Three-Body Dynamics in QCD

Andrew W. Jackura

William & Mary

Department of Physics

Universality in Strongly-Interacting Systems: From QCD to Atoms

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ECT*

awjackura@wm.edu

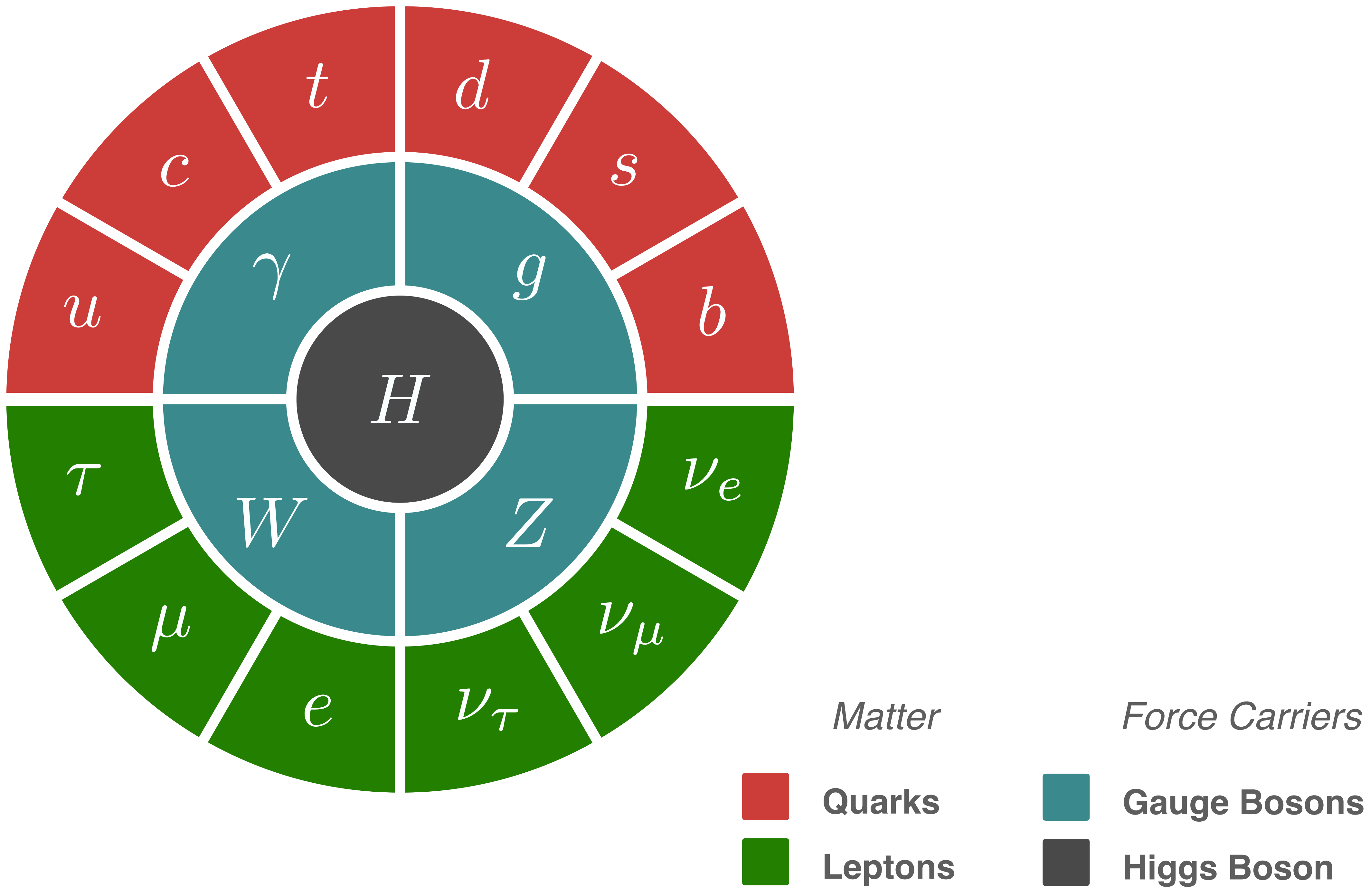
ajackura.github.io

Nuclear Physics and QCD

Nature can be described by a remarkably *simple** theory

- In principle, describes all of matter and interactions we observe (except gravity)
- This simple theory leads to *many* interesting and puzzling phenomena

The Standard Model of Particle Physics



** simple = An anomaly-free renormalizable relativistic quantum gauge field theory, invariant under the gauge group* $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ *which spontaneously breaks via a scalar field to* $SU(3)_C \otimes U(1)_Q$

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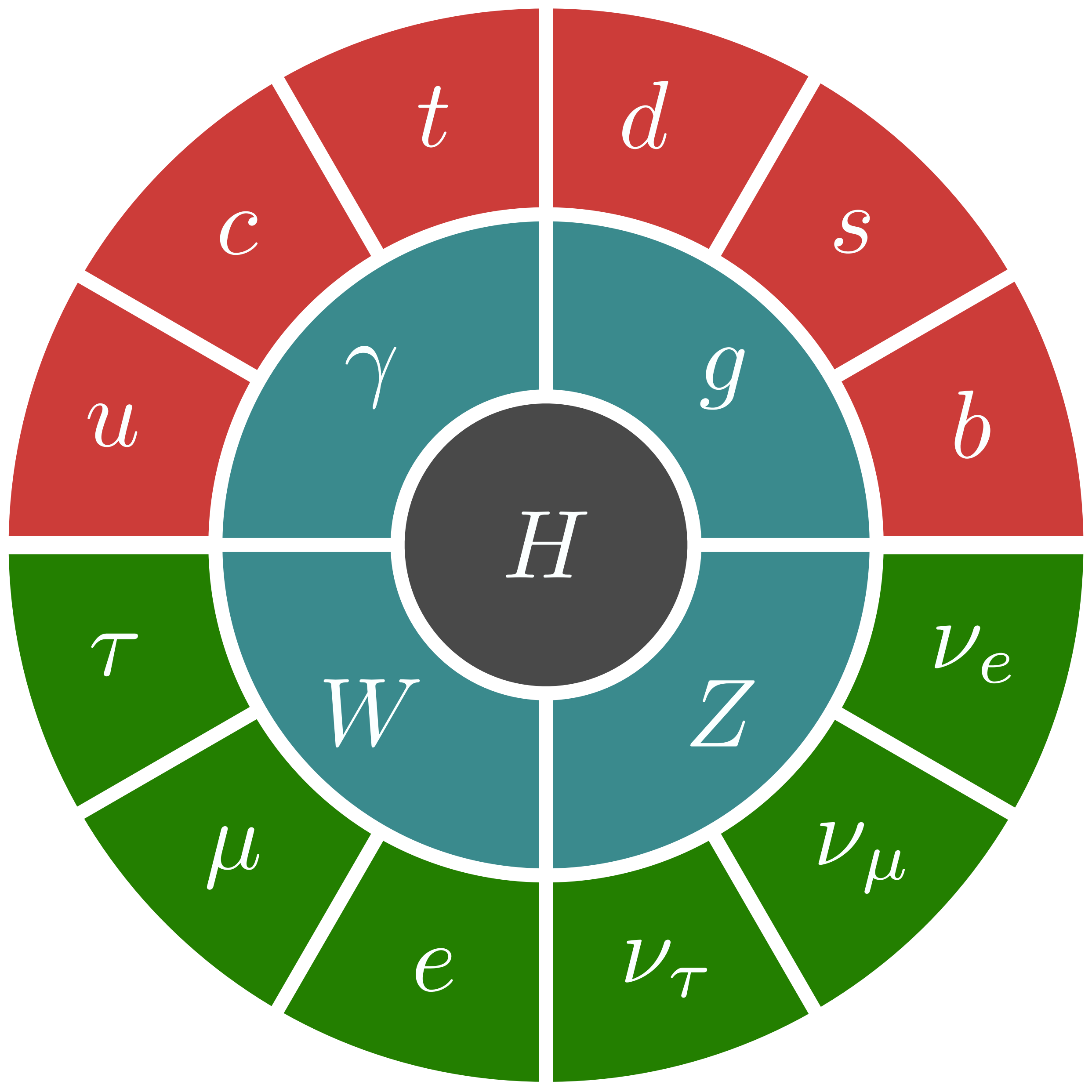
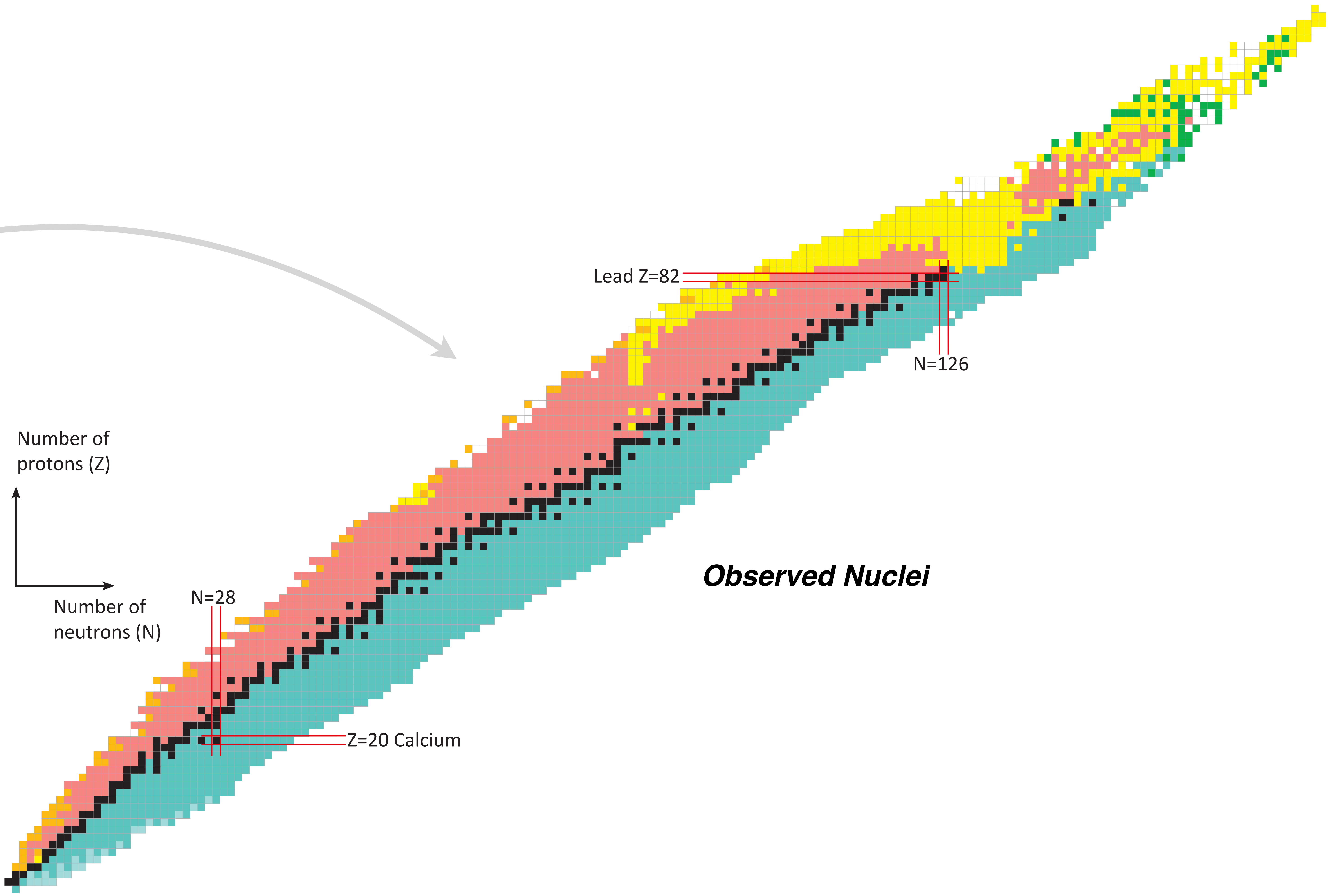




Observed Nuclei

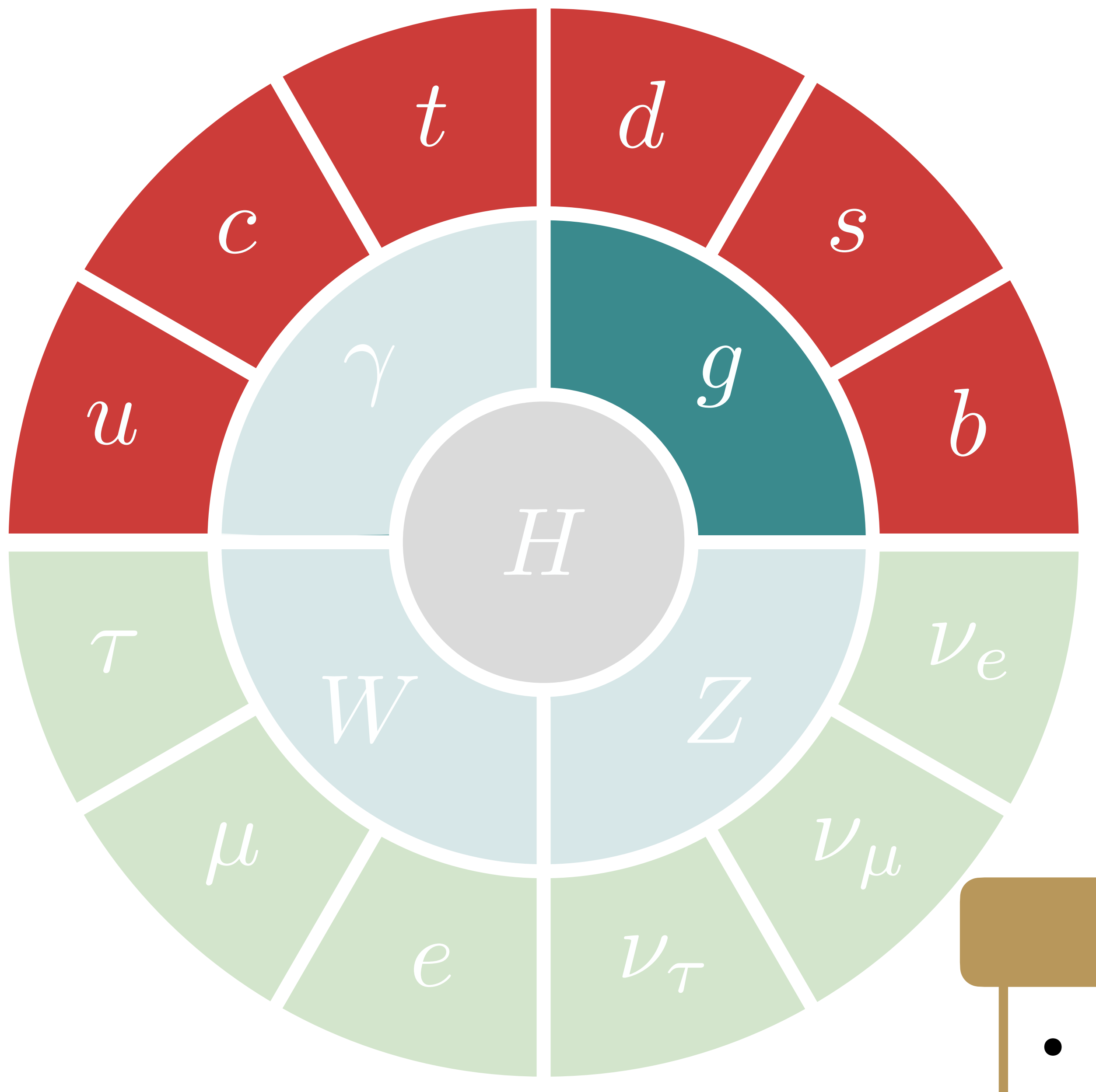
Nuclear Physics and QCD

Nature can be described by a remarkably *simple** theory

- In principle, describes all of matter and interactions we observe (except gravity)
- This simple theory leads to *many* interesting and puzzling phenomena
- **Strong interactions** governed by **Quantum ChromoDynamics (QCD)**

Theory of quark-gluon interactions

The Standard Model of Particle Physics



$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \bar{\psi}_f \left(i\not{D} - m_f \right) \psi_f - \frac{1}{2} \mathrm{tr} \left(\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu} \right)$$

gluon fields

quark fields

Assumptions

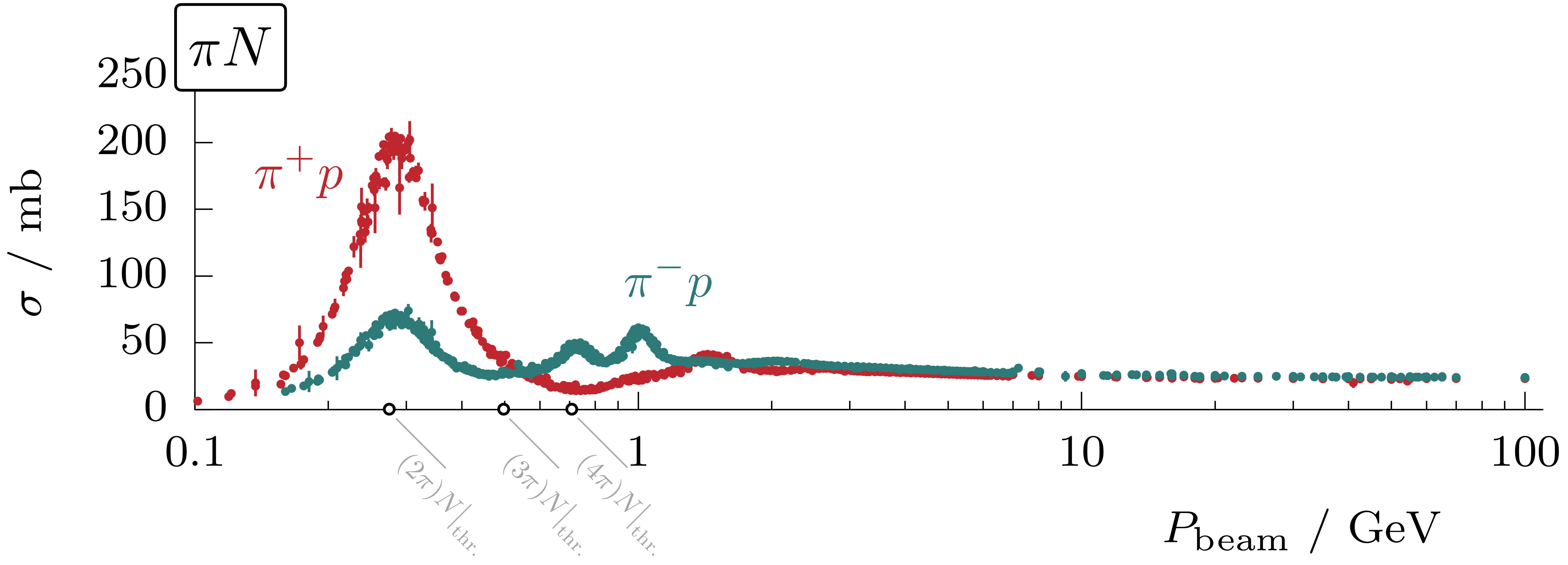
- QCD is the correct description of *all* nuclear phenomena
- Work in Natural Units, $\hbar = c = 1$, energy scales MeV / GeV
- Ignore electroweak interactions

malizable relativistic quantum gauge field theory,
e group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$
eaks via a scalar field to $SU(3)_C \otimes U(1)_Q$

Nuclear Physics and QCD

QCD is a deceptively simple looking theory

- The interactions are strong, meaning perturbative approximations often fail
- We observe **Hadrons** and **Nuclei**, not quarks and gluons
- Often interested in multi-particle **reactions**



