Thank you, Rimas!

Towards the Unitarity Limit in EFTs with Pions

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How to root Nuclear Physics in QCD?
What is the underlying principle that makes simple structures emerge from complex nuclear dynamics?

Office of Nuclear Physics
Exploring Nuclear Matter - Quarks to Stars

König/hg/Hammer/van Kolck: *Phys. Rev. Lett.* **118** (2017) 202501 [1607.04623 [nucl-th]]

Teng/hg: MSc Thesis GW 2023 and [2410.09653 [nucl-th]]

1. Emergent Phenomena in Nuclear Physics: "Order From Chaos"

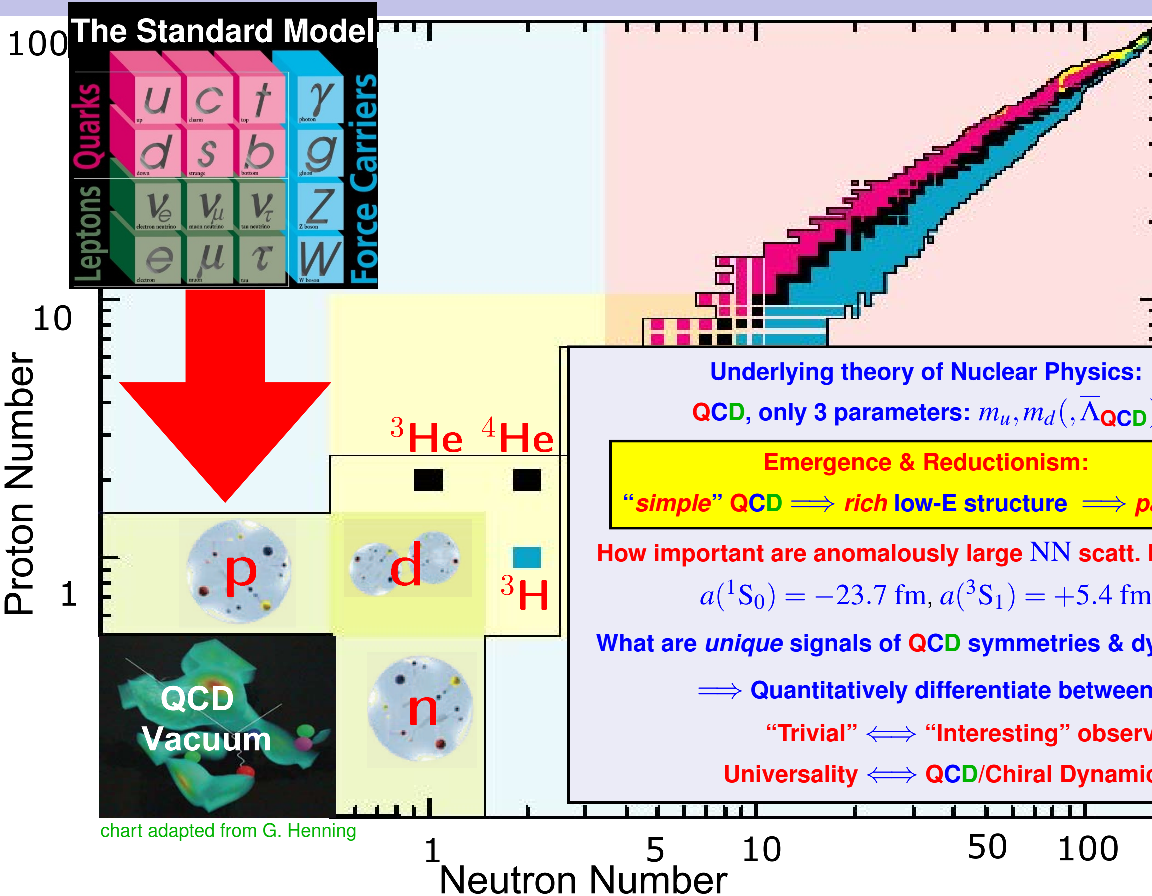


chart adapted from G. Henning

Underlying theory of Nuclear Physics:

QCD, only 3 parameters: $m_u, m_d(, \overline{\Lambda}_{\mathrm{QCD}})$

Emergence & Reductionism:

"*simple*" QCD $\Longrightarrow$ *rich* low-E structure $\Longrightarrow$ *patterns*

How important are anomalously large NN scatt. lengths?:

$a(^1\mathrm{S}_0) = -23.7\ \mathrm{fm}$, $a(^3\mathrm{S}_1) = +5.4\ \mathrm{fm}$

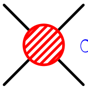
What are *unique* signals of QCD symmetries & dynamics?

$\Longrightarrow$ Quantitatively differentiate between:

"Trivial" $\Longleftrightarrow$ "Interesting" observables

Universality $\Longleftrightarrow$ QCD/Chiral Dynamics

2. What Is The Unitarity Limit? And Why Should I Care?

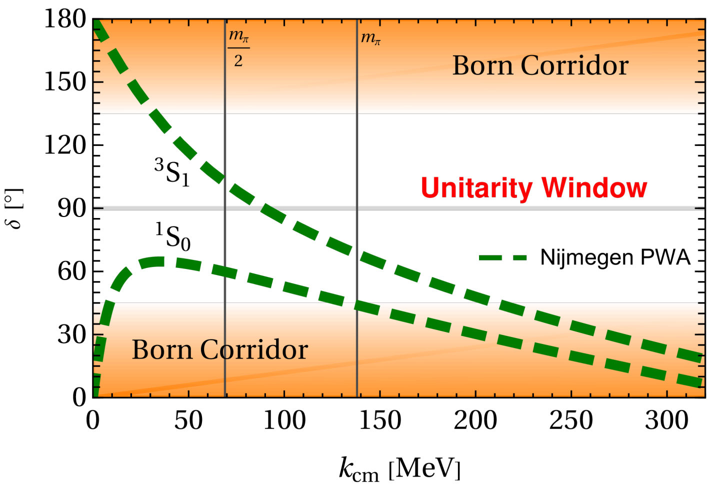


$$\propto \frac{1}{\underbrace{k\cot\delta}_{\text{interaction}\downarrow} - ik} \rightarrow \begin{cases} \dfrac{1}{k\cot\delta}\left[1+\dfrac{i}{\cot\delta}+\dots\right] & \text{for } \cot\delta \gg |i| \text{ i.e. } \delta\to 0 \implies \\ \dfrac{1}{-ik}\left[1+\dfrac{\cot\delta}{i}+\dots\right] & \text{for } \cot\delta \ll |i| \text{ i.e. } \delta\to 90^\circ \implies \end{cases}$$

Unitarity

Born Approximation:
interactions small & perturbative,
their details & scales drive A_{NN}
no bound states

Unitarity Limit implies Universality:
interaction strong: *non-perturbative*,
details irrelevant, unitarity drives A_{NN};
Unitarity Expansion at LO:
no scales in A_{NN}, bound state at $k=0$.



Unitarity Window: $|\cot\delta| \lesssim 1$ ($45^\circ \lesssim \delta \lesssim 135^\circ$)

$\implies$ LO NN nonperturbative in $^1S_0$ & $^3S_1$ for

$30\text{ MeV} \lesssim k_{cm} \lesssim [1.5\dots2]m_\pi$

Outside: **Born Corridors**

LO perturbative for $|\cot\delta| \gtrsim 1$ ($|\delta| \lesssim 45^\circ$)

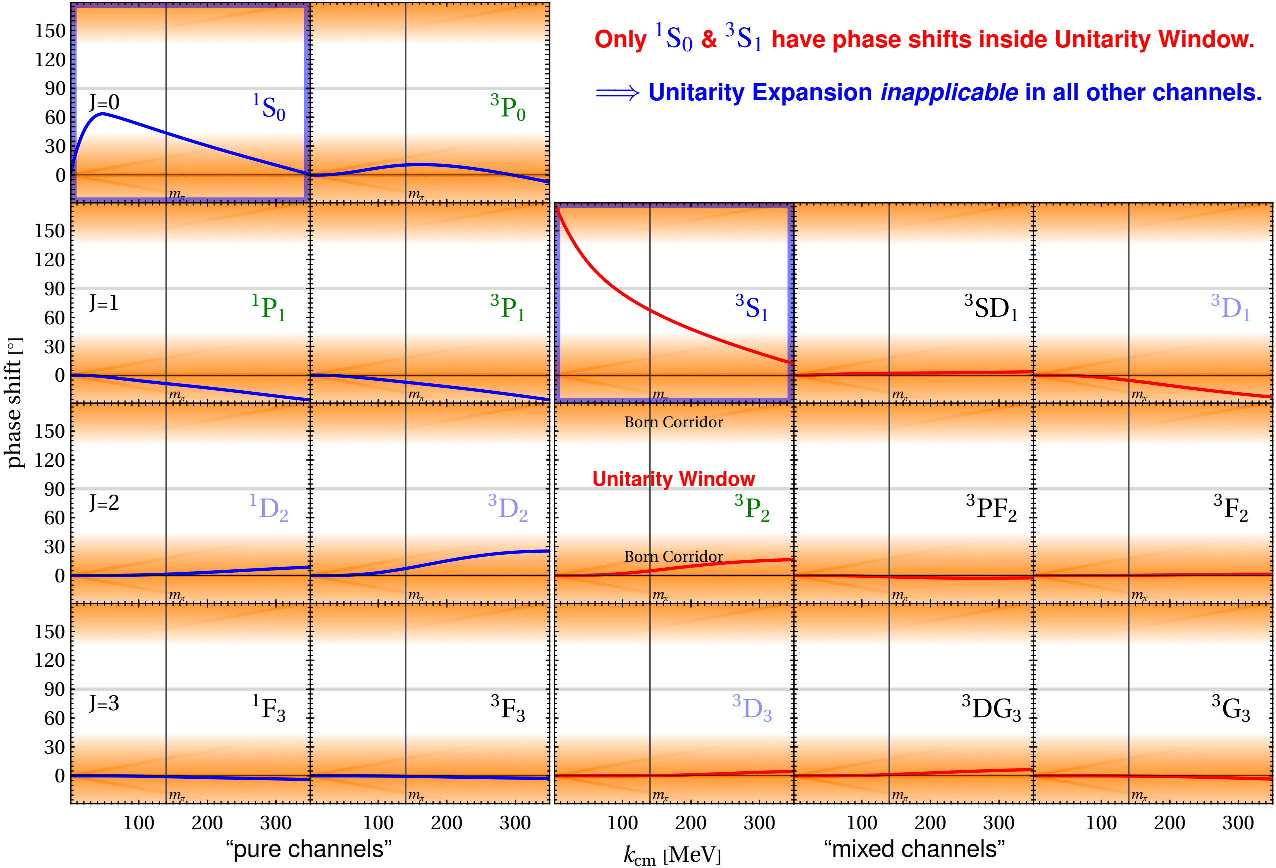
How much of Nuclear Physics does really depend on details of QCD?

How much just from (corrections to) universal aspects around Unitarity?

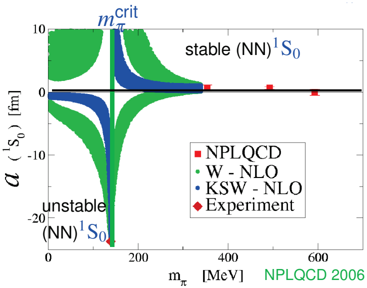
(a) Use Unitarity Expansion Only for Channels with Large NN Phase Shifts!

Only $^1S_0$ & $^3S_1$ have phase shifts inside Unitarity Window.

$\Longrightarrow$ Unitarity Expansion *inapplicable* in all other channels.



