Towards the Unitarity Limit in EFTs with Pions

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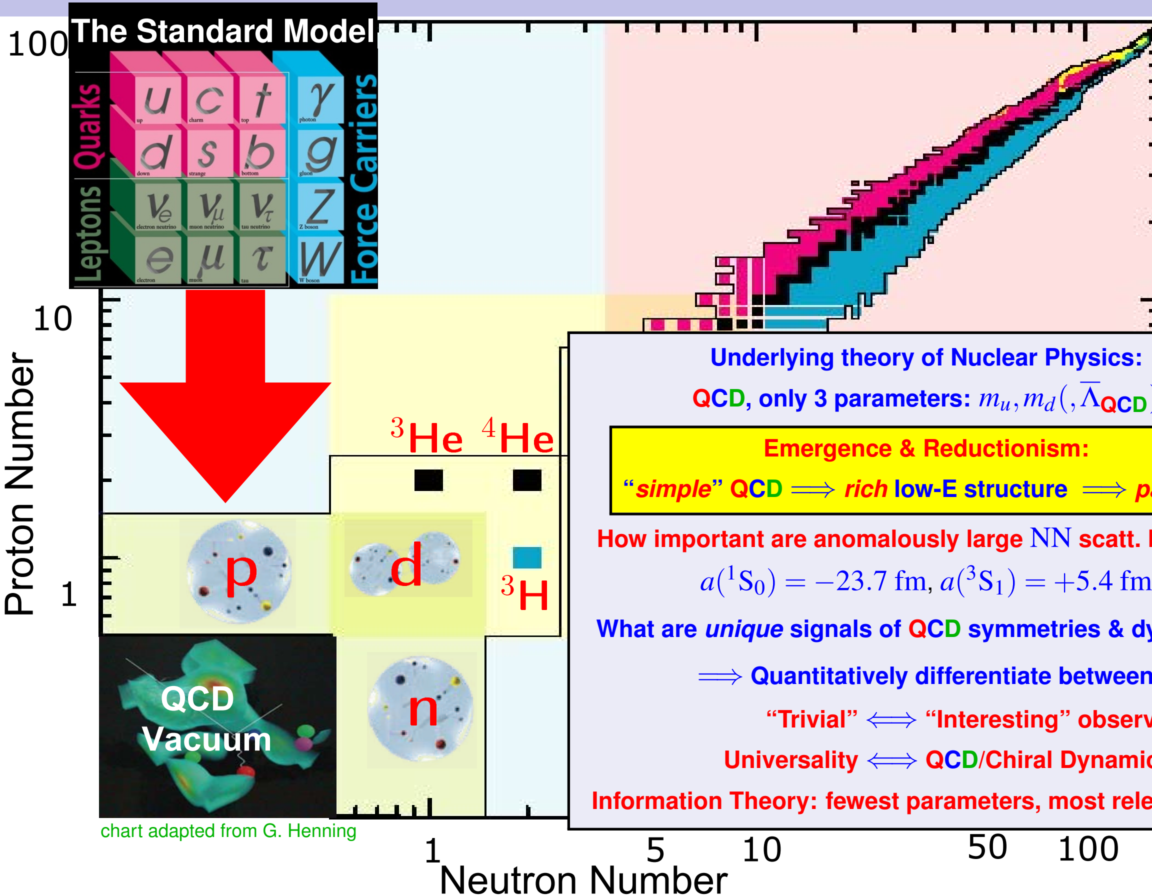
1. Emergent Phenomena in Nuclear Physics: “Order From Chaos”
2. What Is The Unitarity Limit? And Why Should I Care?
3. Unitarity Expansion With Perturbative Pions in NN S-Waves
4. Concluding Conjecture and Questions

How to root Nuclear Physics in QCD?
What is the underlying principle that makes simple structures emerge from complex nuclear dynamics?

König/hg/Hammer/van Kolck: *Phys. Rev. Lett.* **118** (2017) 202501 [1607.04623 [nucl-th]]

Teng/hg: MSc Thesis GW 2023 and [2410.09653 [nucl-th]]

1. Emergent Phenomena in Nuclear Physics: “Order From Chaos”



Underlying theory of Nuclear Physics:

QCD, only 3 parameters: $m_u, m_d (, \overline{\Lambda}_{\text{QCD}})$

Emergence & Reductionism:

“*simple*” QCD $\Longrightarrow$ *rich* low-E structure $\Longrightarrow$ *patterns*

How important are anomalously large NN scatt. lengths?:

$$a(^1\text{S}_0) = -23.7 \text{ fm}, \ a(^3\text{S}_1) = +5.4 \text{ fm}$$

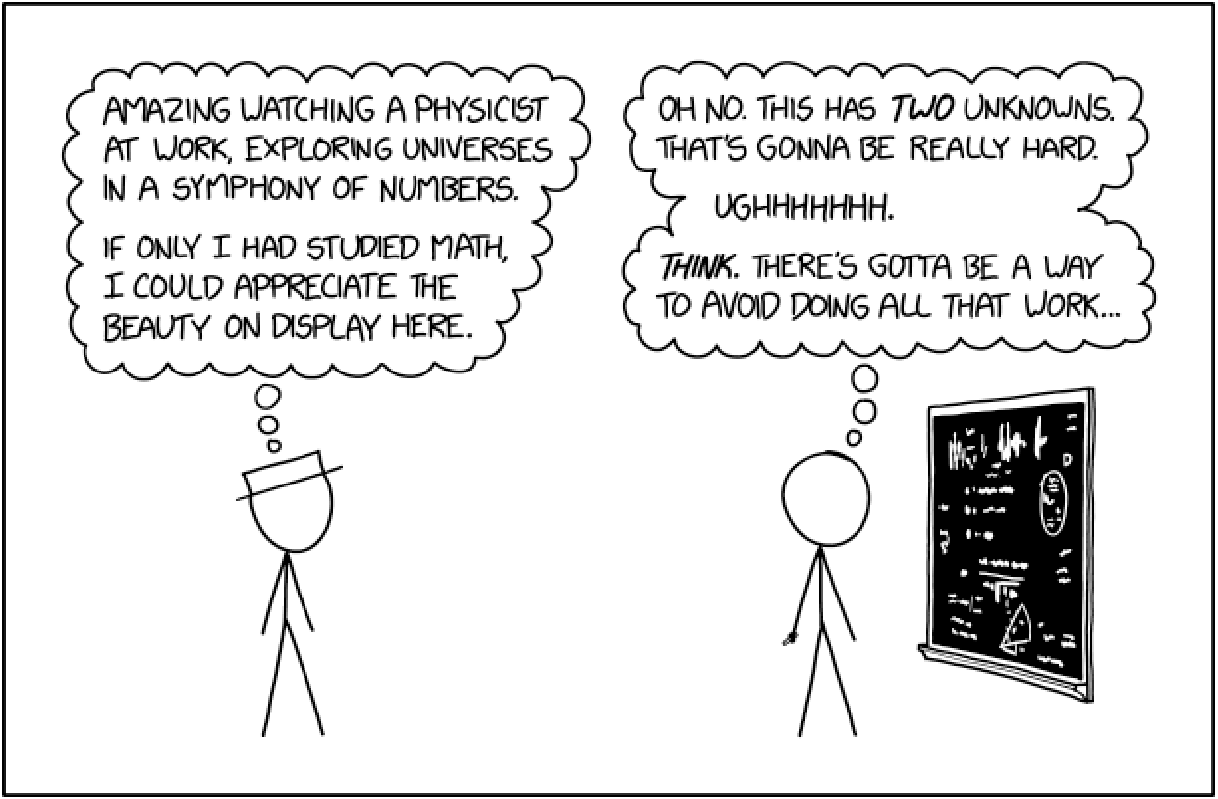
What are *unique* signals of QCD symmetries & dynamics?

$\Longrightarrow$ Quantitatively differentiate between:

“Trivial” $\Longleftrightarrow$ “Interesting” observables

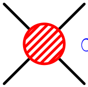
Universality $\Longleftrightarrow$ QCD/Chiral Dynamics

Information Theory: fewest parameters, most relevant info!


AMAZING WATCHING A PHYSICIST AT WORK, EXPLORING UNIVERSES IN A SYMPHONY OF NUMBERS.
IF ONLY I HAD STUDIED MATH, I COULD APPRECIATE THE BEAUTY ON DISPLAY HERE.
OH NO. THIS HAS TWO UNKNOWNS. THAT'S GONNA BE REALLY HARD.
UGHHHHHHH.
THINK. THERE'S GOTTA BE A WAY TO AVOID DOING ALL THAT WORK...

xkcd 2019

2. What Is The Unitarity Limit? And Why Should I Care?

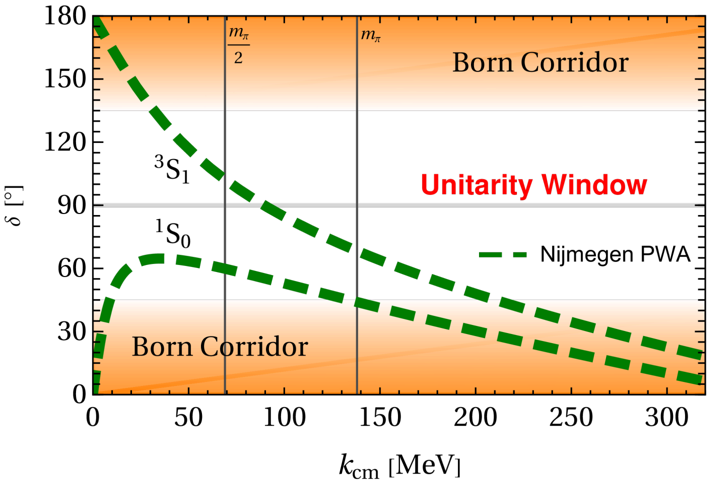


$$\propto \frac{1}{\underbrace{k\cot\delta}_{\text{interaction}} - \mathrm{i}k} \to \begin{cases} \dfrac{1}{k\cot\delta}\left[1+\dfrac{\mathrm{i}}{\cot\delta}+\dots\right] & \text{for } \cot\delta \gg |\mathrm{i}| \text{ i.e. } \delta\to 0 \Longrightarrow \\ \dfrac{1}{-\mathrm{i}k}\left[1+\dfrac{\cot\delta}{\mathrm{i}}+\dots\right] & \text{for } \cot\delta \ll |\mathrm{i}| \text{ i.e. } \delta\to 90^\circ \Longrightarrow \end{cases}$$

interaction ↓ Unitarity

Born Approximation:
interactions small & perturbative,
their details & scales drive A_{NN}
no bound states

Unitarity Limit implies Universality:
interaction strong: *non-perturbative*,
details irrelevant, unitarity drives A_{NN};
Unitarity Expansion at LO:
no scales in A_{NN}, bound state at $k=0$.