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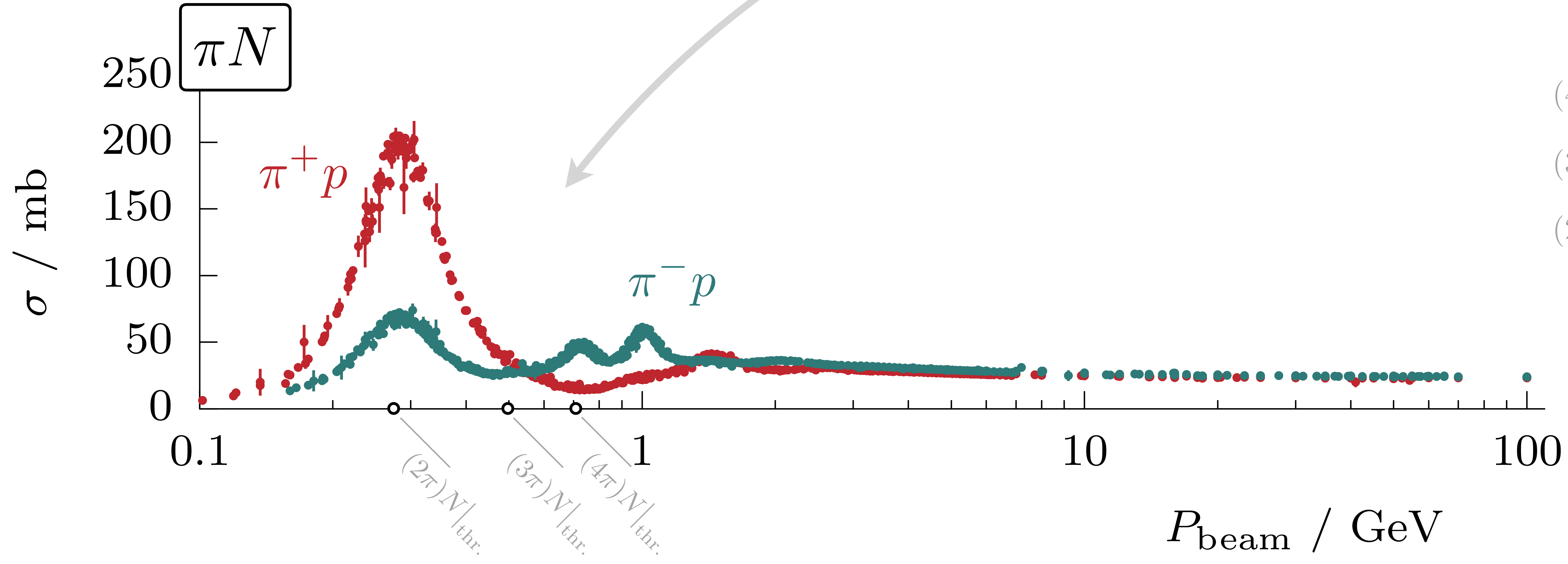
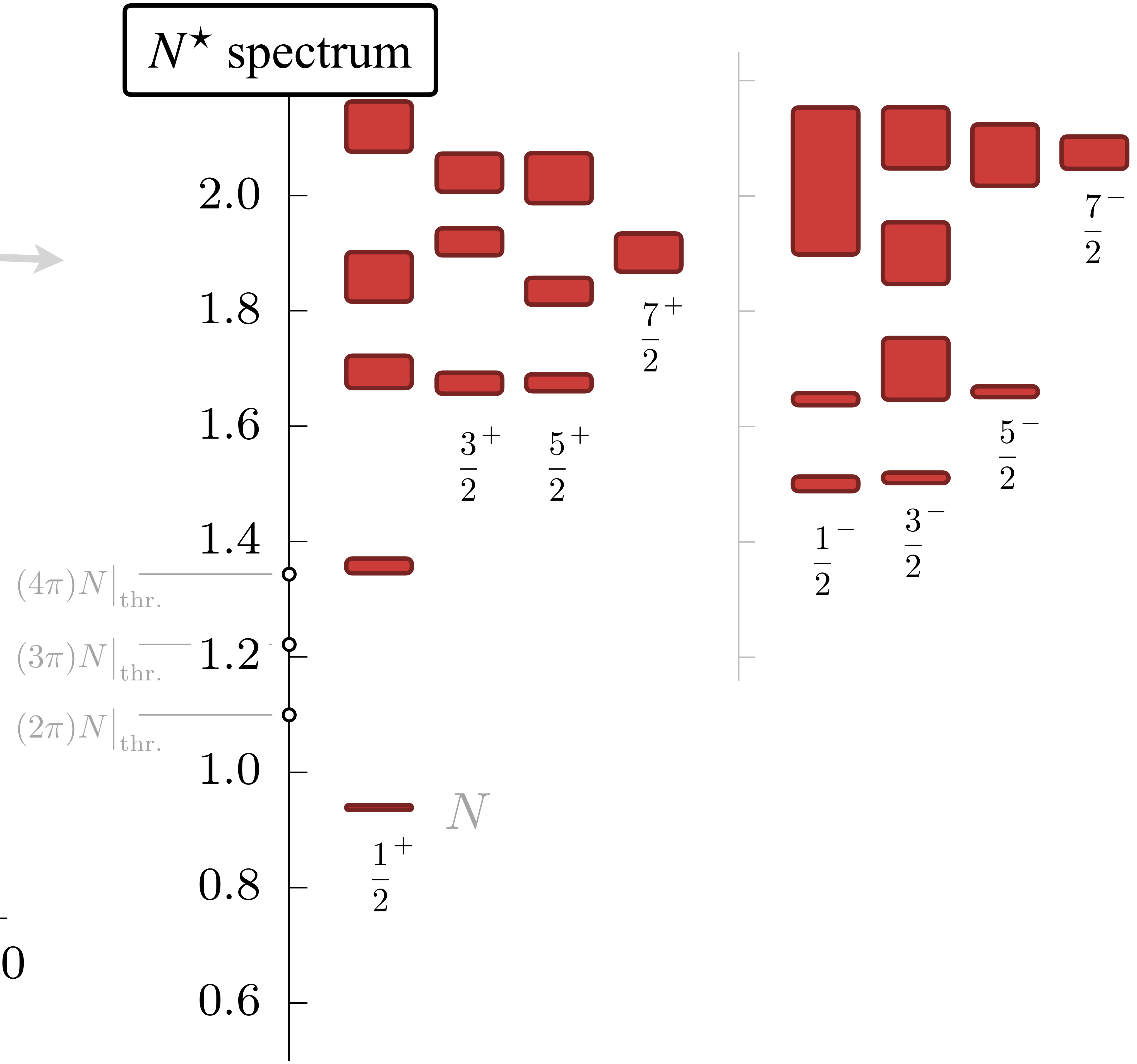
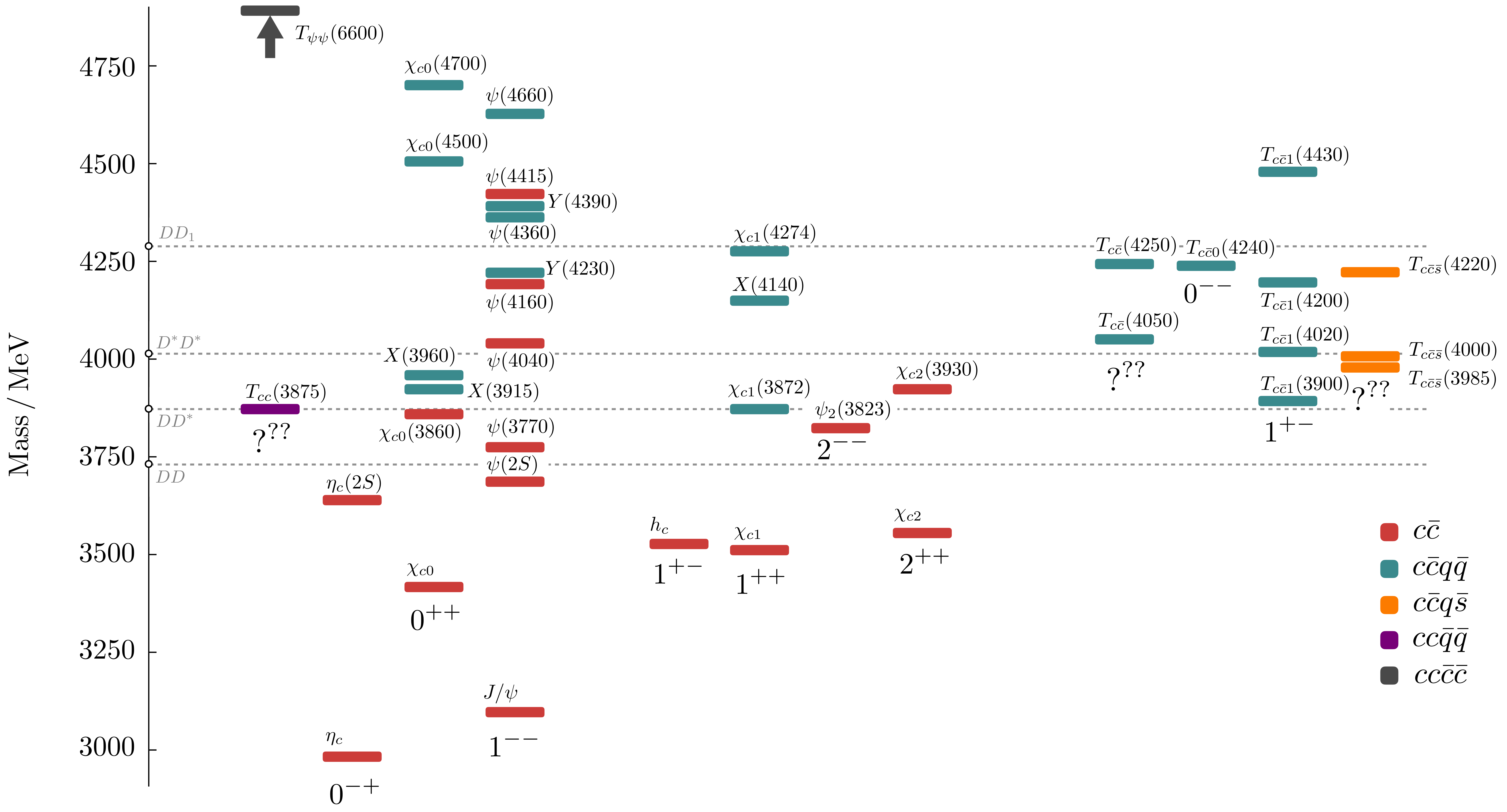
Reactions are window to understanding spectrum and structure of QCD





Three-Body Dynamics

Many interesting cases involve three-body physics



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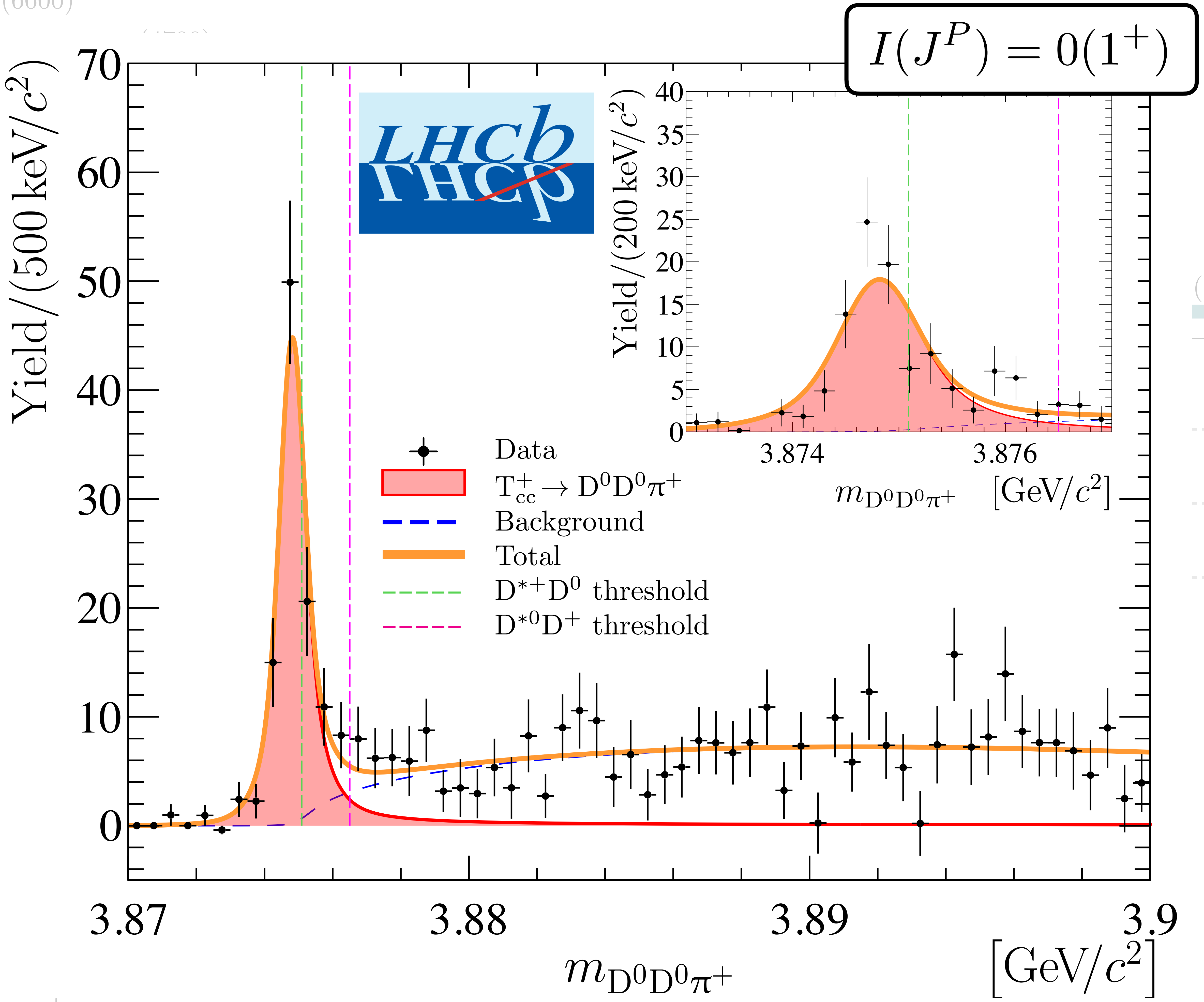
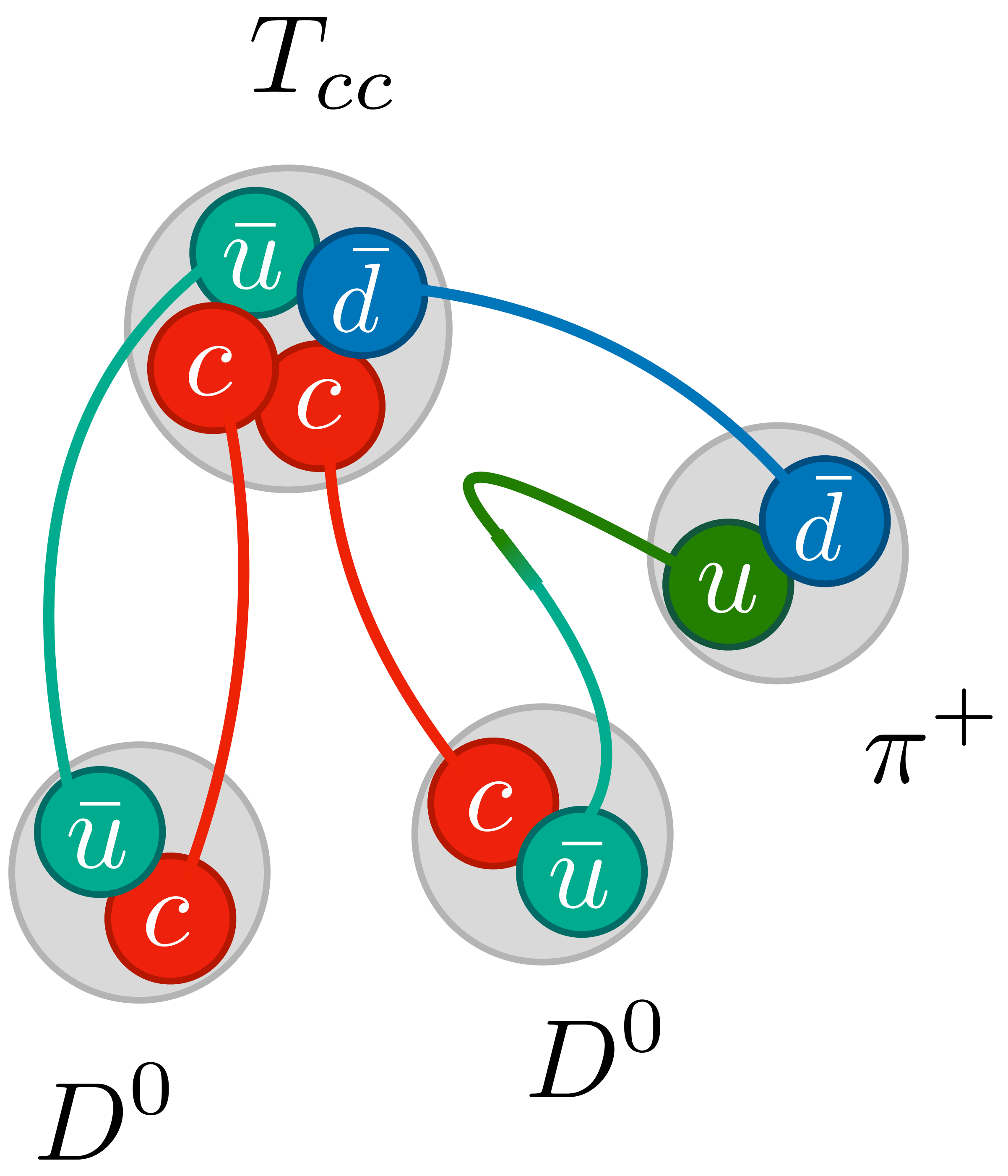




R. Aaij et al., [LHCb Collaboration]
Nature Physics **18**, 751–754 (2022)

Hadrons from QCD

Must resort to non-perturbative methods to probe hadron physics

- Lattice QCD offers systematically improvable, non-perturbative approach
- Stochastically evaluate QCD path integral and correlation functions

Hadron/Multi-Hadron operators constructed from QCD fields

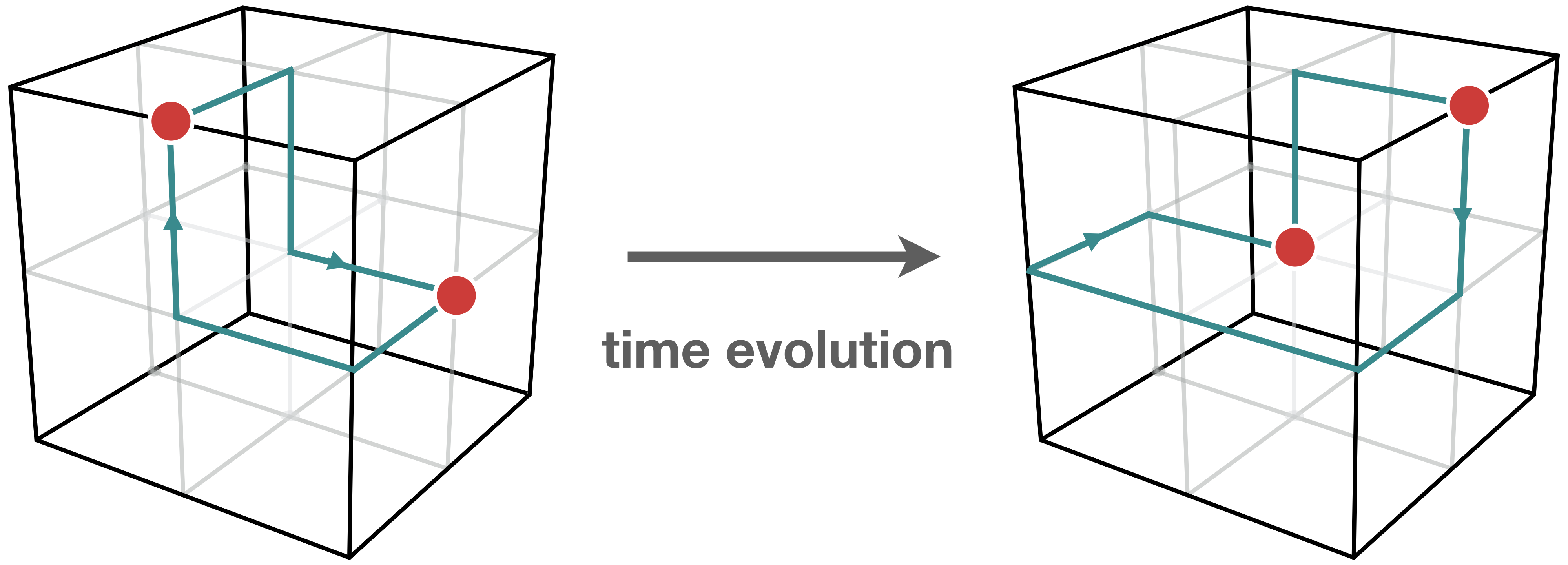
$$\mathcal{C}_L^{(\mathrm{E})}(\tau) = \langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle_L$$

$$= \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \int \mathcal{D}U \, e^{-S_{\mathrm{QCD}}^{(\mathrm{E})}} \, \mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

$$\approx \frac{1}{N_{\mathrm{cfg}}} \sum_{j=1}^{N_{\mathrm{cfg}}} \mathcal{C}_j(\tau) + \sigma_{\mathcal{C}}$$

Lattice Systematics

- Discrete spacetime lattice, a
- Finite volume, L
- Euclidean spacetime, $t \to -i\tau$
- $m_q > m_q^{(\mathrm{phys.})}$



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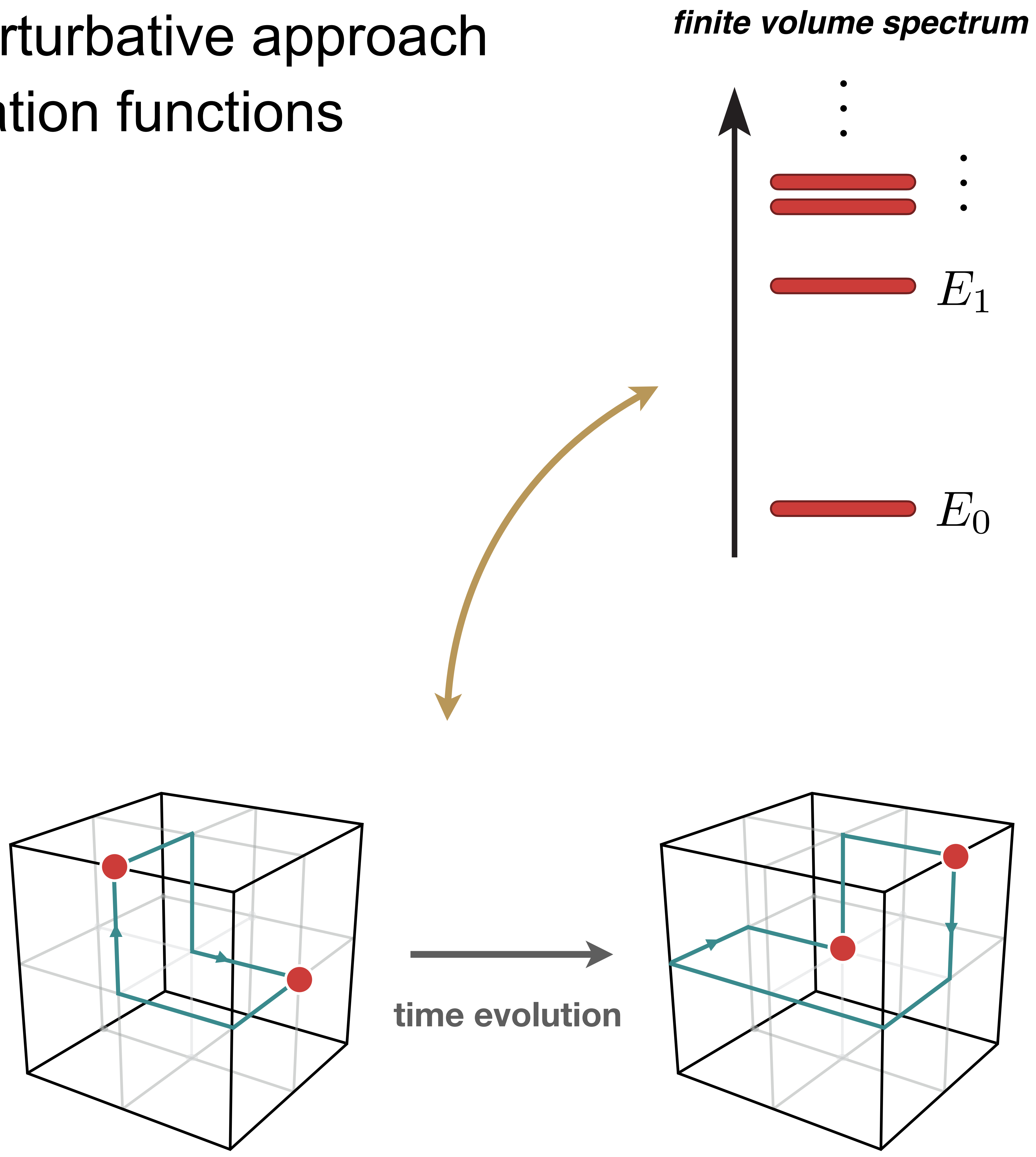
$$\mathcal{C}_L^{(\mathsf{E})}(\tau) = \langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle_L$$