(b) Symmetries in the Unitarity Limit



NPLQCD 2006

χEFT cannot explain anomalous scatt. lengths/shallow binding: Worlds with $a \lesssim \frac{1}{m_\pi}$!

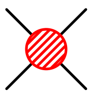
Noether Theorem 1918 [physics/0503066]:

Symmetries and their breaking result in conserved quantities.

(1) **Amplitude saturated at Unitarity Limit:** $\sigma = \frac{4\pi}{k^2}$ maximal (probability conservation).

(2) **Scale Invariance:** $\vec{k} \to e^{\lambda}\,\vec{k}$. actually nonrel. Conformal/Schrödinger Symmetry... Mehen/Stewart/Wise 2000 Nishida/Son 2007

(3) **Wigner-$\mathrm{SU}(4)$ Symmetry of combined spin-isospin rotations** $\begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix} \to U \begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix}$ Wigner, Hund 1937 for heavy nuclei cf. Mehen/Stewart/Wise 1999

In NN:  $= \frac{4\pi}{M}\,\frac{1}{-\frac{1}{a} - ik} = A_{\mathrm{NN}}(^3\mathrm{S}_1) = A_{\mathrm{NN}}(^1\mathrm{S}_0)$ if $a(^3\mathrm{S}_1) = a(^1\mathrm{S}_0)$.

Theorists love Unitarity Limit as Nontrivial Fixed Point characterised by high symmetry: Wigner-$\mathrm{SU}(4)+$ scale-invariance close to FP protected in renormalisation.

What About Nature?

(c) Unitarity Expansion in EFT($\not\pi$)

EFT($\not\pi$)/ERE: $\propto \frac{1}{-ik}\left[1+\frac{k\cot\delta = -\frac{1}{a}+\frac{r}{2}k^2+\dots}{ik}+\dots\right] \to \underbrace{\frac{1}{-ik}}_{\text{LO}}\left[1+\mathrm{i}\left(\underbrace{\frac{1}{ka}}_{<1?!}-\underbrace{\frac{kr}{2}}_{<1?!}\right)+\dots\right]$ — NLO correction

***a priori* justified if** inverse scatt. length/ NN system size/ NN binding momentum $0 \leftarrow \frac{1}{a} \ll$ **typ. momentum** $k \ll \Lambda_{\not\pi} \overset{???}{\sim} m_\pi \sim \frac{1}{r}$ **breakdown/ resolution scale.**

LO: No NN scale. $\Longrightarrow$ Nuclear Physics correlated to *just one* 3N *RG scale fixed by* B_3 *via Efimov effect*.

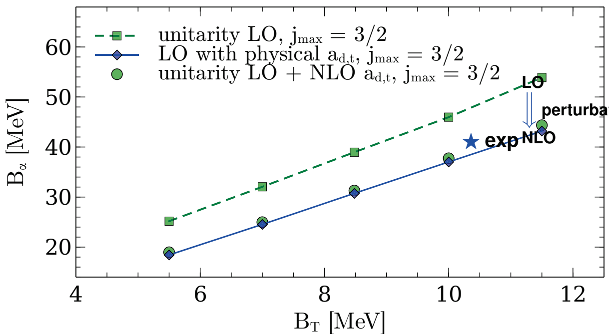
PARADIGM SHIFT*: Unitarity* de-emphasises details of NN & pions, emphasises 3N scale & Universality.

Information Theory in EFT: lossless compression into smallest number of parameters at given accuracy.

$\Longrightarrow$ Explore *Sweet Spot* for patterns, unique signals of QCD:

bound weakly enough to be insensitive to interaction details ($\frac{kr}{2} \ll 1$),

but strongly enough to be insensitive to exact large system size ($ka \gg 1$).

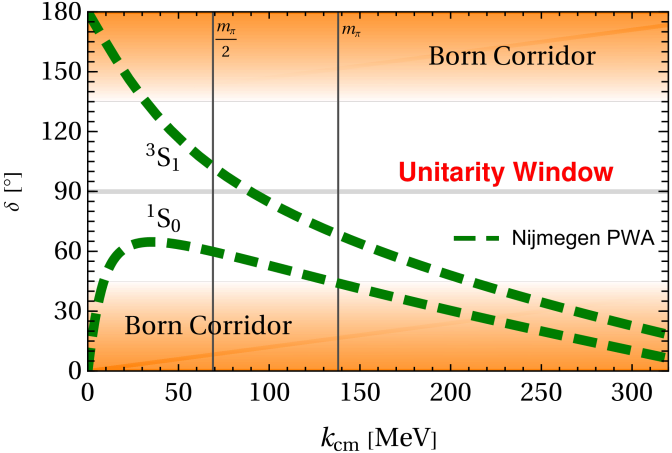


$B_{^3\mathrm{H}} - B_{^3\mathrm{He}}$: NLO: $[0.92 \pm 0.18]$ MeV; exp: 0.764 MeV

	Fermion Unitarity LO $\to$ NLO	exp ^{4}He/^{3}H
ground: B_4/B_3	$4.6 \to 3.8 \pm 0.2$	3.66
excited: B_4^*/B_3	$\sim 1.1 \to \sim 0.98 \pm 0.05$	0.96

Symm. Nucl. Matter	ρ_0 [fm^{-3}]	B/A [MeV]	E_{sym} [MeV]	L[MeV] slope of E_{sym}	K_∞[MeV] compressib.
Kievsky/... EFT($\not\pi$)-inspired	0.15	-16	35	70	251
exp	0.16	-16	≈ 30	$[40\dots60]$	210

(d) χEFT Should Work In the Unitarity Expansion!



NN S waves well in **Unitarity Window** $|\cot\delta| < 1$
for $30\ \mathrm{MeV} \lesssim k_{cm} \lesssim [1.5\ldots2]\,m_\pi$.

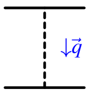
Window's upper limit close to scale

$$\overline{\Lambda}_{NN} = \frac{16\pi f_\pi^2}{g_A^2 M} \approx 300\ \mathrm{MeV}$$

where OPE becomes nonperturbative KSW 1999, FMS 2000.

$\Longrightarrow$ **How to embed pions/χiral symmetry inside Unitarity Window?**

Problem: Pions break **scaling by f_π, m_π, Wigner by mixing.**



$$: \; -\frac{g_A^2}{4 f_\pi^2}\,\frac{1}{\vec{q}^{\,2}+m_\pi^2}\left[\underbrace{\frac{1}{3}\,(\vec{\tau}_1\cdot\vec{\tau}_2)\,(\vec{\sigma}_1\cdot\vec{\sigma}_2)\,\vec{q}^{\,2}}_{\text{central: Wigner-Irrep}} + \underbrace{(\vec{\tau}_1\cdot\vec{\tau}_2)\left((\vec{\sigma}_1\cdot\vec{q})(\vec{\sigma}_2\cdot\vec{q}) - \frac{1}{3}\,(\vec{\sigma}_1\cdot\vec{\sigma}_2)\,\vec{q}^{\,2}\right)}_{\text{tensor: Wigner-breaking, mixes } S\leftrightarrow D,\ D\to D}\right]$$

dictated by Ciral Symmetry!

Explore transition "no$\longrightarrow$nonperturbative pions" via Perturbative ("KSW") Pions (only undisputedly consistent χEFT).

$\Longrightarrow$ **Clash of symmetries: Wigner/Unitarity vs. Chiral??**

χEFT(pπ)$_{\mathbf{UE}}$: χEFT with Perturbative Pions in the Unitarity Expansion: $Q \sim \frac{1}{ka}, \frac{k, m_\pi}{\overline{\Lambda}_{NN}} \ll 1$

3. Unitarity Expansion With Perturbative Pions in NN

(a) χEFT(pπ)$_{\mathbf{UE}}$ at N^{2}LO with $Q \sim \frac{1}{ka}, \frac{kr}{2}, \frac{k, m_\pi}{\Lambda_{NN}} \ll 1$

based on Rupak/Shoresh [nucl-th/9902077] ($^1S_0$),
Fleming/Mehen/Stewart [nucl-th/9911001] ($^1S_0$, $^3S_1$)
mod. for unitarity Teng/hg MSc thesis GW 2023, [2410.09653]

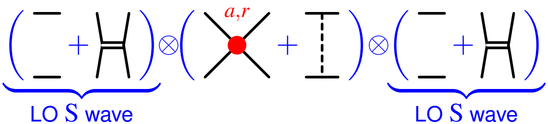
$\mathcal{O}(Q^{-1})$ **(LO):** Nonperturbative; no scale, perfect Wigner, pure S wave.

$A^{(S)}_{-1} = \frac{4\pi i}{M}\frac{1}{k} =$ 

from structureless contacts $C\,(N^\dagger N)^2$

$\mathcal{O}(Q^{0})$ **(NLO):** Scaling and Wigner broken by contacts determined to reproduce PWA values of a, r.

Non-iterated OPE: central only, does not break Wigner but scaling; first non-analyticity: branch point $\pm i\frac{m_\pi}{2}$.

$A^{(S)}_{0} =$ 

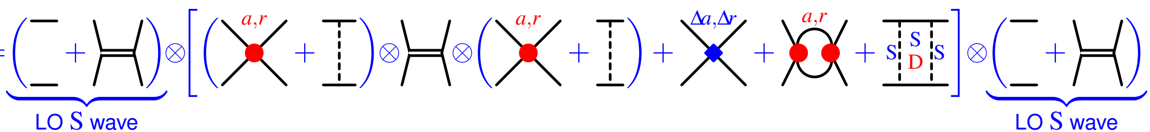
$\Longrightarrow$ **Unitarity, Wigner-SU(4) spin-isospin symmetry align naturally for Perturbative Pions at NLO.**

$\mathcal{O}(Q^{1})$ **(N^{2}LO):** Contacts adjusted to keep a, r at PWA values; multiplied with non-iterated OPE (central only).

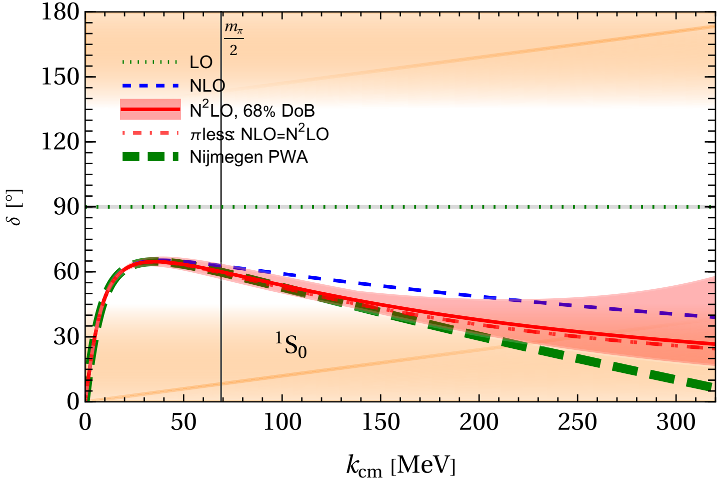
Once-iterated OPE added: first and second non-analyticity: branch points $\pm i\frac{m_\pi}{2}, \pm i m_\pi$.

$A_{1\text{sym}}$: Central $S \to S \to S$ does not break Wigner but scaling: identical in $^1S_0$ and $^3S_1$.

$A_{1\text{break}}$: Tensor $S \to D \to S$ breaks Wigner and scaling: only in $^3S_1$.

$A^{(S)}_{1} =$ 

$\Longrightarrow$ **Is breaking of Wigner-SU(4) spin-isospin symmetry for Perturbative Pions at N^{2}LO indeed small?**



$^1S_0$: central OPE $\Longrightarrow$ Wigner-symmetric.

f_π, m_π break scaling.

