Dark matter contributions as non-topological excitations of a scalar SO(3)-theory

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Tower of Babel,
Pieter Brueghel the Elder,
Museum of Fine Arts,
Vienna.
St. Stephen's Cathedral,

ATOMINSTITUT

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The Imperial Crown,
Imperial Treasury,
Vienna

Imagine we have only Space-Time, $x \in \mathbb{M}^4$

What can we explain?

- Non-trivial metric: $g_{\mu\nu} \quad \rightarrow \quad$ Gravitation
- Rotating frames in $\mathbb{R}^3 : D(x) \in SO(3) \quad \rightarrow \quad$???

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  Topological excitations $\leftrightarrow$ Topological quantum numbers,

  $\Pi_3(\mathbb{S}^3) = \mathbb{Z} \quad \leftrightarrow$ spin?

  $\Pi_2(\mathbb{S}^3) = \mathbb{Z} \quad \leftrightarrow$ charge?

  $\Pi_2(\mathbb{S}^2) = \mathbb{Z} \quad \leftrightarrow$ photon number?

  Non-topological excitations:

  dark matter?

  dark energy?

Field variables in 3+1D

describe field of rotations of spatial Dreibein in $\mathbb{M}^4 \ni x$

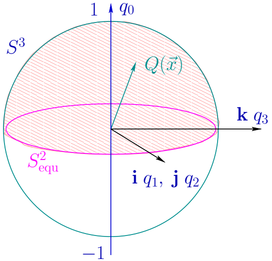
Use rotational group $D(x) \in SO(3)$

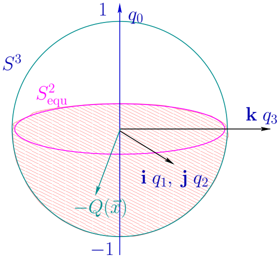
or simpler double covering group of SO(3): SU(2) $\ni Q(x)$

SO(3) versus SU(2)$\simeq \mathbb{S}^3$, $D(x) \leftrightarrow \pm Q(x)$

Field configurations $\pm\{Q(x)\}$ are identical

Two hemispheres of $\mathbb{S}^3$



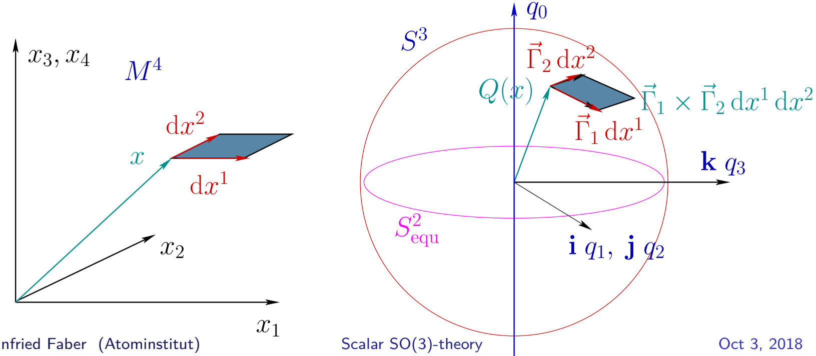


$$Q(\vec{x}) = q_0(\vec{x}) - \mathrm{i}\vec{\sigma}\vec{q}(\vec{x}) = \cos\alpha(\vec{x}) - \mathrm{i}\vec{\sigma}\vec{n}(\vec{x})\sin\alpha(\vec{x}), \qquad q_0^2 + \vec{q}^{\,2} = 1$$

Lagrangian

basic field: $\pm Q(x) \in SU(2)$
Connection one-form: $\partial_\mu Q(x) Q^\dagger(x) =: -\mathrm{i}\vec{\Gamma}_\mu(x)\,\vec{\sigma}$
Curvature: $\vec{R}_{\mu\nu} := \vec{\Gamma}_\mu(x) \times \vec{\Gamma}_\nu(x)$
Lagrangian: $\mathcal{L} = -\kappa\left(\frac{1}{4}\,\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \Lambda\right)$

Skyrme term: $-\frac{\kappa}{4}\,\vec{R}_{\mu\nu}(x)\vec{R}^{\mu\nu}(x)$
potential term: $\Lambda(x)$



Maurer-Cartan equation

from

$$\partial_\mu \partial_\nu Q(x) = \partial_\nu \partial_\mu Q(x)$$

we get the Maurer-Cartan equation

$$\partial_\mu \vec{\Gamma}_\nu - \partial_\nu \vec{\Gamma}_\mu - 2\vec{\Gamma}_\mu \times \vec{\Gamma}_\nu = 0$$

Interpretation: vector field $\vec{\Gamma}_\mu(x)$ can be derived from $Q(x)$,
12 dof of $\vec{\Gamma}_\mu(x)$ can be derived from 3 dof of $Q(x)$.

Therefore (in analogy to QCD and Gravitation)
tensor field $\vec{R}_{\mu\nu} := \vec{\Gamma}_\mu \times \vec{\Gamma}_\nu = \partial_\mu \vec{\Gamma}_\nu - \partial_\nu \vec{\Gamma}_\mu - \vec{\Gamma}_\mu \times \vec{\Gamma}_\nu$.

$\vec{R}_{\mu\nu}$ is gauge covariant (gauge = choice of tangential basis on $\mathbb{S}^3$)

$\vec{\mathcal{A}}_\mu = 2\,\vec{\Gamma}_\mu$ is a trivial connection.

Lagrangian and Hobart-Derrick theorem

$$\mathcal{L}_{\mathrm{curv}} = -\frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} \quad \text{Skyrme term}$$

scaling argument

scale $x \to \lambda x$,

$\int \mathrm{d}^3 x \mathcal{L}_{\mathrm{curv}} \to \frac{1}{\lambda} \int \mathrm{d}^3 x \mathcal{L}_{\mathrm{curv}}$

no stability of solitons $\to$ solitons dissolve

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Hobart-Derrick theorem: compressing term needed

$\mathcal{L}_{\rm kin} = \kappa(\vec{\Gamma}_\mu)^2, \qquad \int \mathrm{d}^3x \mathcal{L}_{\rm kin} \rightarrow \lambda \int \mathrm{d}^3x \mathcal{L}_{\rm kin}$

Skyrme model: $\mathcal{L} = \mathcal{L}_{\rm curv} - \mathcal{L}_{\rm kin}$

stable solitons = Skyrmions $\rightarrow$ short range interaction

monopoles (long range interaction) get infinite energy from $r = \infty$

$\Rightarrow$ NO monopoles

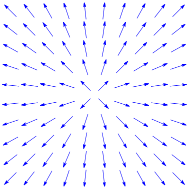
Dirac monopoles

two types of singularities in Dirac description: $A_\mu(x)$

- Dirac string
- Center singularity

one type of singularities in Wu-Yang description: $\vec{n}(x) = \frac{\vec{x}}{|\vec{x}|}$

- Center singularity



no singularities in SU(2)-field description: $Q(\vec{x})$

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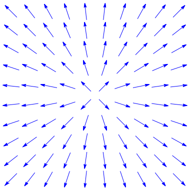

