Transfer reactions: the DWBA method

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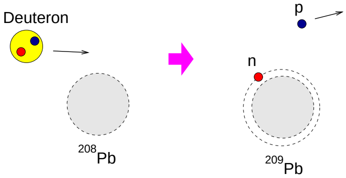
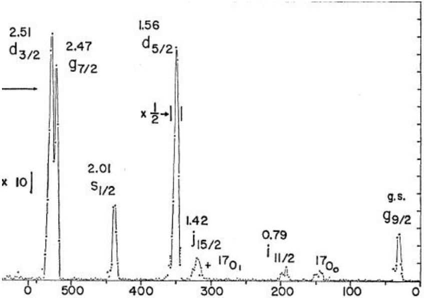
Material available at: `https://github.com/ammoro/R4E`

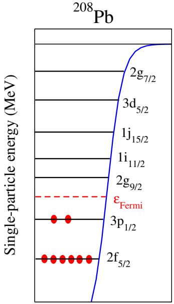
Example: $d + {}^{208}Pb \rightarrow p + {}^{209}Pb$



1. What do we measure in a transfer reaction?
    1. For a typical transfer reaction (e.g. $d + {}^{208}Pb \rightarrow p + {}^{209}Pb$), one measures the angular and energy distribution of outgoing fragments (e.g. protons).
    2. Additionally, one may collect information of decay products of ^{209}Pb (e.g. γ-rays, n, p, etc)
2. What information can we infer from a transfer reaction?
    1. Excitation energies of the residual nucleus (^{209}Pb).
    2. Angular momentum assignment.
    3. Single-particle content of populated states (i.e. spectroscopic factors).

Example: $d + {}^{208}Pb \rightarrow p + {}^{209}Pb$ *Phys. Rev. 159 (1967) 1039*





- ☞ The proton energy spectrum shows some peaks which reflect the energy spectrum of the residual nucleus (^{209}Pb).
- ☞ Each peak has a characteristic angular distribution, which depends on the structure of the associated state.
- ☞ The population probability will depend on the reaction dynamics and on the structure properties of these states.

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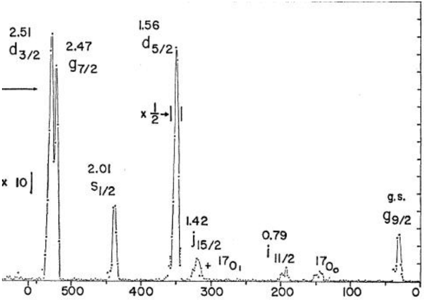

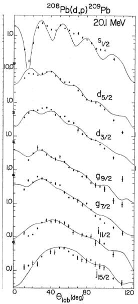

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Consider: $a + A \rightarrow b + B$

- Energy balance (in CM frame):

$$\boxed{E^i_{\rm cm} + M_a c^2 + M_A c^2 = E^f_{\rm cm} + M_b c^2 + M_B c^2}$$

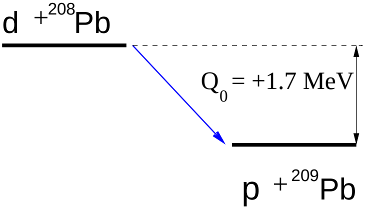
- Q_0 value:

$$\boxed{Q_0 = M_a c^2 + M_A c^2 - M_b c^2 - M_B c^2}$$

$$\boxed{E^f_{\rm cm} = E^i_{\rm cm} + Q_0}$$

  - $Q_0 > 0$: the system gains kinetic energy (exothermic reaction)
  - $Q_0 < 0$: the system loses kinetic energy (endothermic reaction)

Example: d+^{208}Pb → p + ^{209}Pb



$$\boxed{Q_0 = M_d c^2 + M(^{208}\mathrm{Pb})c^2 - M_p c^2 - M(^{209}\mathrm{Pb})c^2 = +1.7\,\mathrm{MeV}}$$

☞ $Q_0 > 0$: the outgoing proton will gain energy with respect to the incident deuteron.