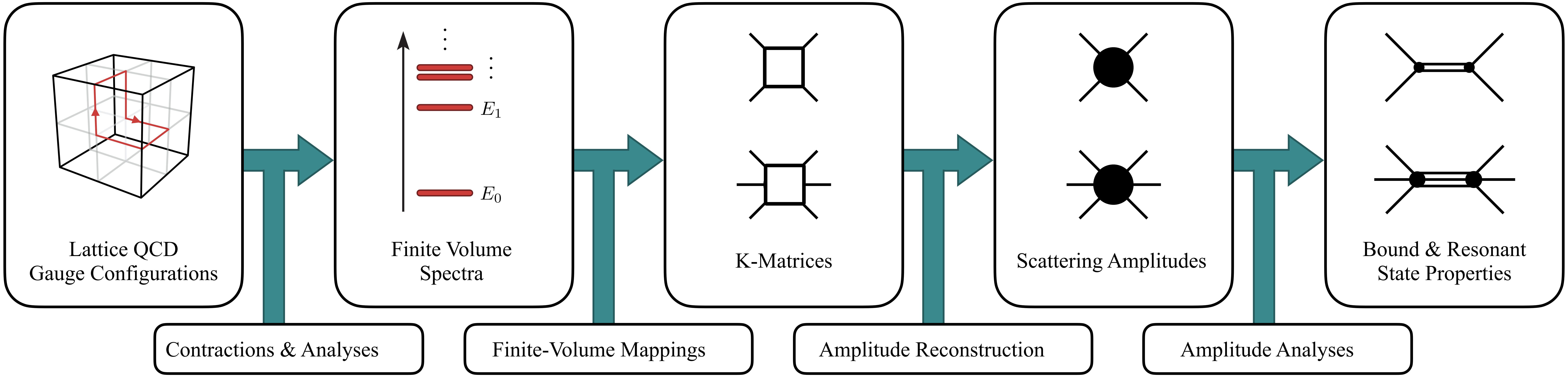
$$= \sum_{\mathfrak{n}} Z_{\mathfrak{n}} Z_{\mathfrak{n}}^\dagger \, e^{-E_{\mathfrak{n}}\tau}$$



Hadronic Reactions from QCD

Lattice QCD is a numerical tool to estimate low-energy QCD observables

- Formulated on **finite**, discretized, Euclidean spacetime — No real-time dynamics
- However, can extract amplitudes from lattice QCD
- Key idea: Map finite-volume energies to infinite-volume objects (via Lüscher)
- A path toward hadronic **reactions** from QCD

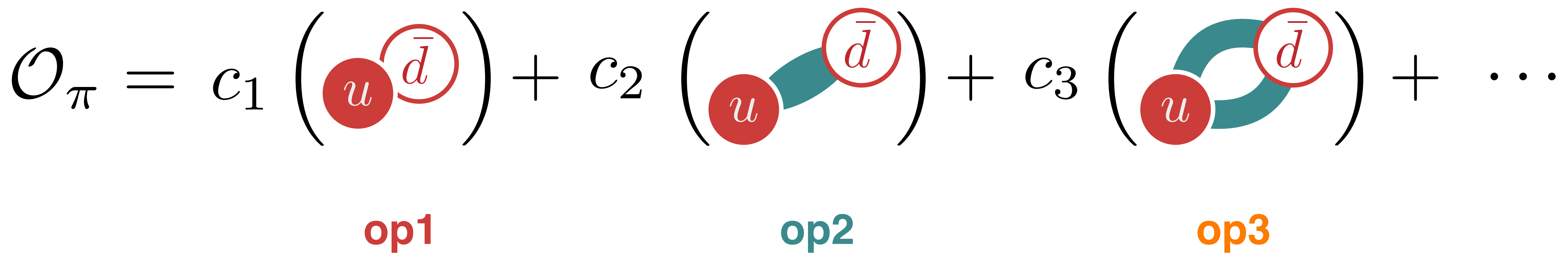


Finite-Volume Spectroscopy

Determine finite-volume spectrum with lattice QCD

- Create large basis of operators with target quantum numbers
- Form matrix of correlation functions, diagonalize (GEVP)

e.g., the pion mass



variationally optimize coefficients to maximize overlaps to desired states

$$\langle \mathcal{O}_\pi(\tau)\mathcal{O}_\pi^\dagger \rangle = e^{-m_\pi \tau} + \sum_{\mathfrak{n}\neq\pi} \varepsilon_{\mathfrak{n}} e^{-m_{\mathfrak{n}}\tau}$$

Finite-Volume Spectroscopy

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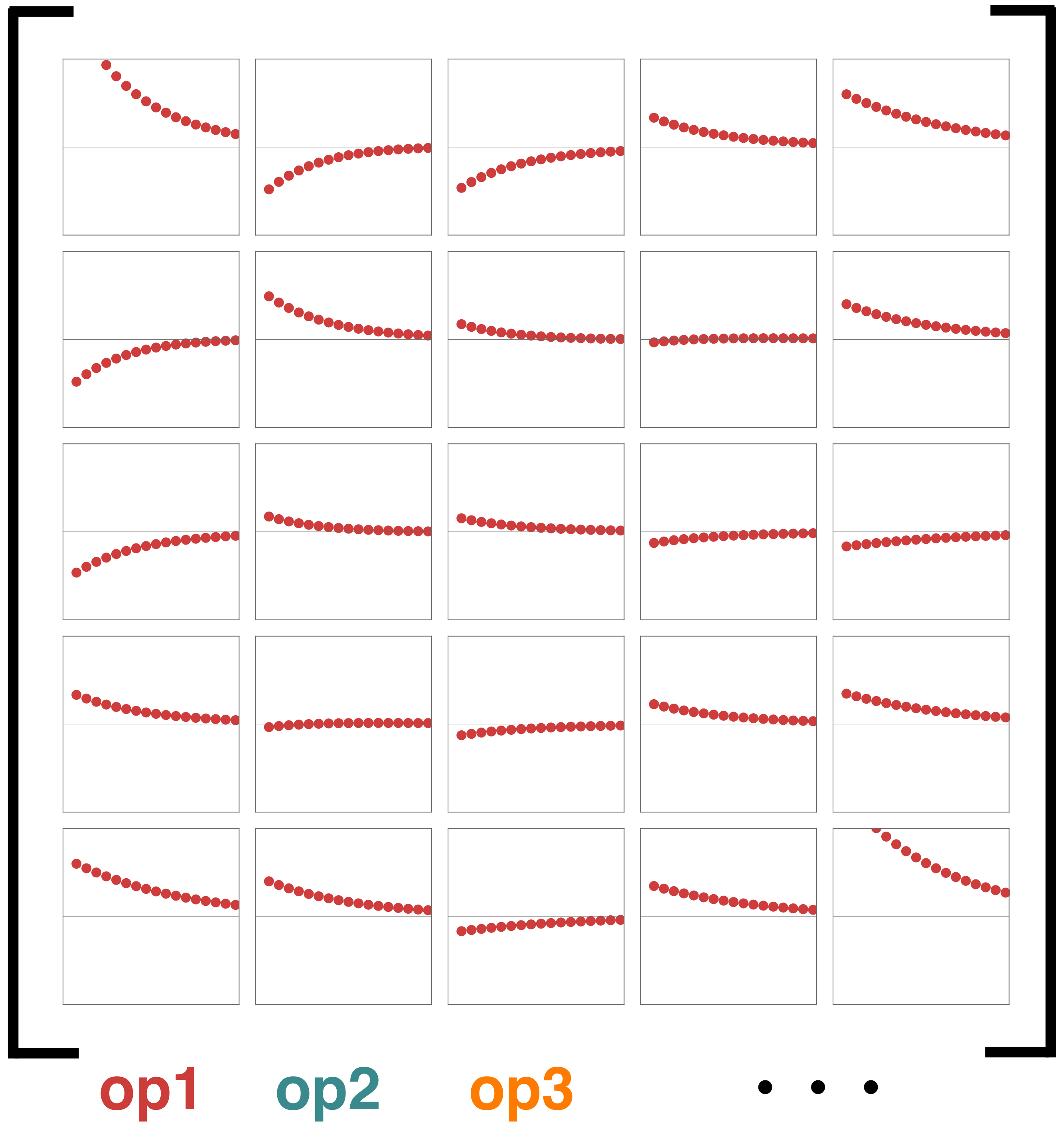
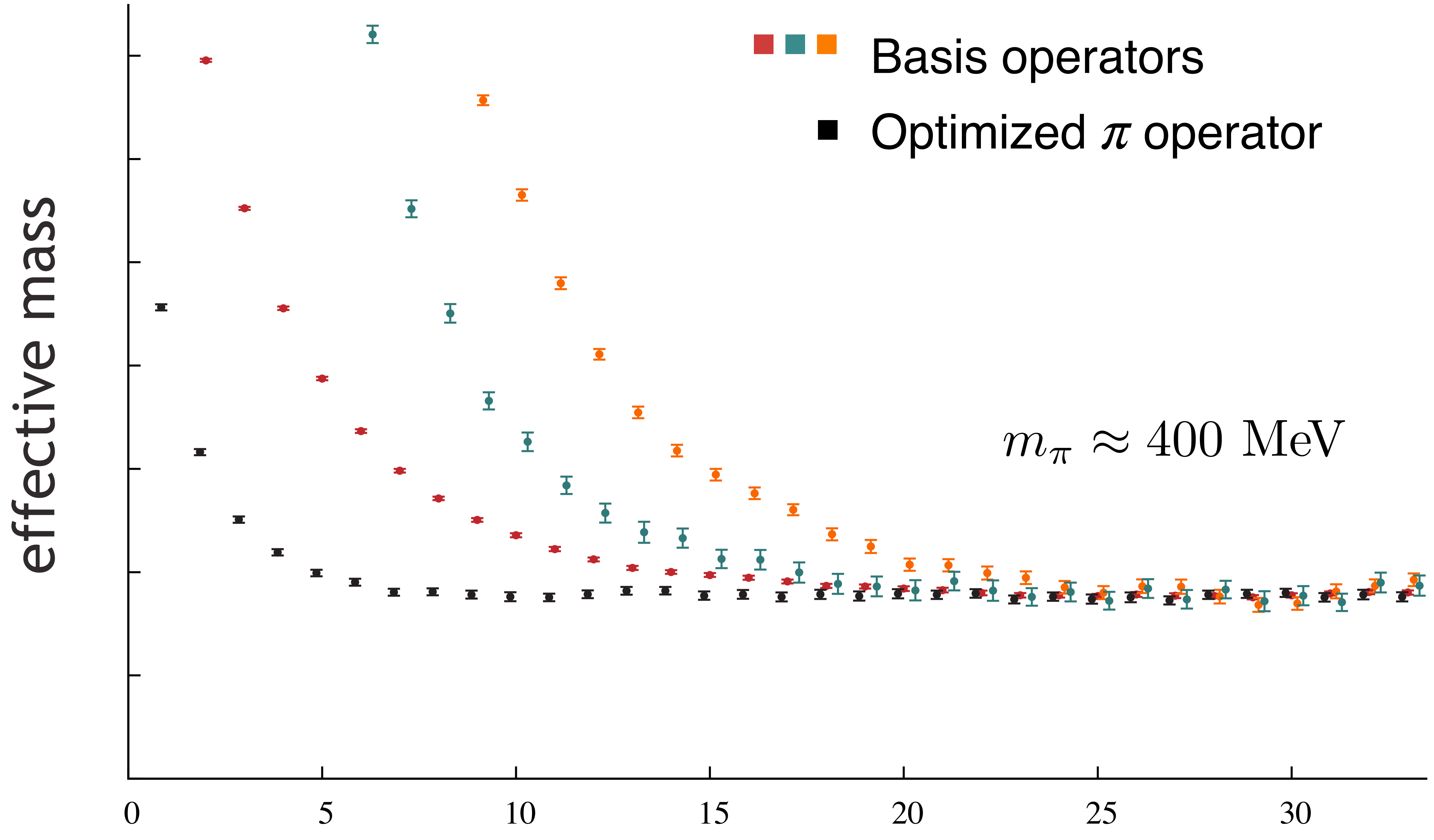
$$\mathcal{C}_{ij}(\tau) =$$





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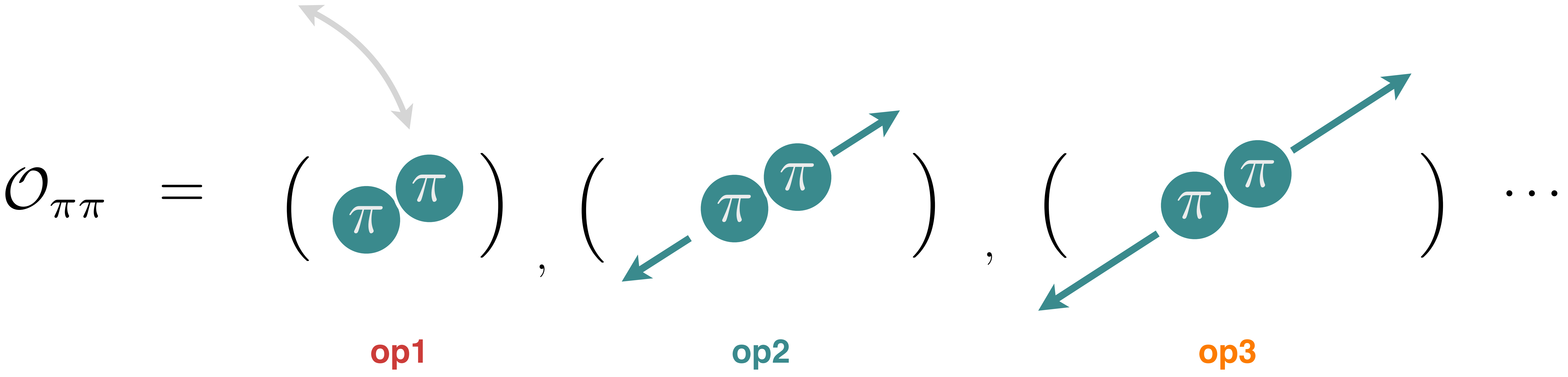
- For extracting multi-body physics, follow similar path

Optimal pion operator



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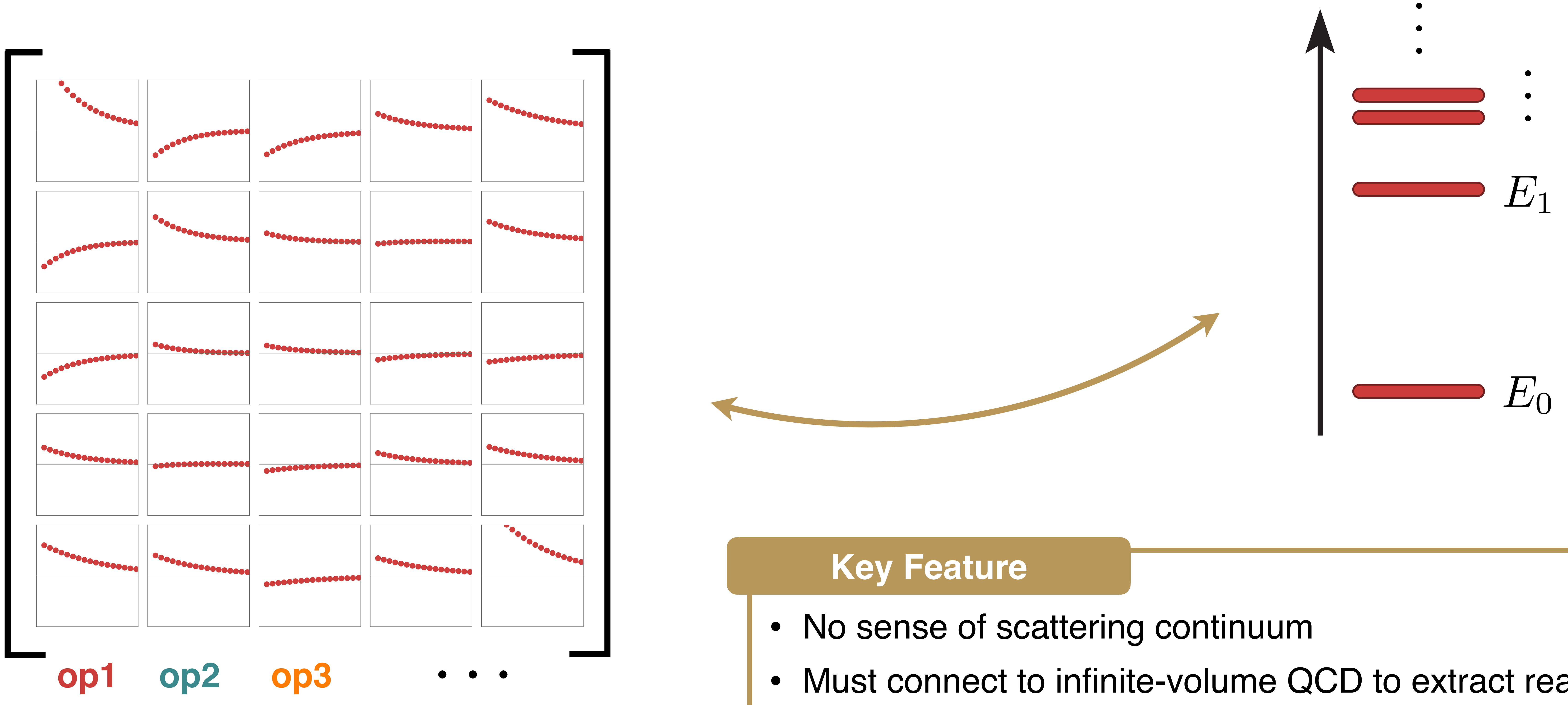

Key Feature

- No sense of scattering continuum
- Must connect to infinite-volume QCD to extract reaction observables

Hadronic Reactions from QCD

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- However, can extract amplitudes from lattice QCD
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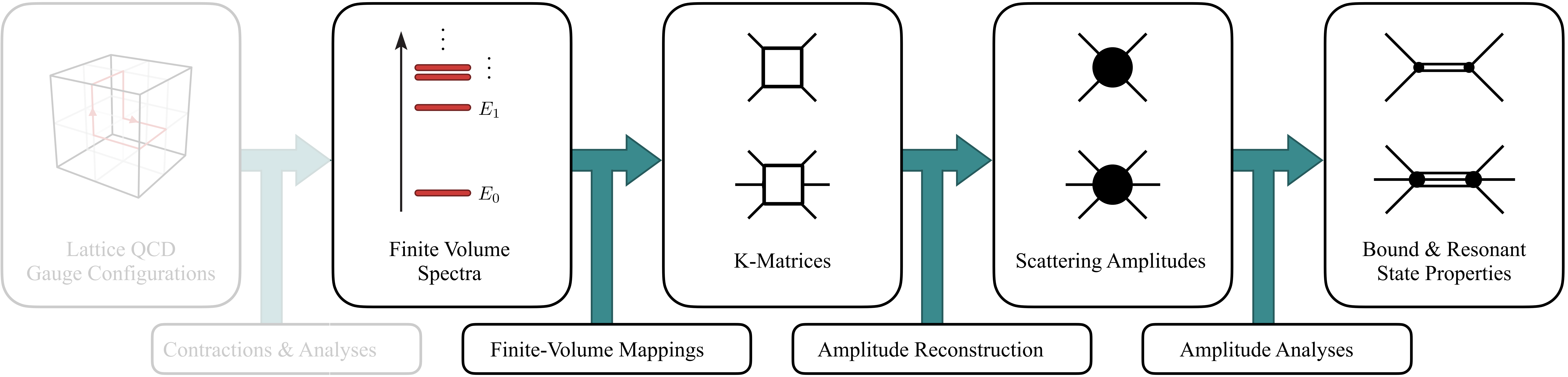
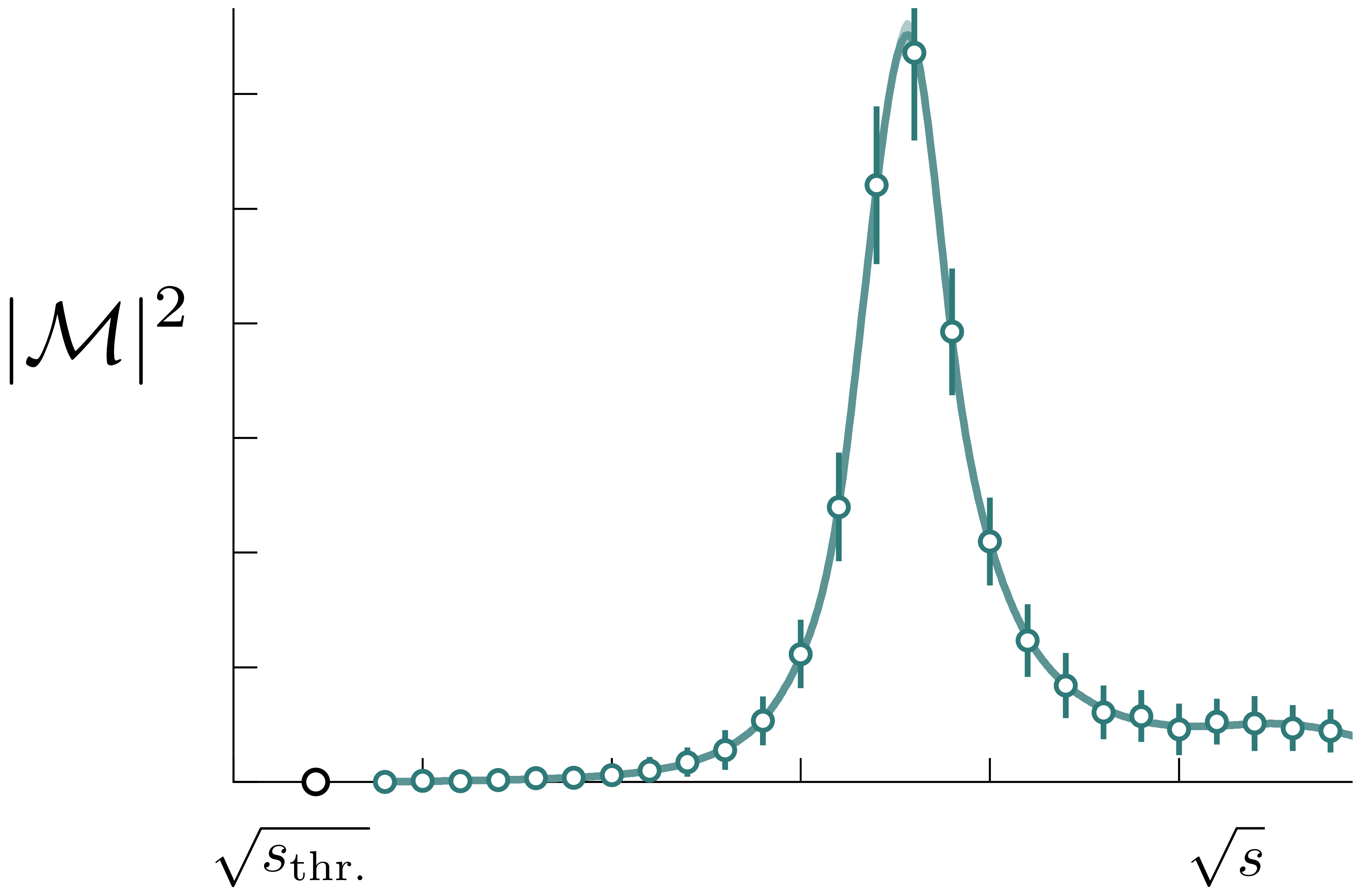