Unitarity Window: $|\cot\delta| \lesssim 1$ ($45^\circ \lesssim \delta \lesssim 135^\circ$)

$\Longrightarrow$ LO NN nonperturbative in $^1S_0$ & $^3S_1$ for

$30\ \mathrm{MeV} \lesssim k_{\mathrm{cm}} \lesssim [1.5\dots2]m_\pi$

Outside: **Born Corridors**

LO perturbative for $|\cot\delta| \gtrsim 1$ ($|\delta| \lesssim 45^\circ$)

How much of Nuclear Physics does really depend on details of QCD?

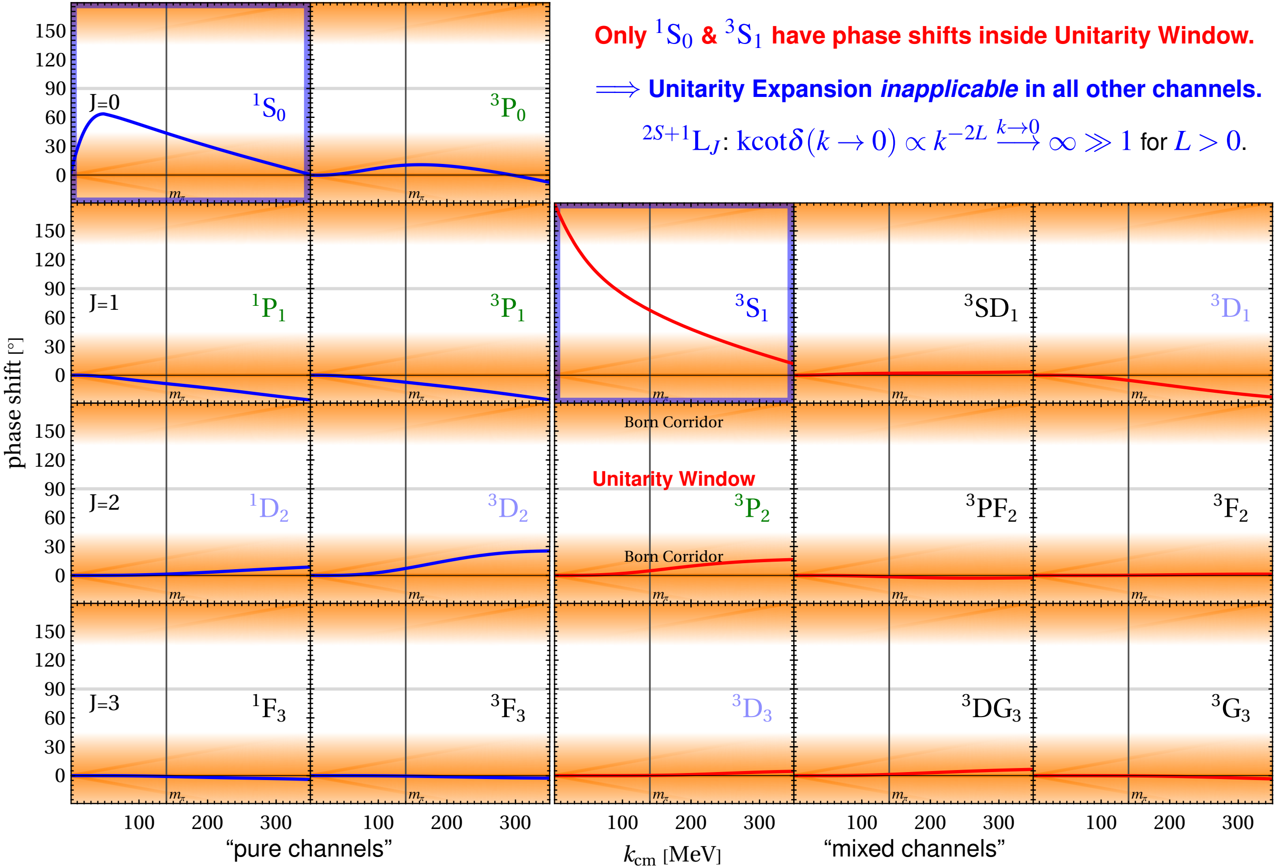
How much just from (corrections to) universal aspects around Unitarity?

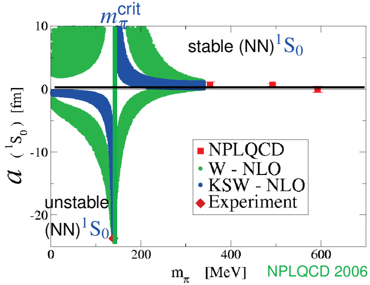
(a) Use Unitarity Expansion Only for Channels with Large NN Phase Shifts!

Only $^1S_0$ & $^3S_1$ have phase shifts inside Unitarity Window.

$\Longrightarrow$ **Unitarity Expansion *inapplicable* in all other channels.**

$^{2S+1}L_J$: $k\cot\delta(k\to 0) \propto k^{-2L} \stackrel{k\to 0}{\longrightarrow} \infty \gg 1$ for $L>0$.





χEFT cannot explain anomalous scatt. lengths/shallow binding: Worlds with $a \lesssim \dfrac{1}{m_\pi}$!

Accident or Symmetry?

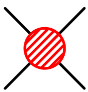
(1) **Amplitude saturated at Unitarity Limit:** $\sigma = \dfrac{4\pi}{k^2}$ maximal (probability conservation).

(2) **Scale Invariance:** $\vec{k} \to e^\lambda\,\vec{k}$. actually nonrel. Conformal/Schrödinger Symmetry... Mehen/Stewart/Wise 2000 Nishida/Son 2007

(3) **Wigner-$\mathrm{SU}(4)$ Symmetry of combined spin-isospin rotations** $\begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix} \to U \begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix}$ Wigner, Hund 1937 for heavy nuclei cf. Mehen/Stewart/Wise 1999

In NN: $^1\mathrm{S}_0 - {}^3\mathrm{S}_1$ forms Wigner-super-sextuplet $\mathbf{6}_A(L=0)$

if $a(^3\mathrm{S}_1) = a(^1\mathrm{S}_0)$:

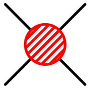


$$= \frac{4\pi}{M}\,\frac{1}{-\frac{1}{a} - ik} = A_{\mathrm{NN}}(^3\mathrm{S}_1) = A_{\mathrm{NN}}(^1\mathrm{S}_0)$$

Theorists love Unitarity Limit as Nontrivial Fixed Point characterised by high symmetry:
Wigner-$\mathrm{SU}(4)+$ scale-invariance close to FP protected in renormalisation.
What About Nature?

(c) Unitarity Expansion in EFT($\not\pi$)

König/hg/Hammer/van Kolck: PRL 2017 [1607.04623]
reviews: van Kolck [2003.09974]; Kievsky/... [2102.13504]

EFT($\not\pi$)/ERE:  $\propto \frac{1}{-ik}\left[1+\frac{k\cot\delta = -\frac{1}{a}+\frac{r}{2}k^2+\dots}{ik}+\dots\right] \to \underbrace{\frac{1}{-ik}}_{\text{LO}}\underbrace{\left[1+i\left(\underbrace{\frac{1}{ka}}_{<1?!}-\underbrace{\frac{kr}{2}}_{<1?!}\right)+\dots\right]}_{\text{NLO correction}}$

***a priori* justified if** $0 \leftarrow \frac{1}{a}$ (inverse scatt. length/ NN system size/ NN binding momentum) $\ll$ typ. momentum $k \ll \Lambda_{\not\pi} \overset{???}{\sim} m_\pi \sim \frac{1}{r}$ (breakdown/ resolution) scale.

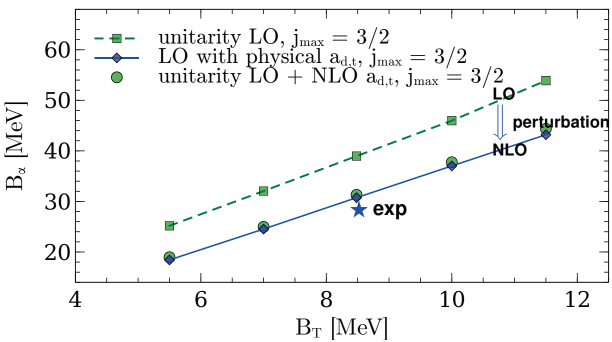
LO: No NN scale. $\Longrightarrow$ Nuclear Physics correlated to *just one* 3N *RG scale fixed by* B_3 *via Efimov effect*.

PARADIGM SHIFT: *Unitarity* de-emphasises details of NN & pions, emphasises 3N scale & Universality.

$\Longrightarrow$ Explore *Sweet Spot* for patterns, unique signals of QCD:

bound weakly enough to be insensitive to interaction details ($\frac{kr}{2} \ll 1$),

but strongly enough to be insensitive to exact large system size ($ka \gg 1$).