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$\Rightarrow$ singularities of inappropriate fields

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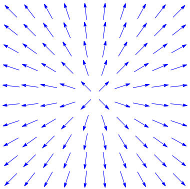

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observe: monopole charges are quantised

Potential $\Lambda(Q)$

term without derivatives $\quad\Rightarrow\quad \int \mathrm{d}^3x\,\Lambda(Q) \stackrel{x\to\lambda x}{\rightarrow} \lambda^3 \int \mathrm{d}^3x\,\Lambda(Q)$

$\Lambda(Q(\infty)) = 0, \quad Q(\vec{x}) = \cos\alpha(\vec{x}) - \mathrm{i}\vec{\sigma}\vec{n}(\vec{x})\sin\alpha(\vec{x})$

$\alpha(\infty) = \frac{\pi}{2}, \quad Q(\infty) = -\mathrm{i}\vec{\sigma}\vec{n}, \quad \vec{n} = \frac{\vec{x}}{|\vec{x}|}$ Dirac (Wu-Yang) monopole

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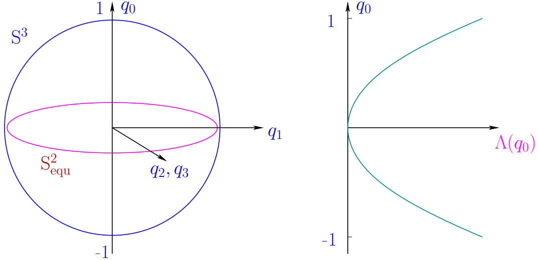

$\Lambda(q_0) = \frac{q_0^{2m}}{r_0^4},\ m = 1, 2, 3, \cdots$ scale r_0, $\rho = r/r_0$

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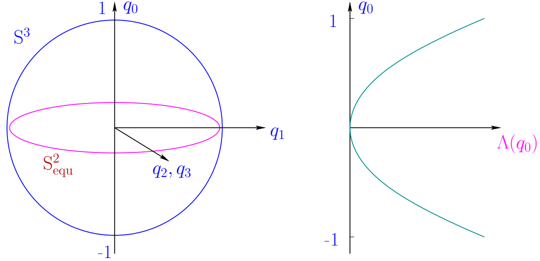

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Consequences: two-dimensional degeneracy of vacuum $\rightarrow$

$\rightarrow$ two Goldstone bosons = photons

Stable minima of energy (topological Solitons)

- hedgehog ansatz: $\vec{n}(x) = \frac{\vec{r}}{r}$, $\qquad x = (ct, \mathbf{r})$

  $Q(x) = \cos\alpha(x) + \mathrm{i}\vec{\sigma}\vec{n}(x)\sin\alpha(x), \quad \text{with} \quad \alpha = \alpha(\rho), \qquad \rho = r/r_0$
- minimisation of energy leads to non–linear differential equation

$$\partial_\rho^2 \cos\alpha + \frac{(1-\cos^2\alpha)\cos\alpha}{\rho^2} - m\rho^2\cos^{2m-1}\alpha = 0$$

- solution for $m = 3$

$$\alpha(\rho) = \arctan(\rho).$$

- energy of soliton $\qquad E = \kappa\frac{\pi^2}{r_0}$
- compare with monopoles?

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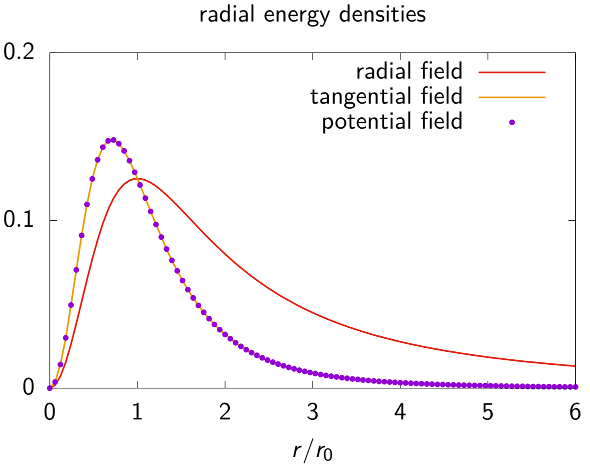
$$\kappa = \frac{\alpha_f \hbar c}{4\pi} = 0.116\,\mathrm{MeV\,fm},\ m_e c^2 = 0.511\ \mathrm{MeV},\ r_0 = 2.21\ \mathrm{fm}$$

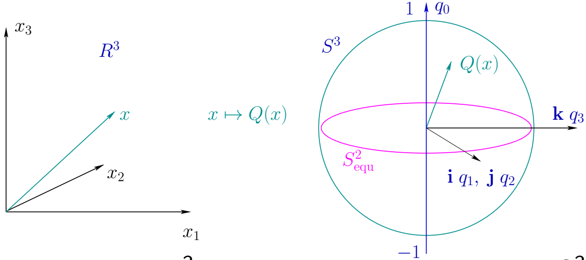
Energy densities

$$\mathcal{L} = -\kappa\left(\frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \frac{q_0^6}{r_0^4}\right), \quad q_0(\rho) = \cos\alpha(\rho) = \frac{1}{\sqrt{1+\rho^2}}$$



particle and field are indistinguishable

Topological quantum numbers



Volume element on R^3	$\mapsto$	Volume element on S^3
$\mathrm{d}r\,\mathrm{d}\vartheta\mathrm{d}\varphi$	$\mapsto$	$\vec{\Gamma}_r(\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi)\,\mathrm{d}r\,\mathrm{d}\vartheta\mathrm{d}\varphi$

Topological charge $\mathcal{Q}$ $\equiv$ number of coverings of S^3

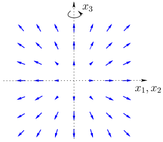
$$\mathcal{Q} = \frac{1}{V(\mathbb{S}^3)} \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \vec{\Gamma}_r(\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi)$$

$$V(\mathbb{S}^3) = \int_{S^2} \mathrm{d}^2 n \int_0^\pi \mathrm{d}\alpha \sin^2\alpha = 2\pi^2$$

Topological charge is conserved: $\frac{\mathrm{d}\mathcal{Q}(t)}{\mathrm{d}t} = 0$

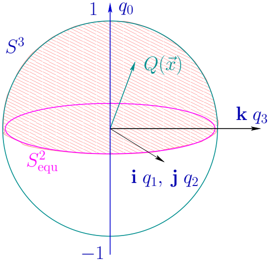
Hedge-hog configuration: $|\mathcal{Q}| = \frac{1}{2}$

Spin, a topological quantum number



Field of $\vec{q}$ vectors

$R^3 \;\to\; \mathbb{S}^3$



Field configuration $Q(\mathbf{r})$ of unit charge covers hemisphere of $\mathbb{S}^3$, $s = \frac{1}{2}$.