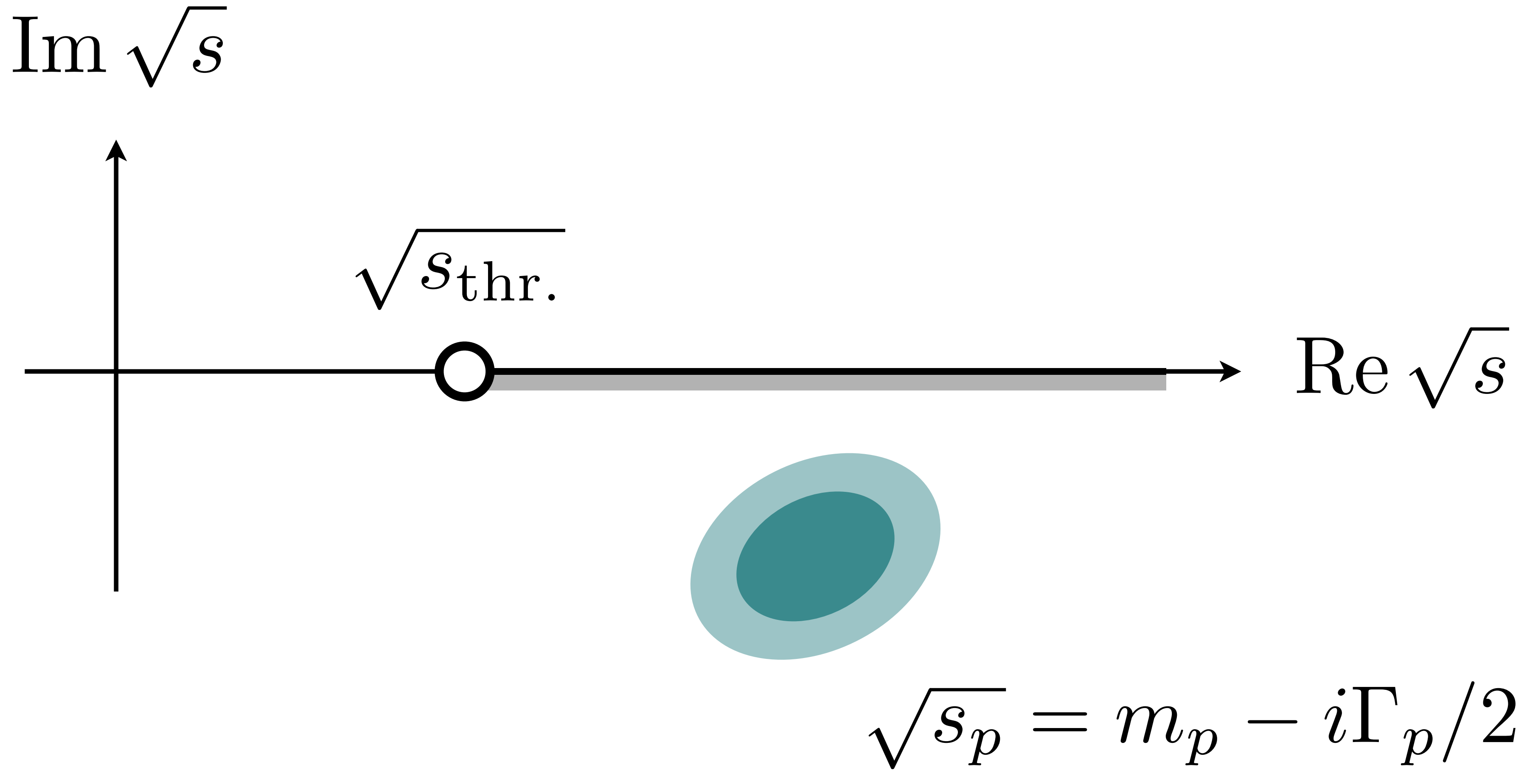
Connecting to infinite-volume physics

Key concept — use fundamental S matrix principles to constrain amplitudes

- Allows for unbiased amplitude construction
- Can determine spectrum via analytic continuation



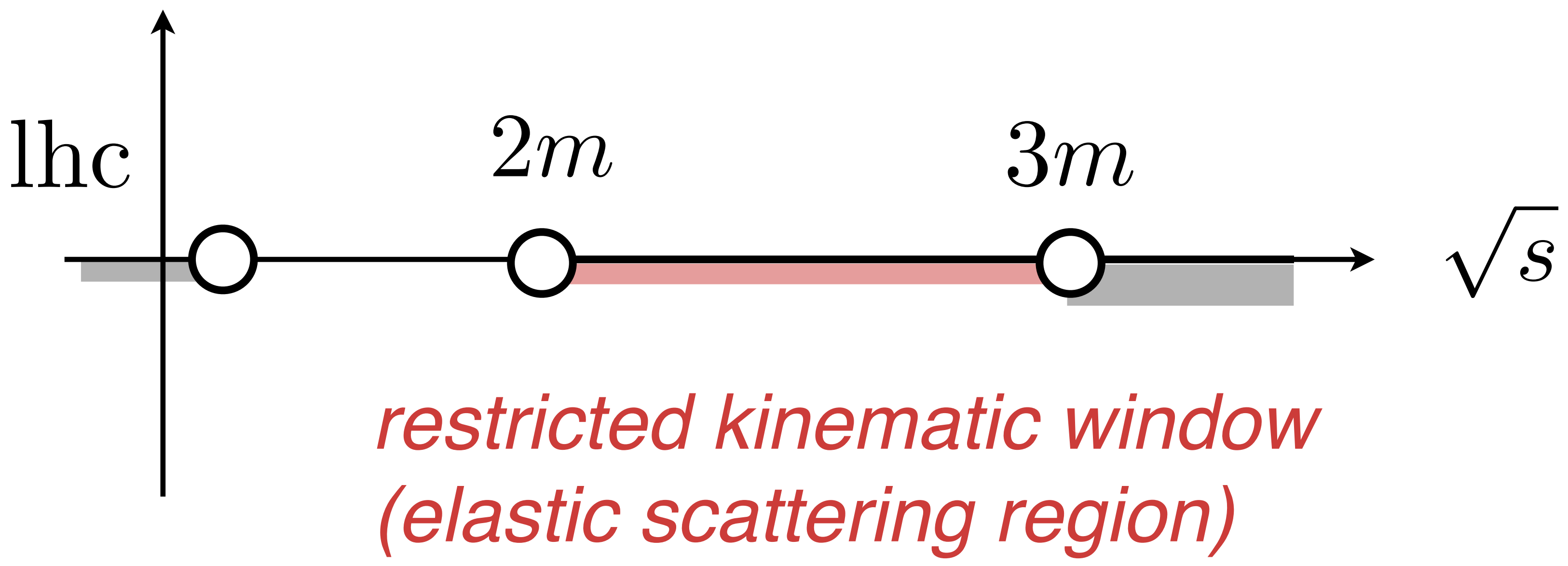
$$\mathcal{M} = \mathcal{K}\frac{1}{1 - i\rho\mathcal{K}}$$

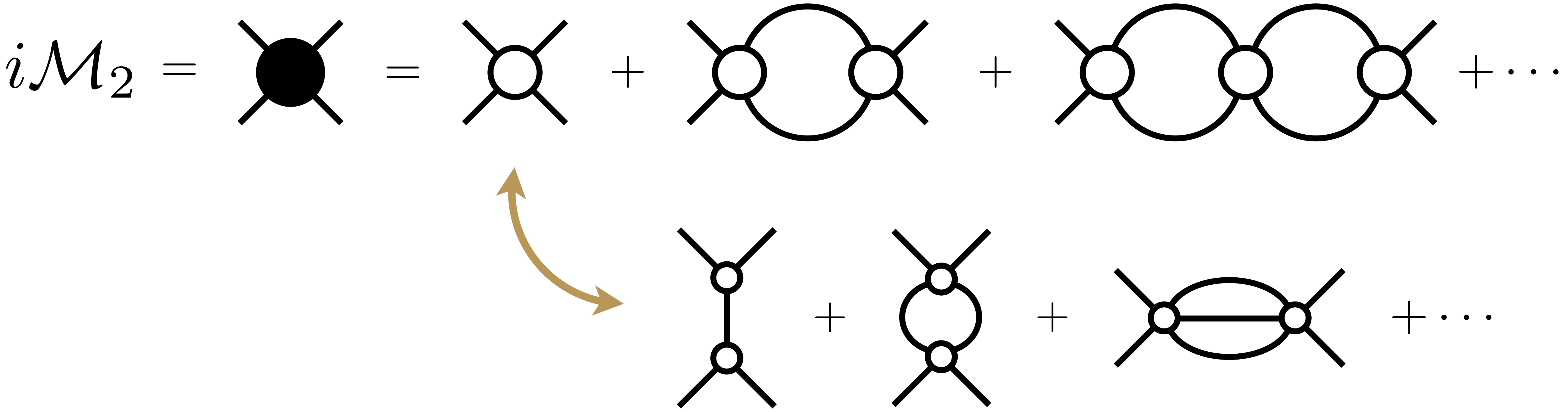


Two-Hadron Systems

Let's review the two-hadron case as a template for three-hadron systems

- e.g., elastic scattering of spinless particles, with restricted energy region





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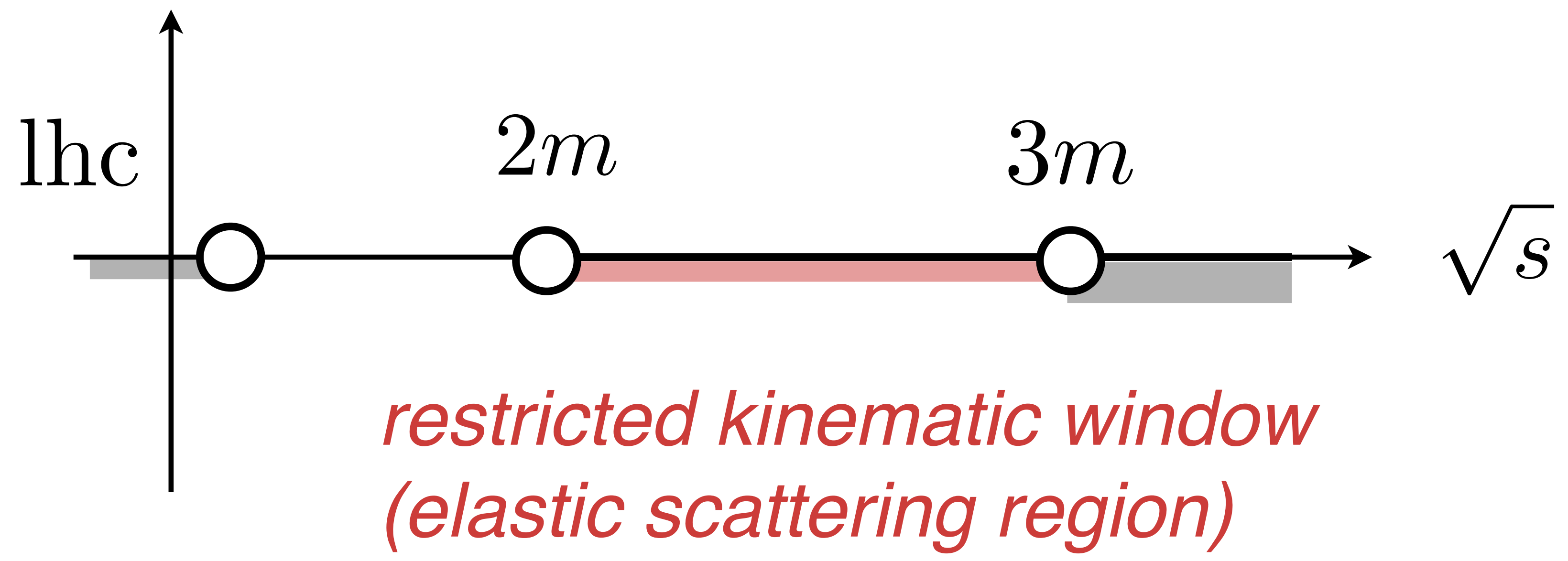
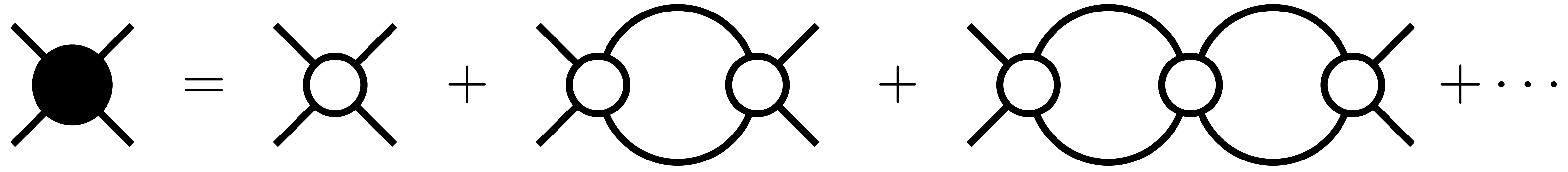
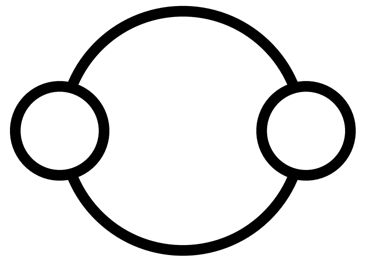


$i\mathcal{M}_2 =$ 

$=$ isolate singular behavior, re-sum short-distance 'stuff' into new real function

 $\sim$ real function $- \; i\rho$

two-body phase space

$$\rho \sim \sqrt{1 - \frac{4m^2}{s}}$$

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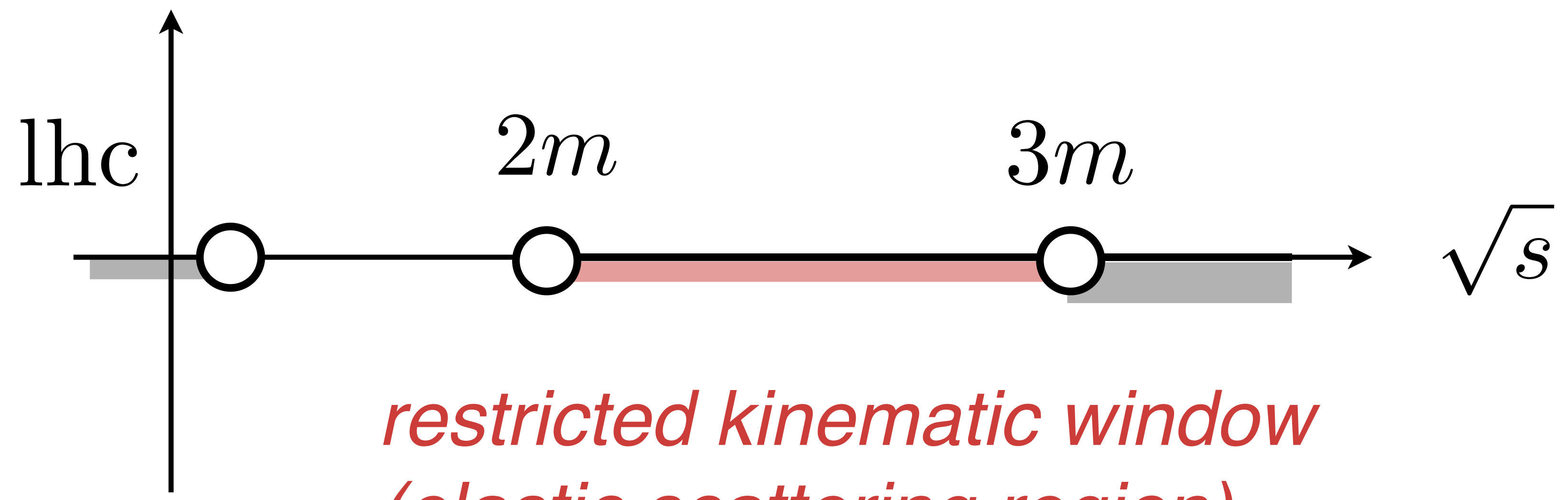
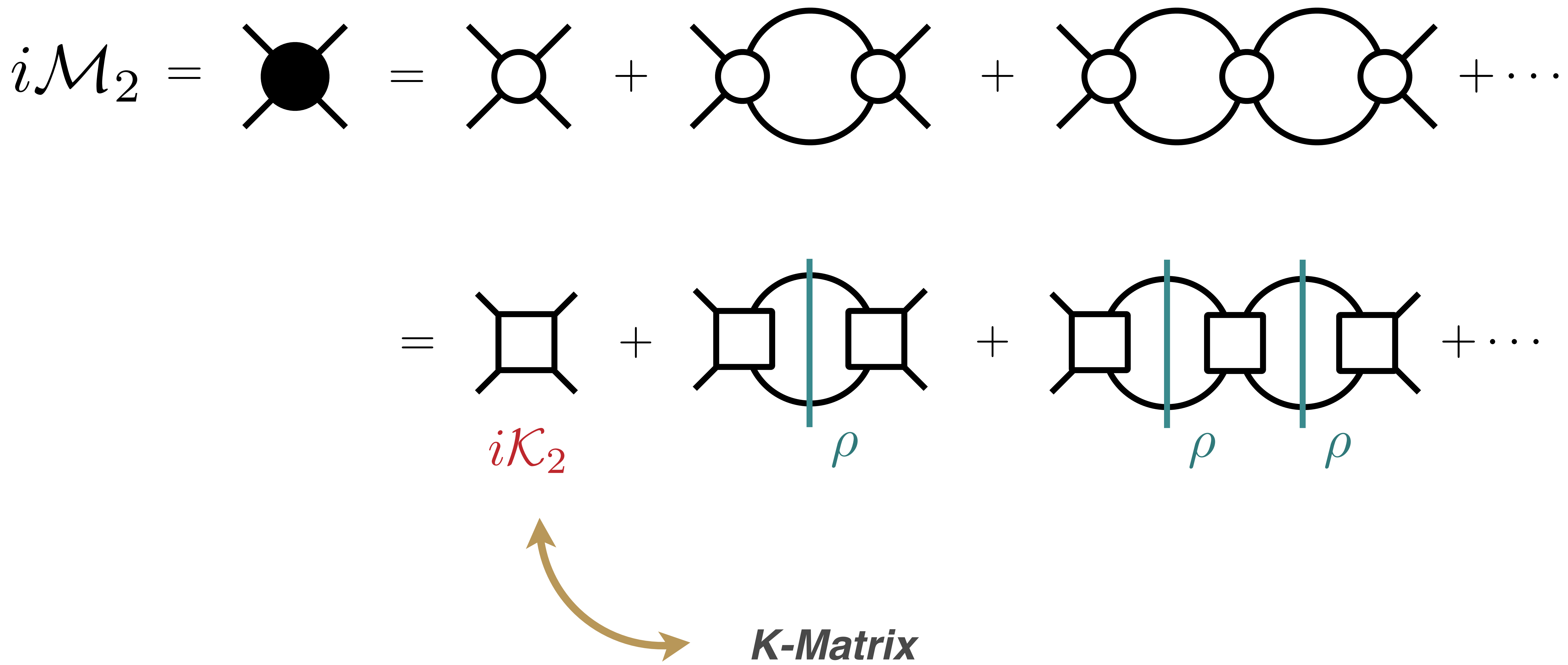


restricted kinematic window (elastic scattering region)



K-Matrix

— short-distance dynamics

— constrained from data (exp. or th.)

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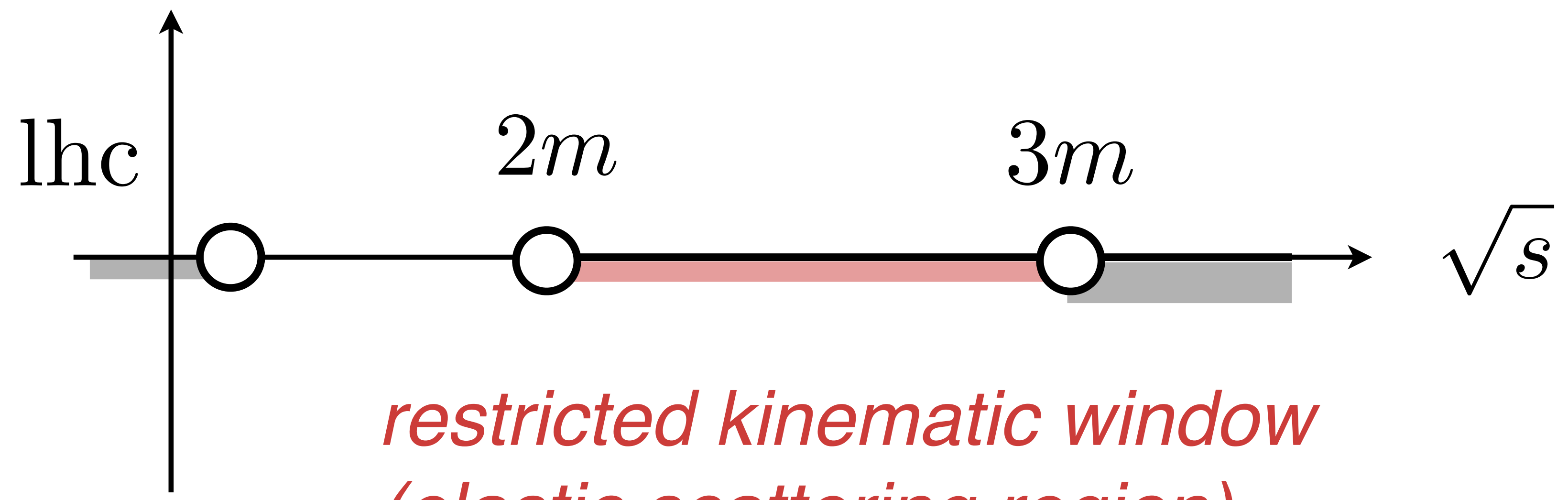

