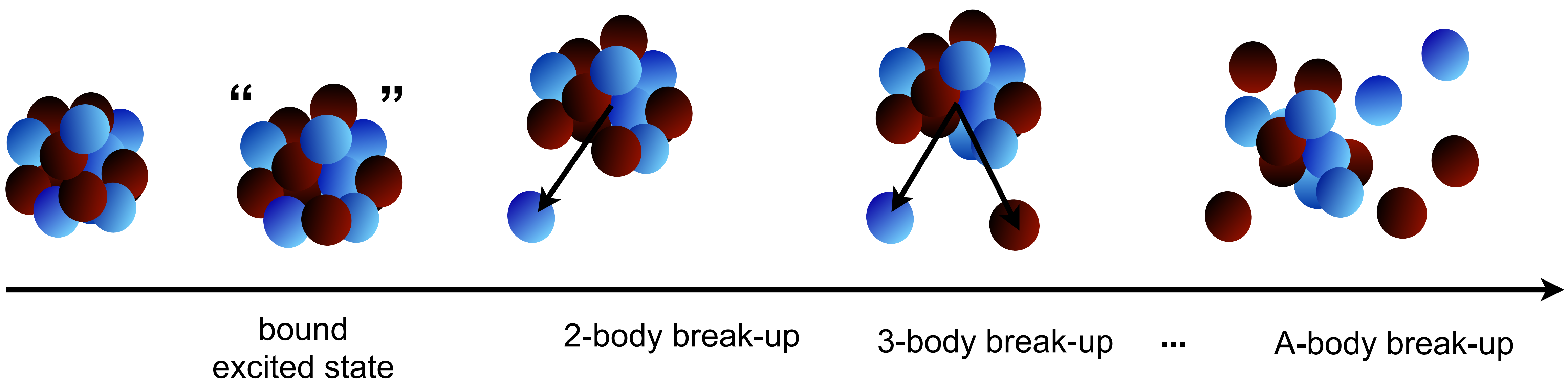
Few-body methods

Lorentz Integral Transform
and
Hypersperical Harmonics expansion

Response functions

$$S(\omega) \rightarrow |\langle N J || \hat{O} || N_0 J_0 \rangle|^2$$

Exact knowledge limited



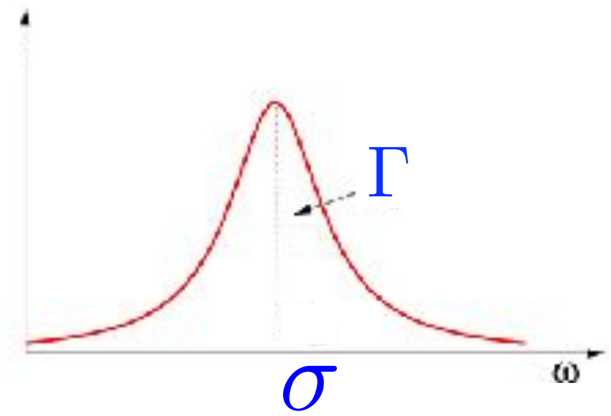
Depending on the energy, many channels involved

Lorentz Integral Transform

Efros, *et al.*, JPG.: Nucl.Part.Phys. **34** (2007) R459

inversion

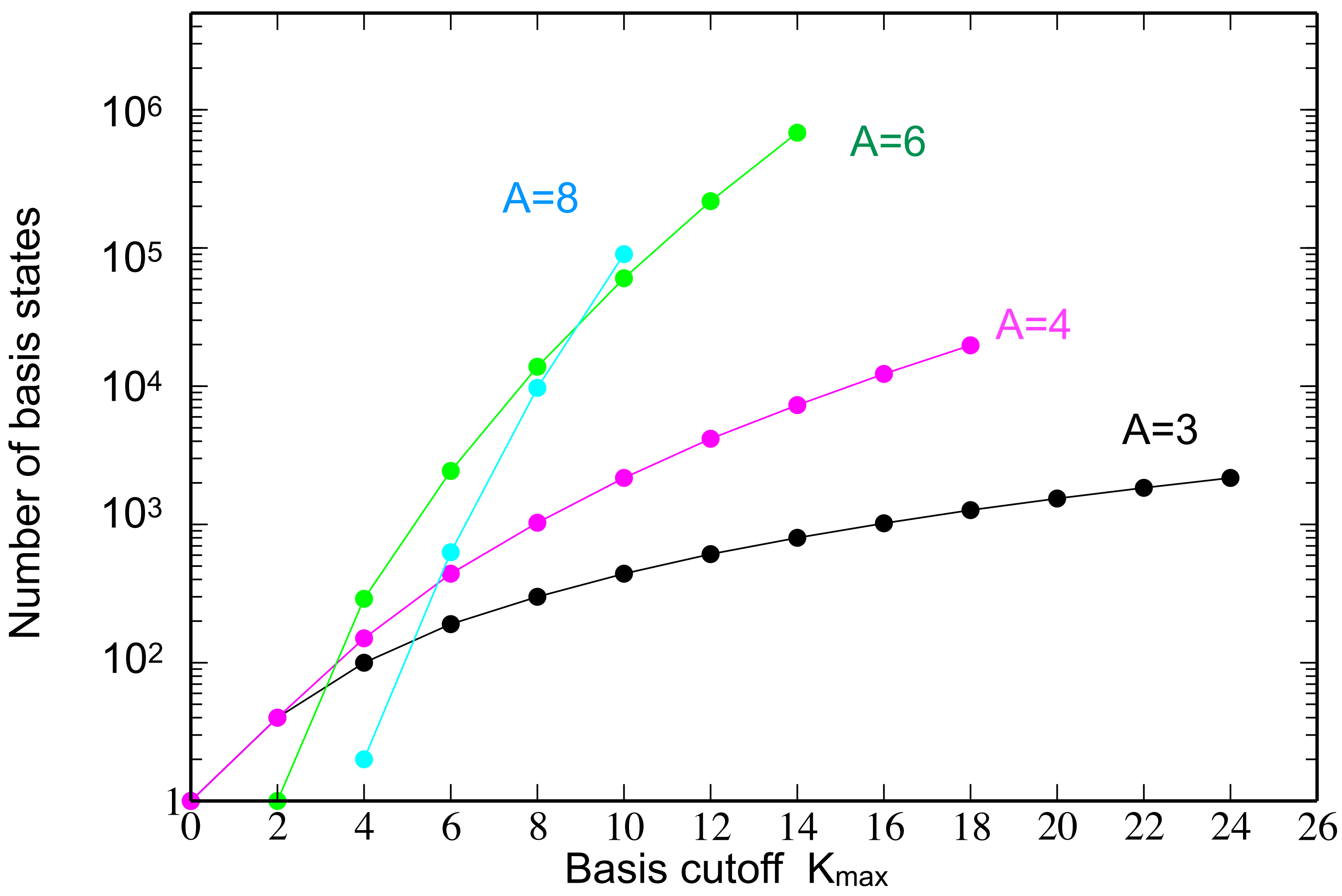
$$L(\sigma,\Gamma) = \frac{\Gamma}{\pi}\int d\omega \frac{S(\omega)}{(\omega-\sigma)^2+\Gamma^2} = \langle\tilde{\psi}|\tilde{\psi}\rangle$$