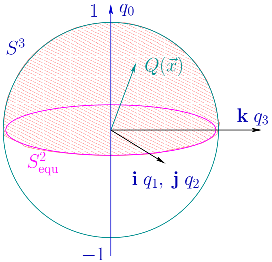
Spin quantum number s

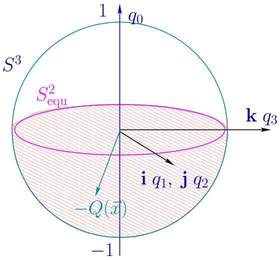
$$s := |\mathcal{Q}| = \left| \frac{1}{V(\mathbb{S}^3)} \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \vec{\Gamma}_r (\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi) \right|$$

Magnetic quantum numbers $m_s = \pm 1/2$: Upper and lower hemisphere

SO(3) versus SU(2)

Two hemispheres of $\mathbb{S}^3$





topologically different Hedge-hog configurations

Use rotational group SO(3)

$\Rightarrow$ $\pm\{Q(x)\}$ are identical,

$\Rightarrow$ interpretation: rotations of spatial Dreibein.

Calculations simpler with SU(2)

no double counting of $\pm\{Q(x)\}$!

the above two configurations are identical

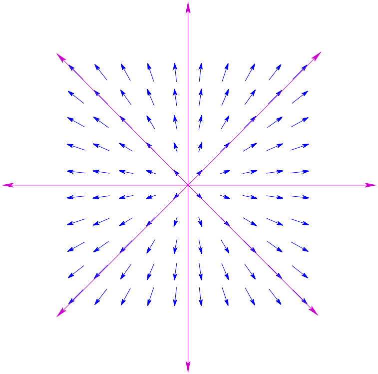
Monopole is wired to surrounding space

flux lines $\equiv$ lines of constant $\vec{n}$-field

flux lines $\equiv$ strings

they connect the soliton with the surrounding,

with other charges

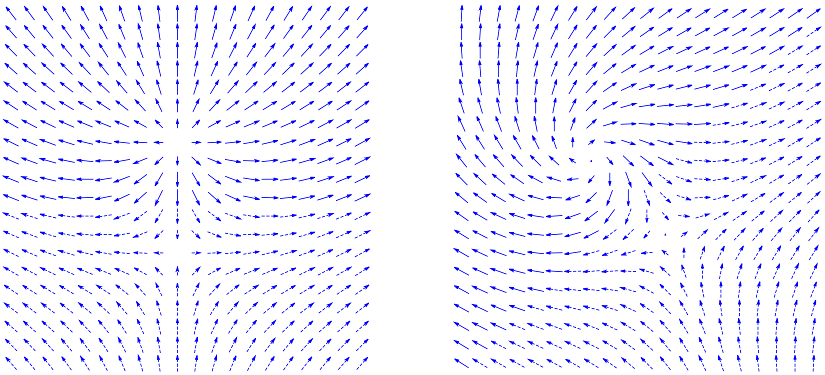


after 4π-rotation
soliton configuration is restored

a property of spin-$1/2$-particles

Spin, an angular momentum

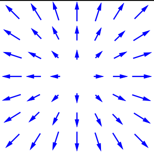
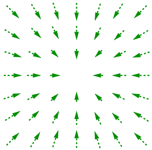
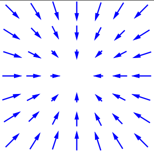
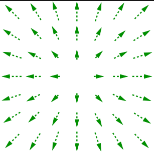
Symmetry broken vacuum, $Q(\infty) = -\mathrm{i}\sigma_3$, Field at infinity is constant
No rigid rotation possible



$S = 1$, Charge Zero

Types of solitons

up to global rotations

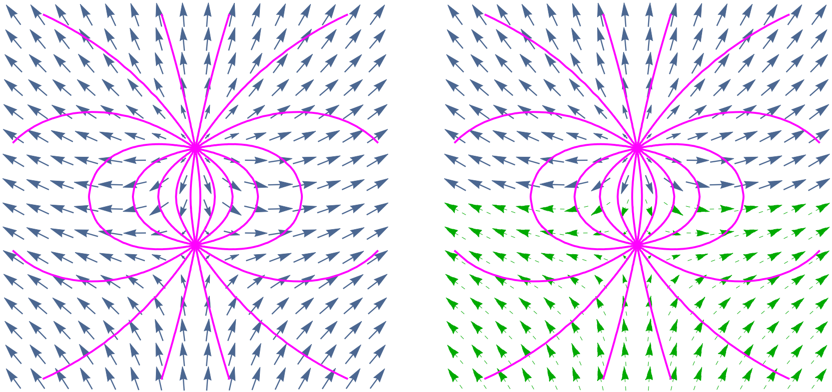
Transf.	1	z	Π_n	$z\Pi_n$
$\vec{n}$	$\vec{r}/r$	$-\vec{r}/r$	-$\vec{r}/r$	$\vec{r}/r$
q_0	≥ 0	≤ 0	≥ 0	≤ 0
Q	1	-1	-1	1
$\mathcal{Q}$	$1/2$	$1/2$	$-1/2$	$-1/2$
diagram				

Crossing the soliton $\rightarrow$ Rotation of spatial Dreibein by 2π,
chirality χ of this rotation given by sign of topological charge $\mathcal{Q}$

$$\mathcal{Q} = \chi \cdot S$$

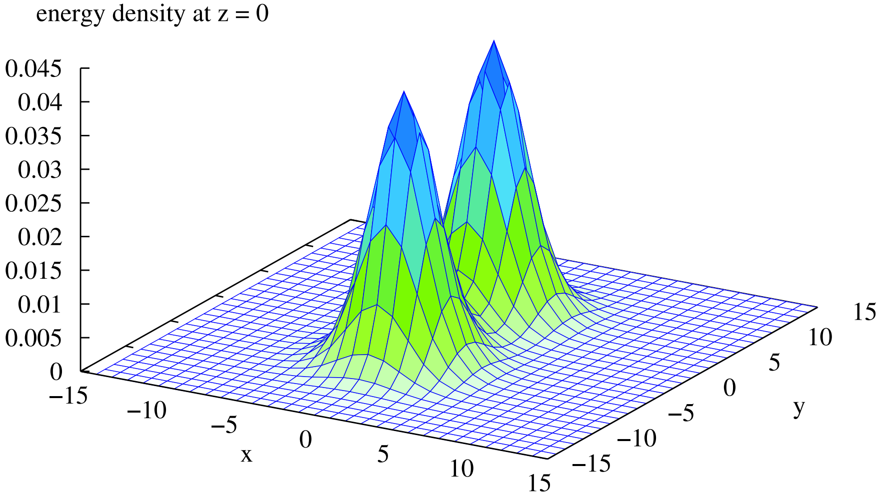
T.D.Lee: Why does the mass violate chiral symmetry?