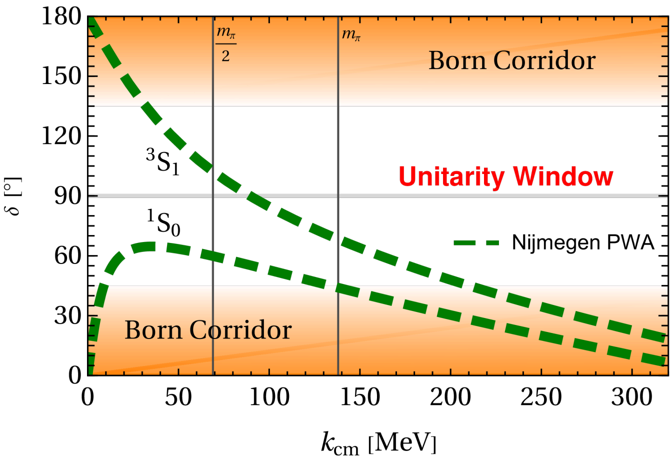


$B_{^3\text{H}} - B_{^3\text{He}}$: NLO: $[0.92 \pm 0.18]$ MeV; exp: 0.764 MeV

	Fermion Unitarity LO $\to$ NLO	exp ^{4}He/^{3}H
ground: B_4/B_3	$4.6 \;\to\; 3.8 \pm 0.2$	3.66
excited: B_4^*/B_3	$\sim 1.1 \to \sim 0.98 \pm 0.05$	0.96

Symm. Nucl. Matter	ρ_0 [fm^{-3}]	B/A [MeV]	E_{sym} [MeV]	L[MeV] slope of E_{sym}	K_∞[MeV] compressib.
Kievsky/... EFT($\not\pi$)-inspired	0.15	-16	35	70	251
exp	0.16	-16	≈ 30	$[40\dots60]$	210

(d) χEFT vs. Unitarity Expansion: Clash Of The Symmetries



NN S waves well in **Unitarity Window** $|\cot\delta|<1$ for $30\text{ MeV} \lesssim k_{cm} \lesssim [1.5\dots2]m_\pi$.

Upper limit close to $\overline{\Lambda}_{NN} = \frac{16\pi f_\pi^2}{g_A^2 M} \approx 300\text{ MeV}$ where OPE becomes nonperturbative KSW 1999 FMS 2000.

$\Longrightarrow$ **How to merge Unitarity and χiral symmetry?**

Problem: Both fundamental & incompatible?!

Pion breaks Unitarity: scaling by f_π, m_π, Wigner by SD mixing.

Unitarity breaks Chiral Symmetry: tensor-OPE$\to 0$.

"Wigner-breaking": split inside $\mathbf{6}_A(L=0)$, or mix with $L\neq 0$

Unitarity dictates *absence/irrelevance*!

$$\downarrow\vec{q}\;:\; -\frac{g_A^2}{4f_\pi^2}\frac{1}{\vec{q}^2+m_\pi^2}\left[\underbrace{\frac{1}{3}(\vec{\tau}_1\cdot\vec{\tau}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2)\vec{q}^2}_{\text{central: Wigner-symmetric}} + \underbrace{(\vec{\tau}_1\cdot\vec{\tau}_2)\left((\vec{\sigma}_1\cdot\vec{q})(\vec{\sigma}_2\cdot\vec{q})-\frac{1}{3}(\vec{\sigma}_1\cdot\vec{\sigma}_2)\vec{q}^2\right)}_{\text{tensor: Wigner-breaking, mixes } S\leftrightarrow D,\ D\to D}\right]$$

Chiral Symmetry dictates *presence* and relative size!

Each symmetry hidden from other: *"accidental"* $\Longrightarrow$ How do they manifest/emerge together in observables?

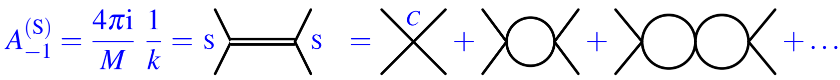
Explore transition "no$\to$nonperturbative pions" via Perturbative ("KSW") Pions (only undisputedly consistent χEFT).

3. Unitarity Expansion With Perturbative Pions in NN S-Waves

(a) χEFT(pπ)$_{\text{UE}}$ at N^{2}LO with $Q \sim \frac{1}{ka}, \frac{k, m_\pi}{\overline{\Lambda}_{NN}} \ll 1$

based on Rupak/Shoresh [nucl-th/9902077] ($^1S_0$),
Fleming/Mehen/Stewart [nucl-th/9911001] ($^1S_0$, $^3S_1$)
mod. for unitarity Teng/hg MSc thesis GW 2023, [2410.09653]

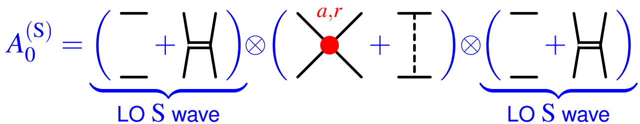
$\mathcal{O}(Q^{-1})$ **(LO):** Nonperturbative; no scale, perfect Wigner, pure S wave.



from structureless contacts $C\ (N^\dagger N)^2$

$\mathcal{O}(Q^{0})$ **(NLO):** Scaling and Wigner broken by contacts determined to reproduce PWA values of a, r.

Non-iterated OPE: central only, does not break Wigner but scaling; first non-analyticity: branch point $\pm i \frac{m_\pi}{2}$.



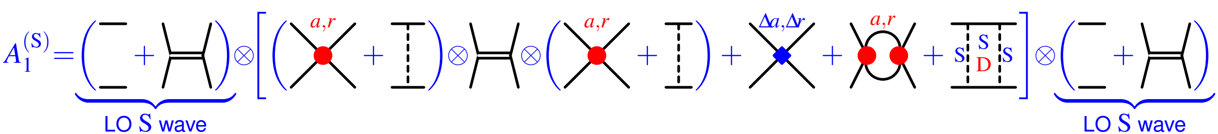
$\Longrightarrow$ **Unitarity, Wigner-SU(4) spin-isospin symmetry align naturally for Perturbative Pions at NLO.**

$\mathcal{O}(Q^{1})$ **(N^{2}LO):** Contacts adjusted to keep a, r at PWA values; multiplied with non-iterated OPE (central only).

Once-iterated OPE added: first and second non-analyticity: branch points $\pm i \frac{m_\pi}{2}, \pm i m_\pi$.

$A_{1\text{sym}}$: Central $S \to S \to S$ does not break Wigner but scaling: identical in $^1S_0$ and $^3S_1$.

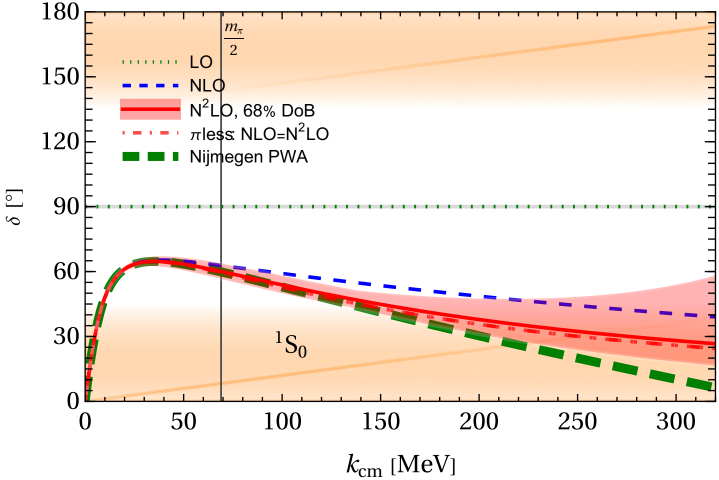
$A_{1\text{break}}$: Tensor $S \to D \to S$ breaks Wigner and scaling: only in $^3S_1$.



$\Longrightarrow$ **Is breaking of Wigner-SU(4) spin-isospin symmetry for Perturbative Pions at N^{2}LO indeed small?**

(b) Perturbative Pions at N^{2}LO: $^1S_0$

perturbative pions to N^{2}LO: Rupak/Shoresh 2000, Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



$^1S_0$: central OPE $\Longrightarrow$ Wigner-symmetric.

f_π, m_π break scaling.

Strict perturbation in "basic interaction part"

$$k\cot\delta = 0_{\text{LO}} + k\cot\delta_{\text{NLO}} + k\cot\delta_{\text{N}^2\text{LO}}$$
$$= 0_{\text{LO}} - \frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2}\right].$$

$\Longrightarrow$ Get δ from $k\cot\delta$.

$^1S_0$ is **"boring"** partial wave: no tensor int.

Bayesian truncation uncertainty at 68% DoB.

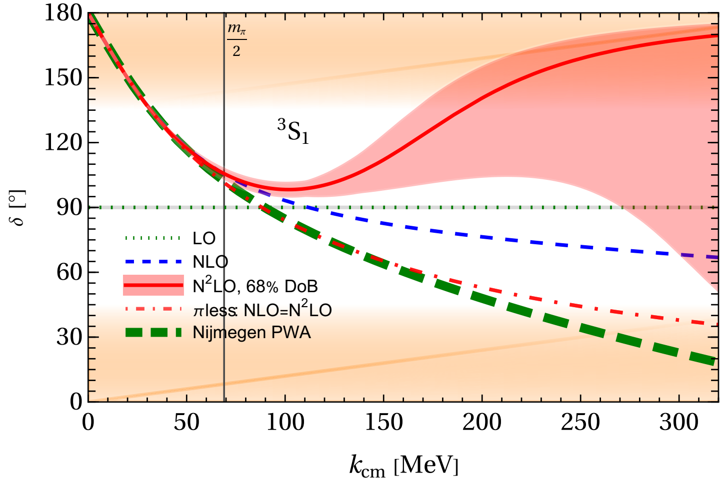
$\Longrightarrow$ Converges order-by-order $\lesssim 300$ MeV.

Agrees within uncertainties with PWA for $\lesssim 250$MeV (even outside Unitarity Window).

Compare to EFT($\not\pi$): minuscule impact of π.

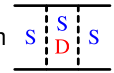
(c) Perturbative Pions at N^{2}LO: $^3S_1$

perturbative pions to N^{2}LO: Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



$^3S_1$: pions break Wigner-$SU(4)$ & scale inv.

$^3S_1$ is **"interesting"** partial wave:

tensor-OPE $\Longrightarrow$ SD mixing from 

$\Longrightarrow$ Terrible convergence (already in FMS):

Converges order-by-order $\lesssim 80$ MeV.