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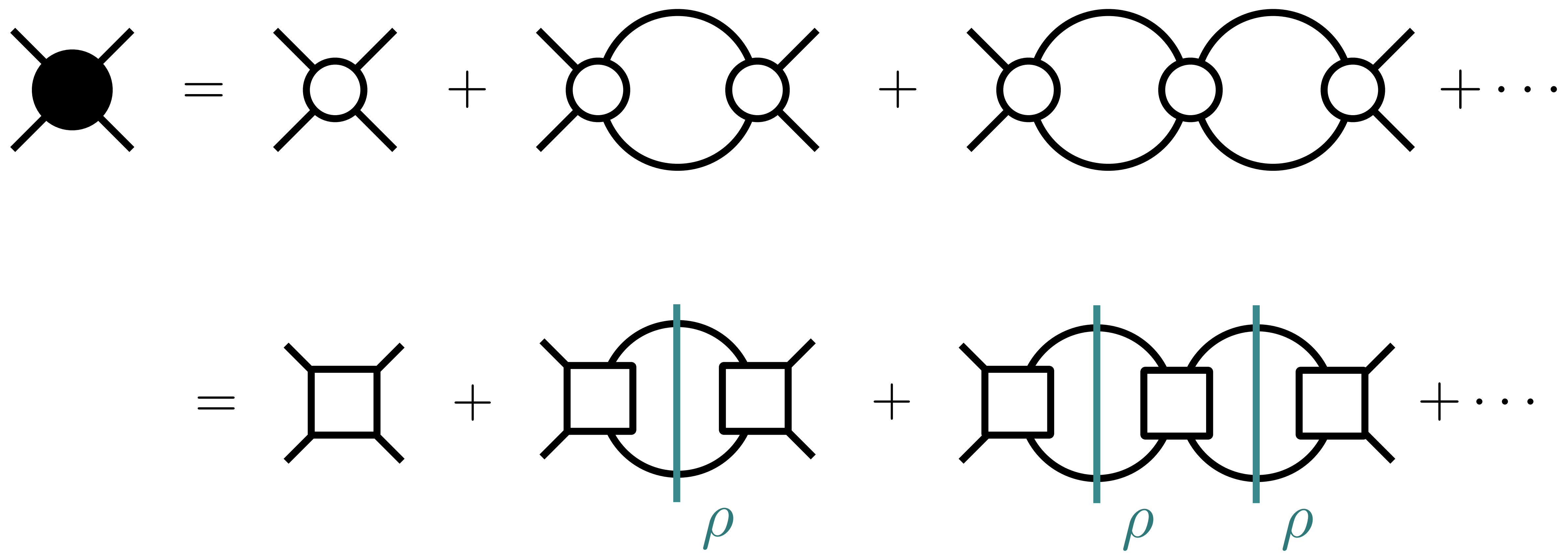


$$= i\mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

S matrix unitarity

$$\mathrm{Im}\,\mathcal{M}_2 = \rho|\mathcal{M}_2|^2$$

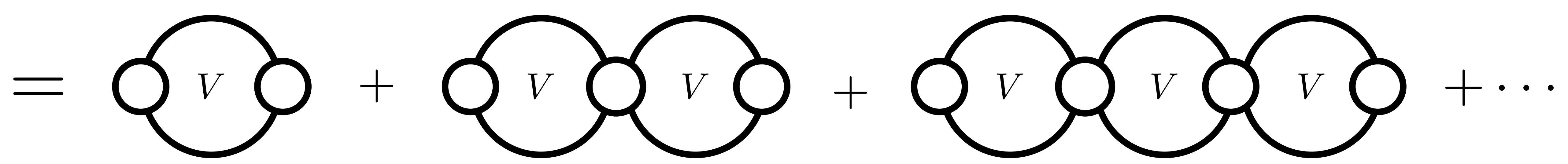
Two-Hadron Systems

Let's review the two-hadron case as a template for three-hadron systems

- e.g., elastic scattering of spinless particles, with restricted energy region
- Connect K matrix to finite-volume through correlation function

$$\mathcal{C}_L^{(\mathsf{M})}(E) = i \sum_{\mathfrak{n}} \frac{Z_{\mathfrak{n}} Z_{\mathfrak{n}}^\dagger}{E - E_{\mathfrak{n}}}$$

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}$$



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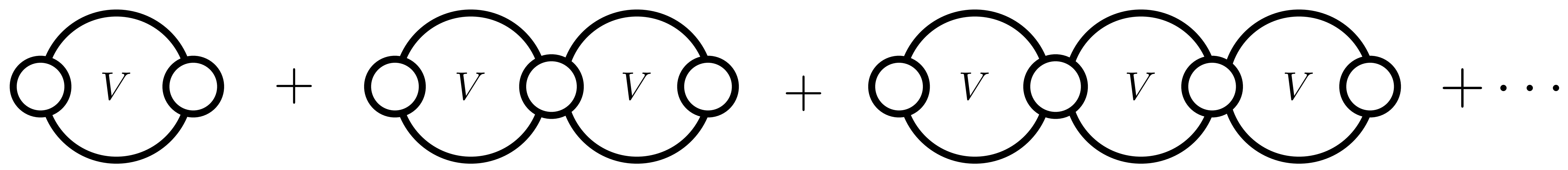
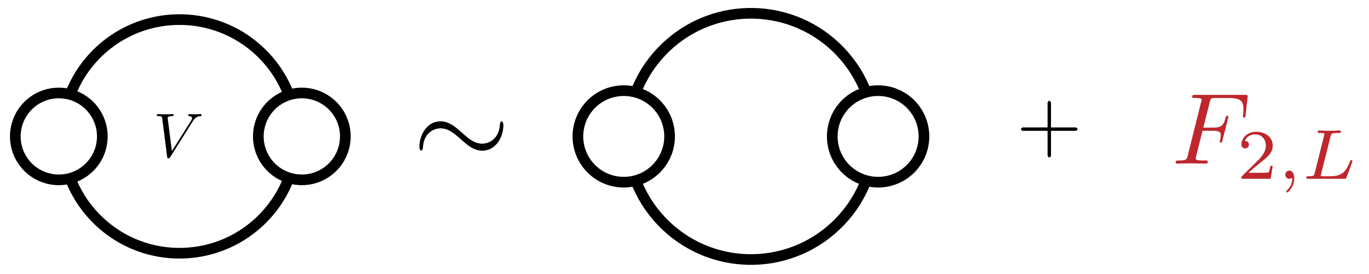
$=$ 

$=$ Follow similar procedure as before …

… additional correction for finite-volume effects



as before

Finite-volume correction (known)

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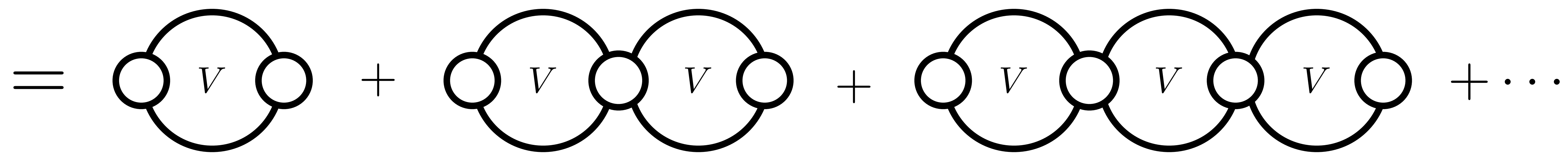
what we compute via lattice QCD



$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_2 \cdot F_{2,L}}$$

what we want

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what we compute via lattice QCD

$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_2 \cdot F_{2,L}}$$

what we want

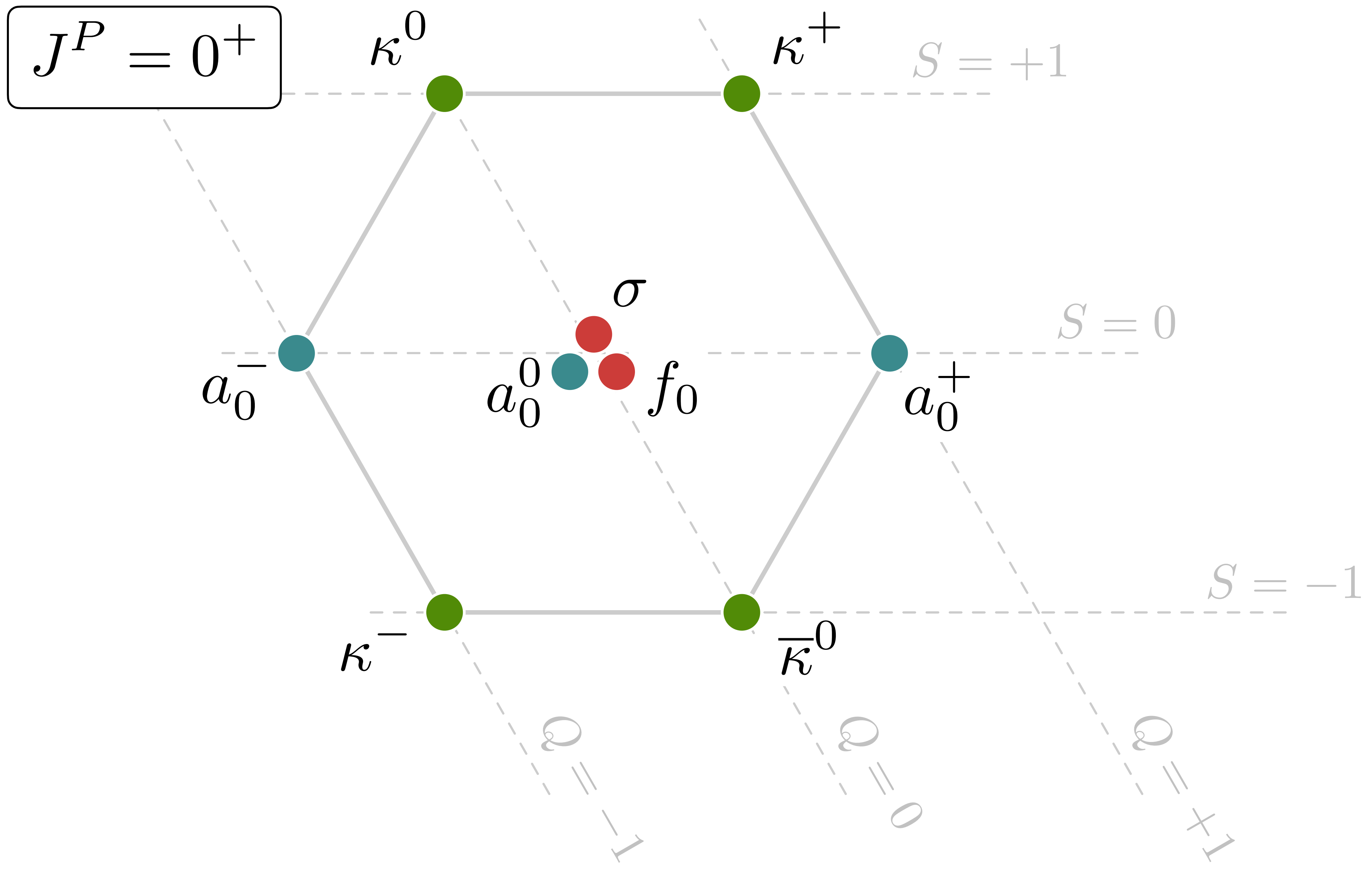
Spectrum satisfies

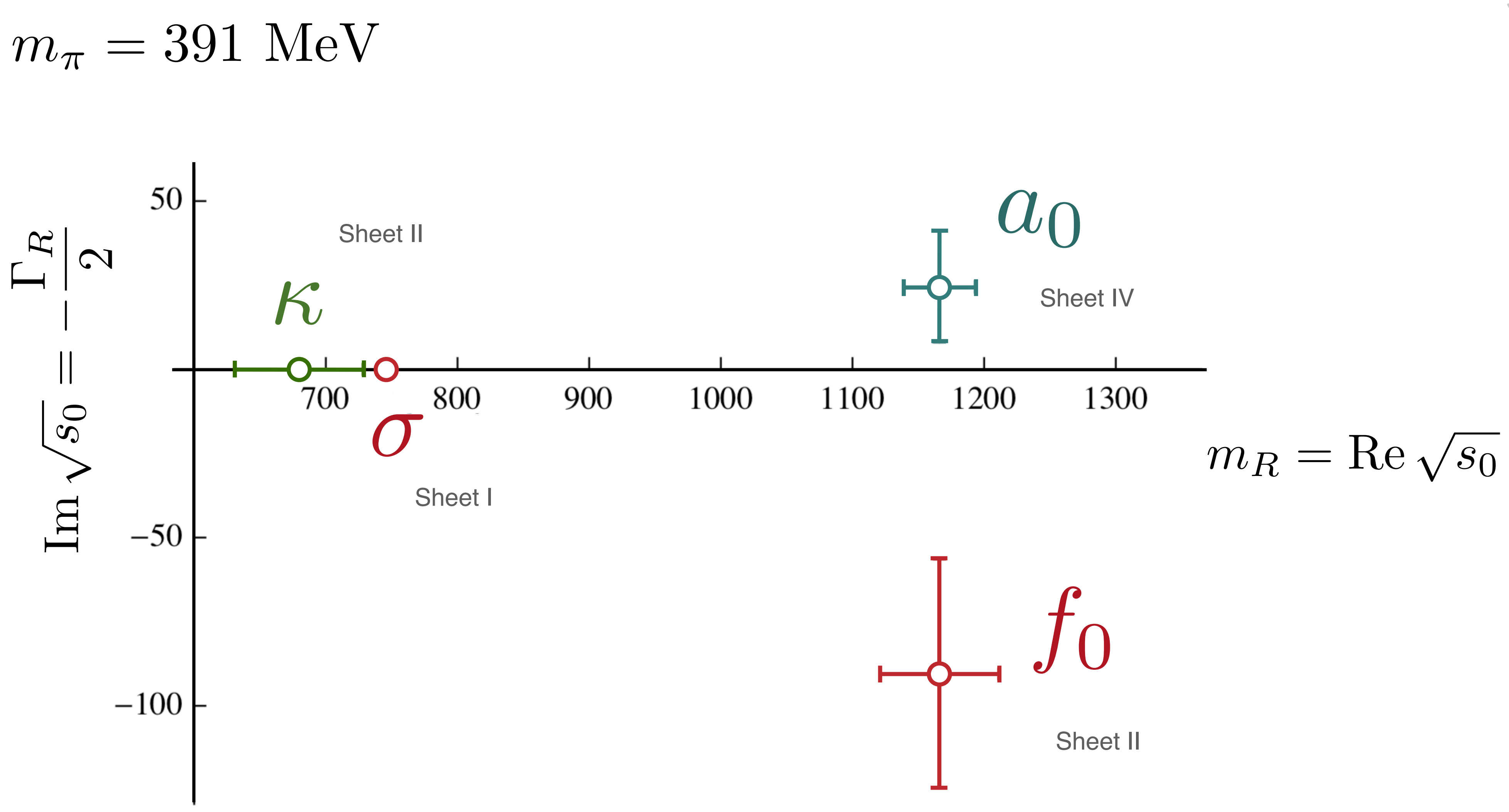
$$\det[1 + \mathcal{K}_2 \cdot F_{2,L}]\Big|_{E=E_{\mathfrak{n}}} = 0$$

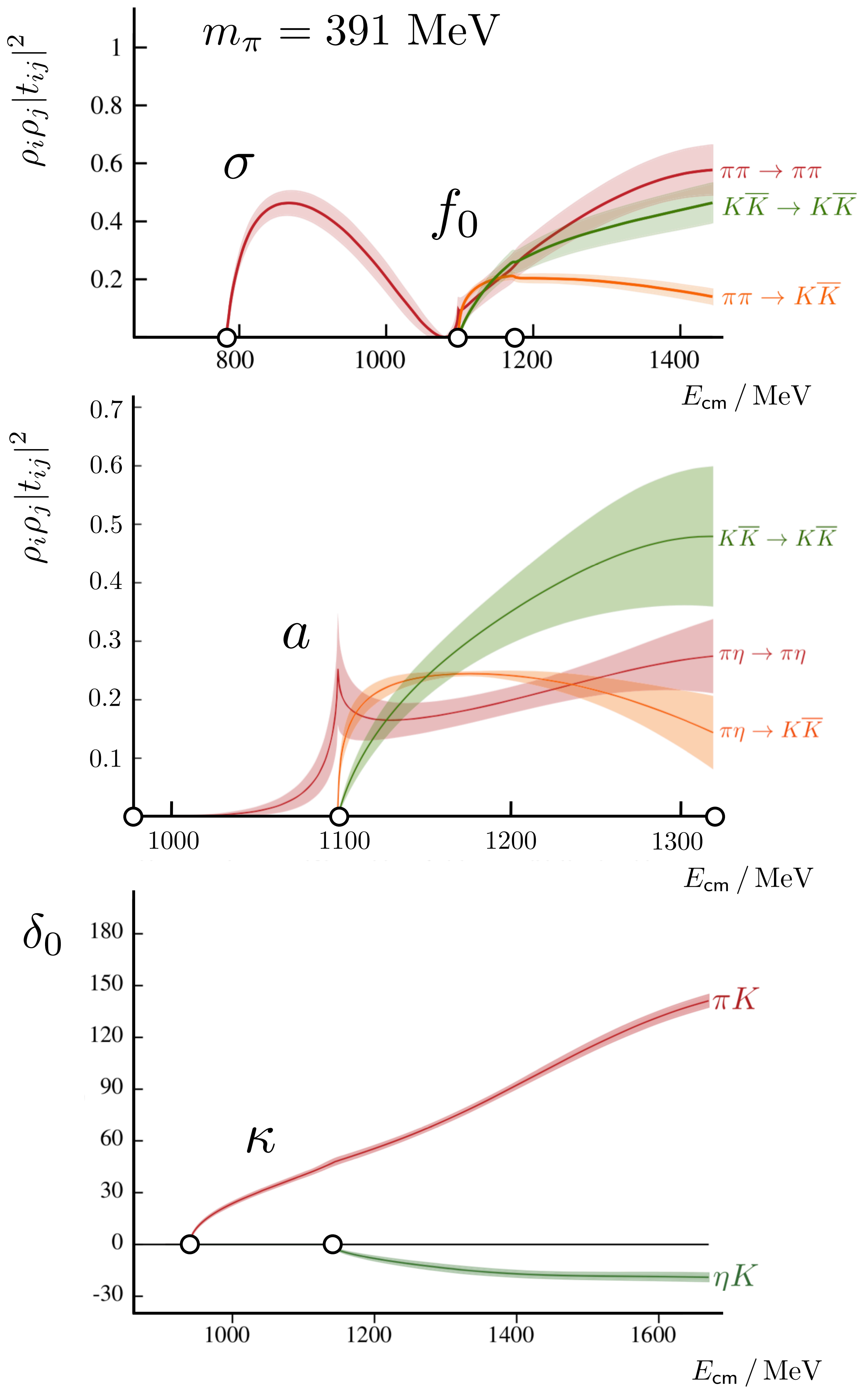
Two-Hadron Systems

Many systems investigated, including coupled channels

e.g., light scalars







R.A. Briceño et al. [HadSpec]
Phys. Rev. **D97**, 054513 (2018)

J.J. Dudek et al. [HadSpec]
Phys. Rev. **D93**, 094506 (2016)

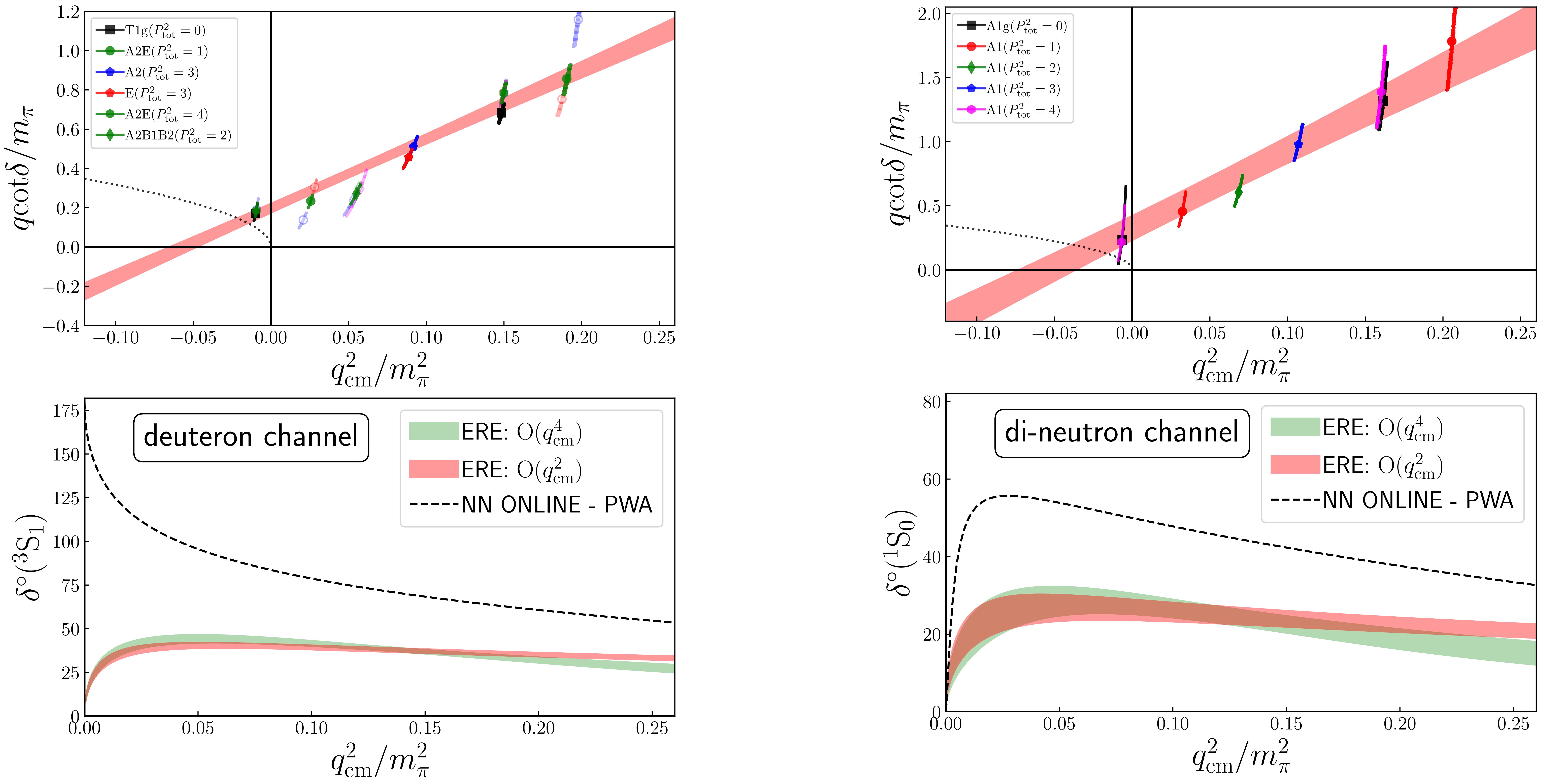
J.J. Dudek et al. [HadSpec]
Phys. Rev. Lett. **113**, 182001 (2014)