Reduce to a bound-state-like equation

$$(H - E_0 - \sigma + i\Gamma) \mid \tilde{\psi}\rangle = O \mid \psi_0\rangle$$

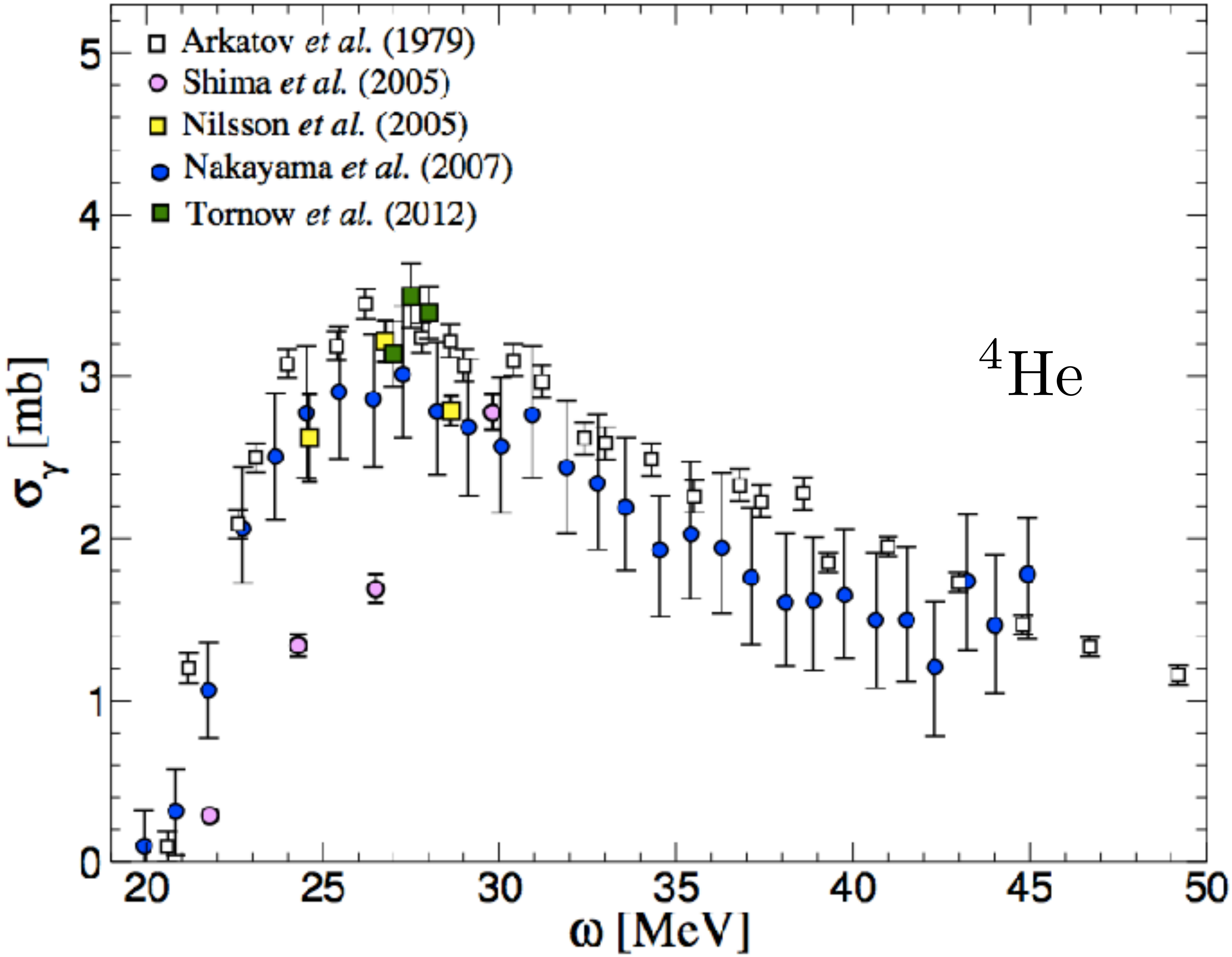
Hyperspherical Harmonics



Exact method

Bad computational scaling

An example

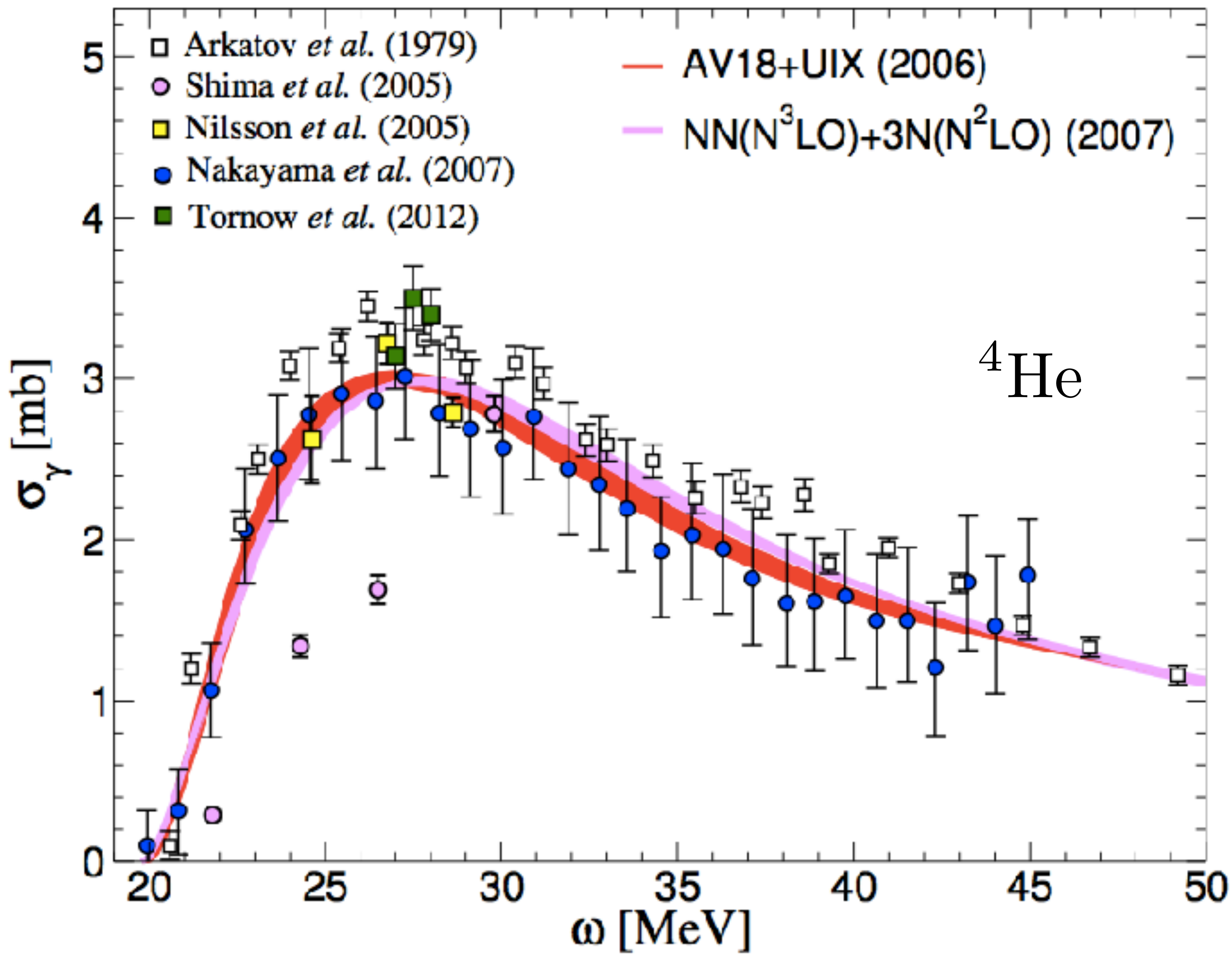


$$\delta_{D1}^{(0)} \rightarrow S_{\mathrm{D1}}(\omega)$$