Field lines of dipol



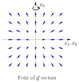
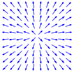
Left: $S = 0$ configuration, Right: $S = 1$ configuration
field lines = lines of constant $\vec{n}$-field

Energy density



Dirac (Electrodynamic) limit

artificial separation between particle and field

  bald $\rightarrow$ cowlick(hair whorl)  

$$r_0 \rightarrow 0 \quad \Longleftrightarrow \quad q_0 = \cos\alpha = 0 \quad \Longleftrightarrow \quad \alpha = \frac{\pi}{2}$$

Dual Dirac monopoles (dual Wu-Yang monopoles)

soliton has singularity in the center

$$Q(x) = -\mathrm{i}\vec{\sigma}\vec{n}(x),\ \vec{\Gamma}_\mu(x) = \vec{n}(x) \times \partial_\mu \vec{n}(x),\ \vec{R}_{\mu\nu}(x) = \partial_\mu \vec{n}(x) \times \partial_\nu \vec{n}(x)$$

$$^*F_{\mu\nu}(x) = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{R}_{\mu\nu}\vec{n} = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{n}(x)[\partial_\mu \vec{n}(x) \times \partial_\nu \vec{n}(x)]$$

$$\mathcal{L}_{\mathrm{ED}} = -\frac{1}{4\mu_0}{}^*F_{\mu\nu}(x){}^*F^{\mu\nu}(x)$$

$$Q_{\mathrm{el}}(\mathcal{S}) = -\frac{e_0}{4\pi}\oint_{\mathcal{S}(u,v)} \mathrm{d}u\mathrm{d}v\, \vec{n}[\partial_u \vec{n} \times \partial_v \vec{n}]$$

world-lines of singularities

Maxwell equations?

world-lines of singularities are closed

$j^\mu = -e_0 c \sum_{i=1}^N \int \mathrm{d}\tau_i \frac{\mathrm{d}X^\mu(\tau_i)}{\mathrm{d}\tau_i}\, \delta^4(x - X(\tau_i)) = (c\rho, \mathbf{j})$

Electric charge is conserved, a topological quantum number

Gauß-law = inhomogeneous Maxwell equations

$\frac{1}{2\mu_0} \oint_{\partial V} \mathrm{d}x^\mu \mathrm{d}x^\nu {}^* F_{\mu\nu} = \frac{1}{6} \int_V \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\rho \epsilon_{\mu\nu\rho\sigma} j^\sigma$

magnetic currents $\hat{=}$ homogeneous Maxwell equations

$$g^\mu = c\, \partial_\nu {}^* F^{\nu\mu} \quad \Leftrightarrow \quad \begin{cases} \rho_{\mathrm{mag}} = \nabla \mathbf{B}, \\ \mathbf{g} = -\nabla \times \mathbf{E} - \partial_t \mathbf{B}, \end{cases}$$

equation of motion

$$\partial_\mu \vec{n}\, g^\mu = 0 \quad \Leftrightarrow \quad \begin{cases} \mathbf{B}\, \mathbf{g} = 0, \\ c^2 \mathbf{B}\, \rho_{\mathrm{mag}} = \mathbf{g} \times \mathbf{E}. \end{cases}$$

solutions of Maxwell equations solve equations of motion

Coulomb and Lorentz forces

Canonical energy-momentum tensor $\Theta^{\mu}{}_{\nu}$ in electrodynamic limit
$T^{\mu}{}_{\nu}(x) = -\frac{1}{\mu_0}{}^*F_{\nu\sigma}(x){}^*F^{\mu\sigma}(x) - \frac{1}{4\mu_0}{}^*F_{\lambda\sigma}(x){}^*F^{\lambda\sigma}(x)\,\delta^{\mu}_{\nu}$
is symmetric

We split total force density
$f_{\nu} = \partial_{\mu}\Theta^{\mu}{}_{\nu} = f^{\mu}_{\mathrm{charges}} + \partial^{\nu} T^{\mu}{}_{\nu} = 0$

interaction is a consequence of topology
Coulomb and Lorentz forces

$$f^{0}_{\mathrm{charges}} = \frac{1}{c}\,\mathbf{j}\,\mathbf{E},$$
$$\mathbf{f}_{\mathrm{charges}} = \rho\,\mathbf{E} + \mathbf{j}\times\mathbf{B}.$$

U(1) gauge invariance

in color space, rotate all vectors with
$\Omega(x) = \mathrm{e}^{-\mathrm{i}\vec{\omega}(x)\vec{L}} = \mathrm{e}^{\vec{\omega}(x)\times}, \quad \vec{\omega} = 2\alpha\vec{n}$
$\vec{n}\,' = \Omega\vec{n} = (\vec{n}\vec{e}_\omega)\vec{e}_\omega + \sin\omega\ \vec{e}_\omega \times \vec{n} - \cos\omega\ \vec{e}_\omega \times (\vec{e}_\omega \times \vec{n})$
$\vec{\Gamma}_\mu = \vec{n} \times \partial_\mu \vec{n}$
$\Gamma_\mu = \vec{L}\,\vec{\Gamma}_\mu \ \mapsto\ \Gamma'_\mu = \vec{L}\,\vec{\Gamma}'_\mu = \Omega(\Gamma_\mu - \mathrm{i}\partial_\mu)\,\Omega^\dagger$

there is a rotational invariance around $\vec{n}$-direction

far-field of hedge-hog soliton: align radial electric field along z-axis
$\vec{E}'_r = \frac{e_0}{4\pi\varepsilon_0}\frac{\vec{e}_3}{r^2}, \qquad \frac{\vec{\Gamma}_\varphi(\vartheta)}{r\sin\vartheta} = \frac{1-\cos\vartheta}{r\sin\vartheta}\vec{e}_3,$ potential of Dirac monopole
rotational invariance of local Dreibein around 3-axis

U(1) gauge invariance is rotational invariance around $\vec{n}$ axis.

define lines of constant $\vec{n}$-field, fibres.

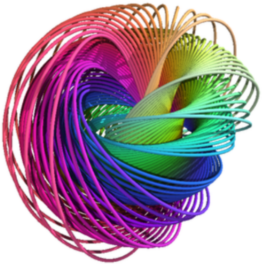
Hopf map

map $\mathbb{S}^3 \to \mathbb{S}^2$, $\quad q = (q_0, q_1, q_2, q_3) \mapsto n = (n_1, n_2, n_3)$

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \text{ and } n_1^2 + n_2^2 + n_3^2 = 1$$

circles of $\mathbb{S}^3$ mapped to linked fibres of $\mathbb{S}^2$

visualisation after stereographic projection: $\mathbb{S}^2 \to \mathbb{R}^3$



$$n_1 = 2(q_0 q_2 + q_1 q_3)$$
$$n_2 = 2(q_2 q_3 - q_0 q_1)$$
$$n_3 = -q_1^2 - q_2^2 + q_3^2 + q_0^2$$

from: http://nilesjohnson.net/etcetera.html

define fibres, $\vec{n} = \text{const.} \in \mathcal{S}^2$

$\mathbb{R}^3 \cup \infty \sim \mathcal{S}^3, \quad \pi_3(\mathcal{S}^2) = \mathbb{Z}$

linking fibres

Gauß's linking number v (Verschlingungszahl)

