The three-nucleon parameter

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Universality in strongly-interacting systems: from QCD to atoms

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Outline

- Appearance of universal behavior
  $\rightarrow$ independence of the interaction details
  $\rightarrow$ equal long-range behavior but different short-range behavior
- Definition of the universal window for weakly bound systems
  $\rightarrow$ Weakly bound systems
  $\rightarrow$ Correlation between bound and scattering states
- Dynamics governed by a few parameters (control parameters)
  $\rightarrow$ Continuous (or discrete) scale invariance

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Interplay of two aspects

- Weakly bound systems are strongly correlated
- In the universal regime details of the interaction are not important
  $\rightarrow$ Effective interactions
  $\rightarrow$ Gaussian (or other) characterization
- Are correlated systems and universal properties compatible?

The universal window

Low energy quantities

- We consider a short-range interaction: $V(r > r_0) \to 0$

  In this case low energy means $E = k^2\hbar^2/m < \hbar^2/mr_0^2$

- In this regime the s-wave phase shift is well described by the effective range expansion up to second order

  $k \cot \delta_0 = -1/a + r_e k^2/2 + \ldots$

  with a the scattering length defined from the Schrödinger equation, $H\phi_0 = 0$

  $$\phi_0(r \to \infty) \to u_0 = 1 - a/r$$

  and r_e the effective range

  $$r_e = \frac{2}{a^2} \int_0^\infty (\phi_0^2 - u_0^2) r^2 dr$$

The universal window

The presence of a shallow bound (or virtual) state

A bound (virtual) state corresponds to the S-matrix pole:

$$S = e^{2i\delta_0} = \frac{e^{i\delta_0}}{e^{-i\delta_0}} = \frac{k\cot\delta_0 + ik}{k\cot\delta_0 - ik}$$

So the pole is the solution of $k\cot\delta_0 - ik = 0$

In general all terms in the expansion of $k\cot\delta_0$ are needed.

However, when a shallow state appears (fine tuning), we can use the expansion up to second order ($i\kappa = k$)

$$k\cot\delta_0 = -1/a + r_e k^2/2 \qquad \longrightarrow \qquad \kappa = 1/a + r_e \kappa^2/2 + \dots$$

which introduces a strict correlation between the low energy parameters.

- The S-matrix representing one shallow state, virtual or bound, is

$$S(k) = \frac{k + i/a_B}{k - i/a_B} \frac{k + i/r_B}{k - i/r_B}$$

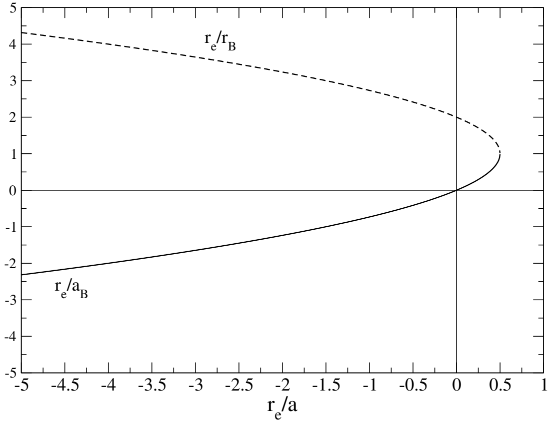
  with the energy of the system $E = \hbar^2 k^2/m$
- The energy pole is described by the energy length

$$1/\kappa = a_B \longrightarrow E_2 = -\hbar^2/m a_B^2$$

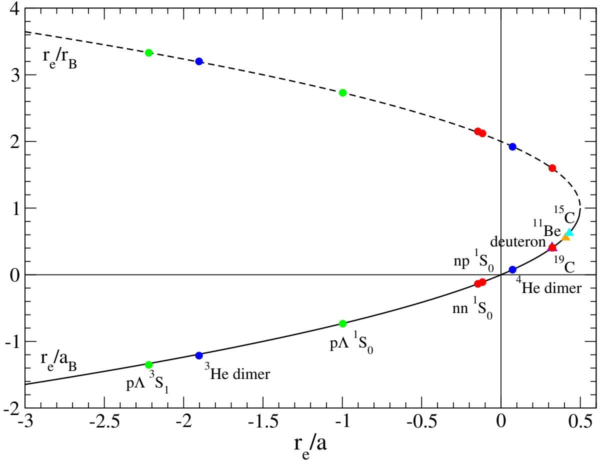
- E_2 is a bound or virtual state when $a_B > 0$ or $a_B < 0$
- the second pole is described by the length $r_B = a - a_B$
- $\frac{r_e}{a_B} = 1 - \sqrt{1 - 2r_e/a}$ $\quad\quad \rightarrow r_e/a < 0.5$
  $\frac{r_e}{r_B} = 1 + \sqrt{1 - 2r_e/a}$