Two-Hadron Systems

Many systems investigated, including coupled channels

e.g., NN scattering

$m_\pi \approx 700\,\mathrm{MeV}$



J. Bulava et al. [BaSc]
arXiv:2505.05547

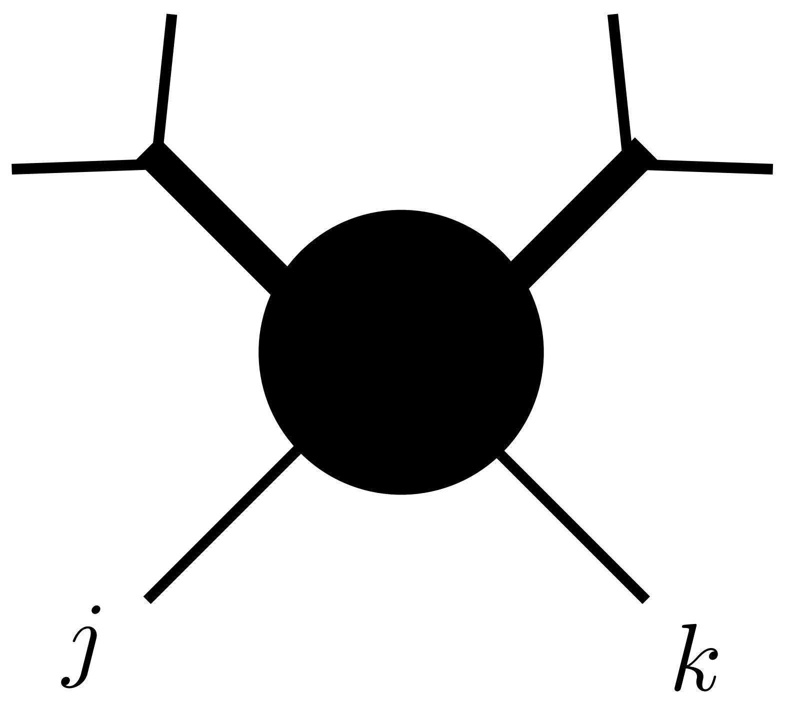
Three-Hadron Systems

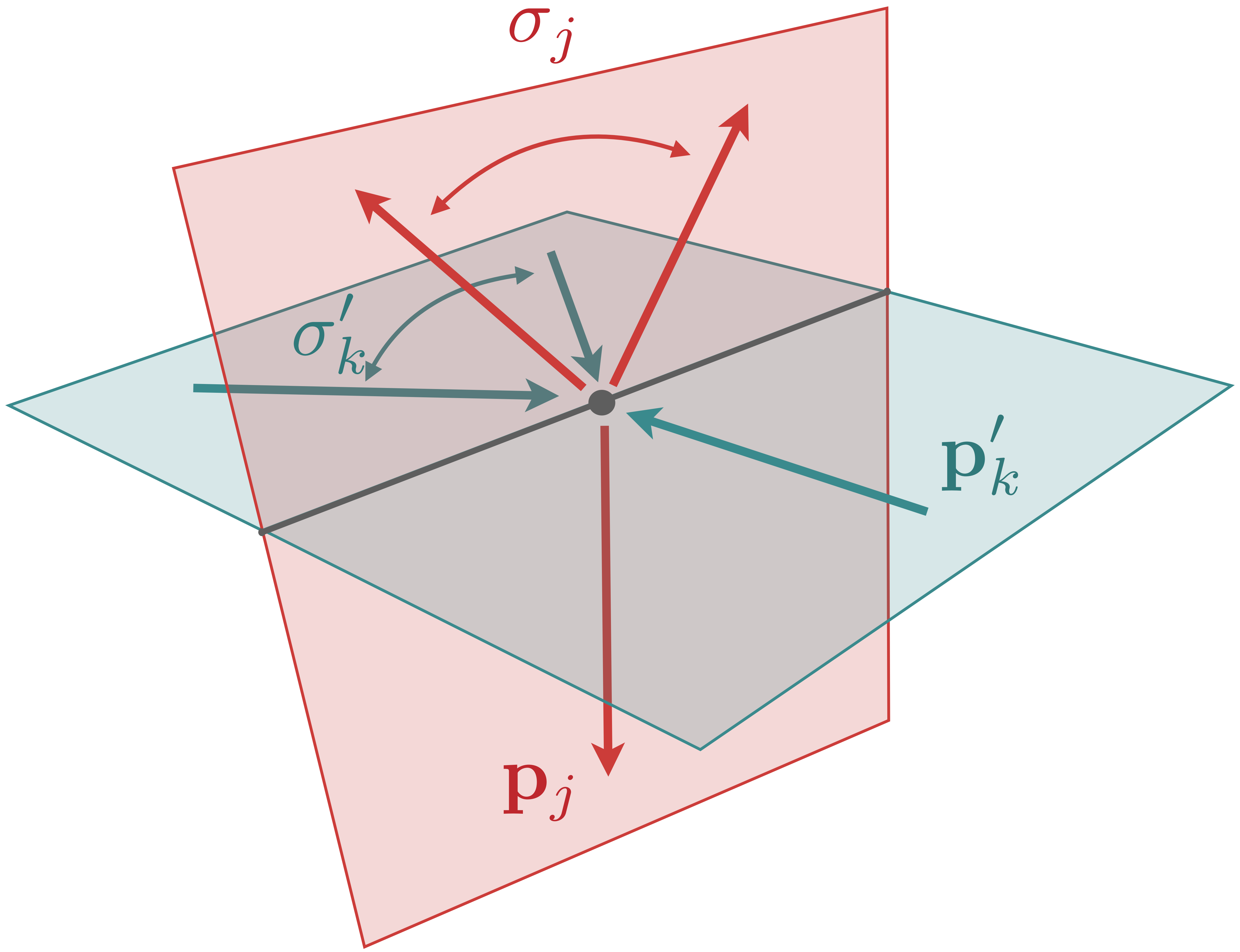
Repeat same procedure for three-body case

- S-matrix unitarity enforces K matrix structure
- Complications from more degrees of freedom (8 vs 2 in two-body)

$$\mathcal{M}_3 = \sum_{j,k} \mathcal{M}_3^{(j,k)}$$

$$= \sum_{j,k}$$



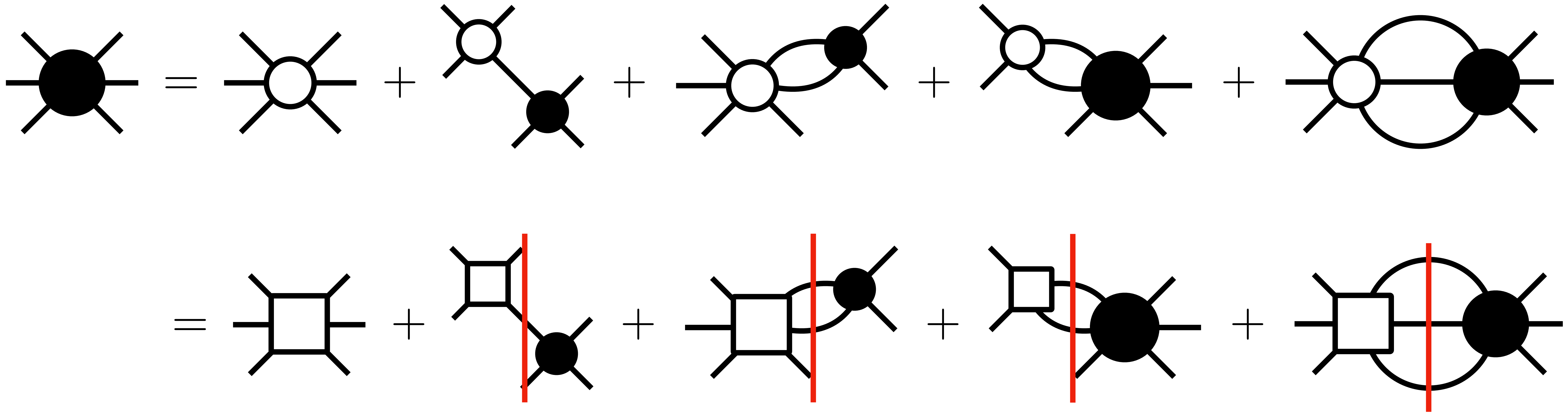


Three-Hadron Systems

Repeat same procedure for three-body case

- S-matrix unitarity enforces K matrix structure
- Have integral equations and more complicated analytic structures

$i\mathcal{M}_3 =$



K-Matrix

— short-distance dynamics

— cutoff-dependent (unphysical)

— constrained from Lattice QCD

Equivalence proofs

AJ et al. [JPAC]
Phys. Rev. D **100**, 034508 (2019)

T. Blanton and S. Sharpe
Phys. Rev. D **102**, 054515 (2020)

AJ
Phys. Rev. D **108**, 034505 (2023)

M. Hansen and S. Sharpe
Phys. Rev. D **90**, (2014) 116003

and many more….

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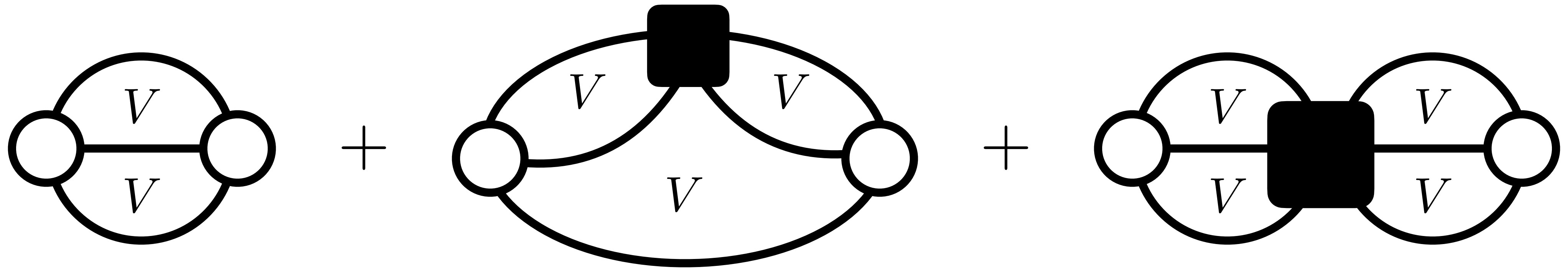
$=$ 

$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_3 \cdot F_{3,L}}$$

Spectrum satisfies

$$\det\left[1 + \mathcal{K}_3 \cdot F_{3,L}\right]\Big|_{E=E_{\mathfrak{n}}} = 0$$

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$$\mathcal{C}_L^{(\mathrm{M})}(E) = i \sum_{\mathfrak{n}} \frac{Z_{\mathfrak{n}} Z_{\mathfrak{n}}^\dagger}{E - E_{\mathfrak{n}}}$$

$=$

$+ \ldots$

$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_3 \cdot F_{3,L}}$$

Given a three-body K matrix, e.g., for three pions …

… we need to reconstruct the three hadron scattering amplitude …

… more challenging than 2-body case …

N.B. constraining the K matrix via lattice QCD is *hard*, but I will assume it can be done

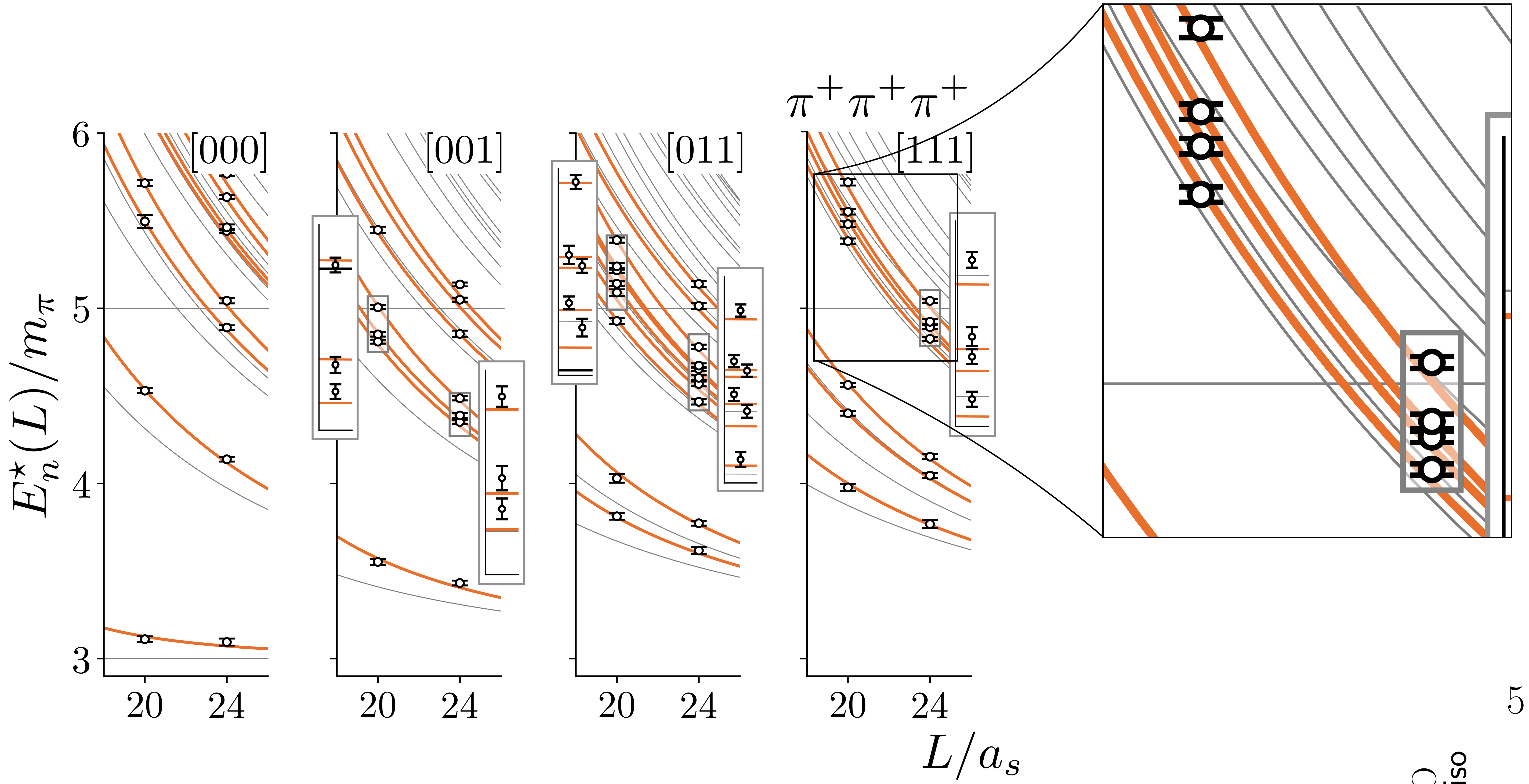
Spectrum satisfies

$$\det\left[1 + \mathcal{K}_3 \cdot F_{3,L}\right]\Big|_{E=E_{\mathfrak{n}}} = 0$$

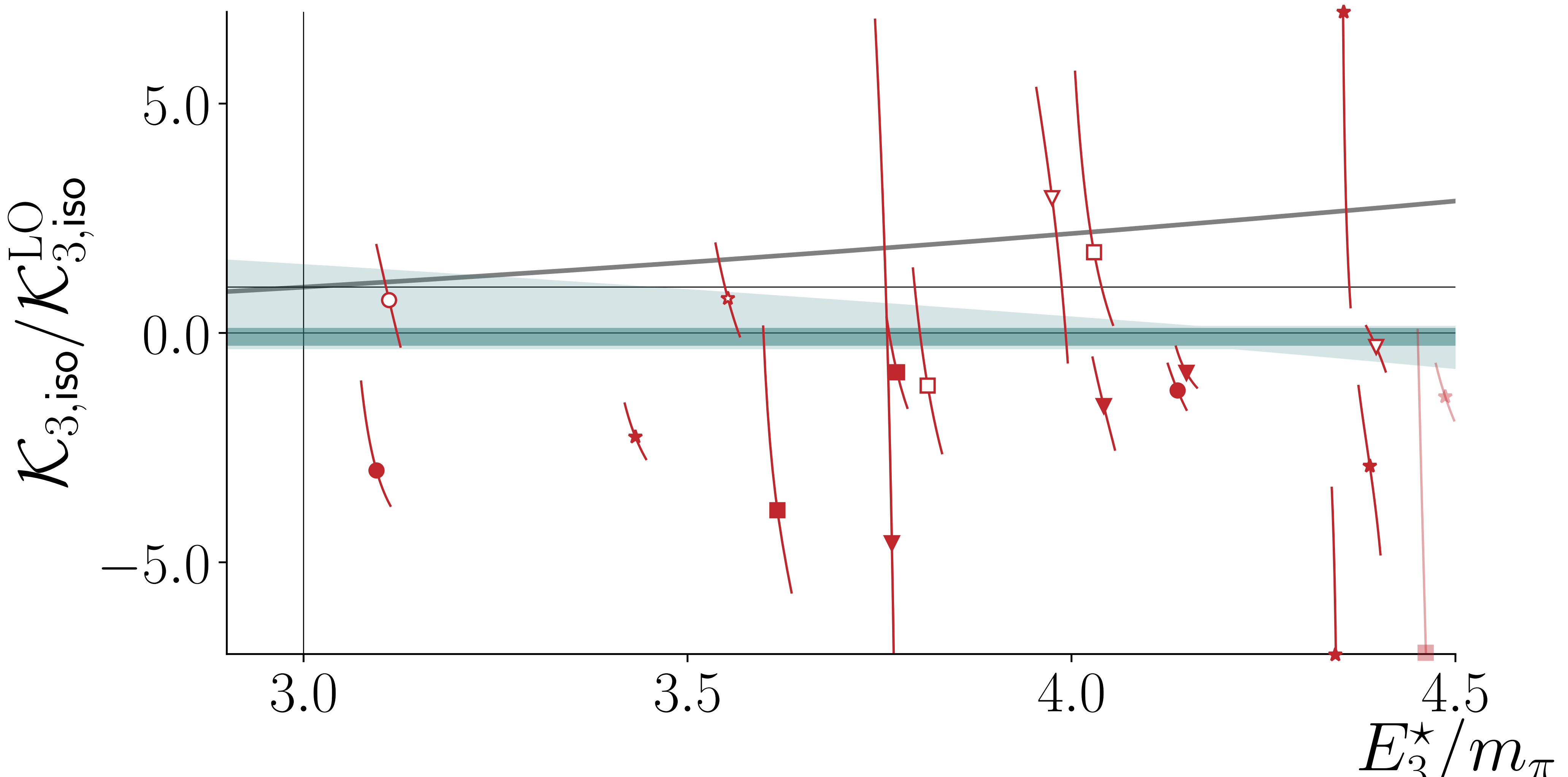
Three Pions

First applications were in $3\pi^+$ in $I(J^P) = 3(0^-)$

- Simple application without contamination from higher partial waves



103 energy levels described by three numbers m_π, $a_{\pi\pi}^{I=2}$, $\mathcal{K}_{3,\mathrm{iso}}^{I=3}$

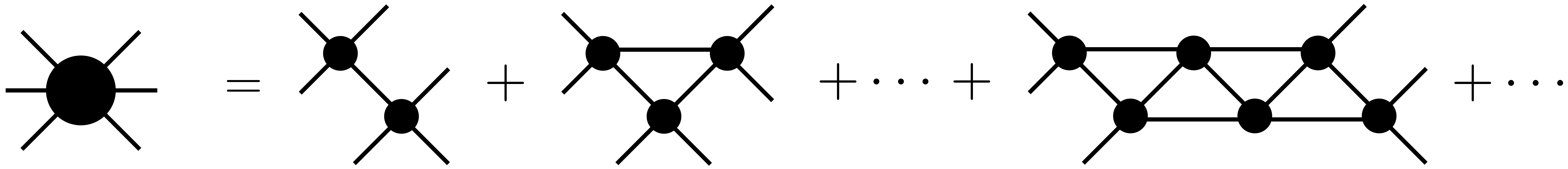


M. Hansen et al. [HadSpec]
Phys. Rev. Lett. **126**, (2021) 012001

Three Pions

First applications were in $3\pi^+$ in $I(J^P) = 3(0^-)$

- Simple application without contamination from higher partial waves
- System dominated by ladder of exchanges



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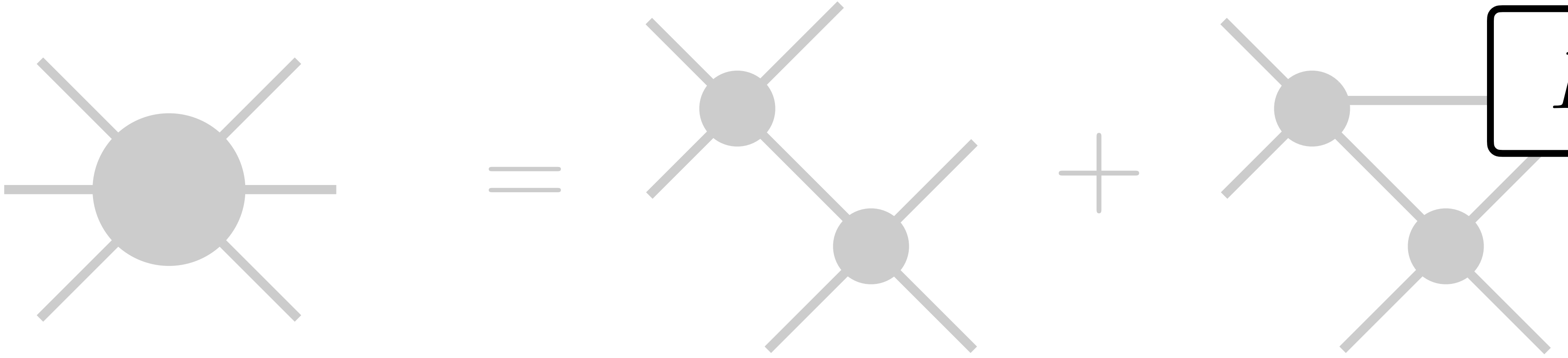

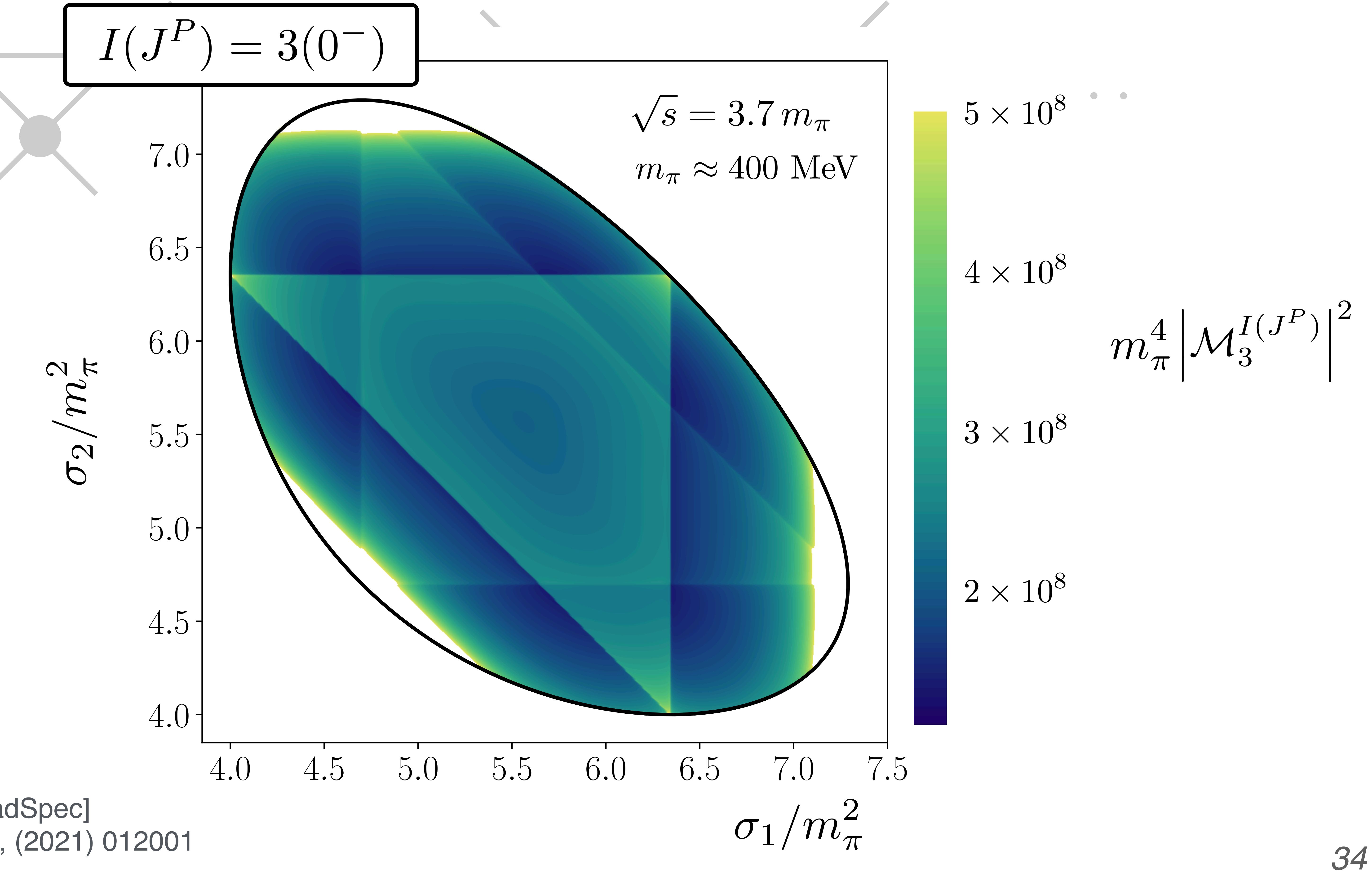

$\sigma_j = (\sqrt{s} - E_j)^2 - p_j^2$

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Higher Partial Waves

Can access higher waves — coupled integral equations

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P} \, \mathcal{R}_{J^P}$$

Rotational Dependence

— Clebsch-Gordan coefficients

— Wigner D matrix elements

— associated factors

Partial Wave Amplitudes

— depends on 'energy' variables only

— Matrix in LS-space for given J^P

$$\mathcal{R}_{J^P} \sim \sum_{m_J} Z_{L'S'}^{Jm_J\,*}(\hat{\mathbf{p}}) Z_{LS}^{Jm_J}(\hat{\mathbf{k}})$$

$$Z_{LS}^{Jm_J}(\hat{\mathbf{k}}) = \sqrt{4\pi(2L+1)} \sum_{\lambda} \langle J\lambda | L0, S\lambda \rangle \, D^{(J)}_{m_J\lambda}(\hat{\mathbf{k}}) \, Y^*_{S\lambda}(\hat{\mathbf{a}})$$