Topological invariant, coverings of $\mathbb{S}^2$

$$v = \frac{1}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot (\mathrm{d}\mathbf{r}_1 \times \mathrm{d}\mathbf{r}_2)$$

linking number v can be positive or negative,

left or right circulating Goldstone bosons

define fibres, $\vec{n} = \text{const.} \in \mathcal{S}^2$

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linking number v can be positive or negative,

left or right circulating Goldstone bosons

conjecture:

Gauß's linking number = Hopf number = photon number

of dual photon field $\vec{C}_\mu = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{\Gamma}_\mu = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{n} \times \partial_\mu \vec{n}$

possible mechanism for inflation?

at big bang:

$Q = 1,$
$\Lambda = 1/r_0^4,$
energy density of $\mathcal{E} = \frac{\alpha_f \hbar c}{4\pi}\Lambda = \frac{\alpha_f \hbar c}{4\pi r_0^4}.$

transition to

$\Lambda = 0,$
$\mathcal{E} = 0,$

release of energy density $\alpha_f \hbar c/(4\pi r_0^4) = 4,8\ \mathrm{keV}/\mathrm{fm}^3 = 7,7 \cdot 10^{29} J/\mathrm{m}^3.$

cosmological function $\Lambda(x)$

potential term $\Lambda(x)$ - a cosmological function

cosmological constant from $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_\mathrm{c}} = 0.69$
$\rho_\mathrm{c} c^2 = \frac{3H^2 c^2}{8\pi G_\mathrm{N}} = 4,9\,\frac{\mathrm{GeV}}{\mathrm{m}^3}$,
prediction of particle physics $\rho_\Lambda c^2 = 1,82 \cdot 10^{121}\,\frac{\mathrm{GeV}}{\mathrm{m}^3}$.

due to Derricks theorem $1/4$ of mass due to Λ,
if this is also valid for nucleons:
14.5 nucleons per $\mathrm{m}^3 \;\to\; \Omega_\Lambda$.

Non-topological excitations

$$Q(x) = \mathrm{e}^{\mathrm{i}\alpha(x)\vec{\sigma}\vec{n}(x)}$$

Fluctuations of α and $\vec{n}$ around solitons and the vacuum $\alpha(x) = \frac{\pi}{2}$.

- $\vec{n}$-waves:
  massless modes,
  propagate with c,
  contribute to dark energy.
- alpha-waves

alpha-waves in the vacuum

Potential $\Lambda(q_0) = \frac{q_0^{2m}}{r_0^4}$, $\quad q_0 = \cos\alpha$

Fluctuations in the angle $\alpha(x)$

in the vacuum: $q_0 = 0,\ \alpha = \frac{\pi}{2}$.

- massive modes,
- contribute to dark matter,
- appear also by superposition of $\vec{n} - \textit{waves}$:

$$Q_1(x) = -\mathrm{i}\vec{\sigma}\vec{n}_1(x),$$
$$Q_2(x) = -\mathrm{i}\vec{\sigma}\vec{n}_2(x),$$
$$Q_1(x)Q_2(x) = \underbrace{-\vec{n}_1(x)\vec{n}_2(x)}_{\cos\alpha(x)} - \mathrm{i}\vec{\sigma}[\underbrace{\vec{n}_1(x)\times\vec{n}_2(x)}_{\sin\alpha(x)}]$$

$\cos\alpha(x) \neq 0, \alpha \neq \frac{\pi}{2} \ \Rightarrow$ massive excitations

spherical alpha-waves around solitons

$\vec{n} = \frac{\vec{r}}{r}$

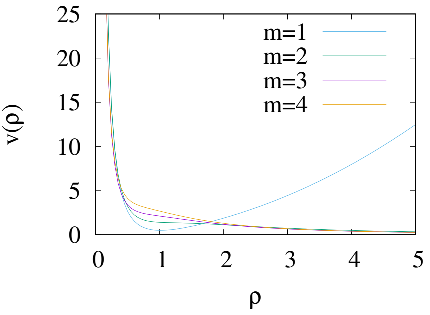
$\vec{\zeta}(t, \vec{r}) = \phi(t, r)\vec{n},$

$Q(x) = \mathrm{e}^{\mathrm{i}\alpha(r)\vec{\sigma}\vec{n}(r)} \rightarrow$

$\rightarrow Q'(x) = \mathrm{e}^{\mathrm{i}\vec{\sigma}\vec{\zeta}(x)} Q(x)$

$\Phi = \frac{\sqrt{2}\sin\alpha}{r}\phi,$

$\rho = \frac{r}{r_0}$



$$\delta\mathcal{L} = \frac{\alpha_f \hbar c}{4\pi}\left\{\frac{1}{2}\,\partial_\mu\Phi\partial^\mu\Phi - \frac{\Phi^2}{r_0^2}\underbrace{\left[\frac{1+3\cos 2\alpha}{4\rho^2} + \frac{m(2m-1)\rho^2}{2}\cos^{2m-2}\alpha\right]}_{v(\rho)}\right\}.$$

For $m = 1$ vibrational excitations,
for $m \geq 2$ radial waves

plain alpha-waves around solitons

$$\vec\sigma\vec\zeta(x) = \sigma_3 A\cos(kx-\omega t)$$

$$Q(x) = \mathrm{e}^{\mathrm{i}\alpha(r)\vec\sigma\vec n(r)},\ \vec n = \frac{\vec r}{r},\ \alpha = \arctan\frac{r}{r_0}$$

$$\to Q'(x) = \mathrm{e}^{\mathrm{i}\vec\sigma\vec\zeta(x)}\mathrm{e}^{\mathrm{i}\alpha(r)\vec\sigma\vec n(r)}.$$

shift of soliton center by $\approx r_0 = 2.2$ fm.

shift much smaller than Compton wave length $\frac{\hbar}{m_e c} \approx 400$ fm.

of negligible influence

Summary

- Particles are topological solitons
- Mass is field energy
- Charge and spin are topological quantum numbers
- Integer multiples of elementary charge only
- Distinction between charges and their field is artificial
- Charges can be described by 2π-rotations of space.
- Spin angular momentum as Eigen-angular momentum,
  Spin as a consequence of orbital motion.
- Two types of massless excitations, $\vec{n}$-waves = electromagnetic waves
- Pauli principle has topological origin
- U(1) gauge invariance =
  = rotational invariance of Dreibein around $\vec{n}$
- Cosmological constant $\Lambda \rightarrow$ Cosmological Function $\Lambda(t, \vec{r})$
- Only 3 rotational degrees of freedom of space were used.
- Only Space and Time.

Problems and speculations

Conjectures

- Stable particles are stable solitons with topological quantum numbers.
- Only topological solitons don't escape our detectors?
- Waves escape our detectors $\rightarrow$ Dark energy and dark matter?
- Waves disturb the pathes of particles -> Quantum Mechanics
- Particles get in resonance with waves -> interference
  (in analogy to Couder's experiments)
- α-waves $\rightarrow$ Dark matter?
- Potential $\Lambda(t, \vec{r}) \rightarrow$ dark energy?
- Inflation: transition of vacuum $Q = 1 \rightarrow Q = \mathrm{i}\vec{n}\vec{\sigma}$?