Strict perturbation in "basic interaction part"

$$k\cot\delta = 0_{\text{LO}} + k\cot\delta_{\text{NLO}} + k\cot\delta_{\text{N}^2\text{LO}}$$
$$= 0_{\text{LO}} - \frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2}\right].$$

$\Longrightarrow$ Get δ from $k\cot\delta$.

$^1S_0$ is **"boring"** partial wave: no tensor int.

Bayesian truncation uncertainty at 68% DoB.

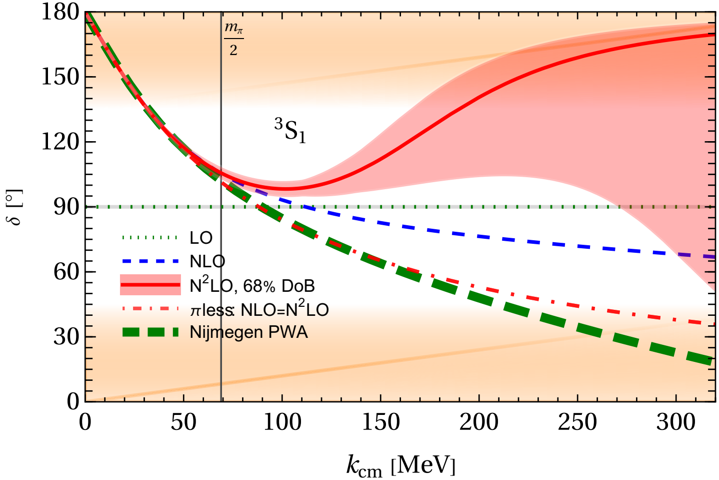
$\Longrightarrow$ Converges order-by-order $\lesssim 300$ MeV.

Agrees within uncertainties with PWA for $\lesssim 250$MeV (even outside Unitarity Window).

Compare to EFT($\not\pi$): minuscule impact of π.

(c) Perturbative Pions at N^{2}LO: $^3S_1$

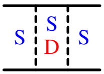
perturbative pions to N^{2}LO: Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



$^3S_1$: pions break Wigner-$SU(4)$ & scale inv.

$^3S_1$ is **"interesting"** partial wave:

tensor-OPE $\Longrightarrow$ SD mixing from



$$k\cot\delta = -\frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2} + \frac{A_{SD}^2}{A_{-1}^3}\right]$$

$\Longrightarrow$ Terrible convergence (already in FMS):

Converges order-by-order $\lesssim 80$ MeV.

Agrees within uncertainties with PWA only for $\lesssim 70$ MeV (not even in Unitarity Window).

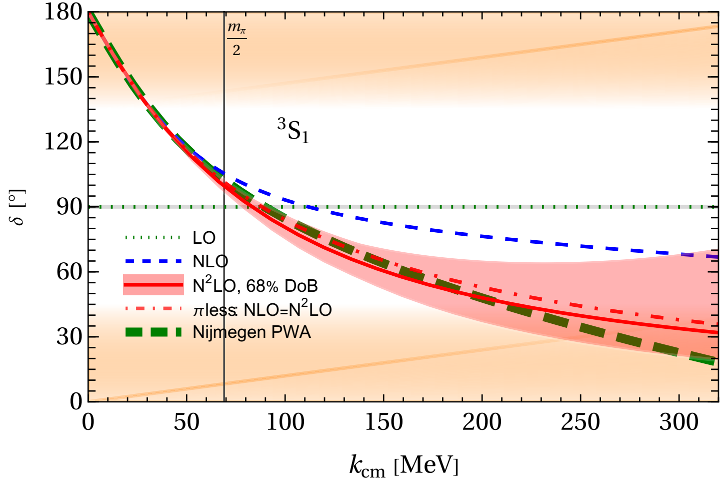
Compare to EFT($\not\pi$): huge impact of pion.

Source of discrepancy: tensor int. (via SD).

(central is identical in $^3S_1$&$^1S_0$)

(c) Perturbative Pions at N^{2}LO: $^3S_1$

perturbative pions to N^{2}LO: Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



$^3S_1$: pions break Wigner-$SU(4)$ & scale inv.

$^3S_1$ is **"interesting"** partial wave:

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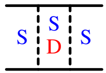


Broken Wigner-$SU(4)$ spoils convergence!

Idea: Use Wigner-$SU(4)$-symmetric pion part.

$\Longrightarrow$ Only $^1S_0$-$^3S_1$ differences of a & r break Wigner-$SU(4)$.

RG-invariant, mildly χsymmetry-breaking.

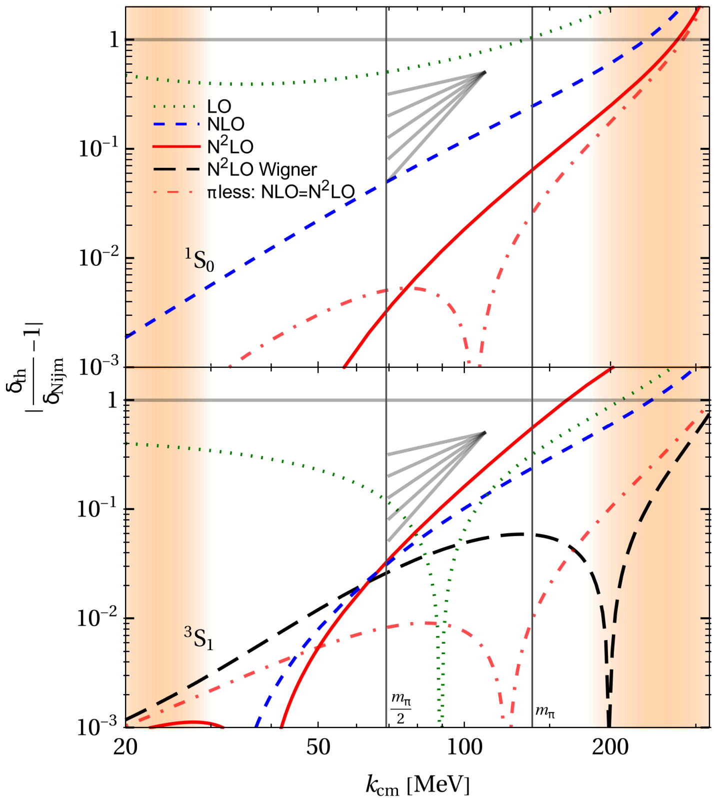
$\Longrightarrow$ Converges order-by-order $\gtrsim 300$ MeV.

Agrees within uncertainties with PWA for $\gtrsim 300$ MeV (even outside Unitarity Window).

Compare to EFT($\not\pi$): tiny impact of pion.

$\Longrightarrow$ **All very similar to $^1S_0$.**

(d) Convergence to Data



$$\frac{\delta(\mathrm{N}^n\mathrm{LO})-\delta(\mathrm{PWA})}{\delta(\mathrm{PWA})}\sim\left(\frac{k,m_\pi}{\overline{\Lambda}}\right)^{n+1}$$

at N^{n}LO with empirical breakdown scale $\overline{\Lambda}$.

$^1S_0$ and Wigner-symmetric $^3S_1$:

consistent slopes and

$\overline{\Lambda}\approx 270\ \mathrm{MeV}\approx\overline{\Lambda}_{\mathrm{NN}}$ OPE scale.

Full $^3S_1$:

N^{2}LO worse than NLO for $\gtrsim 70\ \mathrm{MeV}$.

Picture obscured by points where

theory & PWA identical ("artificial zero"),

or PWA close to zero ("artificial ∞").

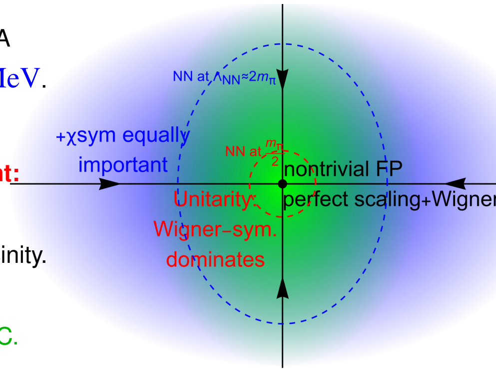
χEFT with Perturbative Pions in Unitarity Expansion $Q \sim \frac{1}{ka}, \frac{kr}{2}, \frac{m_\pi}{\overline{\Lambda}_{NN}} \ll 1$: needs $\delta \to \frac{\pi}{2} \Longrightarrow {}^1S_0, {}^3S_1$ only!

Chiral Physics: m_π, f_π, $(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})(\tau_1 \cdot \tau_2)$ seem opposed to Wigner, but NN/few-N projection forces into it.

Conjecture (at least for Perturbative Pions): Tensor/Wigner-$SU(4)$ symmetry-breaking part of One-Pion Exchange is *super-perturbative* in few-N systems, i.e. does *not enter before* N^{3}LO.

$\Longleftrightarrow$ *Persistence:* Footprint of Symmetries in Unitarity Limit extends far into $p_{typ} \gtrsim m_\pi$, more relevant than χiral symmetry in few-N?! $\Longleftrightarrow$ Better lossless compression of Information!

Evidence: NN S-waves at N^{2}LO converge order-by-order and to PWA inside all of **Unitarity Window** $30\ \text{MeV} \lesssim k \lesssim \overline{\Lambda}_{NN} \approx 300\ \text{MeV}$.
Successful extension of EFT($\not\pi$) to pions.



Appeal: Fine-Tuning $\Longrightarrow$ High Symmetry at Nontrivial Fixed Point:
Universality/scaling $+$ **Wigner-$SU(4)$**
protected in renormalisation at FP $\Longrightarrow$ weakly broken in vicinity.
χiral symmetry not explicit at FP: less protected? $\Longrightarrow$ **Quantify!**
No Wigner in meson/1N sector $\Longrightarrow$ no change to χPT, HBχPT PC.

"Coincidence": N^{2}LO Perturbative Pions overpredict ${}^3SD_1$ mixing, ${}^3D_1$ $\Longrightarrow$ Zero without tensor int. at N^{2}LO.

Some Crucial Tests: If either fails without good reason, Conjecture falsified.

N^{3}LO cf. Beane/Kaplan/Vuorinen 2009, Kaplan 2020

$d\pi \to d\pi$, $\gamma d \to \pi d$
cf. Borasoy/hg 2003

Nd scattering
cf. Bedaque/hg 2000

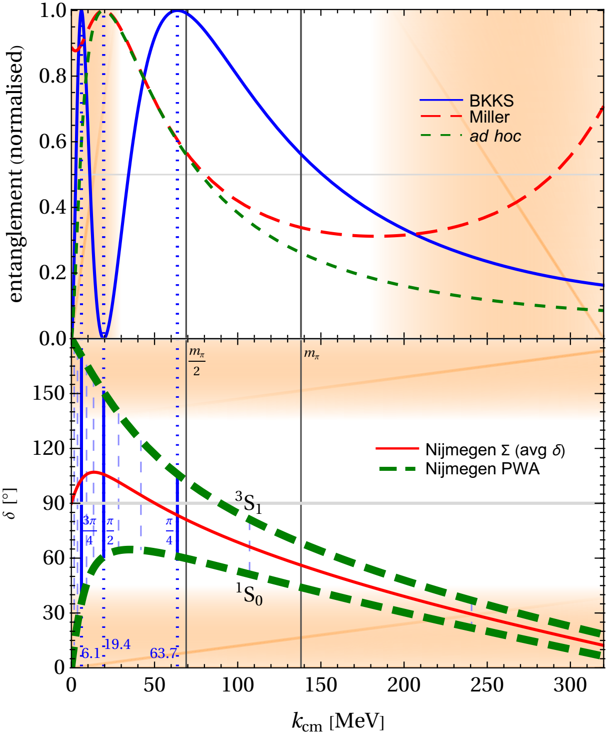
Nonperturbative Pions to N^{2}LO in strict perturbation LO: hg 2023

(a) What is the Small Parameter?: QM Entanglement?