$$S_{\mathrm{D1}}(\omega) = \frac{9}{16\pi^3 \alpha \omega Z^2} \sigma_\gamma(\omega)$$

SB and Saori Pastore, Journal of Physics G.: Nucl. Part. Phys. **41**, 123002 (2014)

An example



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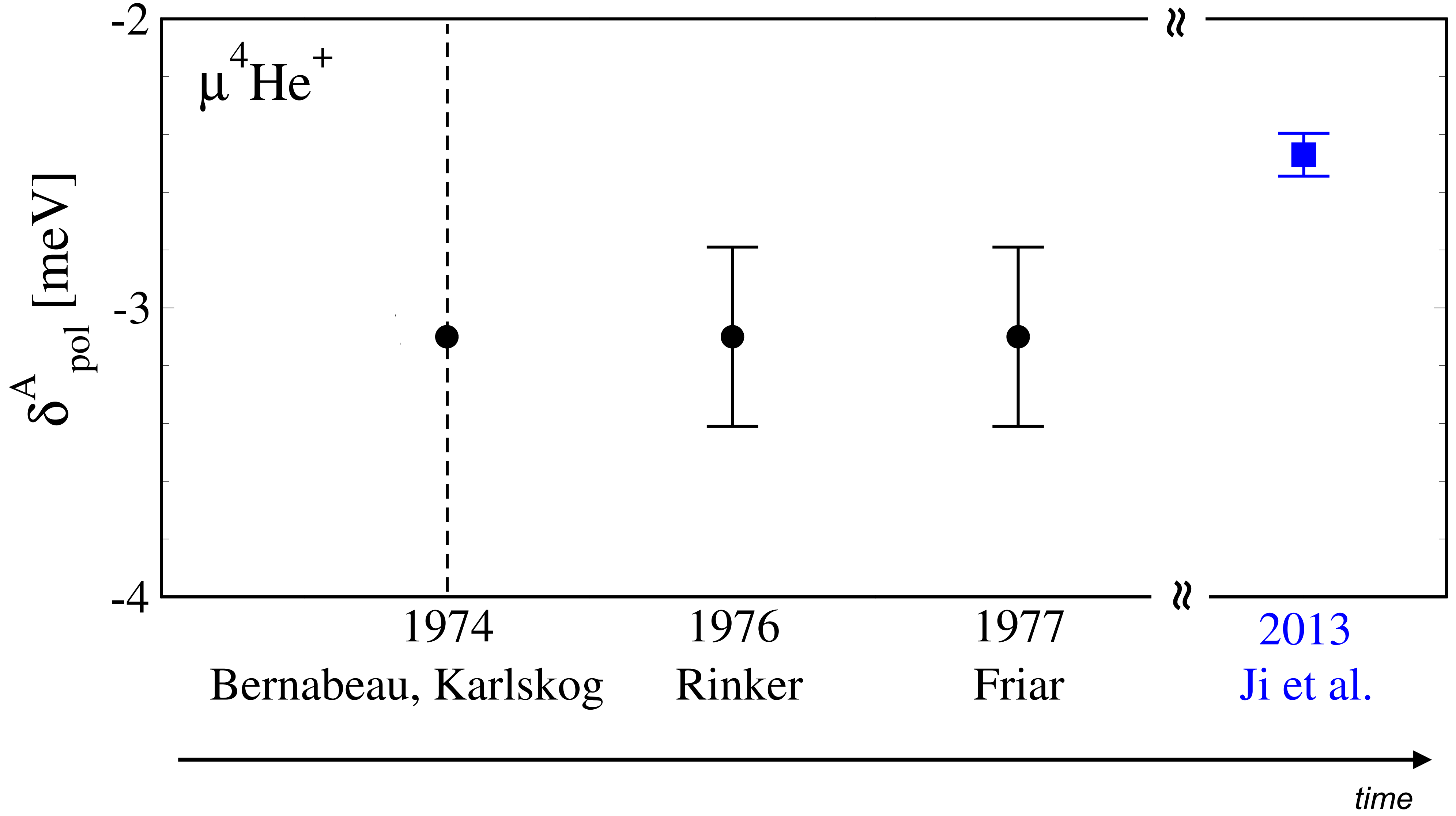
SB and Saori Pastore, Journal of Physics G.: Nucl. Part. Phys. **41**, 123002 (2014)