Problems:

- non-topological magnetic currents
- excited states or α-waves with non-zero mass
- perturbatively non-renormalisable
- quantum properties

My suppositions

- a lot of geometry still hidden in physics
- physics is geometry and not algebra
- we should use algebra to investigate geometry

General relativity:
Wheeler: "Matter tells space how to warp.
And warped space tells matter to move."

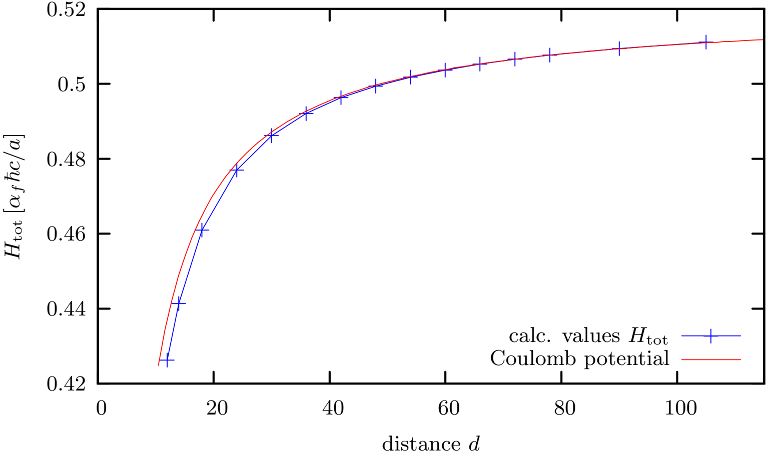
Electro dynamics:
$\cdots$ Charges and electromagnetic fields tell space how to rotate.

Thanks

relations to space-time?

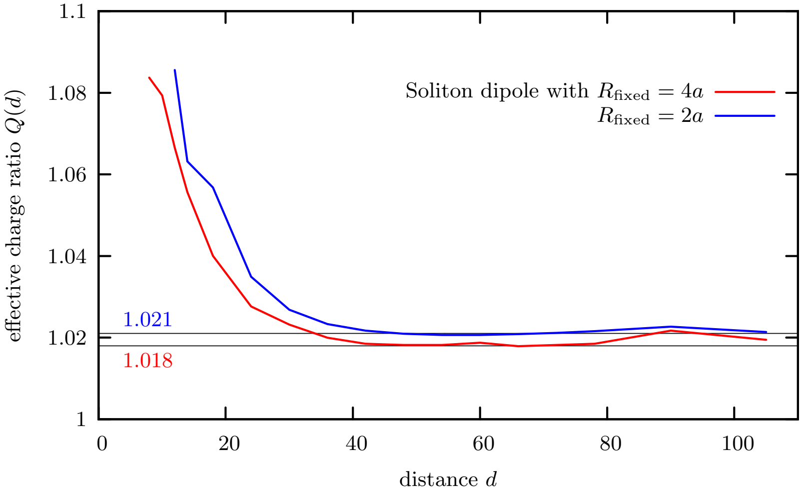
- $SO(3)$-field
  rotation of spatial Dreibein,
  3 degrees of freedom are the three Euler angles,
  crossing unit charge - rotation by 2π.
- potential term $\Lambda(x)$ - a cosmological function
  cosmological constant from $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_\mathrm{c}} = 0.69$
  $\rho_\mathrm{c} c^2 = \frac{3H^2c^2}{8\pi G_\mathrm{N}} = 4,9\,\frac{\mathrm{GeV}}{\mathrm{m}^3}$,
  prediction of particle physics $\rho_\Lambda c^2 = 1,82 \cdot 10^{121}\frac{\mathrm{GeV}}{\mathrm{m}^3}$.
  due to Derricks theorem $1/4$ of mass due to Λ,
  15 nucleons per $\mathrm{m}^3 \;\rightarrow\; \Omega_\Lambda$.
- transition from $Q = 1$ and $\Lambda = 1/r_0^4$ to $\Lambda = 0$
  release of energy density of
  $\alpha_f \hbar c/(4\pi r_0^4) = 4.8\ \mathrm{keV}/\mathrm{fm}^3 = 7.7 \cdot 10^{29} J/\mathrm{m}^3$,
  could be a mechanism for inflation.

Soliton-antisoliton-potential



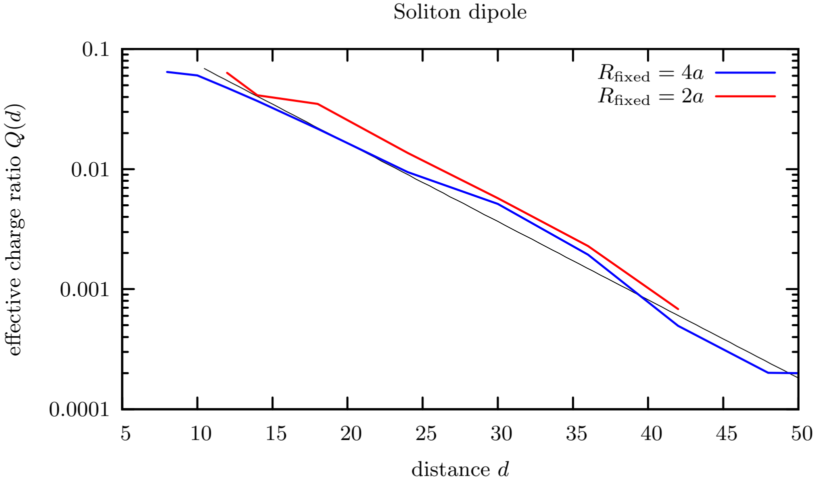
Total energy of a soliton pair for varying distance $d = r/r_0$.
From diploma work of Dominik Theuerkauf (2016).

Effective charge ratio



From diploma work of Dominik Theuerkauf (2016).

Effective charge ratio



From diploma work of Dominik Theuerkauf (2016).

Sine-Gordon model

field variable: $\vartheta(x,t), \quad \mathrm{e}^{\mathrm{i}\vartheta(x,t)} \in \mathbb{S}^1$
in natural units of length, time and energy

hamiltonian density: $h(x,t) = \frac{1}{2}\left(\frac{\partial\vartheta}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\vartheta}{\partial x}\right)^2 + (1-\cos\vartheta)$

degenerate vacua: $h(x,t) \equiv 0, \quad \vartheta(x,t) = \mathrm{const} = 2\pi n, \quad n \in \mathbb{Z}$