Einstein/Podolsky/Rosen 1935
Bell 1964, 1981

Need expansion parameter related to Wigner-$\mathrm{SU}(4)$ to characterise tensor suppression near Fixed Point.

Entanglement Power $\mathcal{E}$: Deviation of QM *states* from direct product $\underbrace{\text{position} \otimes \text{spin} \otimes \text{isospin}}_{\text{classical}}$ $\Longrightarrow$ *operators!*



$S = \mathrm{e}^{2\mathrm{i}\Sigma}\left[\mathbb{1}\cos\Delta + \mathrm{i}\ \mathrm{SWAP}_\sigma\ \sin\Delta\right]$, Σ : phase avg. Δ : phase diff.

spin swap $1 \leftrightarrow 2$: $\mathrm{SWAP}_\sigma := \frac{1}{2}(\mathbb{1} + \vec{\sigma}_1 \cdot \vec{\sigma}_2) = \begin{cases} +1: {}^3S_1 \\ -1: {}^1S_0 \end{cases}$

Unitarity: $S = \mathrm{e}^{2\mathrm{i}(\Sigma=\frac{\pi}{2},\Delta=0)} = -\mathbb{1} \Longrightarrow \mathcal{E} = 0$: classical

How to Define Entanglement Power $\mathcal{E}$ of *Operator*?:

$\mathcal{E}_{\rm BKKS} = \sin^2[2\Delta]$ Rényi entropy of $1\mathrm{N}$ density matrix of avg. over direct-product $|\text{in}\rangle$ Beane/Kaplan/Klco/Savage 2019

$\mathcal{E}_{\rm Miller} = H\left[\frac{\cos^2\Delta\ (\cos\Delta - \cos 2\Sigma)^2}{(1-\cos\Delta\ \cos 2\Sigma)^2}\right]$ von Neumann entropy Miller 2023

$H[f] = -x\ln x - (1-x)\ln(1-x), x = \frac{1}{2}(1+\sqrt{f})$

$\mathcal{E}_{\rm ad\ hoc} = H[\sin^2\Delta]$ relative von Neumann entropy SWAP vs total hgrie 2024

In Unitarity Window, $\mathcal{E} \in [0;1]$, saturates at $k \approx \dfrac{m_\pi}{2}$.

$\Longrightarrow$ Relevance of Entanglement in Unitarity Window??

How to find $\mathcal{E}$ before computation??

(b) What is the Small Parameter?: Large-N_c Limit of QCD?

't Hooft 1974

Need expansion parameter related to Wigner-$\mathrm{SU}(4)$ to characterise tensor suppression near Fixed Point.

Candidate Expansion of QCD for a large number $N_C \to \infty$ of colours:

Kaplan/Savage [hep-ph/9509371]
Kaplan/Manohar [nucl-th/9612021]
Calle Cordón/Ruiz Arriola [0807.2918]

Predicts that all V_{NN} in S waves are suppressed against central (Wigner-$\mathrm{SU}(4)$) – **except** tensor$\lightning$.

Way out?: Wigner-$\mathrm{SU}(4)$ only realised in long-range parts, strongly broken in short-range?? Calle Cordón/Ruiz Arriola [0807.2918]

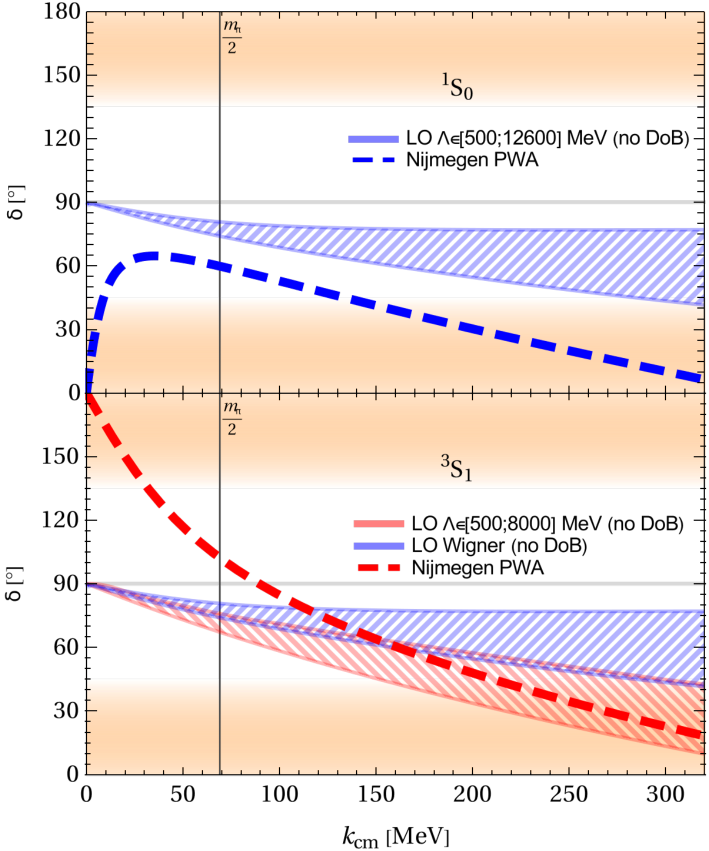
Here: Wigner-$\mathrm{SU}(4)$ breaking only in LECs: short-range – long-range ($k \to 0$) still Wigner-$\mathrm{SU}(4)$ symmetric.

Way out?!: $1/N_c$ expansion assumes that coefficients "of natural size".

Wigner-$\mathrm{SU}(4)$/proximity to Unitarity *forces* leading-$1/N_c$ coefficient of tensor-V_{NN} to be exact zero.

Advantage: Guaranteed to survive renormalisation by Unitarity FP symmetry.

(c) Nonperturbative Pions at LO: Maybe Not Hopeless



LO, 1 mom.-indep. CT, Gaußian regulator.

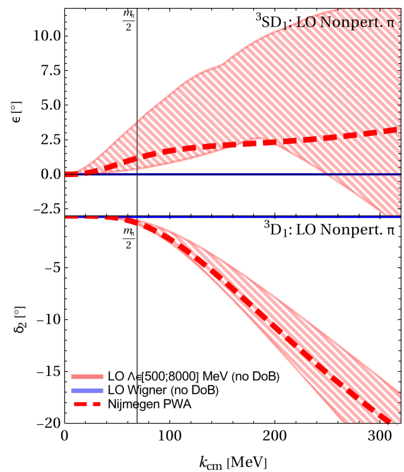
Already deviates from Unitarity $\delta = 90°$.

$\Longrightarrow$ Explicit scale breaking at LO,

$r = \begin{cases} ^1S_0/\text{Wigner } [1\ldots2]\text{fm}; ^3S_1\ [1.2\ldots2.5]\text{fm} \\ \qquad \text{PWA } 2.767(9)\text{fm} \quad 1.852(2)\text{fm} \end{cases}$

Tensor/Wigner-breaking less compatible with unitarity than central/Wigner-symmetric.

Not Bayesian DoBs but cutoff-variation.



4. Concluding Conjecture and Questions

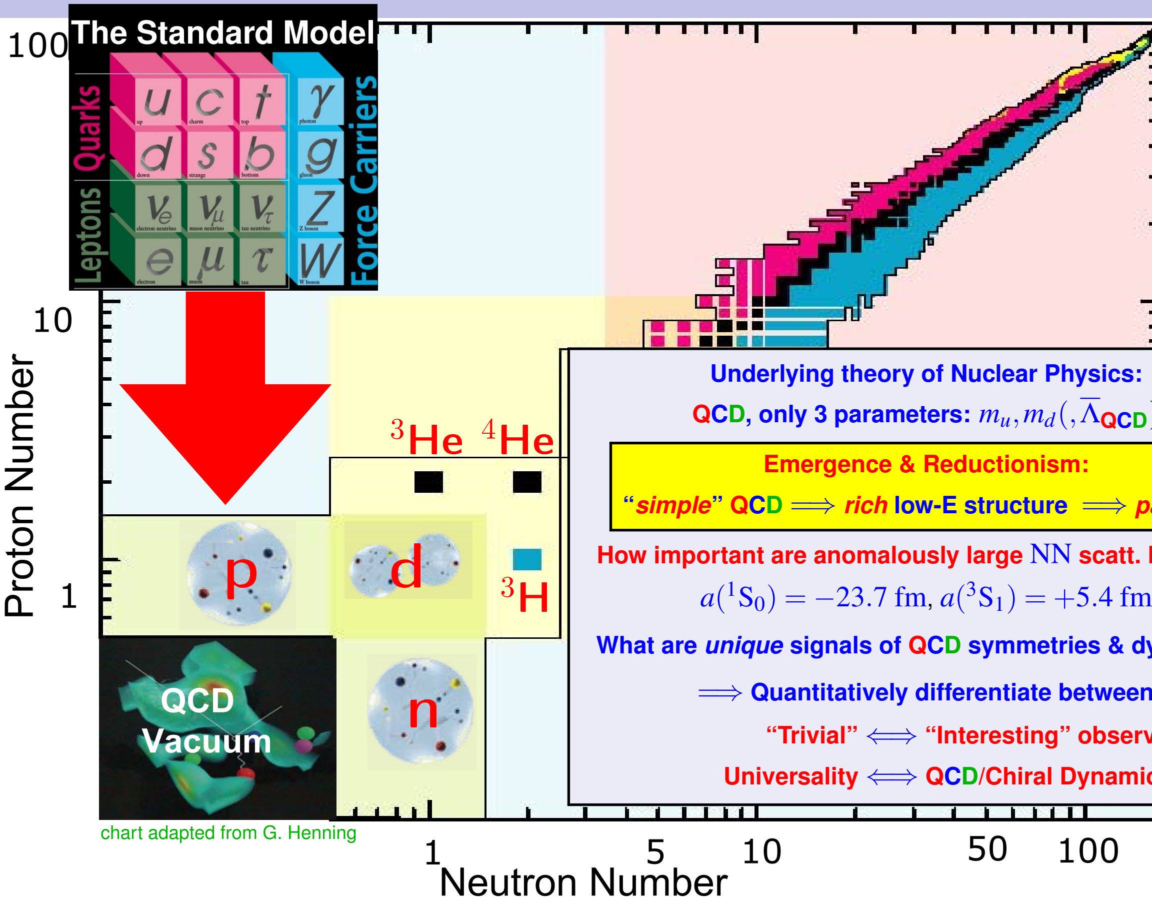


chart adapted from G. Henning

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QCD, only 3 parameters: $m_u, m_d (, \overline{\Lambda}_{\text{QCD}})$

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"*simple*" QCD $\Longrightarrow$ *rich* low-E structure $\Longrightarrow$ *patterns*

How important are anomalously large NN scatt. lengths?:

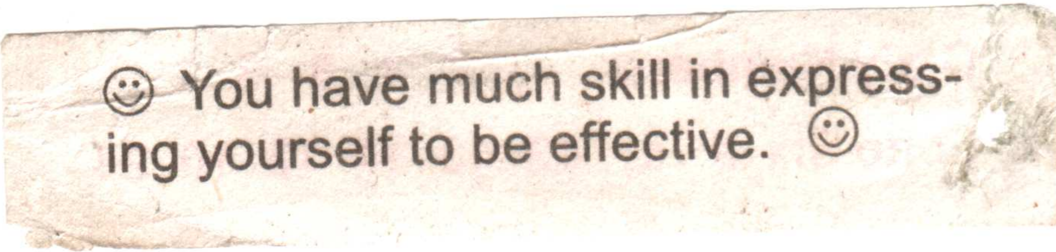
$a(^1\text{S}_0) = -23.7$ fm, $a(^3\text{S}_1) = +5.4$ fm

What are *unique* signals of QCD symmetries & dynamics?