Use these technology to analyze muonic atoms

Impact of ab initio theory

- Reduction of Uncertainties -



C.Ji et al., JPG: Part. Nucl. **45,** 093002 (2018)

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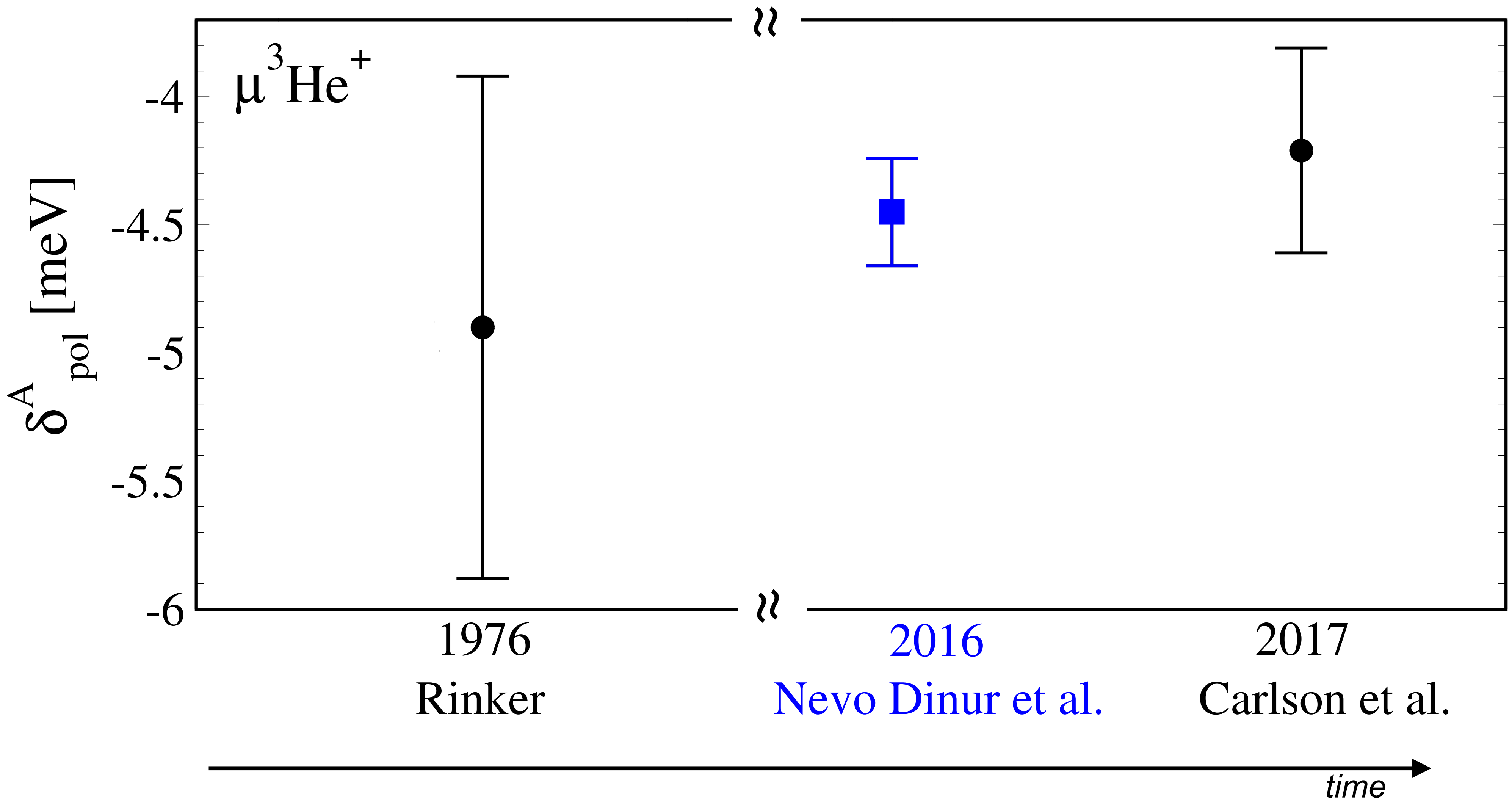
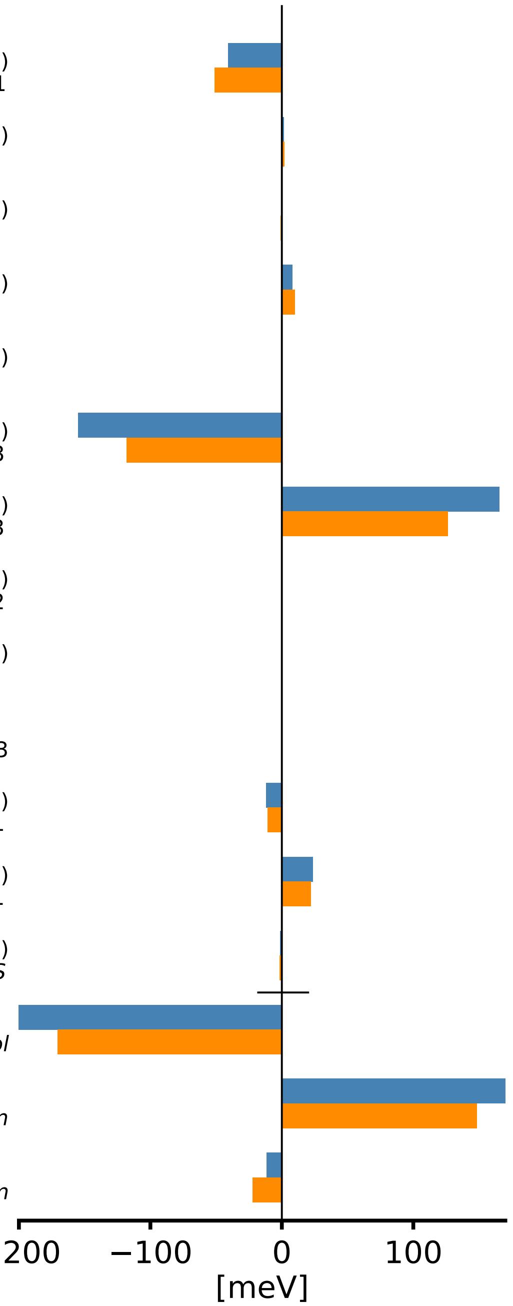