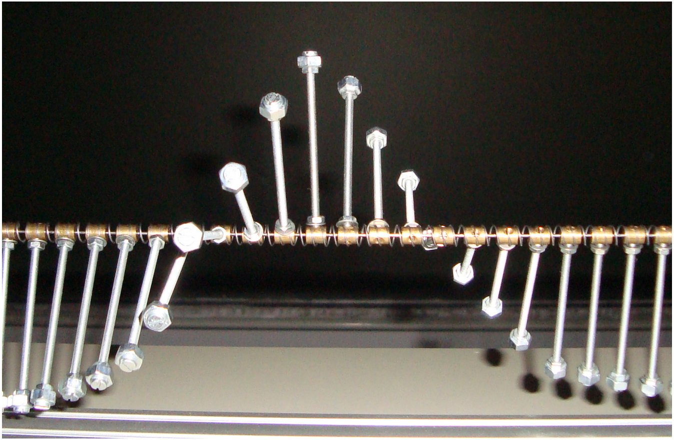
equation of motion a non-linear differential eq.: $\partial_x^2\vartheta - \partial_t^2\vartheta = \sin\vartheta$

soliton solutions: $\vartheta_{\beta\pm}(x,t) = 4\arctan \mathrm{e}^{\pm\gamma(x-\beta t)}$ with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

soliton energy: $E(\beta) = \int \mathrm{d}x\, h(x,t) = 4(\beta^2\gamma + \gamma + \frac{1}{\gamma}) = 8\gamma$

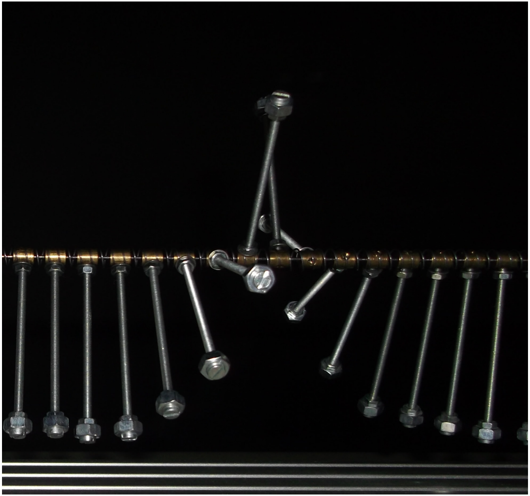
infinitely many time dependent solutions

. . . in the experiment



Picture: Gerald Pechoc

Moving soliton?



Picture: Gerald Pechoc

Topological quantum number Z

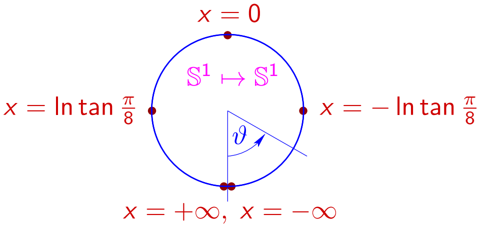
$x \in \mathbb{R}^1$

with boundary condition for $\vartheta(x)$: $\mathrm{e}^{\mathrm{i}\vartheta(-\infty)} = \mathrm{e}^{\mathrm{i}\vartheta(+\infty)}$

$\mathbb{R}^1$ topological equivalent to $\mathbb{S}^1$

field $\vartheta(x)$ is a map: $\mathbb{R}^1 \mapsto \mathbb{S}^1 \xrightarrow{\text{boundary condition}} \mathbb{S}^1 \mapsto \mathbb{S}^1$

winding number Z,
soliton: $Z = 1$



Z is a particle minus antiparticle number

Nice features of Sine-Gordon model

- solitons like particles with relativistic properties
- Mass is field energy
- particle number is a topological quantum number
- particle number is therefore integer
- forces are a consequence of the structure of particles
- distinction between field and particle is artificial

Aim

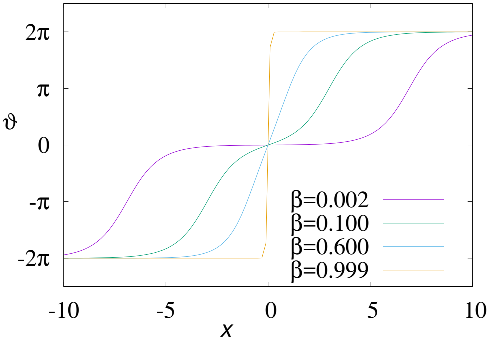
Stable topological solitons,
with long range (Coulombic) interaction,
in 3+1D

Are electrons point-like?

Size of electrons:

- LEP accelerated electrons to 100 GeV,
- Resolution $2 \cdot 10^{-18}$ m, ($\hbar c \approx 200$ MeV fm $= 200$ GeV am),
- Electrons still point-like,
- but $\alpha_f(0 \text{ GeV}) = \frac{1}{137.036} \quad \rightarrow \quad \alpha_f(90 \text{ GeV}) = \frac{1}{127}$.

Sine-Gordon solitons have arbitrary small size:



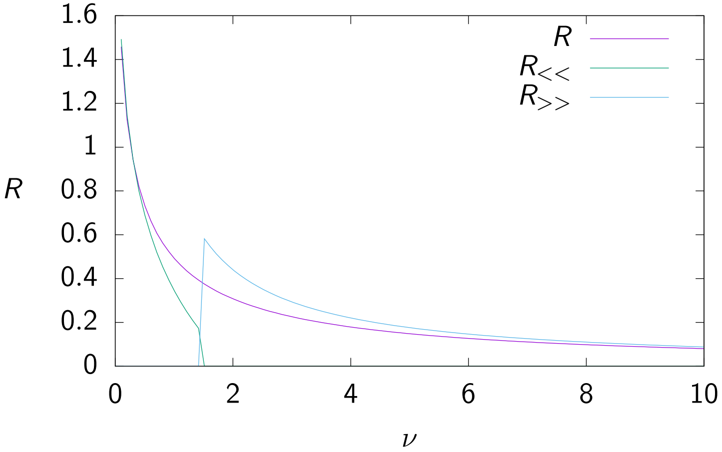
Scattering of solitons:

$$\vartheta(x,t) = 4\,\text{atan}\frac{\beta\sinh(\gamma x)}{\cosh(\beta\gamma t)},$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

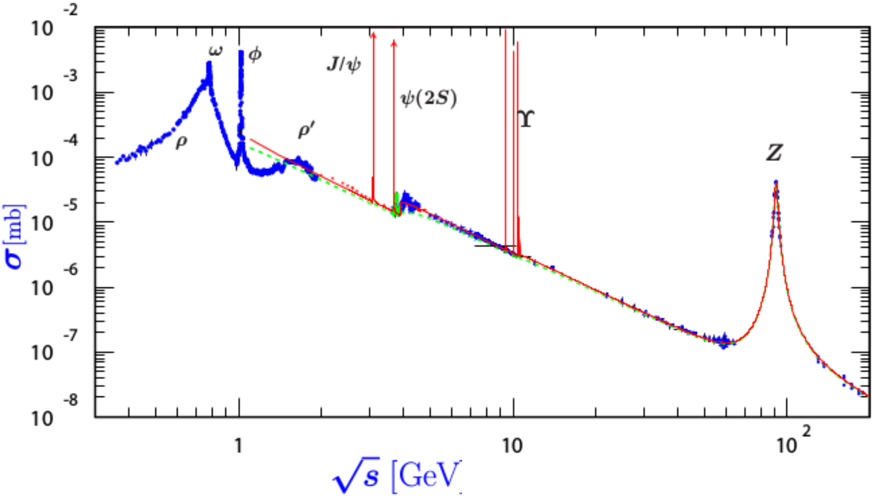
they have arbitrary small size

SG-soliton size in scattering



$$R = \frac{D}{2} = \frac{1}{\gamma}\text{arsinh}\frac{1}{\beta} = \frac{1}{\nu+1}\text{arsinh}\frac{\nu+1}{\sqrt{\nu(\nu+2)}}, \quad \nu = \gamma - 1$$

electron-positron cross section



from http://pdg.lbl.gov/2014/hadronic-xsections/rpp2014-sigma_R_ee_plots.pdf

Are electrons point-like?