Agrees within uncertainties with PWA only for $\lesssim 70$ MeV (not even in Unitarity Window).

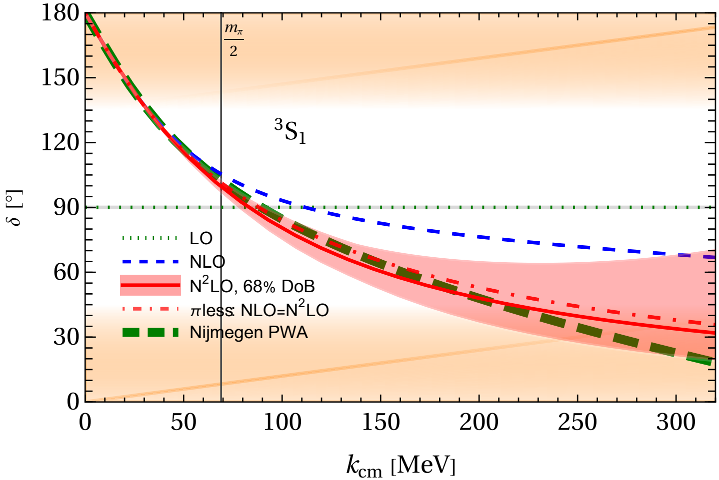
Compare to EFT($\not\pi$): huge impact of pion.

Source of discrepancy: tensor int. (via SD).

(central is identical in $^3S_1$&$^1S_0$)

(c) Perturbative Pions at N^{2}LO: $^3S_1$

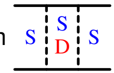
perturbative pions to N^{2}LO: Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



$^3S_1$: pions break Wigner-$SU(4)$ & scale inv.

$^3S_1$ is **"interesting"** partial wave:

tensor-OPE $\Longrightarrow$ SD mixing from



Broken Wigner-$SU(4)$ spoils convergence!

Idea: Use Wigner-$SU(4)$-symmetric pion part.

$\Longrightarrow$ Set tensor OPE to zero.

$\Longrightarrow$ Only $^1S_0$-$^3S_1$ differences of a & r break Wigner-$SU(4)$.

RG-invariant, mildly χsymmetry-breaking.

$\Longrightarrow$ Converges order-by-order $\gtrsim 300$ MeV.

Agrees within uncertainties with PWA for $\gtrsim 300$ MeV (even outside Unitarity Window).

Compare to EFT($\not\pi$): tiny impact of pion.

$\Longrightarrow$ **All very similar to $^1S_0$.**

4. Concluding Conjecture and Questions

χEFT with Perturbative Pions in Unitarity Expansion $Q \sim \frac{1}{ka}, \frac{kr}{2}, \frac{m_\pi}{\Lambda_{\mathrm{NN}}} \ll 1$**: needs** $\delta \to \frac{\pi}{2} \Longrightarrow {}^1\mathrm{S}_0, {}^3\mathrm{S}_1$ **only!**

Chiral Physics: $m_\pi, f_\pi, (\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})(\tau_1 \cdot \tau_2)$ seem opposed to Wigner, but NN/few-N projection forces into it.

> **Conjecture (at least for Perturbative Pions): Tensor/Wigner-$\mathrm{SU}(4)$ symmetry-breaking part of One-Pion Exchange is *super-perturbative* in few-N systems, i.e. does *not enter before* N^{3}LO.**
>
> $\Longleftrightarrow$ ***Persistence:*** **Footprint of Symmetries in Unitarity Limit extends far into** $p_{\mathrm{typ}} \gtrsim m_\pi$**, more relevant than χiral symmetry in few-N?!**
>
> $\Longleftrightarrow$ **Information Theory: Compress to relevant info: fewest parameters at given accuracy!**

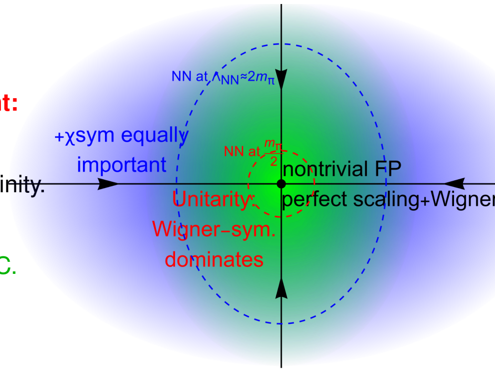
Appeal: Fine-Tuning $\Longrightarrow$ **High Symmetry at Nontrivial Fixed Point:**

Universality/scaling $+$ **Wigner-$\mathrm{SU}(4)$**

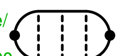
protected in renormalisation at FP $\Longrightarrow$ weakly broken in vicinity.

χiral symmetry not explicit at FP: less protected? $\Longrightarrow$ **Quantify!**

No Wigner in meson/1N sector $\Longrightarrow$ no change to χPT, HBχPT PC.



Some Crucial Tests: If either fails without good reason, Conjecture falsified.

N^{3}LO cf. Beane/Kaplan/Vuorinen 2009, Kaplan 2020 

$\mathrm{d}\pi \to \mathrm{d}\pi$, $\gamma\mathrm{d} \to \pi\mathrm{d}$ cf. Borasoy/hg 2003

Nd scattering cf. Bedaque/hg 2000

Nonperturbative Pions to N^{2}LO in strict perturbation LO: hg 2023

(a) An NN Scenario: What Is LO and If Its Perturbative Depend on Partial Wave

S-Waves

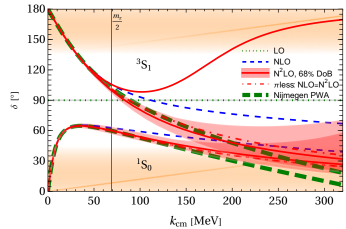
Perturbative about Unitarity $\Longrightarrow$ interaction details irrelevant

Pions of little impact in Unitarity Window (but likely in probes!)

LO: only 1 momentum-indep. contact $C_S\,(\mathrm{N}^\dagger\mathrm{N})^2$ so that $\frac{1}{a}=0$.

Plus central OPE? – tensor-OPE only at high orders...

$\Longrightarrow$ Overall patterns of Nuclear Physics



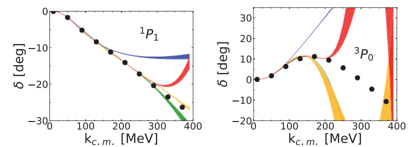
P/D-Waves

Unitarity irrelevant $\Longrightarrow$ interaction details important

LO perturbative or nonperturbative? Birse, Long/..., Kaplan, ...

Pions may be important... $\Longrightarrow$ full OPE at LO?

$\Longrightarrow$ Overall more details in patterns of Nuclear Physics



Wu/Long [1807.04407]

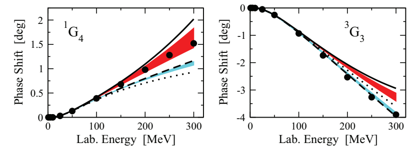
Higher Waves

Unitarity irrelevant $\Longrightarrow$ interaction details important

LO perturbative: Born approximation Kaiser/Brockmann/Weise

OPE details important, but not overall

$\Longrightarrow$ Details of Nuclear Physics



Epelbaum/... [nucl-th/0304037]

Simplify Interactions, reduce computational complexity, concentrate on essentials.

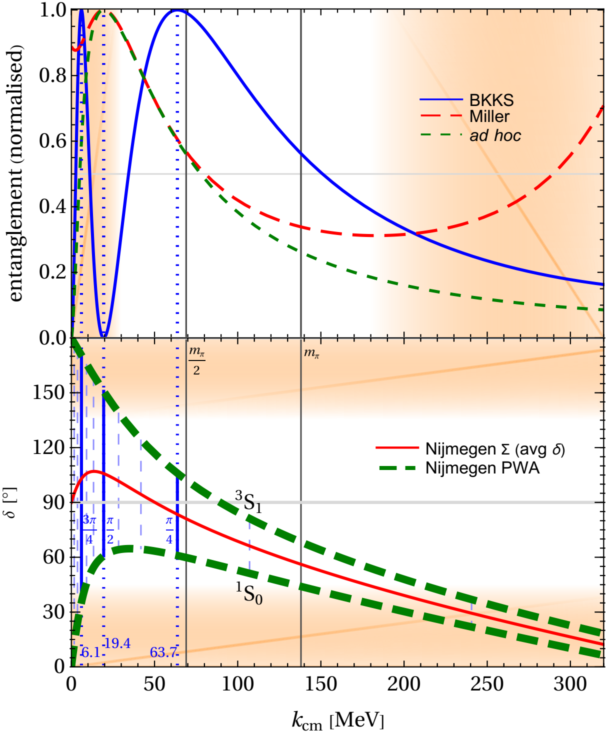
$\Longrightarrow$ **3N interactions, interactions with probes,...**

$\Longrightarrow$ **Explore few-/many-N systems, also with probes: Use Perturbation Theory beyond LO: DWBA, Born.**

(b) What is the Small Parameter?: QM Entanglement?

Need expansion parameter related to Wigner-$\mathrm{SU}(4)$ to characterise tensor suppression near Fixed Point.