C.Ji et al., JPG: Part. Nucl. **45,** 093002 (2018)

Muonic Lithium

Li Muli, SB, Poggialini, SciPost (2020)

With AV4’ potential, in meV



	μ-^{6}Li^{2+}	μ-^{7}Li^{2+}
$\delta_{D1}^{(0)}$	-41.0	-51.0
$\delta_{L}^{(0)}$	1.66	2.04
$\delta_{T}^{(0)}$	-0.75	-0.92
$\delta_{C}^{(0)}$	7.85	9.89
$\delta_{M}^{(0)}$	-	-
$\delta_{R3}^{(1)}$	-155.0	-118.0
$\delta_{Z3}^{(1)}$	165.4	126.5
$\delta_{R2}^{(2)}$	-	-
$\delta_{Q}^{(2)}$	-	-
$\delta_{D1D3}^{(2)}$	-	-
$\delta_{R1}^{(1)}$	-12.0	-11.0
$\delta_{Z1}^{(1)}$	23.47	22.03
$\delta_{NS}^{(2)}$	-1.41	-1.75
δ_{pol}^{A}	-200.65	-170.74
δ_{Zem}^{A}	188.87	148.53
$\delta_{pol}^{A} + \delta_{Zem}^{A}$	-11.78	-22.21
Drake et al., PRA (1985)	-15 ± 4	-21 ± 4

Uncertainty quantification

C.Ji et al., JPG: Part. Nucl. **45,** 093002 (2018)

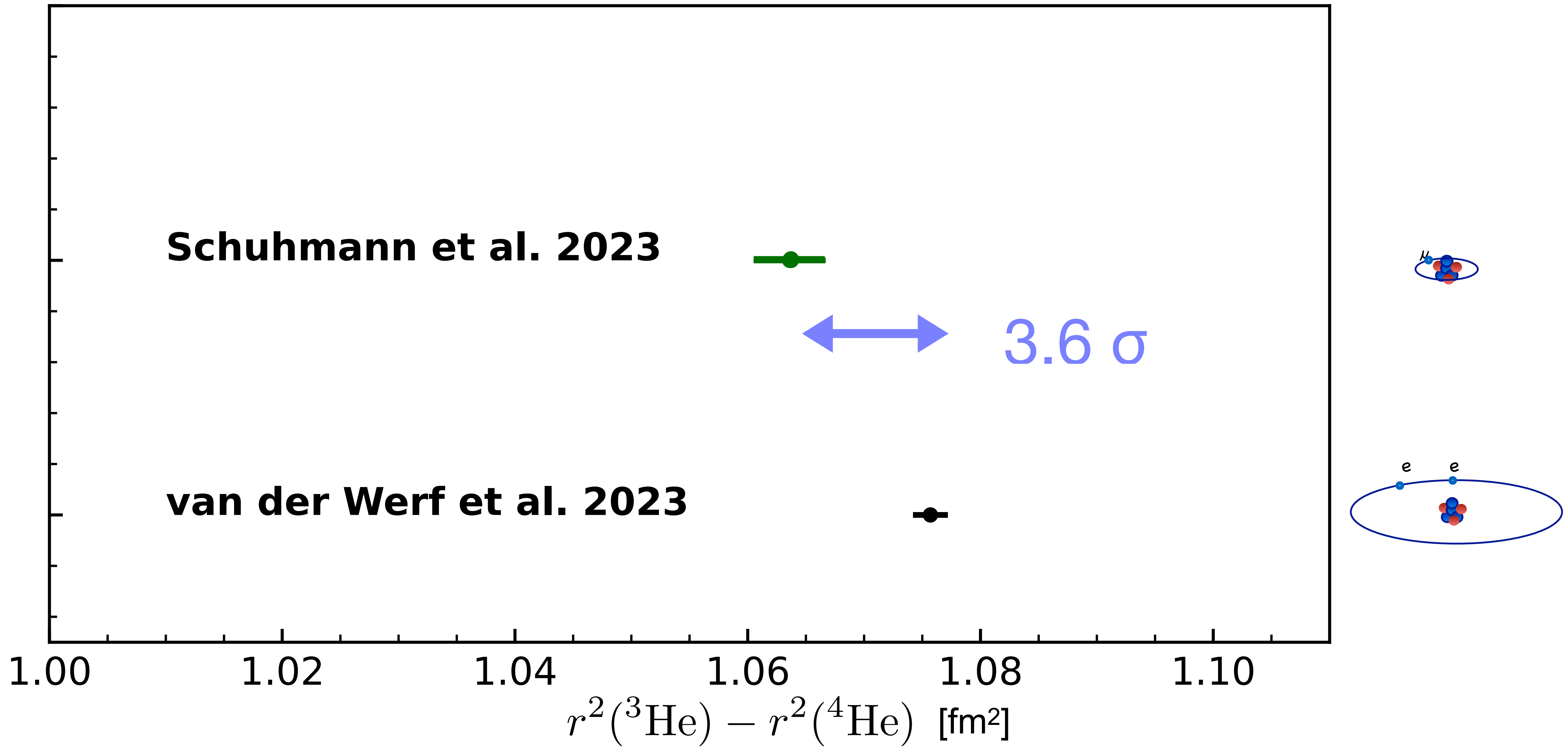
Relative % error

	μ^2H			μ^3H			μ^3He$^+$			μ^4He$^+$		
	$\delta^A_{\rm pol}$	$\delta^A_{\rm Zem}$	$\delta^A_{\rm TPE}$	$\delta^A_{\rm pol}$	$\delta^A_{\rm Zem}$	$\delta^A_{\rm TPE}$	$\delta^A_{\rm pol}$	$\delta^A_{\rm Zem}$	$\delta^A_{\rm TPE}$	$\delta^A_{\rm pol}$	$\delta^A_{\rm Zem}$	$\delta^A_{\rm TPE}$
Numerical	0.0	0.0	0.0	0.1	0.0	0.1	0.4	0.1	0.1	0.4	0.3	0.4
Nuclear-model	0.3	0.5	0.4	1.3	2.4	1.7	0.7	1.8	1.5	3.9	4.6	4.4
ISB	0.2	0.2	0.2	0.7	0.2	0.5	1.8	0.2	0.5	2.2	0.5	0.5
Nucleon-size	0.3	0.8	0.0	0.6	0.9	0.2	1.2	1.3	0.9	2.7	2.0	1.2
Relativistic	0.0	—	0.0	0.1	—	0.1	0.4	—	0.1	0.1	—	0.0
Coulomb	0.4	—	0.3	0.5	—	0.3	3.0	—	0.9	0.4	—	0.1
η-expansion	0.4	—	0.3	1.3	—	0.9	1.1	—	0.3	0.8	—	0.2
Higher $Z\alpha$	0.7	—	0.5	0.7	—	0.5	1.5	—	0.4	1.5	—	0.4
Total	1.0	0.9	0.8	2.3	2.2	2.0	4.2	2.2	2.1	5.5	5.1	4.6