$\Longrightarrow$ Quantitatively differentiate between:

"Trivial" $\Longleftrightarrow$ "Interesting" observables

Universality $\Longleftrightarrow$ QCD/Chiral Dynamics


☺ You have much skill in express-
ing yourself to be effective. ☺

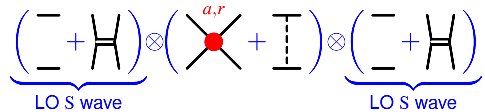
(a) Analytic Answers Shorter By Unitarity

based on Rupak/Shoresh [nucl-th/9902077] ($^1S_0$), Fleming/Mehen/Stewart [nucl-th/9911001] ($^1S_0$, $^3S_1$)
mod. for unitarity Teng/hg MSc thesis GW 2023, [2410.09653]

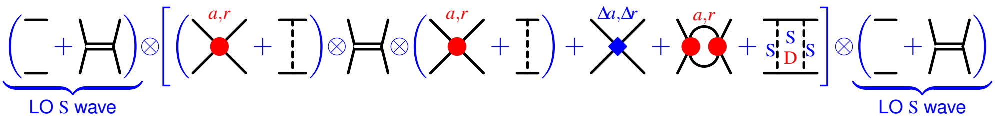
LO: $A^{(S)}_{-1}(k)=\dfrac{4\pi\mathrm{i}}{M}\,\dfrac{1}{k}$ is only S wave

NLO: $A^{(S)}_{0}(k)=-\dfrac{4\pi}{Mk}\left(\dfrac{1}{ka}-\dfrac{kr}{2}\right)-\dfrac{g_A^2}{4f_\pi^2}\left(1-\dfrac{m_\pi^2}{4k^2}\,\overbrace{\ln[1+\dfrac{4k^2}{m_\pi^2}]}^{1\pi\text{ cut}}\right)$

Non-iterated OPE does not break Wigner.



N^{2}LO:



Once-iterated OPE breaks Wigner: S $\to$ D $\to$ S

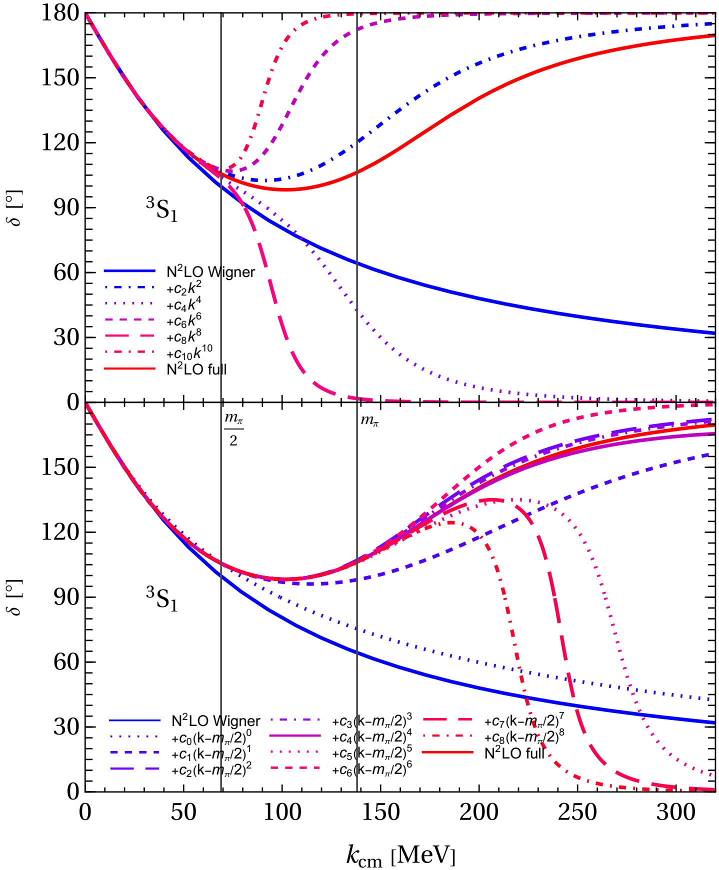
$$A^{(^1S_0)}_{1}(k)\equiv A^{(S)}_{1\text{sym}}(k)=\frac{\left[A^{(S)}_0(k)\right]^2}{A^{(S)}_{-1}(k)}+\frac{g_A^2}{2f_\pi^2}\left[\frac{4}{3am_\pi}-\frac{m_\pi}{k}\left(\frac{1}{ka}-\frac{kr}{2}\right)\right]$$
$$-\frac{g_A^2Mm_\pi}{16\pi f_\pi^2}\left\{A^{(S)}_0(k)\,\frac{m_\pi}{k}\,\underbrace{\arctan[\frac{2k}{m_\pi}]}_{1\pi\text{ cut}}+\frac{g_A^2}{2f_\pi^2}\left[\frac{1}{12}+\left(\frac{m_\pi^2}{4k^2}-\frac{1}{3}\right)\ln 2-\underbrace{F_\pi(\frac{k}{m_\pi})}_{1,2\pi\text{ cut}}\right]\right\}$$

$$A^{(^3S_1)}_{1}(k)=A^{(S)}_{1\text{sym}}(k)+A^{(S)}_{1\text{break}}(k)$$

$$A^{(S)}_{1\text{break}}(k)=-\frac{\left[A^{(SD)}_0(k)\right]^2}{A^{(S)}_{-1}}+\frac{g_A^2}{f_\pi^2}\,\frac{g_A^2Mm_\pi}{16\pi f_\pi^2}\left\{\frac{571-352\ln 2}{210}-\left(1+\frac{3m_\pi^2}{2k^2}+\frac{9m_\pi^4}{16k^4}\right)\overbrace{F_\pi(\frac{k}{m_\pi})}^{1,2\pi\text{ cut}}+\frac{2m_\pi^2}{5k^2}(\ln 4-1)\right.$$
$$\left.+\frac{3m_\pi^4}{16k^4}-\frac{3}{2}\left[\frac{k}{m_\pi}+\frac{m_\pi}{k}-\frac{m_\pi^3}{8k^3}-\frac{3m_\pi^5}{16k^5}\right]\overbrace{\arctan[\frac{k}{m_\pi}]}^{2\pi\text{ cut}}+\frac{3}{16}\left(\frac{m_\pi^4}{k^4}+\frac{3m_\pi^6}{4k^6}\right)\overbrace{\ln[\frac{16(k^2+m_\pi^2)}{4k^2+m_\pi^2}]}^{1,2\pi\text{ cut}}\right\}$$

$$F_\pi(x):=\frac{1}{8x^3}\left(\underbrace{\arctan[2x]\ln[1+4x^2]-\mathrm{Im}\left[\mathrm{Li}_2[\frac{2\mathrm{i}x+1}{2\mathrm{i}x-1}]\right.}_{1\pi\text{ cut}}\left.\left.-2\,\underbrace{\mathrm{Li}_2[\frac{1}{2\mathrm{i}x-1}]}_{2\pi\text{ cut}}\right]\right)$$

(b) Whence the Hockey Stick in $^3S_1$?



Expand Wigner-breaking in Taylor:

$$A^{(S)}_{\text{sym}}(k) + \sum \frac{1}{n!}[A^{(S)}_{\text{break}}]^{(n)}(k_0)(k-k_0)^n$$

Expand about 0:

$k \lesssim \frac{m_\pi}{2}$: convergent, Wigner-breaking tiny

$k \gtrsim \frac{m_\pi}{2}$: no convergence

$\Longrightarrow$ ERE not the problem.

Sorry, no Cohen/Hansen 1999.

Expand about 1st branch point scale $\frac{m_\pi}{2}$:

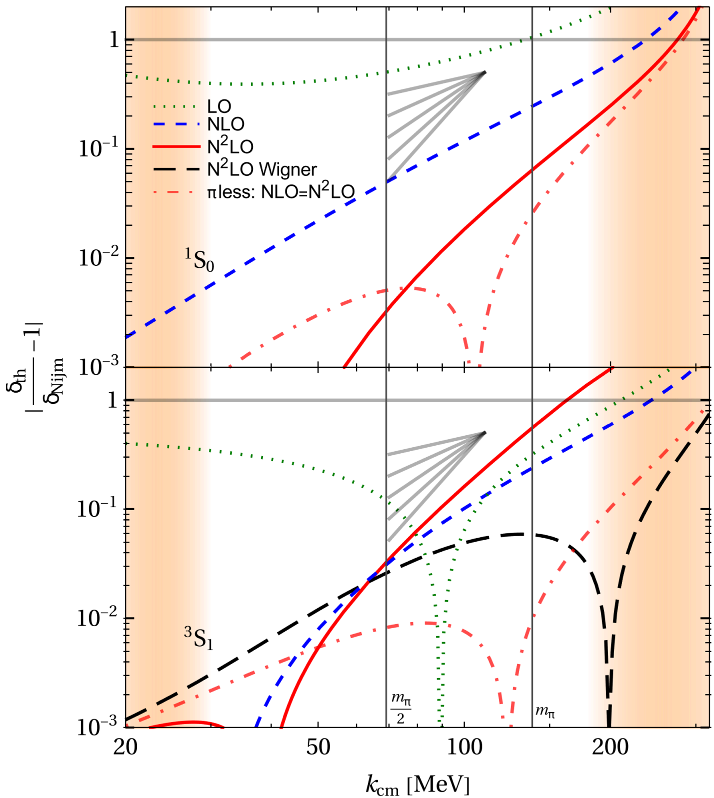
$k \lesssim \frac{m_\pi}{\sqrt{2}}$: convergent, Wigner-breaking tiny

(larger distance to branch point)

$k \lesssim \frac{3}{2}m_\pi$ (>2nd br. pt. scale): convergent

$k \gtrsim \frac{3}{2}m_\pi$: asymptotic (optimal: incl. k^4)

Landau/Páez/Bordeianu: Comp. Phys., Lepage 1997
Teng/hg [2410.09653]



$$\frac{\delta(\mathrm{N}^n\mathrm{LO}) - \delta(\mathrm{PWA})}{\delta(\mathrm{PWA})} \sim \left(\frac{k, m_\pi}{\overline{\Lambda}}\right)^{n+1}$$

at N^{n}LO with empirical breakdown scale $\overline{\Lambda}$.

$^1S_0$ and Wigner-symmetric $^3S_1$:

consistent slopes and

$\overline{\Lambda} \approx 270$ MeV $\approx \overline{\Lambda}_{NN}$ OPE scale.

Full $^3S_1$:

N^{2}LO worse than NLO for $\gtrsim 70$ MeV.