Entanglement: Deviation of QM *states* from direct product $\underbrace{\text{position} \otimes \text{spin} \otimes \text{isospin}}_{\text{classical}}$ $\Longrightarrow$ *Changed by operators!*



$S = \mathrm{e}^{2\mathrm{i}\Sigma}\left[\mathbb{1}\cos\Delta + \mathrm{i}\,\mathrm{SWAP}_\sigma\,\sin\Delta\right]$, Σ : phase avg. Δ : phase diff.

spin swap $1 \leftrightarrow 2$: $\mathrm{SWAP}_\sigma := \frac{1}{2}(\mathbb{1} + \vec{\sigma}_1 \cdot \vec{\sigma}_2) = \begin{cases} +1: {}^3\mathrm{S}_1 \\ -1: {}^1\mathrm{S}_0 \end{cases}$

Unitarity: $S = \mathrm{e}^{2\mathrm{i}(\Sigma=\frac{\pi}{2},\Delta=0)} = -\mathbb{1} \Longrightarrow \mathcal{E} = 0$: classical

How to Define Entanglement Power $\mathcal{E}$ of *Operator*?:

$\mathcal{E}_{\mathrm{BKKS}} = \sin^2[2\Delta]$ Rényi entropy of $1\mathrm{N}$ density matrix of avg. over direct-product $|\text{in}\rangle$ Beane/Kaplan/Klco/Savage 2019

$\mathcal{E}_{\mathrm{Miller}} = H\left[\frac{\cos^2\Delta\,(\cos\Delta - \cos 2\Sigma)^2}{(1-\cos\Delta\,\cos 2\Sigma)^2}\right]$ von Neumann entropy Miller 2023

$H[f] = -x\ln x - (1-x)\ln(1-x), x = \frac{1}{2}(1+\sqrt{f})$

$\mathcal{E}_{\text{ad hoc}} = H[\sin^2\Delta]$ relative von Neumann entropy SWAP vs total Teng/hg 2024

In Unitarity Window, $\mathcal{E} \in [0;1]$, saturates at $k \approx \dfrac{m_\pi}{2}$.

$\Longrightarrow$ Relevance of Entanglement in Unitarity Window??

How to find $\mathcal{E}$ before computation??

(c) What is the Small Parameter?: Large-N_c Limit of QCD?

't Hooft 1974

Need expansion parameter related to Wigner-$\mathrm{SU}(4)$ to characterise tensor suppression near Fixed Point.

Candidate Expansion of QCD for a large number $N_C \to \infty$ of colours:

Kaplan/Savage [hep-ph/9509371]
Kaplan/Manohar [nucl-th/9612021]
Calle Cordón/Ruiz Arriola [0807.2918]

Predicts that all V_{NN} in S waves are suppressed against central (Wigner-$\mathrm{SU}(4)$) – **except** tensor $\lightning$.

Way out?: Wigner-$\mathrm{SU}(4)$ only realised in long-range parts, strongly broken in short-range?? Calle Cordón/Ruiz Arriola [0807.2918]

Here: Wigner-$\mathrm{SU}(4)$ breaking only in LECs: short-range – long-range ($k \to 0$) still Wigner-$\mathrm{SU}(4)$ symmetric.

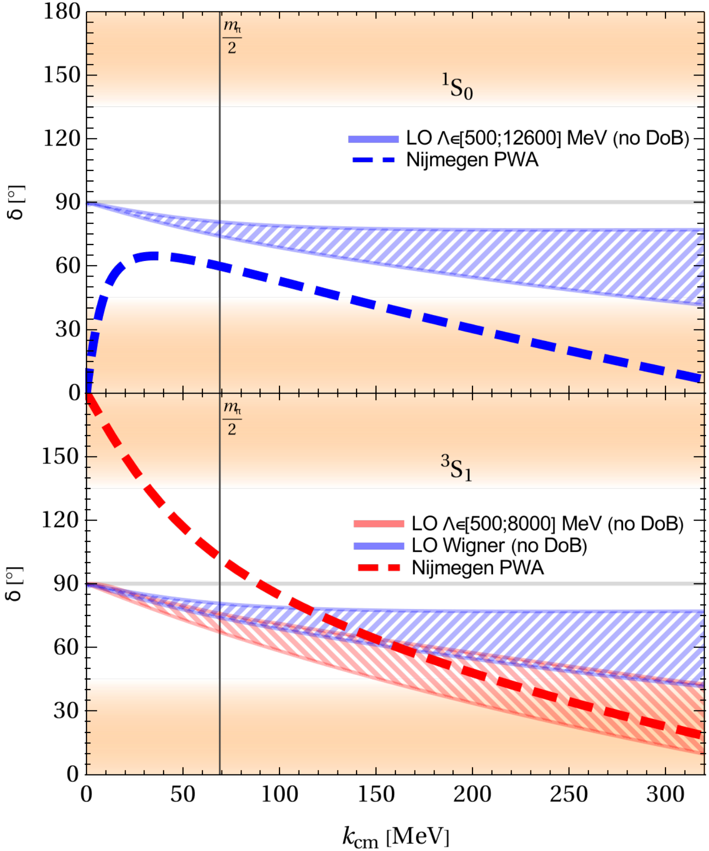
Way out?!: $1/N_c$ expansion assumes that coefficients "of natural size".

Wigner-$\mathrm{SU}(4)$/proximity to Unitarity *forces* leading-$1/N_c$ coefficient of tensor-V_{NN} to be exact zero.

Advantage: Guaranteed to survive renormalisation by Unitarity FP symmetry.

(d) Nonperturbative Pions at LO: Maybe Not Hopeless

hg 2023
Carter/Thiem/hg in preparation



LO, 1 mom.-indep. CT, Gaußian regulator.

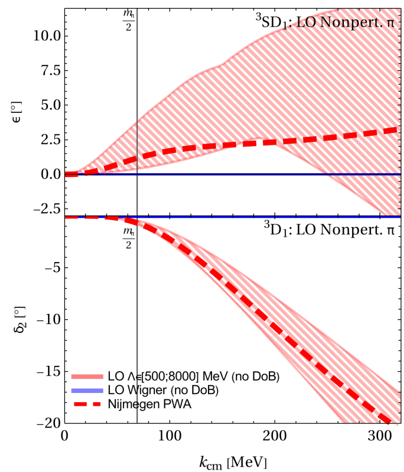
Already deviates from Unitarity $\delta = 90°$.

$\Longrightarrow$ Explicit scale breaking at LO,

$r = \begin{cases} ^1S_0/\text{Wigner } [1\ldots2]\text{fm}; {}^3S_1\ [1.2\ldots2.5]\text{fm} \\ \text{PWA } 2.767(9)\text{fm} \quad 1.852(2)\text{fm} \end{cases}$

Tensor/Wigner-breaking less compatible with unitarity than central/Wigner-symmetric.

Not Bayesian DoBs but cutoff-variation.



4. Concluding Conjecture and Questions

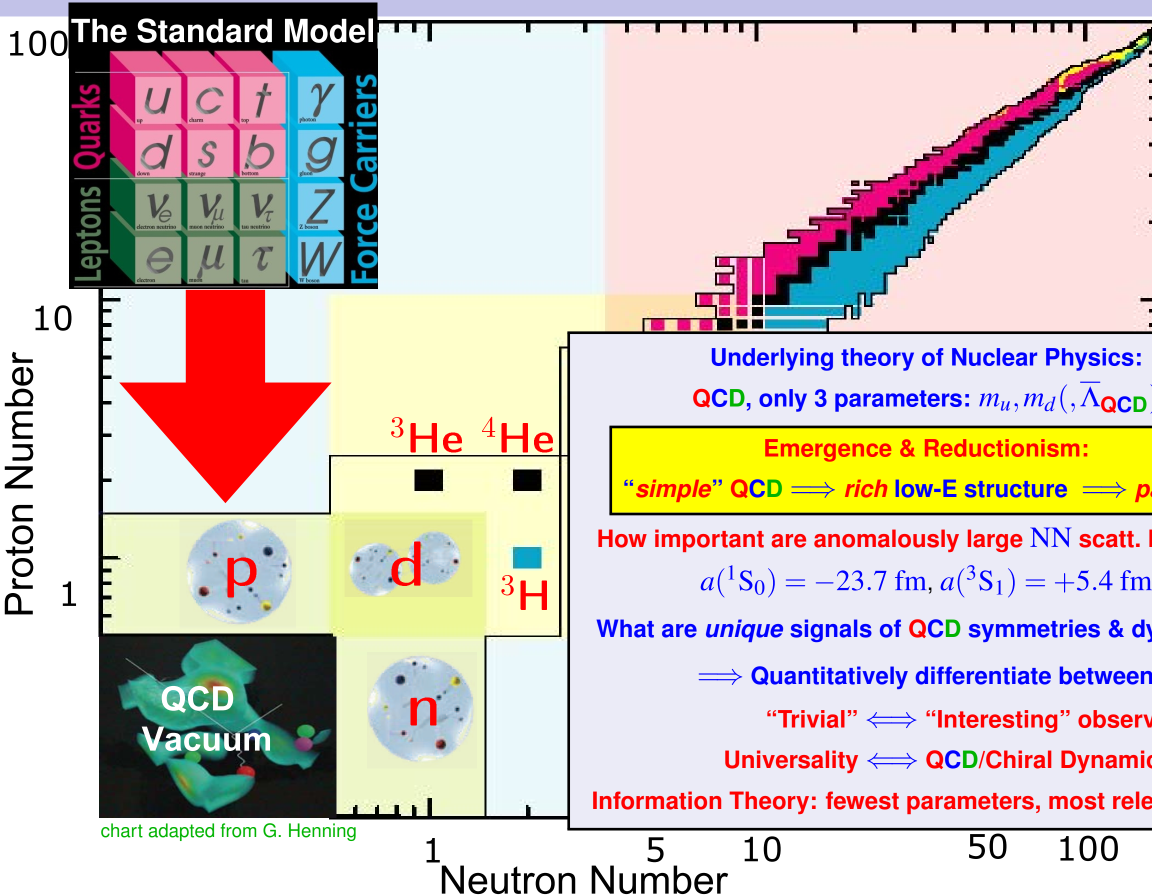


chart adapted from G. Henning

Underlying theory of Nuclear Physics:

QCD, only 3 parameters: $m_u, m_d (, \overline{\Lambda}_{\text{QCD}})$

Emergence & Reductionism:

"*simple*" QCD $\Longrightarrow$ *rich* low-E structure $\Longrightarrow$ *patterns*

How important are anomalously large NN scatt. lengths?:

$$a(^1\text{S}_0) = -23.7\text{ fm},\ a(^3\text{S}_1) = +5.4\text{ fm}$$

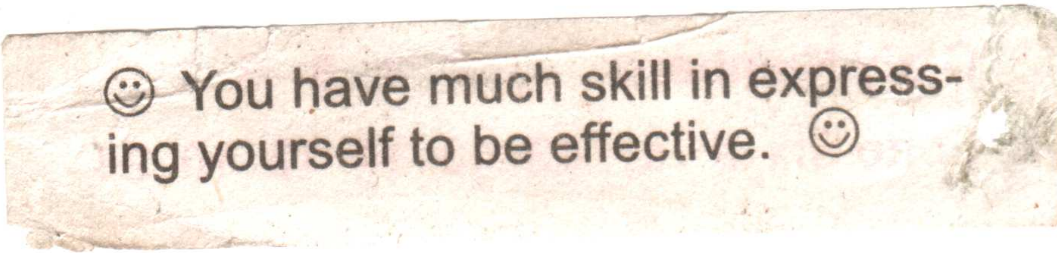
What are *unique* signals of QCD symmetries & dynamics?