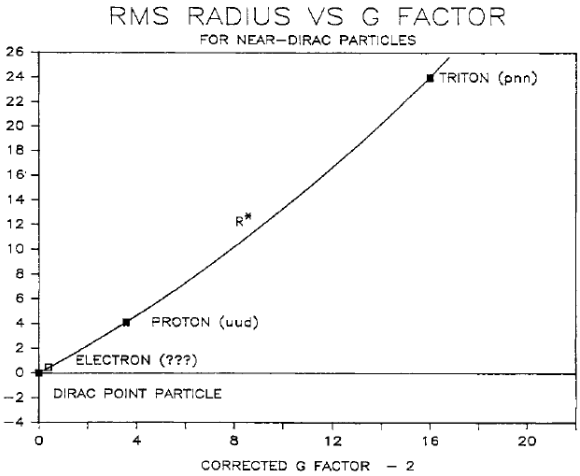
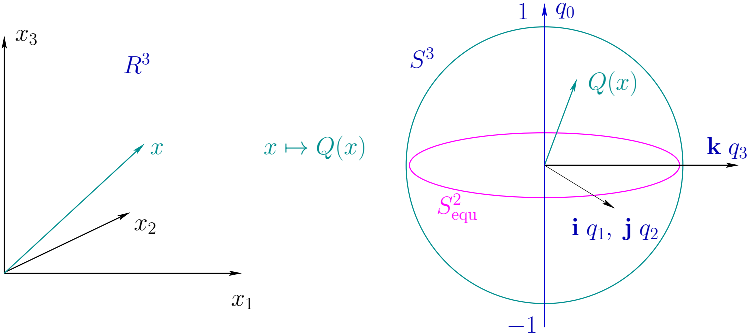


Fig. 8. Normalized RMS radius $R^* = R/\lambda_C$ vs. corrected g-factor minus 2 for near-Dirac particles [3]. A parabola has been fitted to the data points. Recent theories conjecture that the electron, similar to proton and triton, is composed of three smaller fermions. The data point at the origin represents a Dirac point particle of finite arbitrary charge and mass.

Hans Dehmelt, Physica Scripta, Volume 1988, T22 “A Single Atomic Particle Forever Floating at Rest in Free Space: New Value for Electron Radius”

Calorons in SU(2)

a field of Polyakov loop matrices $Q(\vec{x})$, $\quad L(\vec{x}) = \mathrm{Tr}\, Q(\vec{x})$
covering $\mathbb{S}^3 \cong SU(2) \cong$ unit quaternions



$$R^3 \to \mathbb{S}^3, \quad x \mapsto Q(\vec{x}) = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 = q_0 - \mathrm{i}\vec{q}\vec{\sigma}$$

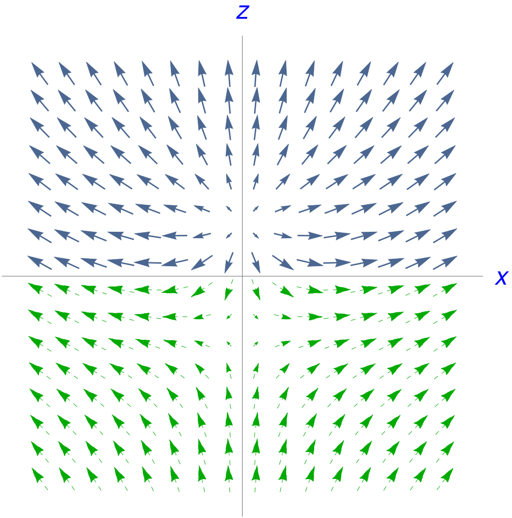
$$\mathbf{i} = -\mathrm{i}\sigma_1, \mathbf{j} = -\mathrm{i}\sigma_2, \mathbf{k} = -\mathrm{i}\sigma_3$$

(an)holonomy = vacuum with broken symmetry, e.g. $Q(\infty) = -\mathrm{i}\sigma_3$

$$Q(\vec{x}) = q_0(\vec{x}) - \mathrm{i}\vec{q}(\vec{x})\vec{\sigma}$$

$$q_0^2 + \vec{q}^2 = 1$$

$\vec{q}(\vec{x})$-field



$q_0 > 0$

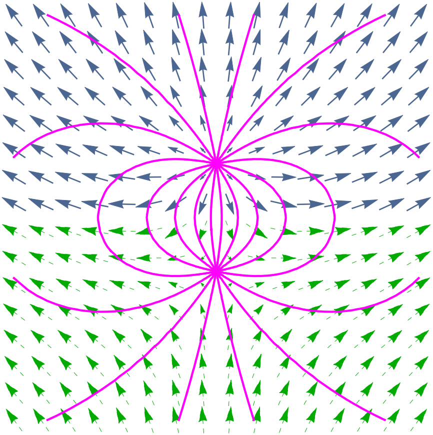
$q_0 < 0$

$$Q(\vec{x}) = q_0(\vec{x}) - \mathrm{i}\vec{q}(\vec{x})\vec{\sigma} = \cos\alpha(\vec{x}) - \mathrm{i}\vec{\sigma}\vec{n}(\vec{x})\sin\alpha(\vec{x})$$

connect points with $\vec{n} = \text{const.}$

$q_0 > 0$

$q_0 < 0$



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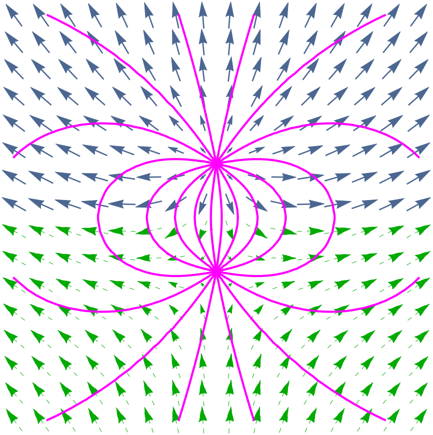

a diplole field $\Rightarrow$ NO SINGULARITY

How describe dynamics of monopoles

with SU(2) scalar field. We relate

- vector field $A_\mu(x) \;\rightarrow\;$ connection $\vec{\Gamma}_\mu(x),\ \partial_\mu Q(x) Q^\dagger(x) =: -\mathrm{i}\vec{\Gamma}_\mu(x)\,\vec{\sigma}$
- tensor field $F_{\mu\nu}(x) \;\rightarrow$ curvature $\vec{R}_{\mu\nu} := \vec{\Gamma}_\mu(x) \times \vec{\Gamma}_\nu(x)$