$I_{3\pi}^G$	J^{PC}	$([\pi\pi]_\ell^I \pi)_L$
3^-	0^{-+}	$([\pi\pi]_S^2 \pi)_S$
	1^{-+}	none
	1^{++}	$([\pi\pi]_S^2 \pi)_P$
2^-	0^{--}	$([\pi\pi]_S^2 \pi)_S$, $([\pi\pi]_P^1 \pi)_P$
	1^{--}	$([\pi\pi]_P^1 \pi)_P$
	1^{+-}	$([\pi\pi]_S^2 \pi)_P$, $([\pi\pi]_P^1 \pi)_S$, $([\pi\pi]_P^1 \pi)_D$
1^-	0^{-+}	$([\pi\pi]_S^{0,2} \pi)_S$, $([\pi\pi]_P^1 \pi)_P$
	1^{-+}	$([\pi\pi]_P^1 \pi)_P$
	1^{++}	$([\pi\pi]_S^{0,2} \pi)_P$, $([\pi\pi]_P^1 \pi)_S$, $([\pi\pi]_P^1 \pi)_D$
0^-	0^{--}	$([\pi\pi]_P^1 \pi)_P$
	1^{--}	$([\pi\pi]_P^1 \pi)_P$
	1^{+-}	$([\pi\pi]_P^1 \pi)_S$, $([\pi\pi]_P^1 \pi)_D$

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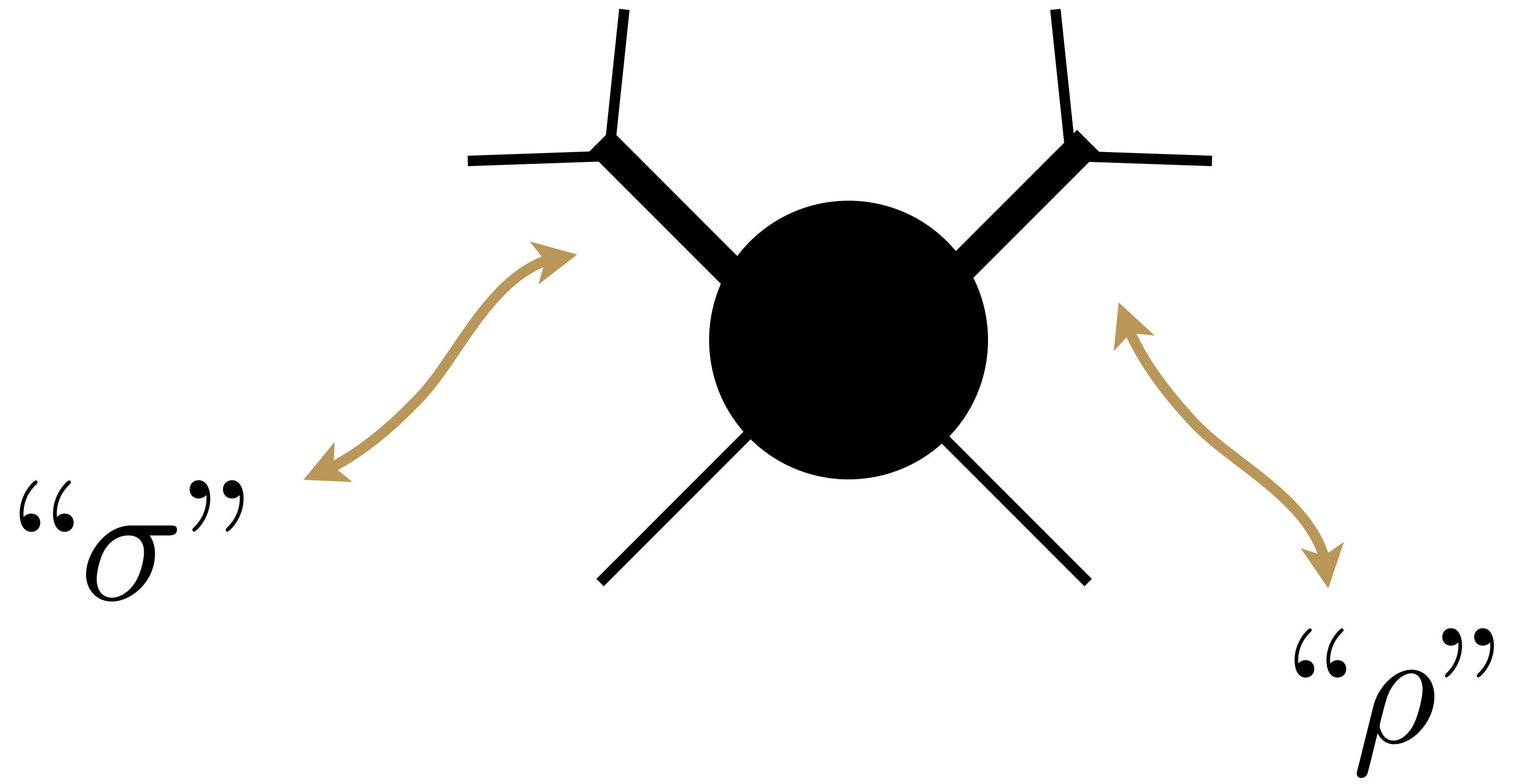
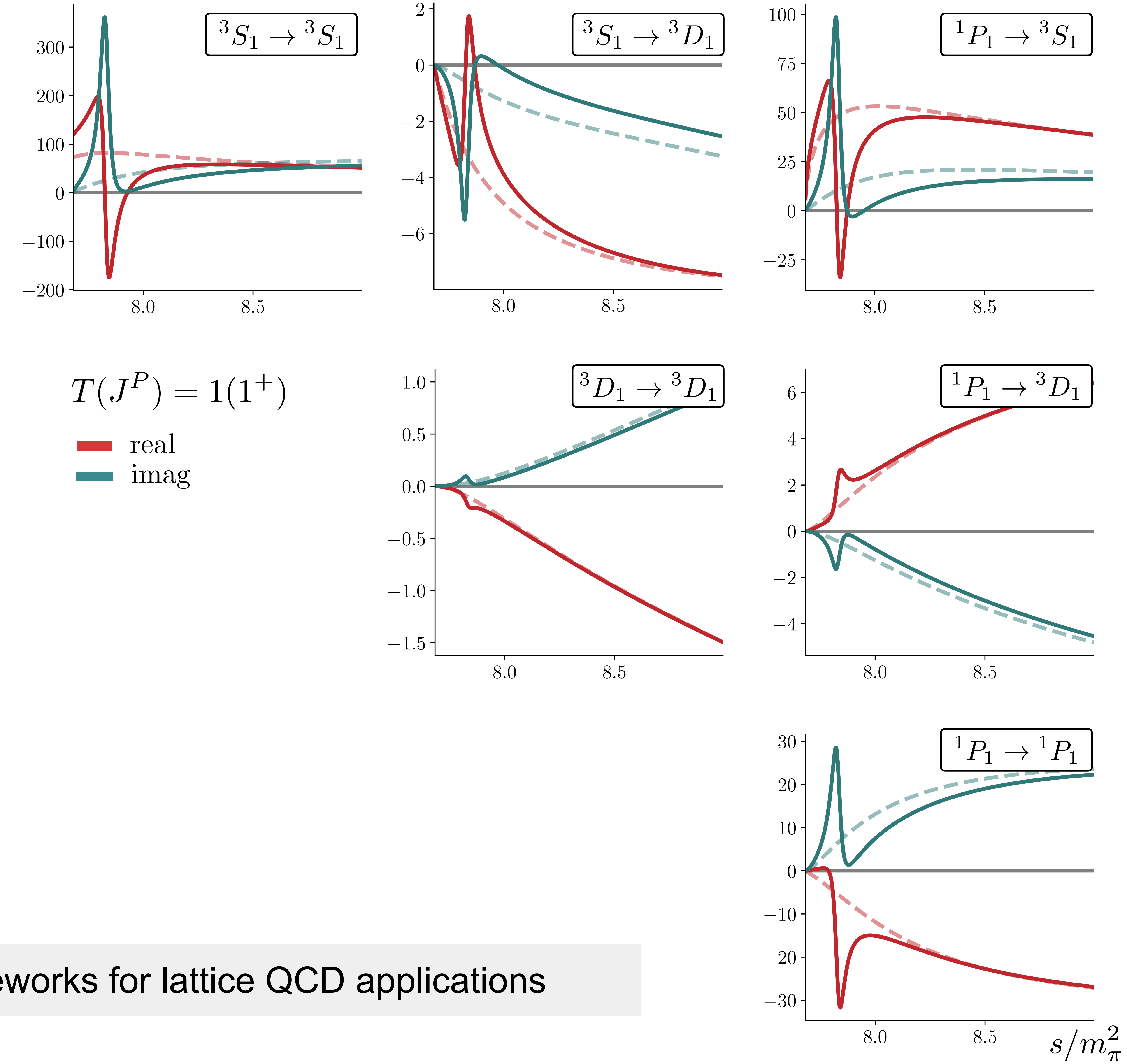




AJ and R. Briceño
Phys.Rev.D **109** (2024) 9, 096030

R. Briceño, C. Costa, AJ
Phys.Rev.D **111** (2025) 3, 036029

Testing frameworks for lattice QCD applications

Higher Partial Waves

Can form observables, e.g., reaction rates*

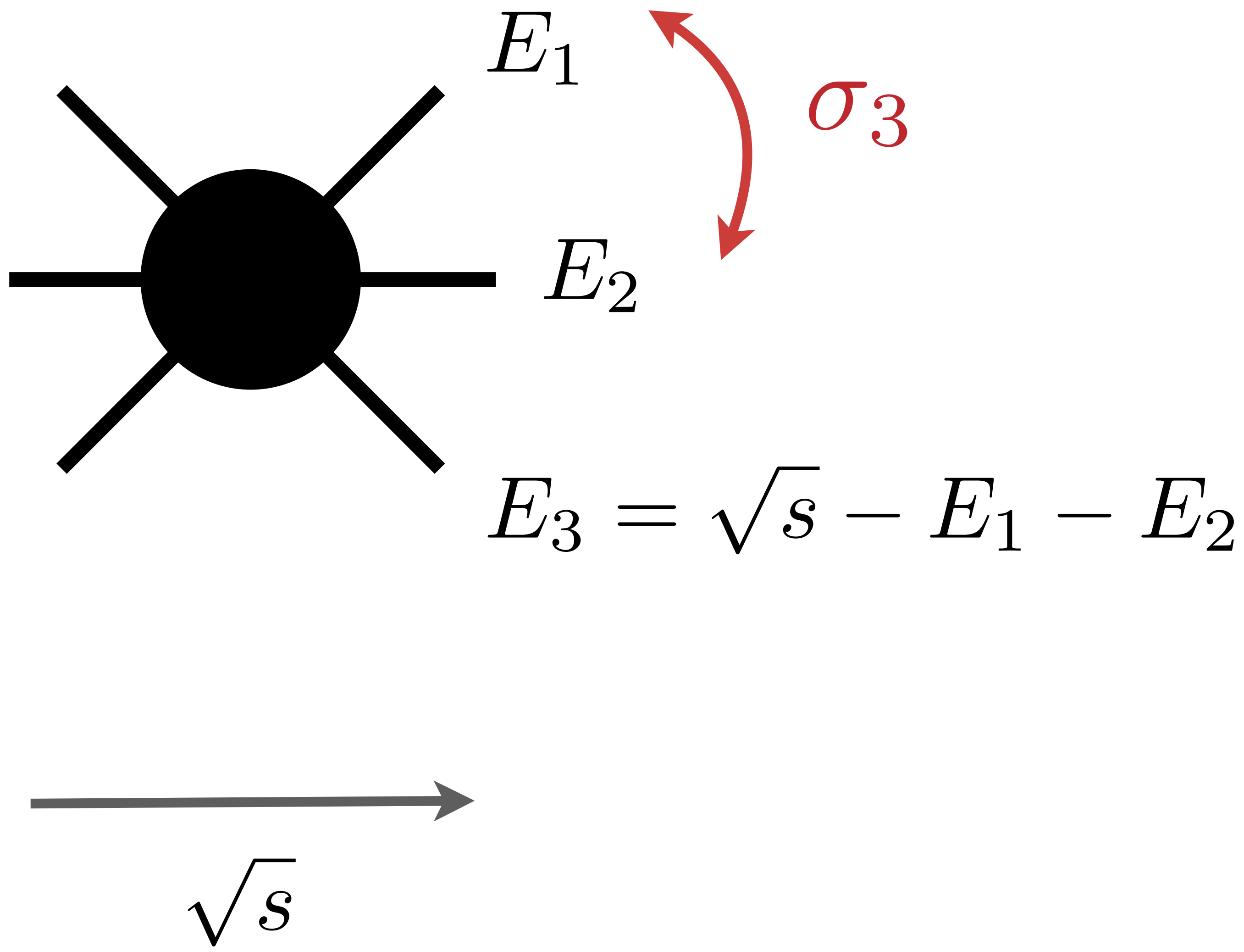
$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_1\mathrm{d}E_2} \propto \sum_{I(J^P)} |\mathcal{M}_3^{I(J^P)}|^2$$

Differential $\mathbf{3} \to \mathbf{3}$ *reaction rate*

$$\mathcal{M}^{\mathfrak{q}}_{\mathfrak{c}'\mathfrak{c}} = \sum_{j,k} \sum_{\mathfrak{c}_j, \mathfrak{c}'_k} \left(\mathcal{R}^{\mathfrak{q}}_{\mathfrak{c}'_k}\right)^{\top} \cdot \mathcal{M}^{\mathfrak{q}}_{\mathfrak{c}'_k \mathfrak{c}_j} \cdot \left(\mathcal{R}^{\mathfrak{q}}_{\mathfrak{c}_j}\right)$$

Full amplitude requires recoupling coefficients between spectator choices

fix initial configuration



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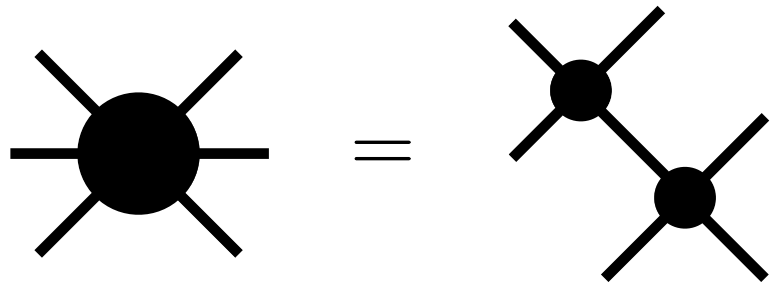
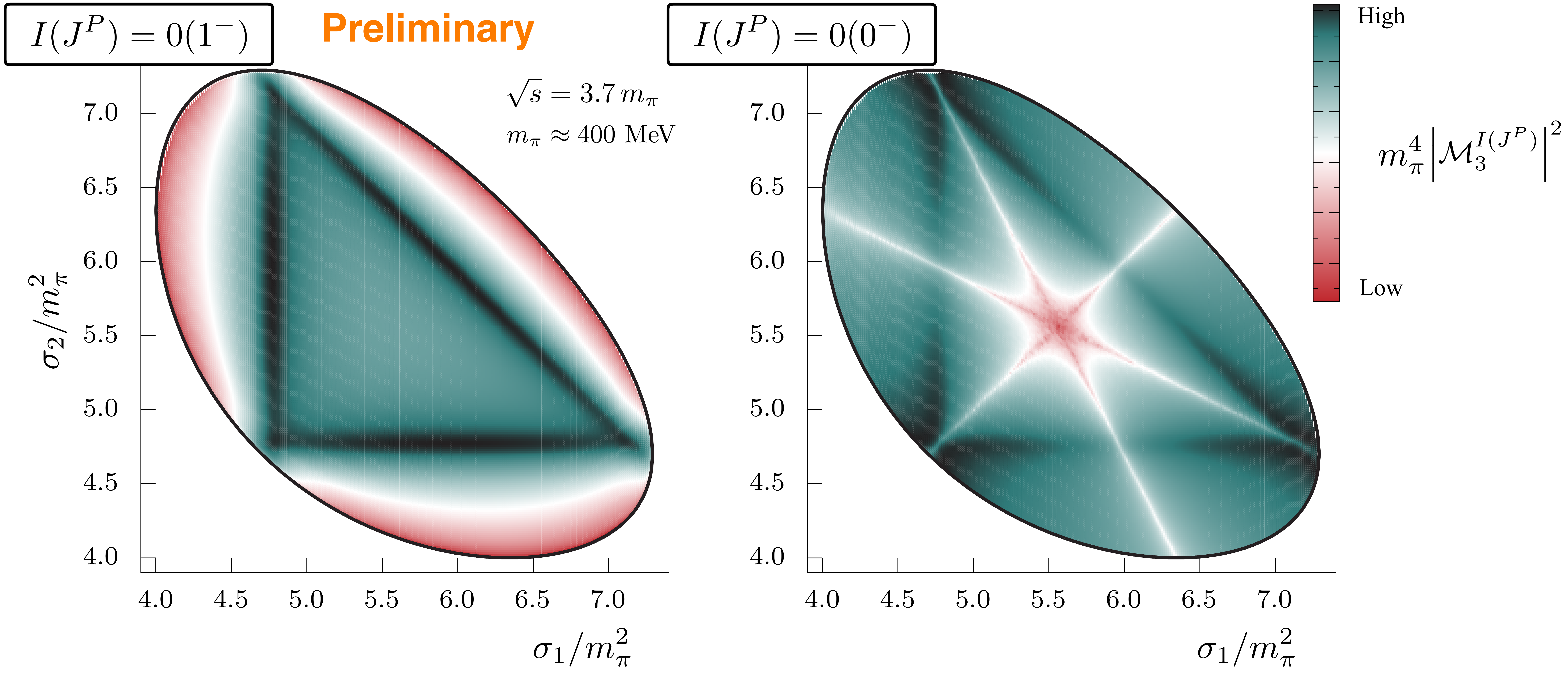
**Ignoring subtleties about defining 3-body reaction reactions, see works by Taylor and Newton*

Higher Partial Waves

Can form observables, e.g., reaction rates*



Testing frameworks for lattice QCD applications



$\sigma_j = (\sqrt{s} - E_j)^2 - p_j^2$

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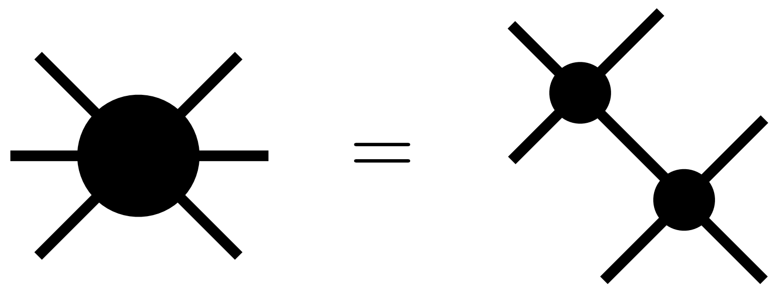
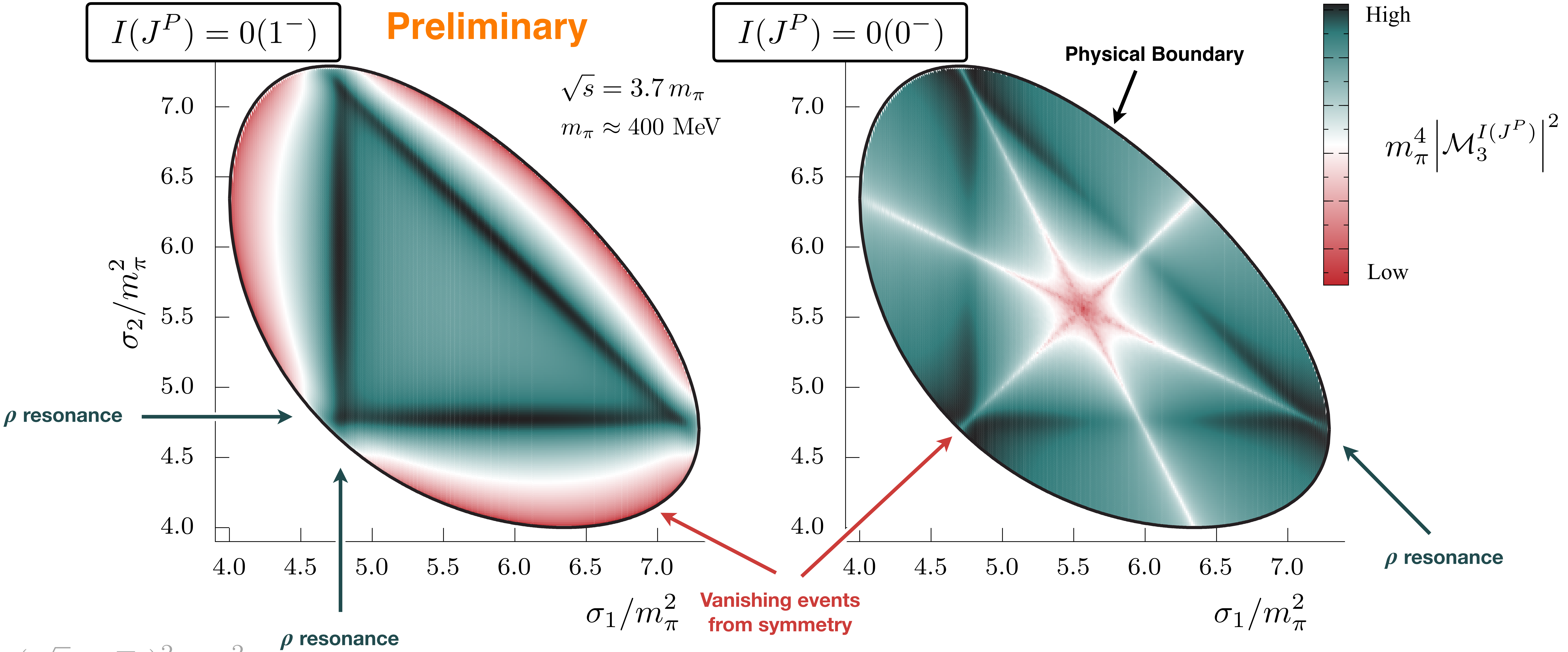