Nuclear model error is dominating for ^{3,4}He, but we had not performed yet an order-by-order analysis in chiral EFT

The Helium Isotope Shift puzzle

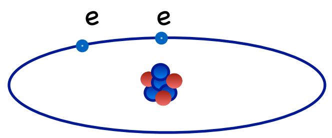
Li Muli, Richardson, SB, PRL **134** 032502 (2025)



The Helium Isotope Shift puzzle

Li Muli, Richardson, SB, PRL **134** 032502 (2025)

Nuclear structure corrections to ordinary atoms



$$\delta^{A}_{\mathrm{TPE},\,e} = -\frac{2}{3} m (Z\alpha)^2 \phi^2_{nS}\ \tilde{\alpha}_{\mathrm{pol,e}}$$

$$\tilde{\alpha}_{\mathrm{pol,e}} = \sum_{N\neq 0} |\langle N|\,\mathbf{D}\,|0\rangle|^2 \left[\frac{19}{6\omega_N} + \frac{5\ln(2\omega_N/m)}{\omega_N}\right]$$

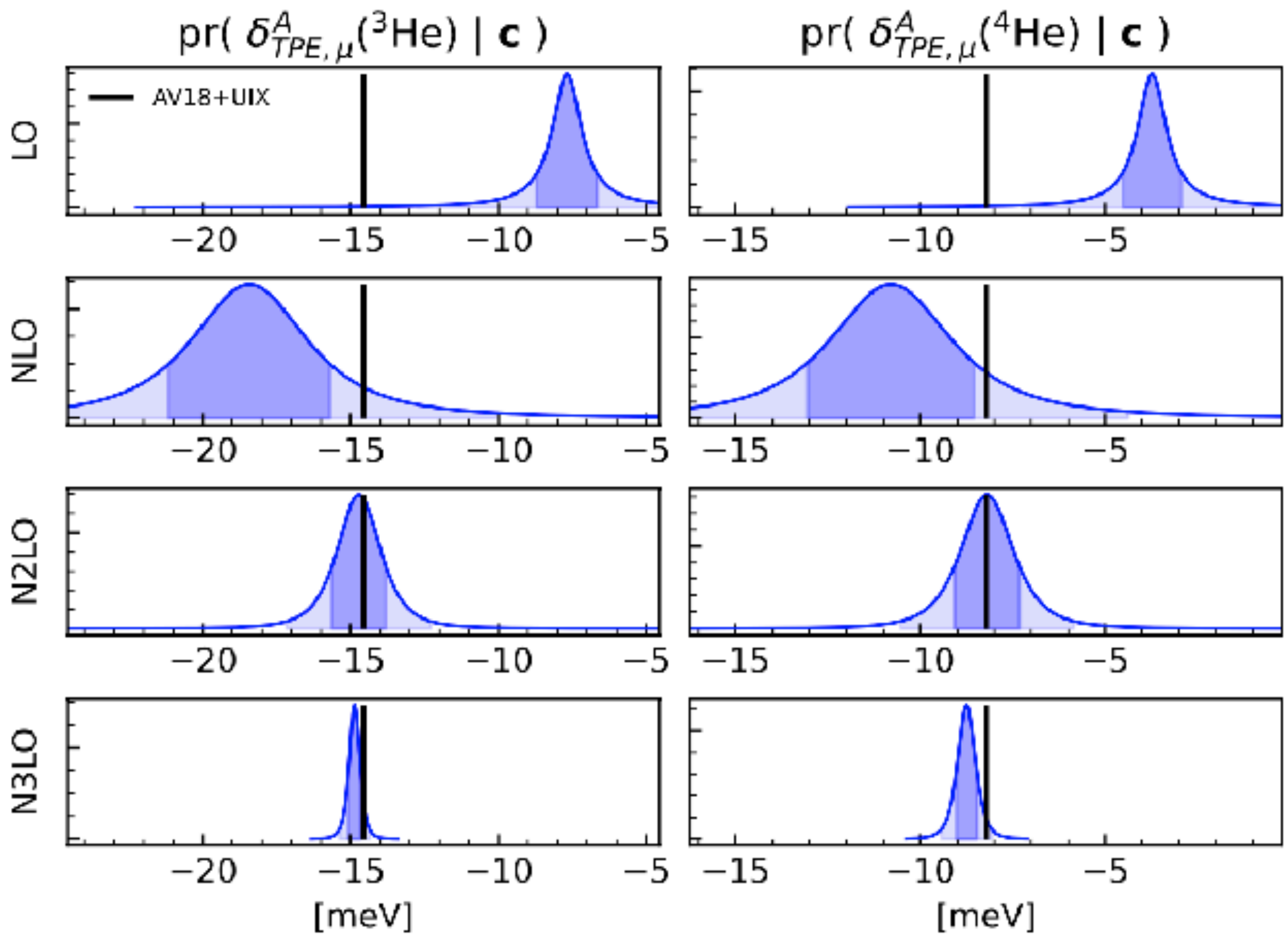
In meV	^{3}He	^{4}He
Our at N3LO	3.514(68)	1.909(96)
Pachucki, Moro (2007)	3.560(360)	2.070(200)

The Helium Isotope Shift puzzle

Li Muli, Richardson, SB, PRL **134** 032502 (2025)