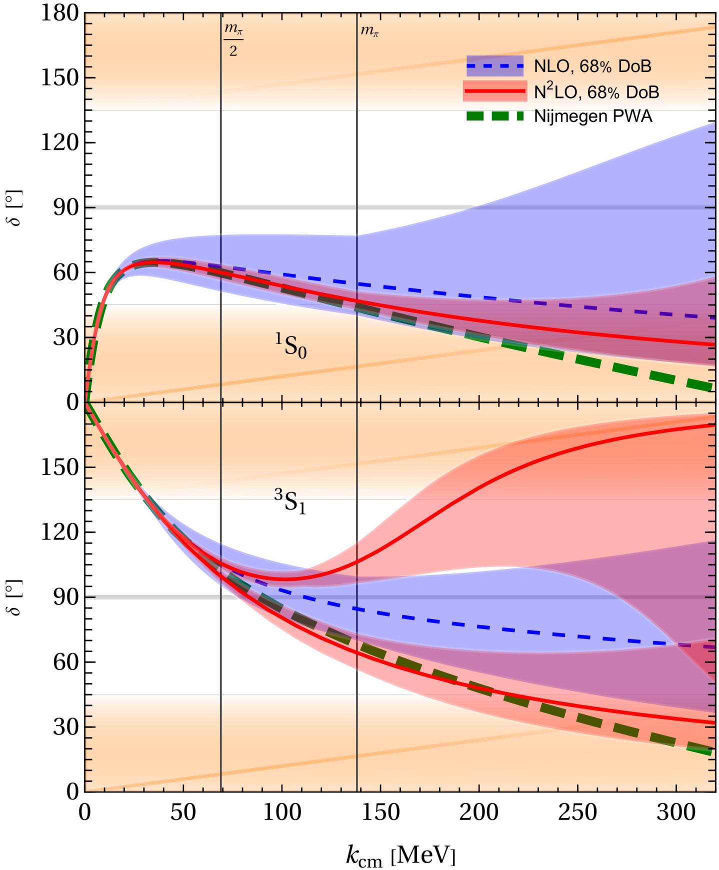
Picture obscured by points where

theory & PWA identical (“artificial zero”),

or PWA close to zero (“artificial ∞”).

(d) NLO & N^{2}LO Bayesian Truncation Uncertainties

hg/...[1203.6834], Cacciari/Houdeau [1105.5152]
BUQEYE [1506.01343], hg/...[1511.01952]
Teng/hg [2410.09653]



Apply "max" criterion to $\cot\delta$ order-by-order:

Unitarity: $k\cot\delta_{LO} = 0 \Rightarrow$ "$-ik$" sets scale.

Bayesian N^{2}LO truncation uncertainty at k:

$$\pm Q^3 \max\left\{\frac{\cot\delta_0(k)-\cot\delta_0(0)}{Q};\frac{\cot\delta_1(k)}{Q^2}\right\}$$

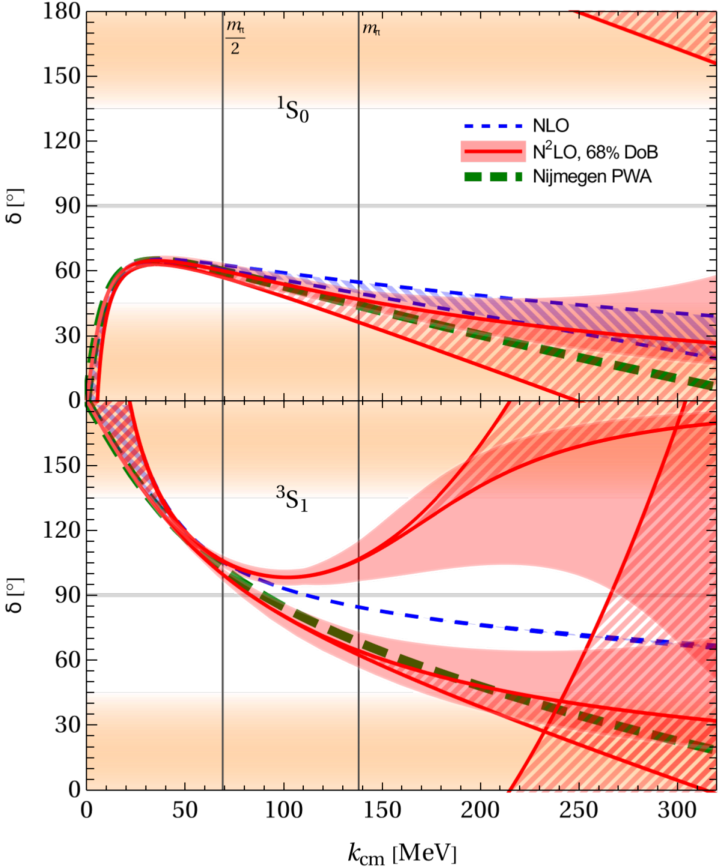
with $Q = \dfrac{\max\{k;m_\pi\}}{\overline{\Lambda}_{NN}\sim 300\ \text{MeV}}$

NLO: rescaled to 68% DoB,
assuming uniform&log-uniform prior.

Only Wigner-symmetric forms have
N^{2}LO uncertainties consistent with NLO,
and NLO&N^{2}LO consistent with PWA.

(e) Different Ways To Extract Phase Shifts at NLO and N^{2}LO

Teng/hg [2410.09653]



So far:

$$k\cot\delta = 0_{LO} + k\cot\delta_{NLO} + k\cot\delta_{N^2LO}$$
$$= -\frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2}\right]$$

is fundamental, derive $\delta(k)$ from it.

$$\xrightarrow{k\to 0} 0_{LO} + \left(-\frac{1}{a} + \frac{r}{2}k^2\right)_{N^{1+2}LO} + \mathcal{O}(k^4)$$

constructed to reproduce ERE.

Hatched: difference to

directly from amplitude KSW 1999,FMS 2000

$$\delta(k) = \frac{\pi}{2}\Big|_{-1} + \left(\frac{1}{ka} - \frac{kr}{2}\right)\Big|_0 + \dots$$

$\to\infty$ for $k\to 0$ outside Unitarity Window.

Methods agree inside Unitarity Window

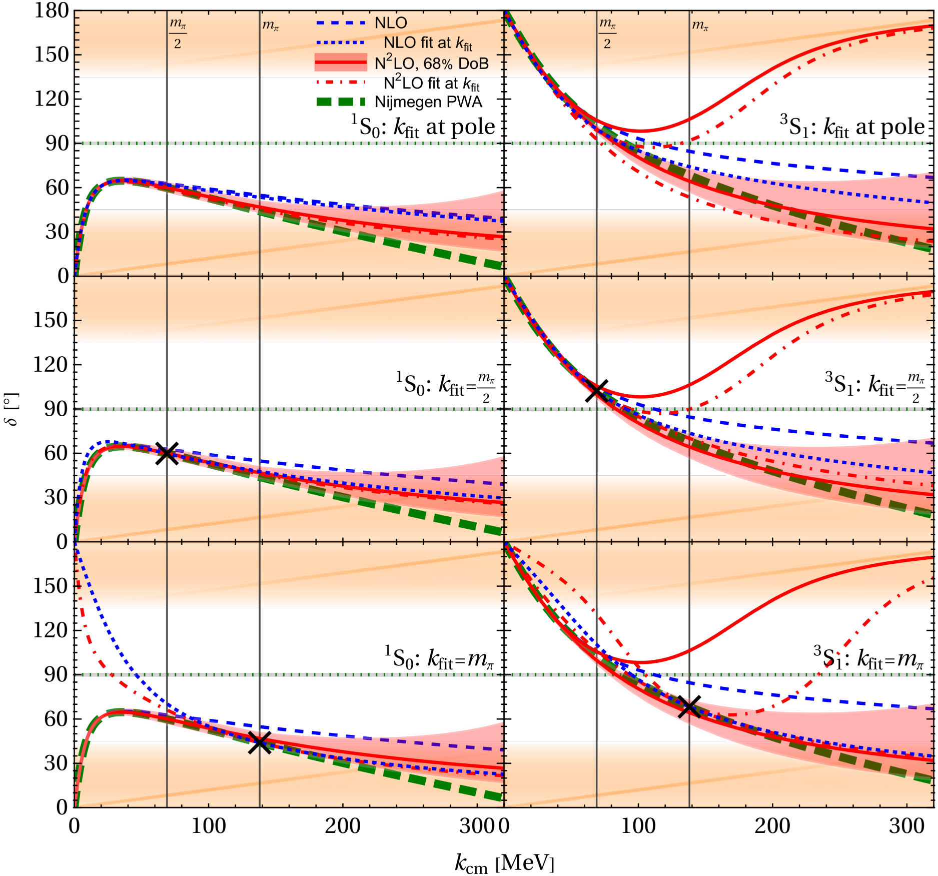
$\frac{1}{ka}, \frac{kr}{2} < 1$ (must in centre $|\cot\delta|\to 0$):

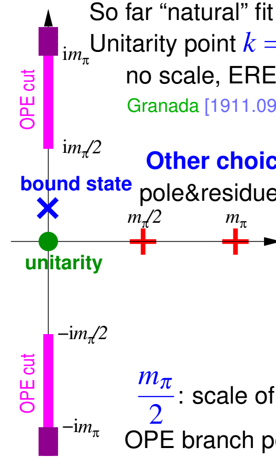
Independent assessment of

truncation uncertainty, consistent with Bayes.

(f) Different Renormalisation/Parameter-Determination Points

Teng/hg [2410.09653]





So far “natural” fit at Unitarity point $k=0$: no scale, ERE

Granada [1911.09637]

Other choices:

pole&residue: ok

$\frac{m_\pi}{2}$: scale of 1st OPE branch point

No cure to hockey-stick.

Uncertainties & breakdown scale very similar.

m_π: 2nd OPE branch point

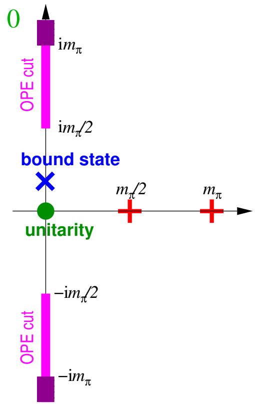
No cure to hockey-stick.

Low-k $^1S_0$ bad inside Unitarity Window.

(g) Virtual/Real Bound-State Pole Positions and Residues

$$\frac{1}{A_{-1}(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+A_0(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+A_1(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+\dots} \stackrel{!}{=} 0$$

$$\frac{1}{Z} = \mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}k}\,(k\cot\delta(k)-\mathrm{i}k)\Big|_{k=\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots}$$



For $k_{\text{fit}}=0$, pions cannot correct a, r since we force the ERE values Granada [1911.09637]

$\Longrightarrow$ pole at binding momentum $\mathrm{i}\gamma = \frac{\mathrm{i}}{a}\left(1+\frac{r}{2a}+\frac{r^2}{2a^2}+\mathcal{O}(\frac{r^3}{a^3})\right)$

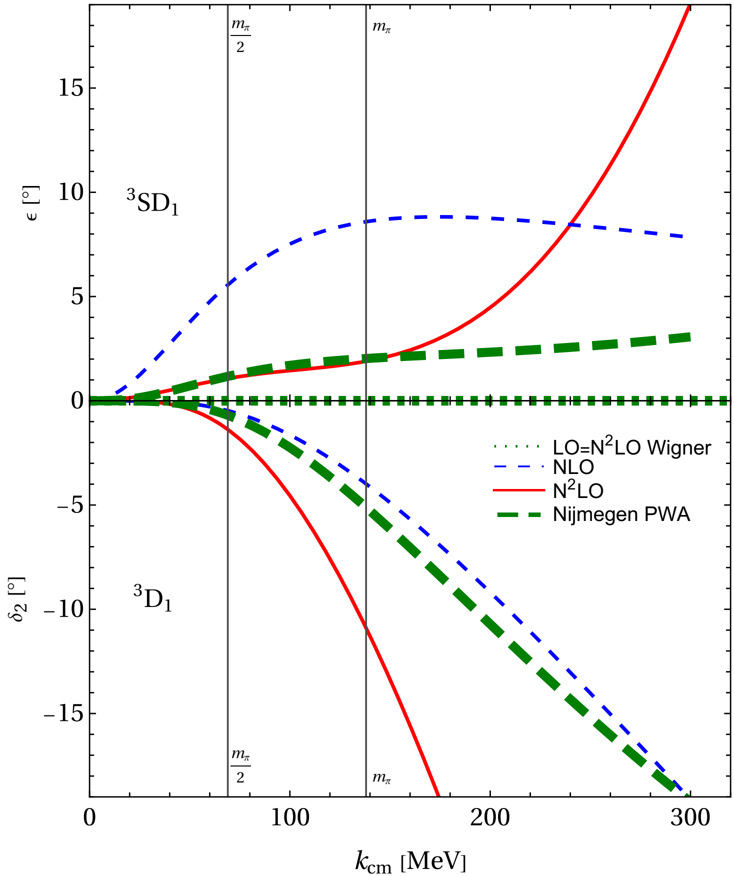
with residue $Z = 1+\frac{r}{a}+\frac{3r^2}{2a^2}+\mathcal{O}(\frac{r^3}{a^3})$.