$\Longrightarrow$ Quantitatively differentiate between:

"Trivial" $\Longleftrightarrow$ "Interesting" observables

Universality $\Longleftrightarrow$ QCD/Chiral Dynamics

Information Theory: fewest parameters, most relevant info!


☺ You have much skill in expressing yourself to be effective. ☺

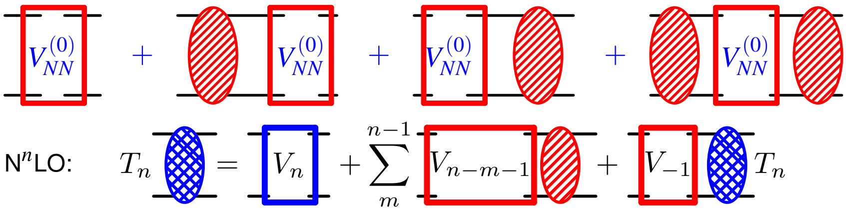
(a) Do Contributions to *Observables* Decrease With Increasing Order?

$\Longrightarrow$ Find radius of convergence $k \lesssim \overline{\Lambda}_{\text{EFT}}$, systematically estimate truncation error (Bayes) – and *only then* compare to data: beware of confirmation bias.

Corrections in $Q \ll 1$ by "strict perturbation" about LO (Distorted-Wave Born; efficient way Vanasse 1305.0283):
$\Longrightarrow$ Power-counting of amplitudes (observables); simple, no resummation artefacts.

NLO:

N^{n}LO:



Use/Develop More Strict-Perturbation Methods! cf. hg notes Trento 2018-21;

$\longrightarrow$ Oliver Thim; $\Delta\mathcal{O}_0 = \dfrac{\mathcal{O}[V_{-1} + \epsilon\, V_0] - \mathcal{O}[V_{-1}]}{\epsilon}$ with $\epsilon \to 0$ Shi/.../Long/... PRC 106 (2022) 015505; [2205.02000] ...

Contrast to Popular "Partially-Resummed Perturbation" Weinberg 1990

Power-count V_{NN} & iterate $\Longrightarrow T = \dfrac{V_{\text{LO}} + V_{\text{NLO}} + \dots}{1 - (V_{\text{LO}} + V_{\text{NLO}} + \dots)\, G_{NN}}$.

$\Longrightarrow$ Obscures PC in observables, unphysical poles around $\overline{\Lambda}_{\text{EFT}}$: artefacts, wrong causal structure.

$\Longrightarrow$ Limited to small cutoff variation range $\Lambda \approx \overline{\Lambda}_{\text{EFT}} \pm 20\%$, implementation & numerics more difficult.

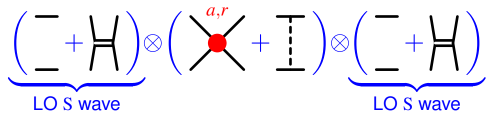
Works *under assumption that* expansion indeed small, e.g. $1 + x + x^2 + x^3 - \dfrac{1}{1-x} = x^4 + \mathcal{O}(x^5)$ *if* $|x| < 1$.
But resummed version loses control over convergence test & over which interactions needed for Λ-independence (UV changed!). May still provide some guidance/insight – but beware of missed CTs!

(b) Analytic Answers Shorter By Unitarity

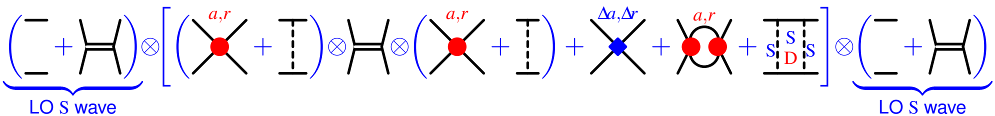
LO: $A^{(S)}_{-1}(k) = \frac{4\pi i}{M}\,\frac{1}{k}$ is only S wave

NLO: $A^{(S)}_0(k) = -\frac{4\pi}{Mk}\left(\frac{1}{ka}-\frac{kr}{2}\right) - \frac{g_A^2}{4f_\pi^2}\left(1-\frac{m_\pi^2}{4k^2}\overbrace{\ln\left[1+\frac{4k^2}{m_\pi^2}\right]}^{1\pi\text{ cut}}\right)$

Non-iterated OPE does not break Wigner.



N^{2}LO:



Once-iterated OPE breaks Wigner: $S \to D \to S$

$$A_1^{(^1S_0)}(k) \equiv A^{(S)}_{1\text{sym}}(k) = \frac{\left[A_0^{(S)}(k)\right]^2}{A^{(S)}_{-1}(k)} + \frac{g_A^2}{2f_\pi^2}\left[\frac{4}{3am_\pi} - \frac{m_\pi}{k}\left(\frac{1}{ka}-\frac{kr}{2}\right)\right]$$

$$-\frac{g_A^2 M m_\pi}{16\pi f_\pi^2}\left\{A_0^{(S)}(k)\,\frac{m_\pi}{k}\,\underbrace{\arctan\left[\frac{2k}{m_\pi}\right]}_{1\pi\text{ cut}} + \frac{g_A^2}{2f_\pi^2}\left[\frac{1}{12}+\left(\frac{m_\pi^2}{4k^2}-\frac{1}{3}\right)\ln 2 - \underbrace{F_\pi\left(\frac{k}{m_\pi}\right)}_{1,2\pi\text{ cut}}\right]\right\}$$

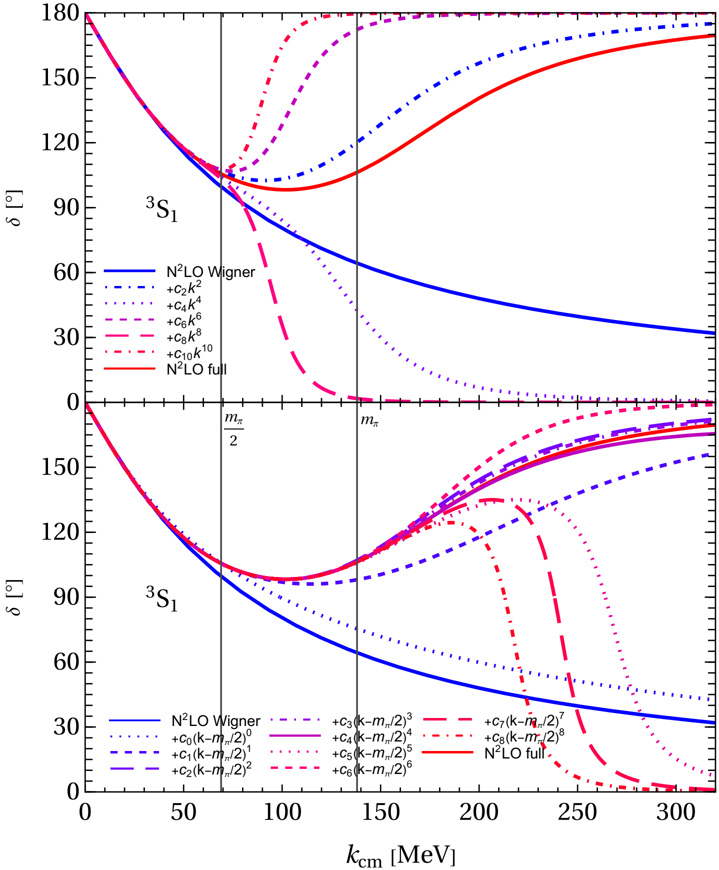
$$A_1^{(^3S_1)}(k) = A^{(S)}_{1\text{sym}}(k) + A^{(S)}_{1\text{break}}(k)$$

$$A^{(S)}_{1\text{break}}(k) = -\frac{\left[A_0^{(SD)}(k)\right]^2}{A^{(S)}_{-1}} + \frac{g_A^2}{f_\pi^2}\,\frac{g_A^2 M m_\pi}{16\pi f_\pi^2}\left\{\frac{571-352\ln 2}{210} - \left(1+\frac{3m_\pi^2}{2k^2}+\frac{9m_\pi^4}{16k^4}\right)\overbrace{F_\pi\left(\frac{k}{m_\pi}\right)}^{1,2\pi\text{ cut}} + \frac{2m_\pi^2}{5k^2}(\ln 4 - 1)\right.$$

$$\left. + \frac{3m_\pi^4}{16k^4} - \frac{3}{2}\left[\frac{k}{m_\pi}+\frac{m_\pi}{k}-\frac{m_\pi^3}{8k^3}-\frac{3m_\pi^5}{16k^5}\right]\overbrace{\arctan\left[\frac{k}{m_\pi}\right]}^{2\pi\text{ cut}} + \frac{3}{16}\left(\frac{m_\pi^4}{k^4}+\frac{3m_\pi^6}{4k^6}\right)\overbrace{\ln\left[\frac{16(k^2+m_\pi^2)}{4k^2+m_\pi^2}\right]}^{1,2\pi\text{ cut}}\right\}$$

$$F_\pi(x) := \frac{1}{8x^3}\left(\underbrace{\arctan[2x]\ln[1+4x^2] - \operatorname{Im}\left[\mathrm{Li}_2\left[\frac{2ix+1}{2ix-1}\right]\right.}_{1\pi\text{ cut}} \left. -2\,\underbrace{\mathrm{Li}_2\left[\frac{1}{2ix-1}\right]}_{2\pi\text{ cut}}\right]\right)$$

(c) Whence the Hockey Stick in $^3S_1$?