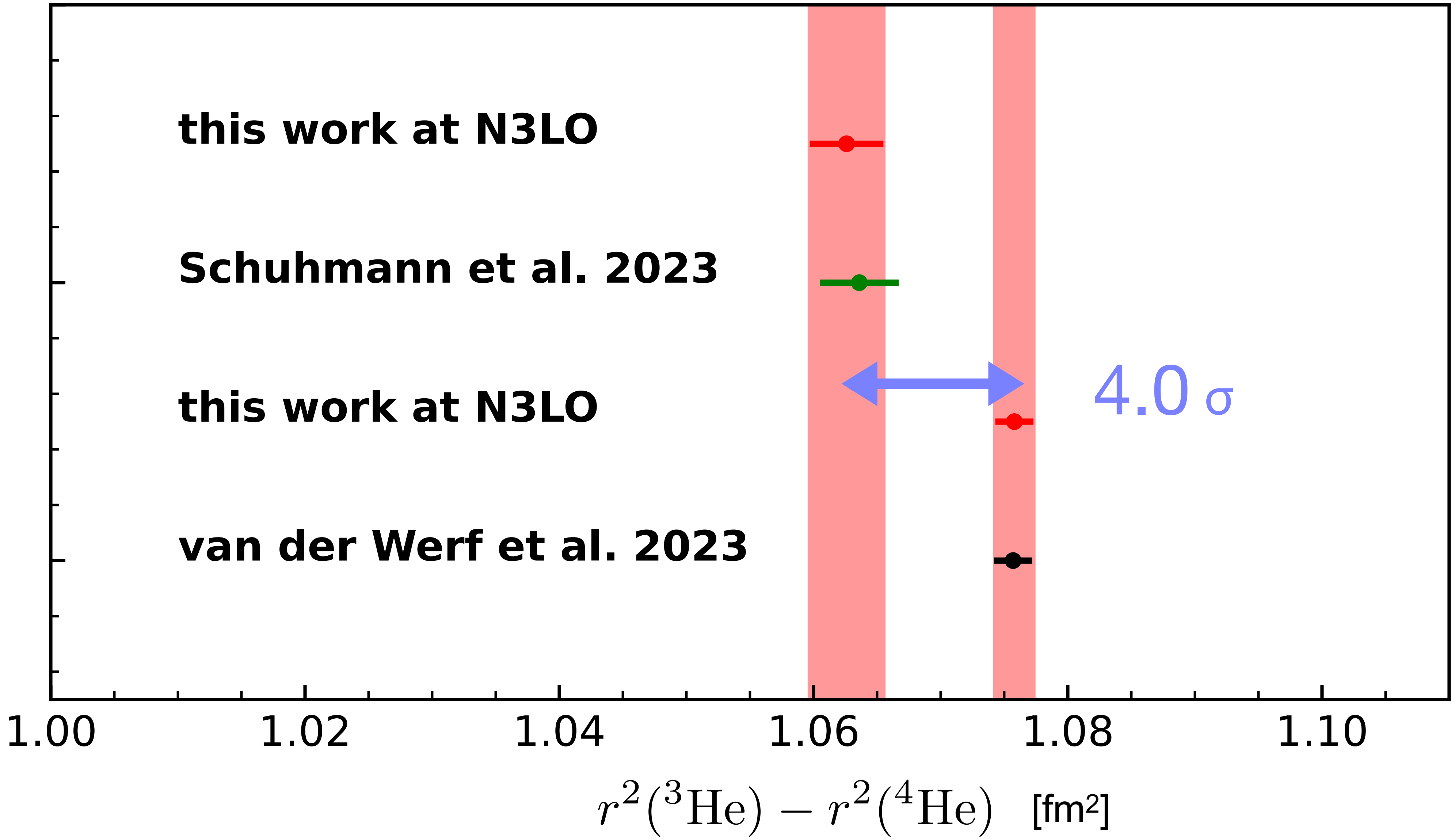
Nuclear structure corrections to muonic atoms

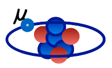
Bayesian uncertainty quantification $\delta = \delta_{\rm ref} \sum_{n=0}^{\infty} c_n (Q/\Lambda)^n$

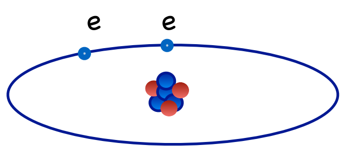


The Helium Isotope Shift puzzle

Li Muli, Richardson, SB, PRL **134** 032502 (2025)







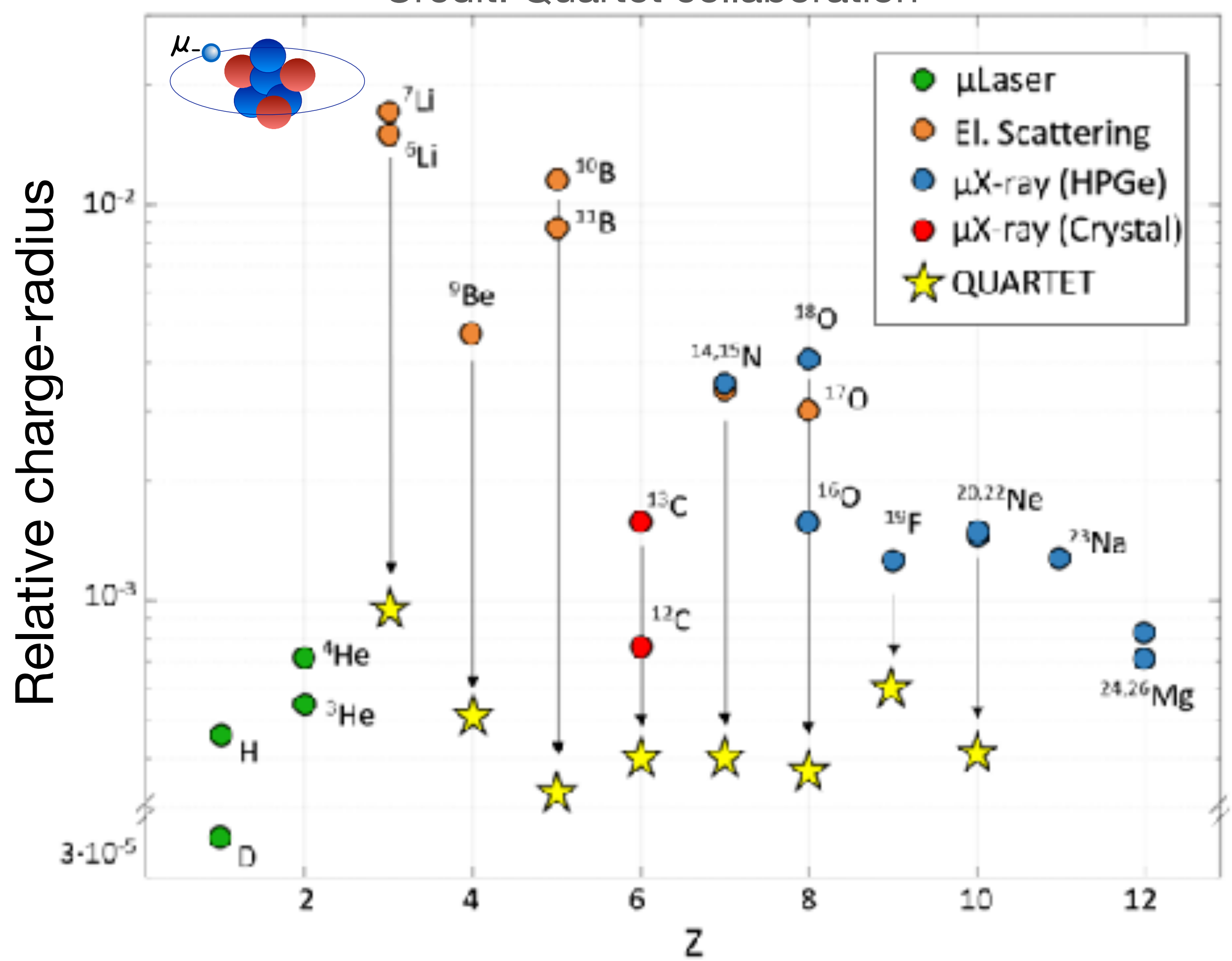
Nuclear structure does not solve the puzzle, it rather slightly enhances it