Testing frameworks for lattice QCD applications



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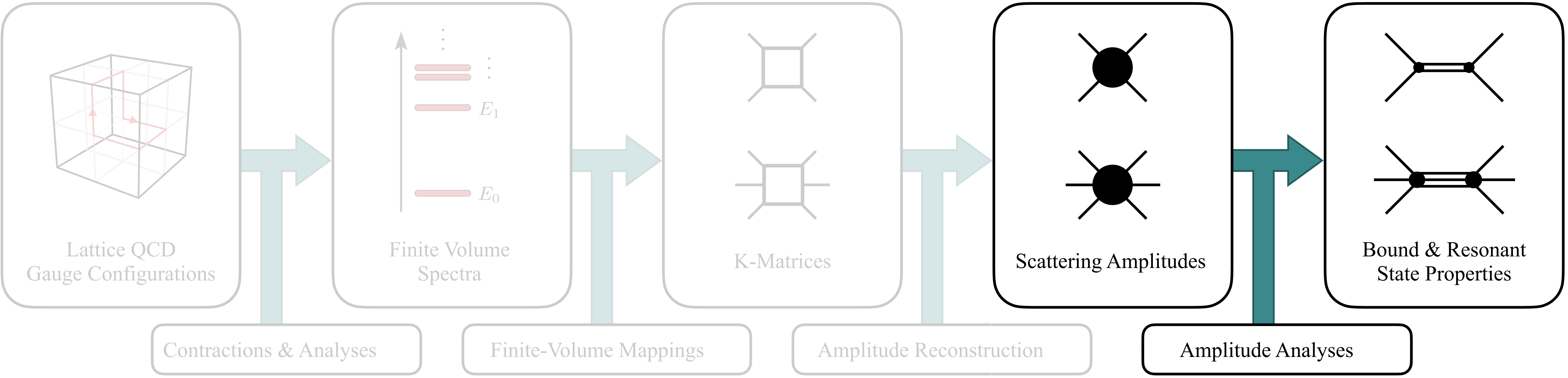
Hadronic Reactions from QCD

Lattice QCD is a numerical tool to estimate low-energy QCD observables

- Formulated on **finite**, discretized, Euclidean spacetime — No real-time dynamics
- However, can extract amplitudes from lattice QCD
- Key idea: Map finite-volume energies to infinite-volume objects (via Lüscher)
- A path toward hadronic **reactions** from QCD

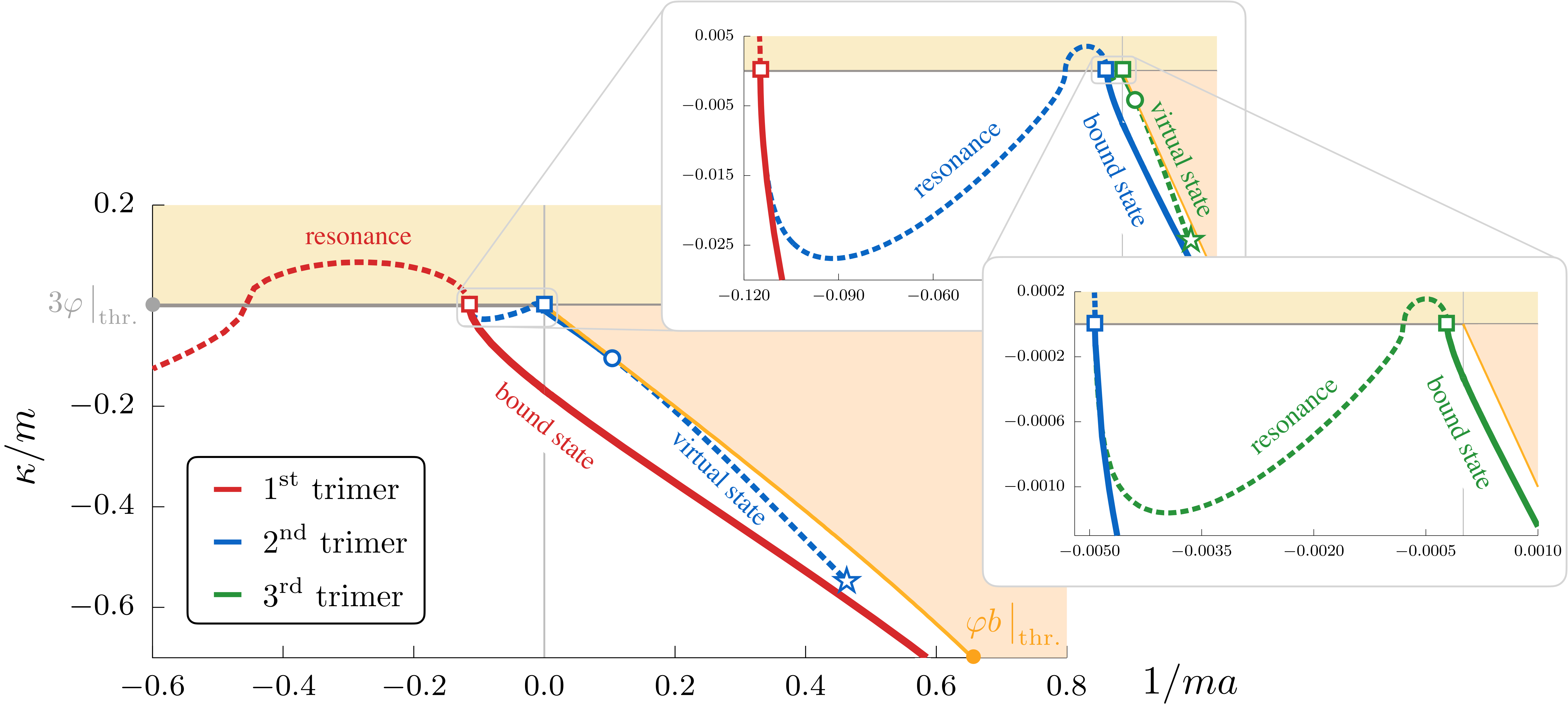


Spectral Studies

Spectral analyses require continuing to pole positions

- Community is building confidence in robust solution strategies
- e.g., exploring Efimov physics with relativistic three-body framework

$$\mathcal{M}_3 \sim -\frac{g^2}{E^2 - E_0^2}$$



S. Dawid, M. Islam, R. Briceño, AJ
Phys. Rev. A **109**, (2024) 043325

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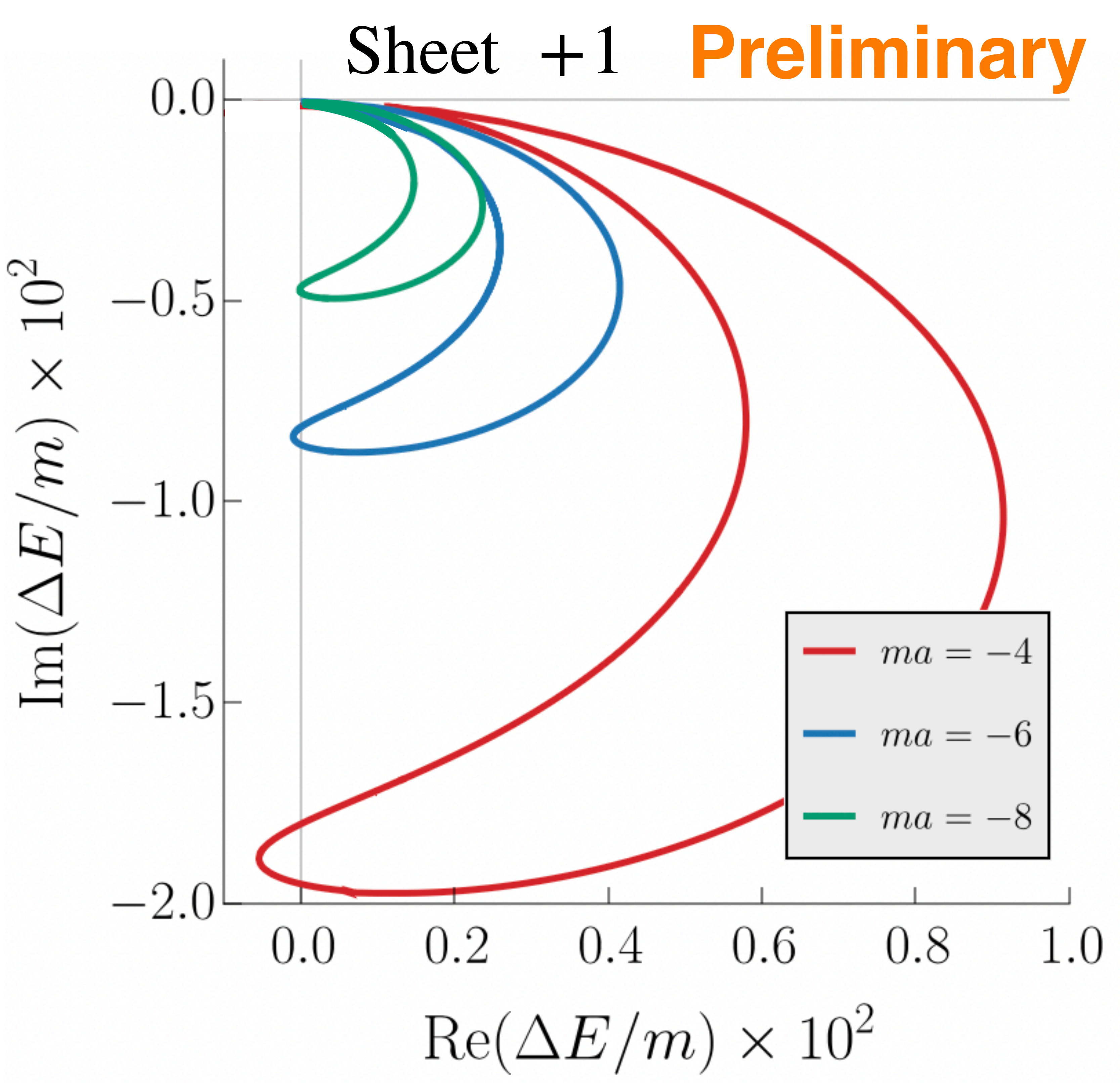

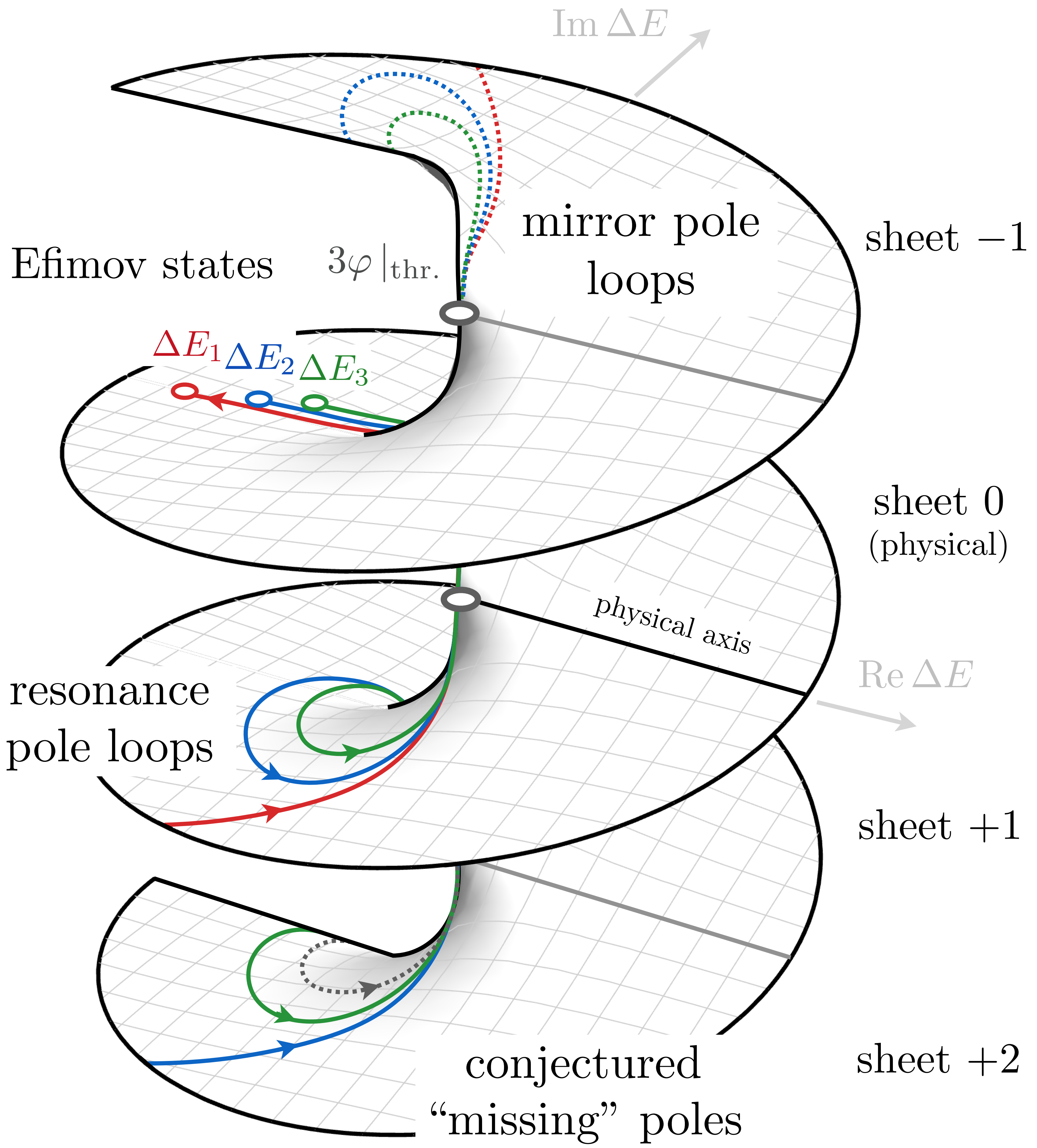

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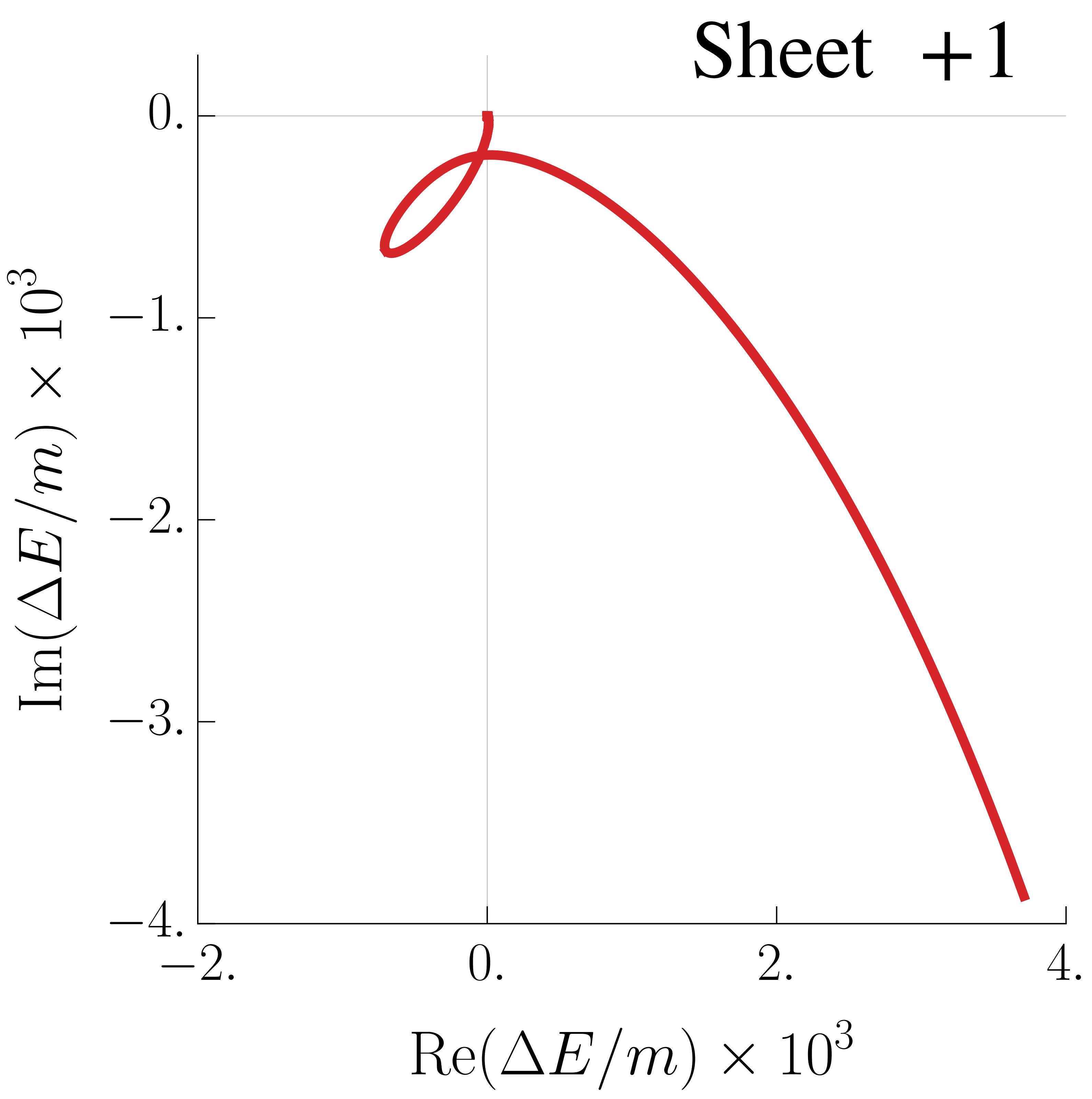

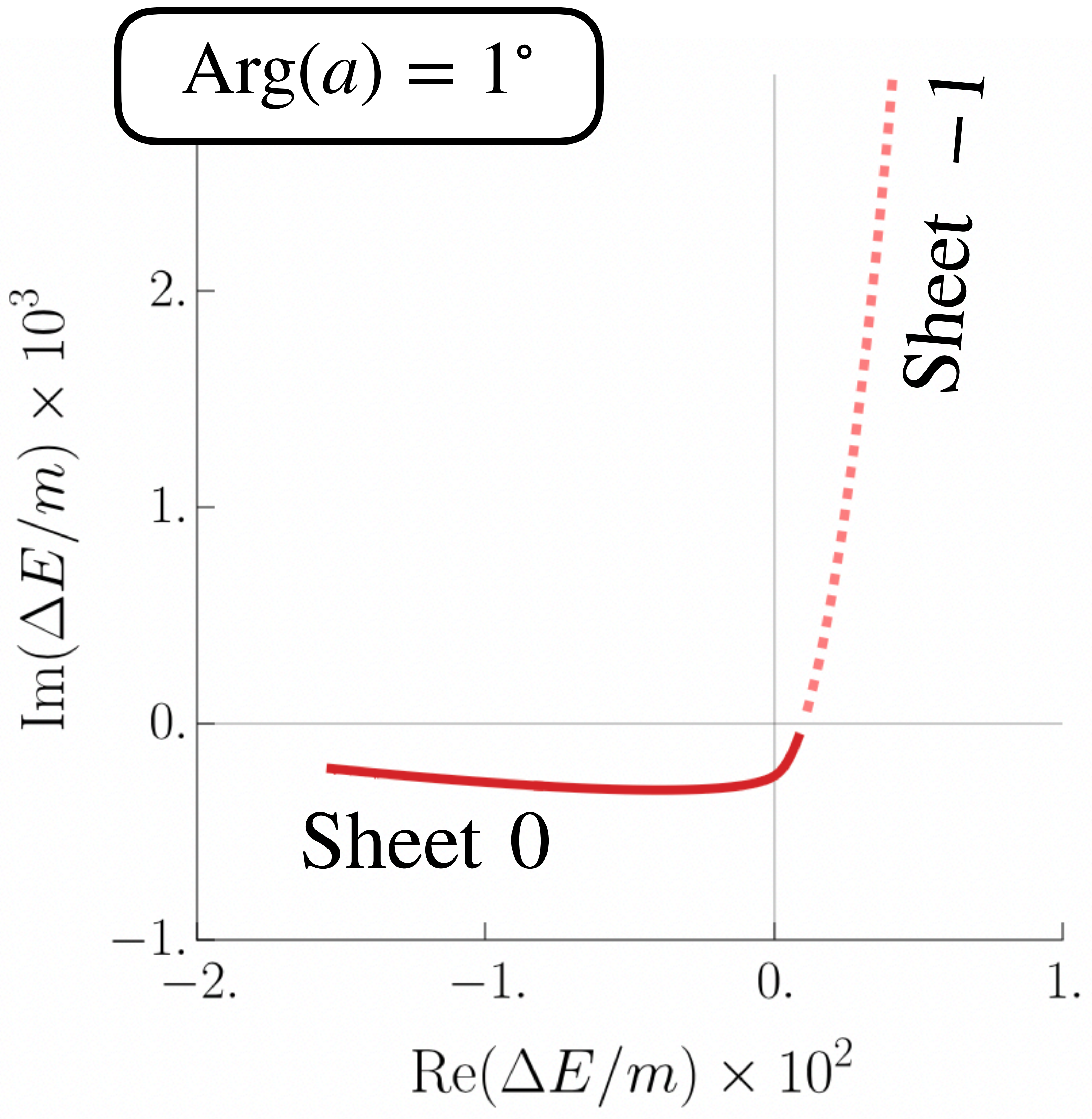

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Summar

Three-Body studies from lattice QCD is maturing

- Accessing three-body reaction rates
- Incorporating higher partial waves, non-zero spin on-going
- Applications to lattice QCD analyses appearing in literature past few years