Expand Wigner-breaking in Taylor:

$$A^{(S)}_{\text{sym}}(k) + \sum \frac{1}{n!}[A^{(S)}_{\text{break}}]^{(n)}(k_0)(k-k_0)^n$$

Expand about 0:

$k \lesssim \frac{m_\pi}{2}$: convergent, Wigner-breaking tiny

$k \gtrsim \frac{m_\pi}{2}$: no convergence

$\Longrightarrow$ ERE not the problem.

Sorry, no Cohen/Hansen 1999.

Expand about 1st branch point scale $\frac{m_\pi}{2}$:

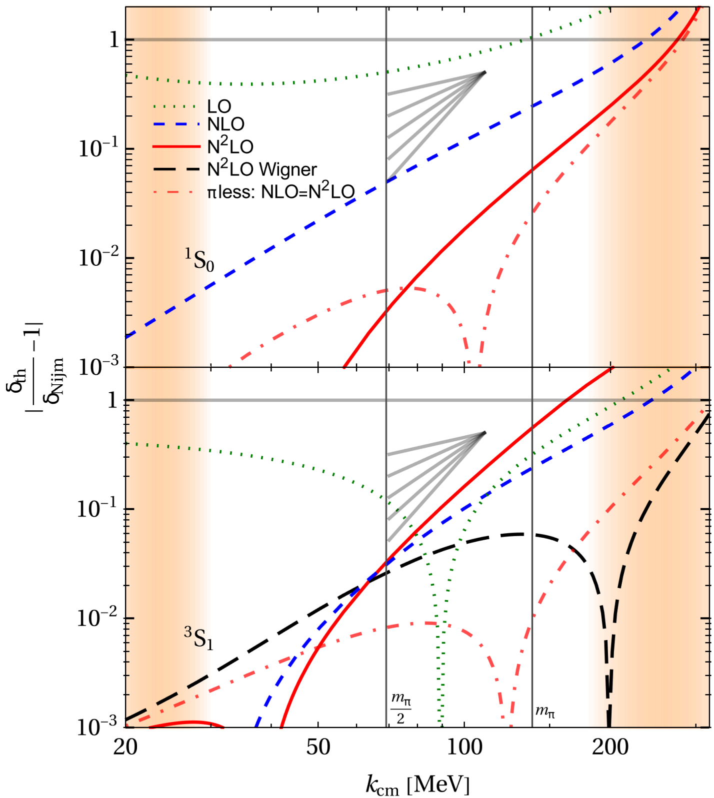
$k \lesssim \frac{m_\pi}{\sqrt{2}}$: convergent, Wigner-breaking tiny

(larger distance to branch point)

$k \lesssim \frac{3}{2}m_\pi$ (>2nd br. pt. scale): convergent

$k \gtrsim \frac{3}{2}m_\pi$: asymptotic (optimal: incl. k^4)

Landau/Páez/Bordeianu: Comp. Phys., Lepage 1997
Teng/hg [2410.09653]



$$\frac{\delta(\text{N}^n\text{LO}) - \delta(\text{PWA})}{\delta(\text{PWA})} \sim \left(\frac{k, m_\pi}{\overline{\Lambda}}\right)^{n+1}$$

at N^{n}LO with empirical breakdown scale $\overline{\Lambda}$.

$^1\mathrm{S}_0$ and Wigner-symmetric $^3\mathrm{S}_1$:

consistent slopes and

$\overline{\Lambda} \approx 270\ \mathrm{MeV} \approx \overline{\Lambda}_{\mathrm{NN}}$ OPE scale.

Full $^3\mathrm{S}_1$:

N^{2}LO worse than NLO for $\gtrsim 70\ \mathrm{MeV}$.

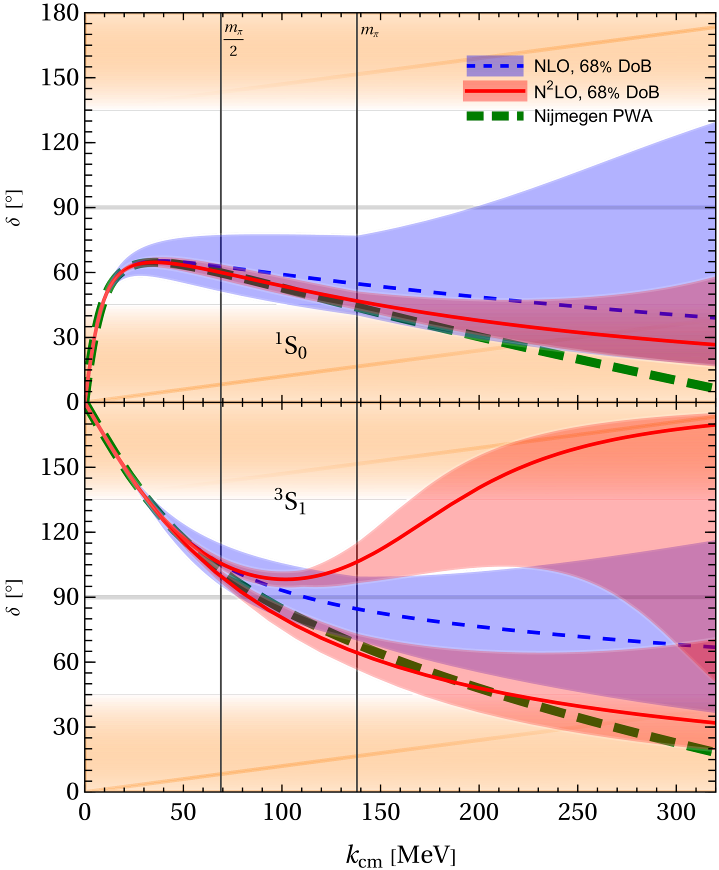
Picture obscured by points where

theory & PWA identical (“artificial zero”),

or PWA close to zero (“artificial ∞”).

(e) NLO & N^{2}LO Bayesian Truncation Uncertainties

hg/... [1203.6834], Cacciari/Houdeau [1105.5152]
BUQEYE [1506.01343], hg/... [1511.01952]
Teng/hg [2410.09653]



Apply "max" criterion to $\cot\delta$ order-by-order:

Unitarity: $k\cot\delta_{\mathrm{LO}} = 0 \Rightarrow$ "$-ik$" sets scale.

Bayesian N^{2}LO truncation uncertainty at k:

$$\pm Q^3 \max\left\{\frac{\cot\delta_0(k)-\cot\delta_0(0)}{Q};\frac{\cot\delta_1(k)}{Q^2}\right\}$$

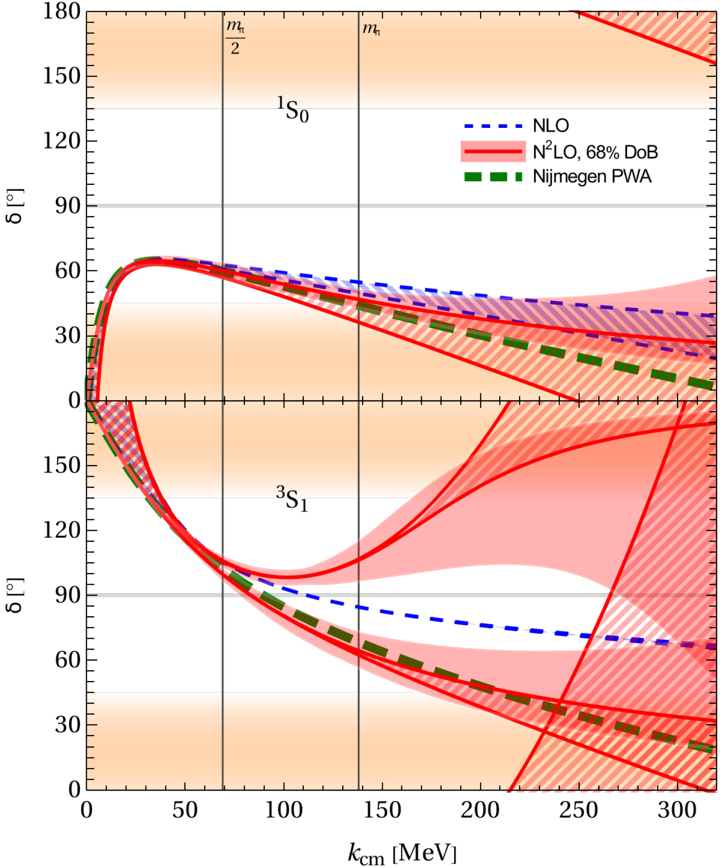
with $Q = \dfrac{\max\{k;m_\pi\}}{\overline{\Lambda}_{\mathrm{NN}} \sim 300\ \ \mathrm{MeV}}$

NLO: rescaled to 68% DoB,
assuming uniform&log-uniform prior.

Only Wigner-symmetric forms have
N^{2}LO uncertainties consistent with NLO,
and NLO&N^{2}LO consistent with PWA.

(f) Different Ways To Extract Phase Shifts at NLO and N^2LO

Teng/hg [2410.09653]



So far:

$$k\cot\delta = 0_{LO} + k\cot\delta_{NLO} + k\cot\delta_{N^2LO}$$
$$= -\frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2}\right]$$

is fundamental, derive $\delta(k)$ from it.

$$\stackrel{k\to 0}{\longrightarrow} 0_{LO} + \left(-\frac{1}{a} + \frac{r}{2}k^2\right)_{N^{1+2}LO} + \mathcal{O}(k^4)$$

constructed to reproduce ERE.

Hatched: difference to

directly from amplitude KSW 1999,FMS 2000

$$\delta(k) = \frac{\pi}{2}\Big|_{-1} + \left(\frac{1}{ka} - \frac{kr}{2}\right)\Big|_0 + \dots$$

$\to\infty$ for $k\to 0$ outside Unitarity Window.