In fact, problem was in QED (missing off-diagonal hyperfine interaction effects) → Qi et al, Phys. Rev. Research **7**, L022020 (2025)

Outlook

- Ab-initio nuclear theory has allowed a strong reduction of uncertainties in the evaluation of TPE for the Lamb shift.
- Progress for HFS → see Chen Ji's talk
- Precise computations of charge and Zemach radii → see Saori Pastore's talk
- New ideas on how to use ANN → see Tim Egert's talk

Credit: Quartet collaboration



Thanks to my collaborators

THANK YOU

Backup Slides

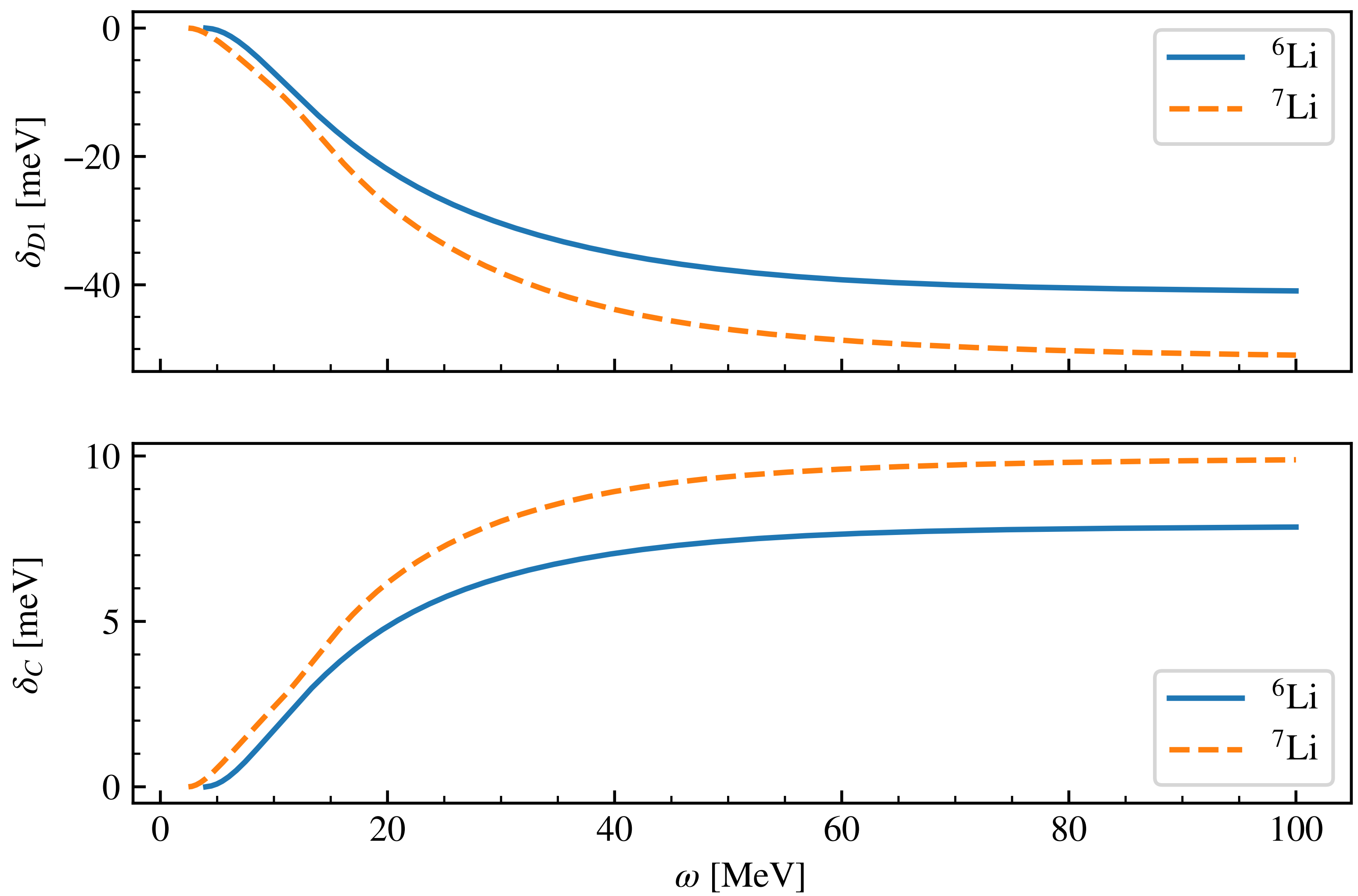
Muonic Lithium

$$\delta^{(0)}_{\mathrm{D1}} \propto \int_0^\infty d\omega \sqrt{\frac{2m_r}{\omega}} S_{\mathrm{D1}}(\omega)$$

$$\delta^{(0)}_{\mathrm{C}} \propto \int_0^\infty d\omega \frac{m_r}{\omega} \ln \frac{2(Z\alpha)^2 m_r}{\omega} S_{\mathrm{D1}}(\omega)$$

S.Li Muli,
A. Poggialini,
S.B,

SciPost (2020)



With AV4'

Semi realistic

potential