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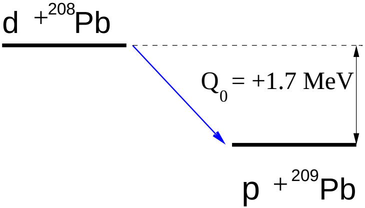

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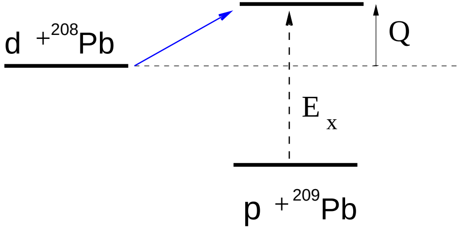
N.b.: For a transfer reaction, the Q value is just the difference in binding energies of the transferred particle/cluster in the initial and final nuclei:

$$Q_0 = \varepsilon_b(f) - \varepsilon_b(i) = 3.936 - 2.224 = +1.7\,\text{MeV}$$

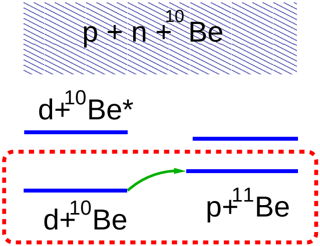
If the transfer leads to an excited state, the Q-value will change accordingly, and hence the kinetic energy of the outgoing nuclei.



Energy balance:

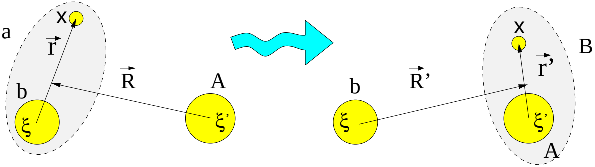
$$\boxed{E^f_{\rm cm} = E^i_{\rm cm} + Q = E^i_{\rm cm} + Q_0 - E_x}$$

☞ *If we know Q_0 we can infer the excitation energies (E_x) measuring the final kinetic energy of outgoing fragments.*



☞*In a transfer calculation, the modelspace will contain states belonging to different mass partitions, and hence to different internal Hamiltonians.*

- Transfer process: $\underbrace{(b+x)}_{a} + A \rightarrow b + \underbrace{(A+x)}_{B}$



- Complications arise with respect to inelastic scattering because now we have two different mass partitions involved

$$\underbrace{a+A}_{\alpha} \rightarrow \underbrace{b+B}_{\beta}$$

- Projectile-target interaction in post representation:

$$V_\beta(\mathbf{R}',\mathbf{r}') = V_{xb} + U_{bA} = \underbrace{U_\beta(\mathbf{R}')}_{\text{Aux. pot.}} + \underbrace{[V_{xb} + U_{bA} - U_\beta(\mathbf{R}')]}_{\text{Resid. inter.}} \equiv U_\beta(\mathbf{R}') + \Delta V_\beta$$

- Differential cross section:. In general, $\left(\frac{d\sigma}{d\Omega}\right)_{(\beta,\alpha)} = \frac{\mu_\alpha \mu_\beta}{(2\pi\hbar^2)^2} |\mathcal{T}_{\beta,\alpha}|^2$
- In DWBA:

$$\boxed{\mathcal{T}_{\beta,\alpha}(\theta) = \int \underbrace{\chi_\beta^{(-)*}(\mathbf{K}_\beta,\mathbf{R}')\,\Phi_\beta^*(\xi_\beta)}_{\text{final state}}\,\Delta V_\beta\,\underbrace{\chi_\alpha^{(+)}(\mathbf{K}_\alpha,\mathbf{R})\,\Phi_\alpha(\xi_\alpha)}_{\text{initial state}}\;\underbrace{d\xi_\beta\,d\mathbf{R}'}_{\text{(all coordinates)}}}$$

- Initial and final internal states:

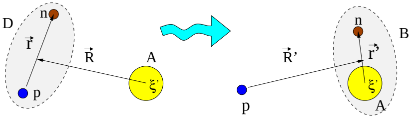
$$\text{Initial state: } \Phi_\alpha(\xi_\alpha) = \varphi_a(\xi,\mathbf{r})\Phi_A(\xi') \qquad \xi_\alpha \equiv \{\xi,\xi',\mathbf{r}\}$$

$$\text{Final state: } \Phi_\beta(\xi_\beta) = \varphi_b(\xi)\Phi_B(\xi',\mathbf{r}') \qquad \xi_\beta \equiv \{\xi,\xi',\mathbf{r}'\}$$

- $\chi_{\alpha,\beta}^{(\pm)}$ are distorted waves for entrance and exit channels, obtained with appropriate optical potentials $U_\alpha(\mathbf{R})$, $U_\beta(\mathbf{R}')$

$$\left[E_{\text{c.m.}} - \hat{T}_{\mathbf{R}} - U_\alpha(\mathbf{R})\right]\chi_\alpha^{(+)}(\mathbf{K}_\alpha,\mathbf{R}) = 0$$

$$\left[E'_{\text{c.m.}} - \hat{T}_{\mathbf{R}'} - U_\beta(\mathbf{R}')\right]\chi_\beta^{(+)}(\mathbf{K}_\beta,\mathbf{R}') = 0$$



- Introduce auxiliary potentials in entrance ($U_\alpha(\mathbf{R})$) and exit ($U_\beta(\mathbf{R}')$) channels.
- Projectile-target interaction: $V_\beta = V_{pn} + U_{pA} = U_{pB}(\mathbf{R}') + \underbrace{V_{pn} + U_{pA} - U_{pB}}_{\Delta V_\beta} \equiv U_\beta(\mathbf{R}') + \Delta V_\beta$
- Internal states:

$$\Phi_\alpha^{(0)}(\xi_\alpha) = \varphi_d(\mathbf{r})\phi_A(\xi') \qquad \xi_\alpha = \{\xi', \mathbf{r}\}$$
$$\Phi_\beta(\xi_\beta) = \Phi_B(\xi', \mathbf{r}') \qquad \xi_\beta = \{\xi', \mathbf{r}'\}$$

- Post-form DWBA transition amplitude:

$$\boxed{\mathcal{T}_{d,p}^{\mathrm{DWBA}} = \int\int \chi_p^{(-)*}(\mathbf{K}_p, \mathbf{R}')\Phi_B(\xi', \mathbf{r}')\,(V_{pn} + U_{pA} - U_{pB})\chi_d^{(+)}(\mathbf{K}_d, \mathbf{R})\varphi_d(\mathbf{r})\phi_A(\xi')\,d\xi_\beta d\mathbf{R}'}$$

- For medium-mass/heavy targets: $U_{pA} \approx U_{pB} \Rightarrow V_{pn} + U_{pA} - U_{pB} \approx V_{pn}(\mathbf{r})$

➩ We need to evaluate the overlap integral

$$\int d\xi' \;\phi_B^*(\xi', \mathbf{r}')\phi_A(\xi') \equiv \langle \phi_B | \phi_A \rangle$$

➩ Use the parentage decomposition of $B \to A + n$

$$\boxed{\Phi_B(\xi', \mathbf{r}') = \mathcal{A}_{BA}^{\ell j I}\phi_A(\xi')\varphi_{nA}^{\ell j I}(\mathbf{r}') + \sum_{A' \neq A} \phi_{A'}(\xi')\varphi_{nA}^{\ell' j' I'}(\mathbf{r}')}$$

$$\Rightarrow \langle \phi_B | \phi_A \rangle = \mathcal{A}_{BA}^{\ell j I}\varphi_{nA}^{\ell j I}(\mathbf{r}')$$

- $\mathcal{A}_{BA}^{\ell j I}$ = spectroscopic amplitude
- $|\mathcal{A}_{BA}^{\ell j I}|^2 = S_{BA}^{\ell j I}$ = spectroscopic factor
- $\varphi_{nA}^{\ell j I}(\mathbf{r}')$ = single-particle wavefunction describing motion of n with respect to A.