For general k_{fit}, match to $k\cot\delta_{\text{PWA}}(k_{\text{fit}})$, $\frac{\mathrm{d}}{\mathrm{d}k}k\cot\delta_{\text{PWA}}(k_{\text{fit}})$. $\Longrightarrow$ Predict a, r.

k_{fit}		$^1S_0$ scatt. length a [fm]	eff. range r [fm]	(bind. mom.,residue) $(\gamma\,[\,\text{MeV}],Z)$	$^3S_1$ scatt. length a [fm]	eff. range r [fm]	(bind. mom.,residue) $(\gamma\,[\,\text{MeV}],Z)$
ERE		$-23.735(6)*$	$2.673(9)*$	$(-7.892, 0.9034)$	$5.435(2)*$	$1.852(2)*$	$(+47.7023, 1.689)*$
pole		-23.7104	2.7783		5.6128	2.3682	
$\frac{m_\pi}{2}$	NLO	-38.988	3.3270	$(-4.86, 0.925)$	4.9310	2.4966	$(+55., 1.9)$
	N^{2}LO	-25.428	2.7281	$(-7.34, 0.910(2))$	4.7768	2.4492	$(+57(3)., 1.9(2))$
	sym.				5.4625	1.6124	$(+43.0(5), 1.42(4))$
m_π	NLO	$+\ 9.2856$	4.2285	$(+28., 1.8)$	$3.3442\dagger$	$3.1886\dagger$	$(+114., 3.)$ ↯
	N^{2}LO	$+34.3335$	2.8956	$(+6.01, 1.10)$	$1.8376\dagger$	$3.3741\dagger$	$(+387(330)., 7(9).)$ ↯
	sym.				4.5344	1.7006	$(+54(1)., 1.5(1))$

Bayesian N^{2}LO uncertainties $*$: input ↯: cannot converge: $Q\sim\frac{1}{ka},\frac{kr}{2}\ll 1 \Longrightarrow \frac{r}{2a}\ll 1$ ↯

(h) $^3\mathrm{SD}_1$ Mixing: Full vs. Wigner



No other channels close to Unitarity Window:

$|\delta_{l\geq1}| < 25^\circ$ ($|\cot\delta_{l\geq1} > 2|$).

$^3\mathrm{SD}_1$ mixing only by tensor/Wigner-breaking.

In Unitarity Expansion very similar to FMS:

$k \gtrsim 70$ MeV:

No order-by-order convergence,

convergence to PWA elusive.

Zero by Wigner at N²LO.

Natural size at N³LO at $k \approx m_\pi$:

$90^\circ \times (Q \approx 0.5)^3 \approx 10^\circ$

$\Longleftrightarrow$ PWA: $\lesssim 10^\circ$.

$\Longrightarrow$ Not inconsistent.

SD & DD contacts at N³LO

$\Longrightarrow$ Reproducing PWA possible.

(a) Two Identical Bosons in Resummed-Range EFT

follow Habashi/Sen/Fleming/van Kolck [2007.07360], [2012.14995], [2209.08432]

If effective range $|r_0| \ll |a| \to \infty$ scattering length $\Longrightarrow r_0$ perturbative: $\mathcal{A}_{\rm LO} \propto \dfrac{1}{\frac{1}{a} + \mathrm{i}\,k}$ **"Short-Range" EFT**

If $|r_0|$ also large, both non-perturbative at LO: $\mathcal{A}_{\rm LO} \propto \dfrac{1}{\frac{1}{a} - \frac{r}{2}k^2 + \mathrm{i}\,k}$ ***"Resummed-Range EFT"***

Let $|r_0|$ set scale. $\Longrightarrow$ Momenta K in $|r_0^{-1}|$ etc. $\Longrightarrow$ Universal, i.e. depend only on dimensionless $\xi := \dfrac{2r_0}{a}$.

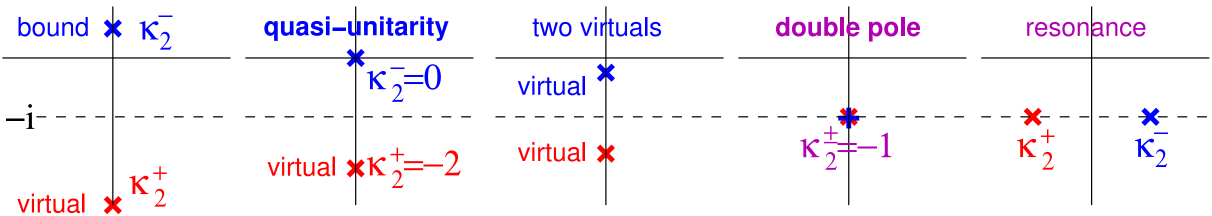
Wigner bound, causality: momentum-dependent BB contact interactions **only renormalisable for** $r_0 \leq 0$.

Fewster [hep-th/9412050], Phillips/Cohen [nucl-th/9607048], Phillips/Beane/Cohen [hep-th/9706070], [nucl-th/9709062],...

Alternative: Second-simplest $S = \dfrac{K + \mathrm{i}\,\kappa_2^-}{K - \mathrm{i}\,\kappa_2^-}\,\dfrac{K + \mathrm{i}\,\kappa_2^+}{K - \mathrm{i}\,\kappa_2^+}$: 2 poles $\mathrm{i}\,\kappa_2^\pm$ in complex-momentum plane ($B_2 = \frac{(\kappa_2^\pm)^2}{M|r_0|^2}$).

Poles of BB propagator $\dfrac{2}{\xi - K^2 - 2\mathrm{i}K}$ from $r_0 \leq 0$, a: $\kappa_2^\pm = -\left[1 \pm \sqrt{1 - \xi(a, r_0)}\right] \xrightarrow[\text{un-scale}]{r_0 \ll a} \left(-\dfrac{2}{|r_0|}, \dfrac{1}{a}\right)$

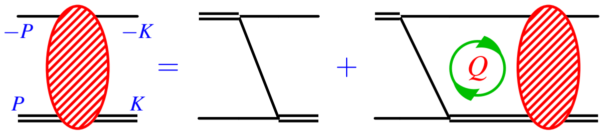
$\xi < 0$	$\xi = 0$	$0 < \xi < 1$	$\xi = 1$	$\xi > 1$
$(r_0 < 0 < a)$	$(r_0 = 0 \vee \lvert a\rvert = \infty)$	$(a < 2r_0 < 0)$	$(a = 2r_0 < 0)$	$(2r_0 < a < 0)$



We Will Deal With These Three Cases

(b) Three Identical Bosons in Resummed-Range EFT

Faddeev integral equation for half off-shell S-wave amplitude $T(\mu^2;K,Q)$, total cm energy μ^2, rescaled by $M^m|r_0^{-1}|^n$:



with $V_{\text{S-ex}}(\mu^2;P,Q) = \frac{1}{PQ}\ln\frac{P^2+Q^2+PQ-\mu^2}{Q^2+Q^2-PQ-\mu^2}$

$$T(\mu^2;K,P) = 4\pi V_{\text{S-ex}}(\mu^2;P,K) + \frac{4}{\pi}\int_0^\infty \mathrm{d}Q\; \overbrace{Q^2\,\frac{V_{\text{S-ex}}(\mu^2;P,Q)}{\xi+\frac{3Q^2-4\mu^2}{4}+\sqrt{3Q^2-4\mu^2}}}^{\text{kernel } \mathcal{K}(\mu^2;P,Q)}\; T(\mu^2;K,Q)$$

3B Binding Expectations from (BB) propagator ($\mu^2=-\kappa_3^2$)
$$\frac{1}{\xi+\underbrace{\frac{3Q^2+4\kappa_3^2}{4}}_{\text{eff. range effect}}+\sqrt{3Q^2+4\kappa_3^2}}:$$

$Q\gg\kappa_3,\xi$ $\Longrightarrow \to \frac{1}{Q^2}$ tames UV, no divergence $\Longrightarrow$ no 3BI $\Longrightarrow$ no limit cycle.

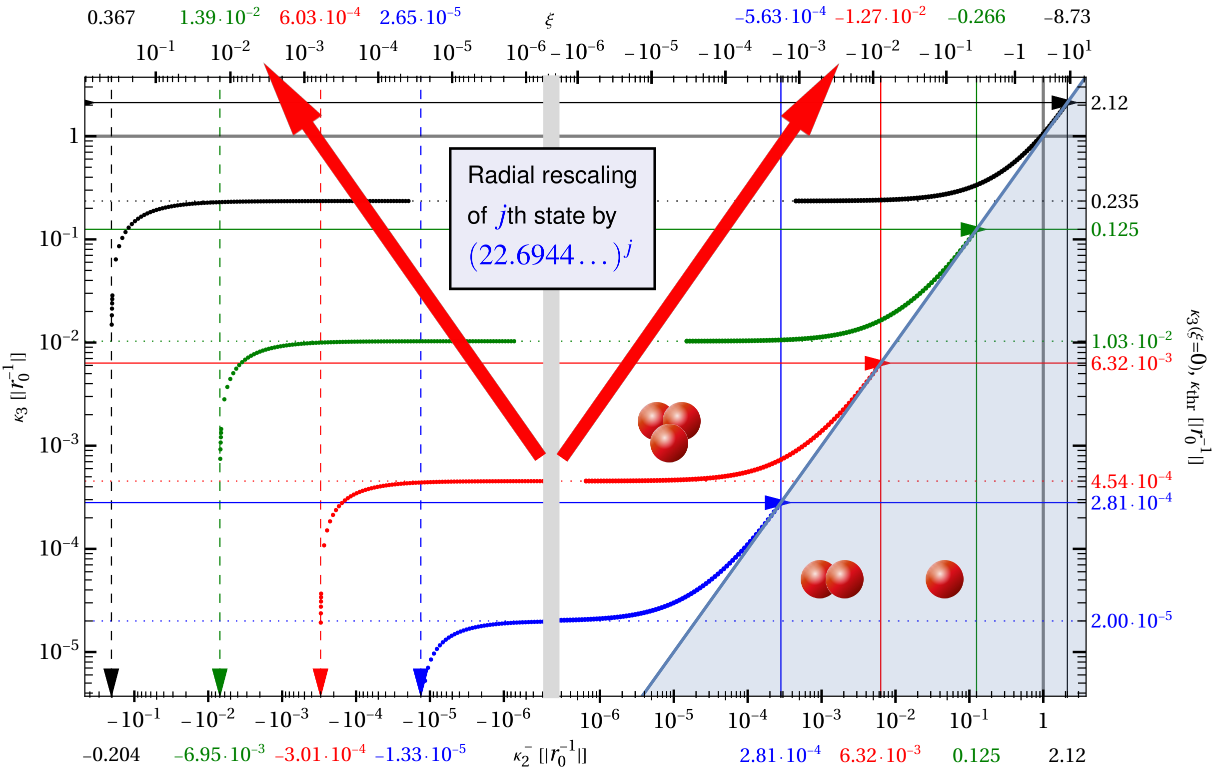
$\frac{3Q^2+4\kappa_3^2}{4}\gg\sqrt{3Q^2+4\kappa_3^2}\gtrsim 4$ $\Longrightarrow$ Quick convergence – can bound state be supported?