Methods agree inside Unitarity Window

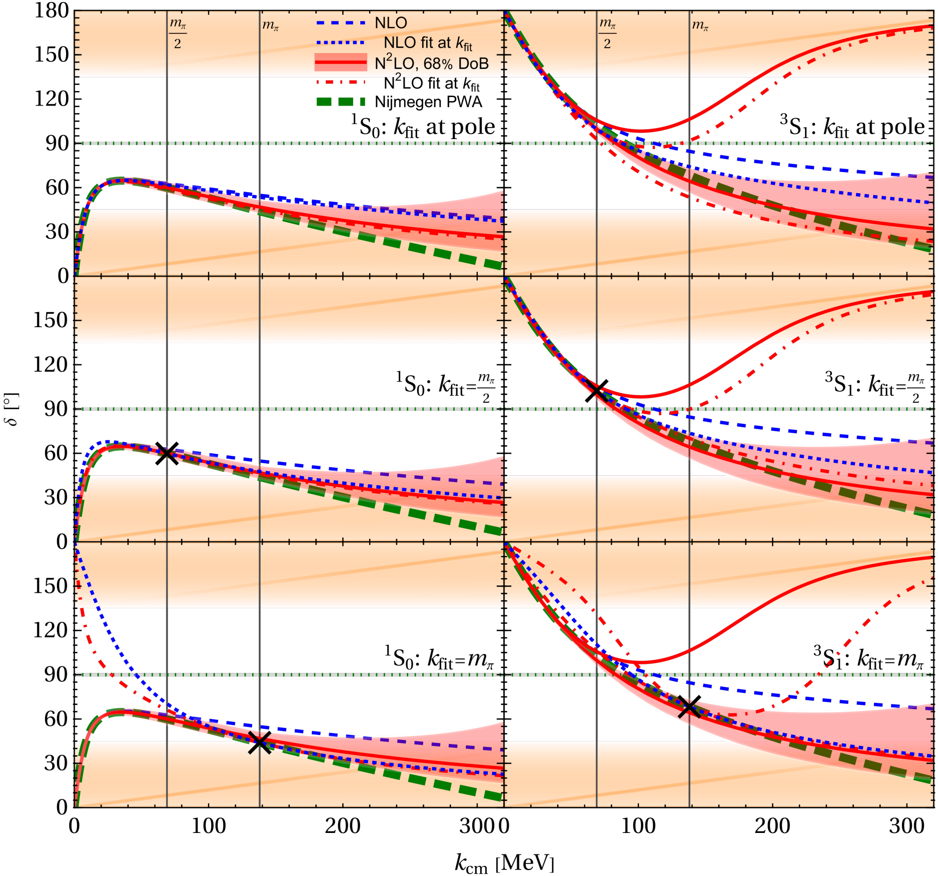
$\frac{1}{ka}, \frac{kr}{2} < 1$ (must in centre $|\cot\delta|\to 0$):

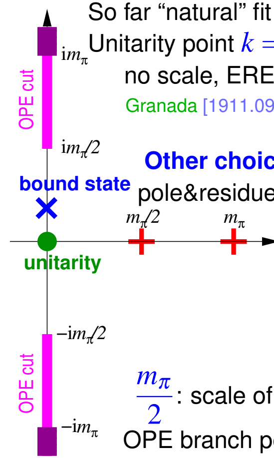
Independent assessment of

truncation uncertainty, consistent with Bayes.

(g) Different Renormalisation/Parameter-Determination Points

Teng/hg [2410.09653]





So far “natural” fit at Unitarity point $k=0$: no scale, ERE

Granada [1911.09637]

Other choices:

pole&residue: ok

$\frac{m_\pi}{2}$: scale of 1st OPE branch point

No cure to hockey-stick.

Uncertainties & breakdown scale very similar.

m_π: 2nd OPE branch point

No cure to hockey-stick.

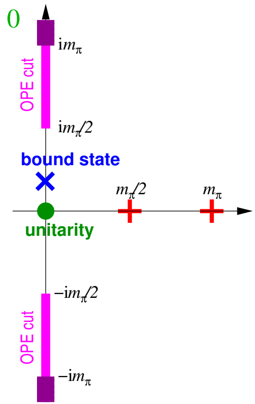
Low-k $^1S_0$ bad inside Unitarity Window.

(h) Virtual/Real Bound-State Pole Positions and Residues

Teng/hg [2410.09653]

$$\frac{1}{A_{-1}(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+A_0(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+A_1(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+\dots}\overset{!}{=}0$$

$$\frac{1}{Z}=\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}k}\,(k\cot\delta(k)-\mathrm{i}k)\Big|_{k=\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots}$$



For $k_{\text{fit}}=0$, pions cannot correct a, r since we force the ERE values Granada [1911.09637]

$\Longrightarrow$ pole at binding momentum $\mathrm{i}\gamma=\frac{\mathrm{i}}{a}\left(1+\frac{r}{2a}+\frac{r^2}{2a^2}+\mathcal{O}(\frac{r^3}{a^3})\right)$

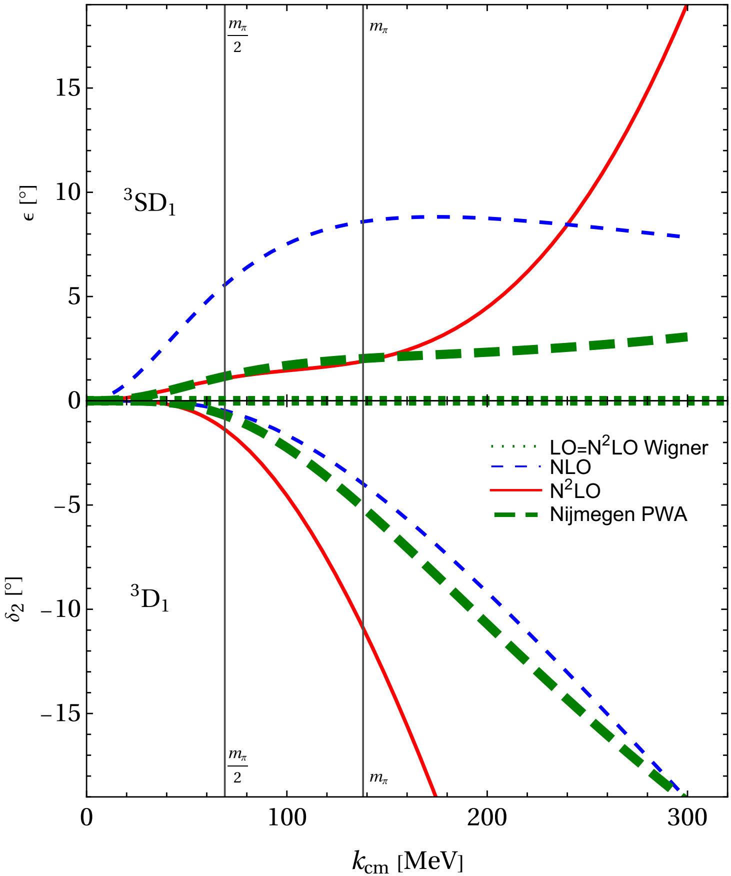
with residue $Z=1+\frac{r}{a}+\frac{3r^2}{2a^2}+\mathcal{O}(\frac{r^3}{a^3})$.

For general k_{fit}, match to $k\cot\delta_{\text{PWA}}(k_{\text{fit}})$, $\frac{\mathrm{d}}{\mathrm{d}k}k\cot\delta_{\text{PWA}}(k_{\text{fit}})$. $\Longrightarrow$ Predict a, r.

	$^1S_0$			$^3S_1$		
k_{fit}	scatt. length a [fm]	eff. range r [fm]	(bind. mom.,residue) $(\gamma$ [MeV]$,Z)$	scatt. length a [fm]	eff. range r [fm]	(bind. mom.,residue) $(\gamma$ [MeV]$,Z)$
ERE	$-23.735(6)*$	$2.673(9)*$	$(-7.892, 0.9034)$	$5.435(2)*$	$1.852(2)*$	$(+47.7023, 1.689)*$
pole	-23.7104	2.7783		5.6128	2.3682	
$\frac{m_\pi}{2}$ NLO	-38.988	3.3270	$(-4.86, 0.925)$	4.9310	2.4966	$(+55., 1.9)$
$\frac{m_\pi}{2}$ N^{2}LO	-25.428	2.7281	$(-7.34, 0.910(2))$	4.7768	2.4492	$(+57(3)., 1.9(2))$
$\frac{m_\pi}{2}$ sym.				5.4625	1.6124	$(+43.0(5), 1.42(4))$
m_π NLO	$+\ 9.2856$	4.2285	$(+28., 1.8)$	$3.3442\dagger$	$3.1886\dagger$	$(+114., 3.)$ ↯
m_π N^{2}LO	$+34.3335$	2.8956	$(+6.01, 1.10)$	$1.8376\dagger$	$3.3741\dagger$	$(+387(330)., 7(9).)$ ↯
m_π sym.				4.5344	1.7006	$(+54(1)., 1.5(1))$

Bayesian N^{2}LO uncertainties $*$: input ↯: cannot converge: $Q\sim\frac{1}{ka},\frac{kr}{2}\ll 1\Longrightarrow\frac{r}{2a}\ll 1$ ↯

(i) $^3\mathrm{SD}_1$ Mixing: Full vs. Wigner



No other channels close to Unitarity Window:

$|\delta_{l\geq1}| < 25^\circ$ ($|\cot\delta_{l\geq1} > 2|$).

$^3\mathrm{SD}_1$ mixing only by tensor/Wigner-breaking.

In Unitarity Expansion very similar to FMS:

$k \gtrsim 70\ \mathrm{MeV}$:

No order-by-order convergence,

convergence to PWA elusive.

Zero by Wigner at N^{2}LO.

Natural size at N^{3}LO at $k \approx m_\pi$:

$90^\circ \times (Q \approx 0.5)^3 \approx 10^\circ$

$\Longleftrightarrow$ PWA: $\lesssim 10^\circ$.

$\Longrightarrow$ Not inconsistent.

SD & DD contacts at N^{3}LO

$\Longrightarrow$ Reproducing PWA possible.