☞ *The spectroscopic factor quantifies how much a given nuclear state (B) resembles a particular configuration of a core nucleus (A) plus a nucleon (or cluster of nucleons) in a given single-particle configuration.*

1. Double-magic nucleus plus a single nucleon:

$$|^{209}\mathrm{Bi(g.s.)}\rangle_{9/2^-} \approx \left[|^{208}\mathrm{Pb}(0^+)\rangle \otimes |\pi 1h_{9/2}\rangle\right]_{9/2^-}$$

☞ *almost* single-particle configuration ($S^{\ell sj}_{IJ} \approx 1$).

2. Deformed core plus an extra nucleon:

$$|^{11}\mathrm{Be(gs)}\rangle_{1/2^+} = \alpha\left[|^{10}\mathrm{Be}(0^+)\rangle \otimes |\nu 2s_{1/2}\rangle\right]_{1/2^+} + \beta\left[|^{10}\mathrm{Be}(2^+)\rangle \otimes |\nu 1d_{5/2}\rangle\right]_{1/2^+} + \ldots$$

with $|\alpha|^2 + |\beta|^2 + \ldots \approx 1$

3. Due to indistinguishability of neutrons (or protons) the SF can be even larger than 1!

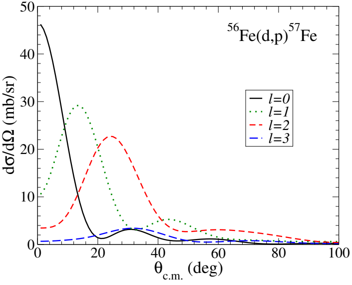
➩ In post form:

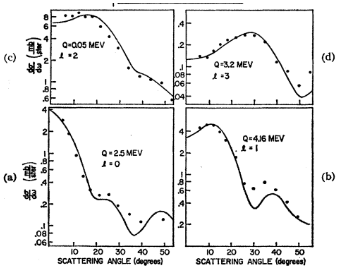
$$\boxed{\mathcal{T}_{d,p}^{\mathrm{DWBA}} = \mathcal{A}_{BA}^{\ell jI} \int\int \chi_p^{(-)*}(\mathbf{K}_p, \mathbf{R}')\varphi_{nA}^{\ell jI,*}(\mathbf{r}')\, V_{pn}(\mathbf{r})\, \chi_d^{(+)}(\mathbf{K}_d, \mathbf{R})\varphi_d(\mathbf{r}) d\mathbf{r}' d\mathbf{R}'}$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(d,p)} = \frac{\mu_\alpha \mu_\beta}{(2\pi\hbar^2)^2} S_{BA}^{\ell jI} \left| \int\int \chi_p^{(-)*}(\mathbf{K}_p, \mathbf{R}')\varphi_{nA}^{\ell jI,*}(\mathbf{r}')\, V_{pn}(\mathbf{r})\, \chi_d^{(+)}(\mathbf{K}_d, \mathbf{R})\varphi_d(\mathbf{r}) d\mathbf{r}' d\mathbf{R}' \right|^2}$$

$$|\mathcal{A}_{BA}^{\ell jI}|^2 = S_{BA}^{\ell jI} = \text{spectroscopic factor}$$

☞*In DWBA, the transfer cross section is proportional to the product of the projectile and target spectroscopic factors. Comparing the data with DWBA calculations, one can extract the values of* $S_{BA}^{\ell jI}$





☞*Angular distributions of transfer cross sections are very sensitive to the single-particle configuration of the transferred nucleon/s.* $\Rightarrow \varphi_{nlj}(\mathbf{r})$

☞ From classical arguments, and assuming an infinite mass target, the angle of the first maximum appears at:

$$\theta_{\max} \approx \arcsin\left(\frac{\sqrt{\ell(\ell+1)}\hbar}{P_i R}\right) = \arcsin\left(\frac{\sqrt{\ell(\ell+1)}}{K_0 R}\right)$$

with P_i the incident momentum of the projectile and R the distance at grazing.