$\frac{3Q^2+4\kappa_3^2}{4}\gtrsim 4\gtrsim\sqrt{3Q^2+4\kappa_3^2}$ $\Longrightarrow$ All of similar size. $\Longrightarrow$ $\kappa_3\lesssim 2$ likely, effective-range effects large.

$\frac{3Q^2+4\kappa_3^2}{4}\ll\sqrt{3Q^2+4\kappa_3^2}\lesssim 4$ $\Longrightarrow \to \frac{1}{\xi+\sqrt{3Q^2+4\kappa_3^2}}$: Effective range perturbative. $\Longrightarrow$ Efimov'ish.

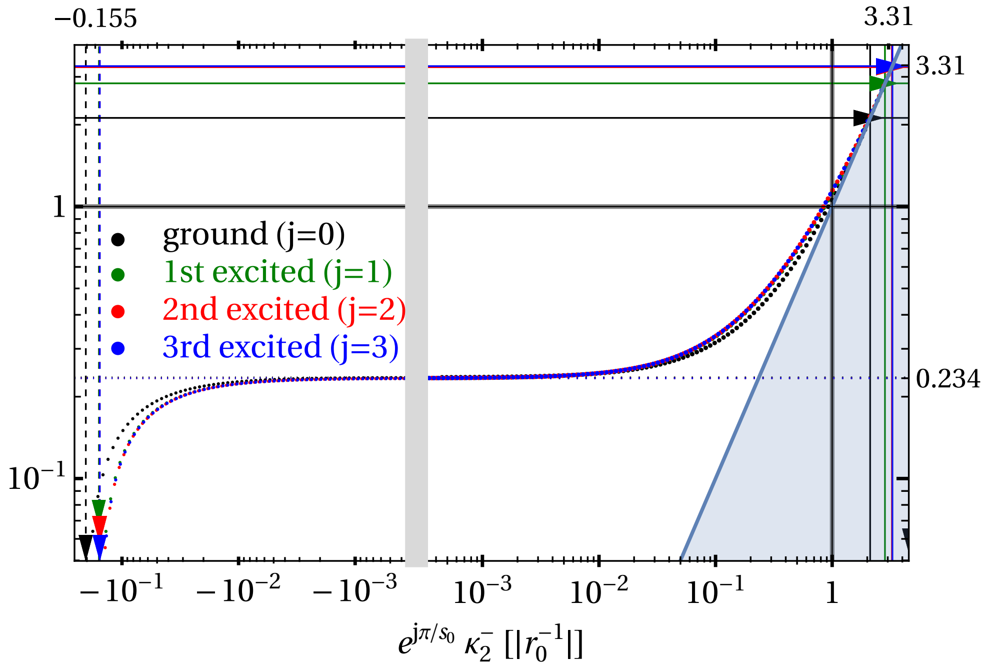
Renormalisable at LO without 3B Interaction. $\Longrightarrow$ Stable ground state, no (new) 3B parameter.

Meets expectations: $\kappa_3^{(0)} \leq 2.1 \leftrightarrow \lesssim 2$; no state for large $|\kappa_2^-|$; Efimov's Discrete Scale Invariance approximate.



(d) Are Trajectories Self-Similar?: Lay On Top Of Each Other!

Radially Rescale jth state by Efimov's Discrete Scale Invariance factor $e^{j\pi/s_0} = (22.6944\dots)^j$, $s_0 = 1.0062\dots$.



Points Of Interest relative to Efimov

Threshold: ground state 30% smaller; 1st excited state 15% *smaller*

Zero Binding: ground state 30% bigger; 1st excited state 10% *bigger*

Meets expectations: $r_0 \neq 0$ has biggest effect on lowest states.

(e) Short-Range EFT as Low-Energy ReΣRangeEFT: Fixing Efimov's Tower

From 2B propagator in 3B kernel to Efimov:

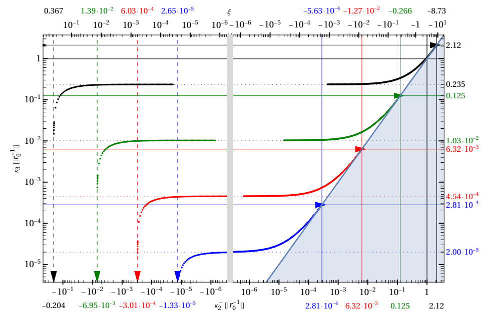
$$\frac{1}{\xi+\underbrace{\frac{1}{4}(3Q^2+4\kappa_3^2)}_{\text{eff. range}}+\sqrt{3Q^2+4\kappa_3^2}}\xrightarrow{Q\sim\kappa_3\to 0}\underbrace{\frac{1}{\xi+\sqrt{3Q^2+4\kappa_3^2}}}_{\text{Short-Range EFT}}\xrightarrow{\xi\to 0}\frac{1}{\underbrace{\sqrt{3}\,Q}_{\text{Efimov's Discrete Scale Inv.}}}$$

$\kappa_2^+(\xi=0)=-2|r_0^{-1}|\neq 0$ still sets scale of 2B. $\Longrightarrow$ Quasi-unitarity: No 2B scale invariance, except around 0.

$\Longrightarrow$ **"Near-Discrete Scale Invariance"** in 3B, Efimov tower fixed in limit $\xi=\frac{2r_0}{a}\to 0$ at $r_0<0$.

$\Longrightarrow$ (Over-accurately) predict binding energies in systems with large $|a|$ and $r_0<0$?!

state	zero binding $\kappa_2^-(\kappa_3=0)$ $[\lvert r_0^{-1}\rvert]$	quasi-unitarity $\kappa_3(\kappa_2^-=0)$ $[\lvert r_0^{-1}\rvert]$	threshold $\kappa_3\stackrel{!}{=}\kappa_2^-$ $[\lvert r_0^{-1}\rvert]$
ground	$-2.04318(6)\cdot 10^{-1}$	$2.35412(3)\cdot 10^{-1}$	$2.11862(2)$
1st exc.	$-6.9517(5)\cdot 10^{-3}$	$1.03030(5)\cdot 10^{-2}$	$1.25108(1)\cdot 10^{-1}$
2nd exc.	$-3.0144(1)\cdot 10^{-4}$	$4.53987(1)\cdot 10^{-4}$	$6.320(1)\cdot 10^{-3}$
3rd exc.	$-1.3269(2)\cdot 10^{-5}$	$2.00039(5)\cdot 10^{-5}$	$2.810(3)\cdot 10^{-4}$
$j\to\infty$	$-0.1551(1)\,\mathrm{e}^{-j\frac{\pi}{s_0}}$	$0.23381(8)\,\mathrm{e}^{-j\frac{\pi}{s_0}}$	$3.31(2)\,\mathrm{e}^{-j\frac{\pi}{s_0}}$



ReΣRangeEFT for $\xi\to 0$ encompasses SREFT. $\Longrightarrow$

Match Efimov scale Λ_* of 3BI in "hard cutoff regularisation"

$$H_0(\Lambda)\simeq -A\,\frac{\sin[s_0\ln\frac{\Lambda}{\Lambda_*}-\operatorname{arccot}s_0]}{\sin[s_0\ln\frac{\Lambda}{\Lambda_*}+\operatorname{arccot}s_0]}$$

Bedaque/ Hammer/ van Kolck [nucl-th/9809025]

to ReΣRangeEFT at same 2B binding $\kappa_2^-(\xi=0)$ per state.

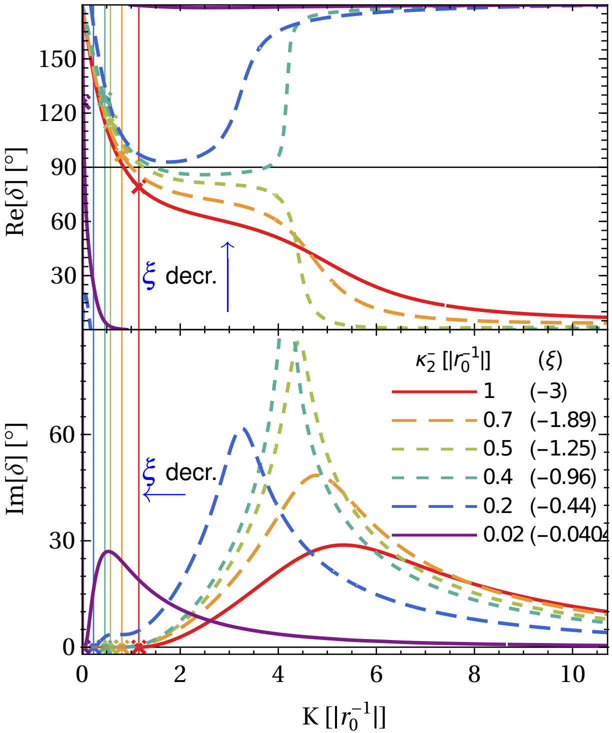
state	Efimov's Λ_* $[\lvert r_0^{-1}\rvert]$	amplitude A
ground	$0.614379(2)$	$0.87866(2)$
1st excited	$0.610223(1)$	$0.87866(1)$
2nd excited	$0.610206(1)$	$0.87866(1)$
3rd excited	$0.610206(1)$	$0.87866(1)$
$j\to\infty$	$0.610206(1)$	$0.87866(1)$

A: cf. Braaten/Kang/Platter[1101.2854]: 3 SigFig, no uncertainy

hg/van Kolck [2308.01394]

(a) Scattering B on Bound (BB) at cm Momentum K, Energy $\frac{3K^2}{4} - (\kappa_2^-)^2$

3B breakup threshold at $\sqrt{\frac{4}{3}}\,\kappa_2^- > 0$ (vertical lines) $\Longrightarrow$ Hetherington-Schick contour deformation.



$\kappa_2^- = 1$ (red) same as previous.

3B binding thresholds $\kappa_{\text{thr}}^{(0)} = 2.12\ldots$, $\kappa_{\text{thr}}^{(1)} = 0.12\ldots$

Plateau continues to flatten & grow to $K \in [1;4]$,

turns into trough at $\kappa_2^- \approx 0.43$ ($\approx$ln-mean of $\kappa_{\text{thr}}^{(0,1)}$).

$\delta(K \in [1;4]) \approx 80°$ smells unitary, but $\kappa_2 \neq 0$ bound.

"Near-Unitarity Window"

Cliff in Re correlates to huge inelasticity (peak in Im).

$\Longrightarrow$ Resonance?? (not BW+const. background)

hg/UvK investigating

$K \searrow \kappa_{\text{thr}}^{(2)}$: trough disappears, $\delta \approx 0$ except for $K \to 0$.

$\delta(K \to \infty)$ flips $0 \to 180°$, more rapid with $K \searrow 0$.

$\Longrightarrow$ Perturbative, BB propagator $\sim \frac{1}{Q^2}$ Coulomb-like.

B(BB) eff. range $r_3 > 0$ always, while BB $r_0 < 0$.