Summary: What do we learn form a transfer experiment

1. Excitation energies of residual nucleus

➩ *The Q-value is related to the masses and excitation energies*

2. Spectroscopic factors (related to occupation numbers)

➩ In DWBA, $\sigma^{\ell j I} \propto S_{BA}^{\ell j I}$

3. Angular momentum of populated states.

➩ For heavy targets, the first maximum occurs at:

$$\theta_{\max} \approx \arcsin\left(\frac{\sqrt{\ell(\ell+1)}\hbar}{P_i R}\right)$$

$$|^{11}\mathrm{Be}\rangle = \alpha\, |^{10}\,\mathrm{Be}(0^+) \otimes \nu 2s_{1/2}\rangle + \beta\, |^{10}\,\mathrm{Be}(2^+) \otimes \nu 1d_{5/2}\rangle + \ldots$$

➩ In DWBA:

$$\sigma(0^+) \propto |\alpha|^2; \quad \sigma(2^+) \propto |\beta|^2$$

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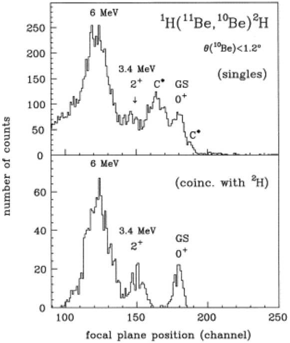
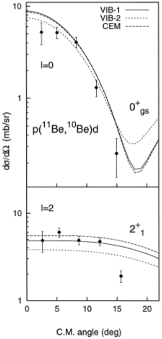
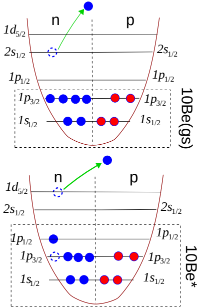
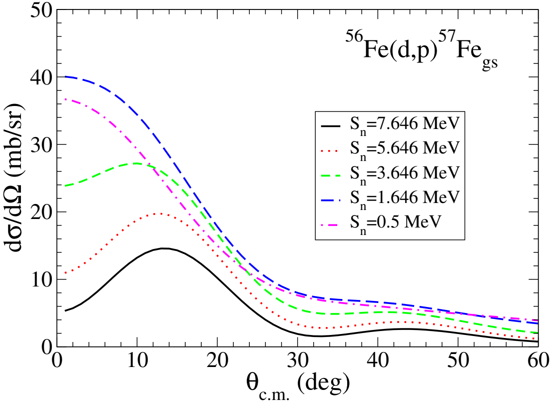
Fortier et al, PLB461, 22 (1999)



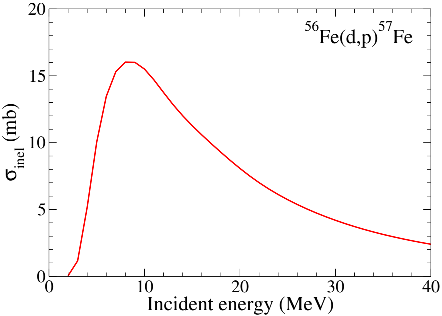




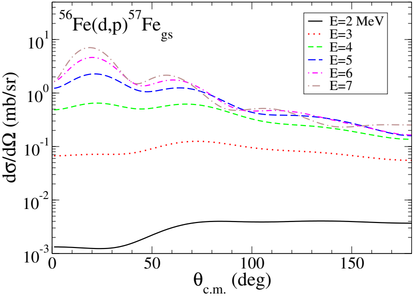
Dependence with binding energy for a fixed incident energy (12 MeV):



Dependence on beam energy



- $E \gg V_b$: diffractive structure, forward peaked.
- $E \ll V_b$: smooth dependence with θ, backward peaked.



⇨*At sub-Coulomb energies, the angular distribution is weakly sensitive to ℓ transfer (but sensitive to other parameters, such as the tail of the bound-state wavefunction)*