The universal window: Protagonists of the story

$a \to$ scattering length, $r_e \to$ effective range
$a_B \to E = \hbar^2/ma_B^2 \to$ energy length, $r_B = a - a_B \to$ second pole



Physical systems inside the universal window



Systems are put on the figure by their experimental data

Effective description

- The S-matrix

$$S(k) = \frac{k + i/a_B}{k - i/a_B} \frac{k + i/r_B}{k - i/r_B}$$

is exactly represented by the Eckart potential:

$$V(r) = -2\frac{\hbar^2}{mr_0^2} \frac{\beta e^{-r/r_0}}{(1 + \beta e^{-r/r_0})^2}$$

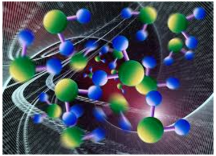
$$\begin{cases} a = 4r_0 \frac{\beta}{\beta - 1} \\ a_B = 2r_0 \frac{\beta+1}{\beta-1} \end{cases} \qquad \begin{cases} r_e = 2r_0 \frac{\beta+1}{\beta} \\ r_B = 2r_0 \end{cases}$$

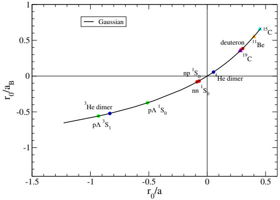
The universal window

- The figure shows the universal character of the window delimited by $-\infty < r_e/a < 0.5$ and $-\infty < r_e/a_B < 1$
- The systems can be related along the curve:
  Systems with similar values of r_e/a, or equivalently similar values of β, are related by scale transformation: $r_0 \to \lambda r_0$
- Many observables depend by the position on the curve:

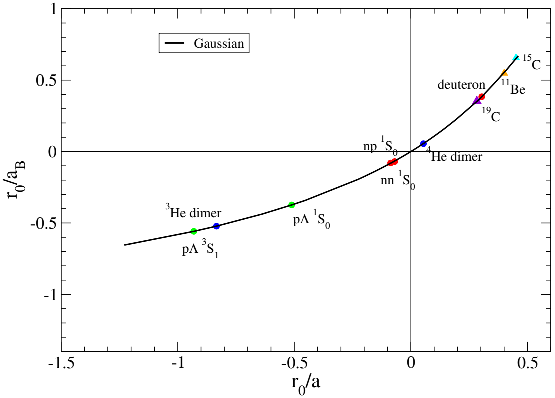
	helium dimer		deuteron	
	exp.	calc.	exp.	calc.
$-\frac{\hbar^2}{ma_B^2} \approx -\frac{\hbar^2}{mr_e^2}(1-\sqrt{1-2r_e/a})^2$	1.3 mK	1.3mK	2.224MeV	2.223 MeV
$< r^2 > \approx \frac{a^2}{8}\left[1+(\frac{r_B}{a})^2\right]$	$67.015 a_0$	$67.017 a_0$	1.967fm	1.955fm
$C_a^2 \approx \frac{2}{a_B}\frac{1}{1-r_e/a_B}$	$0.10898 a_0^{-1/2}$	$0.10899 a_0^{-1/2}$	$0.885\text{fm}^{-1/2}$	$0.883\text{fm}^{-1/2}$

The gaussian interaction as a coordinate system





The universal window in terms of the Gaussian parameters



$$V(r) = -\frac{\hbar^2}{m r_0^2}\beta e^{-(r/r_0)^2}$$

Effective description

- System inside the window have been described using different EFT frameworks
- The nuclear system is currently described using chiral potentials or using pionless EFT
- Atomic helium has been extensively studied using potentials models (Aziz potentials, TTY potential, etc) and also using contact EFT
- Halo nuclei are currently studied using potential models and also Halo EFT
- Hadron systems as $N - \Lambda$ and hypernuclei are studied using potential models and also using chiral or contact EFT
- The above figure suggests a common description of the systems on the plot based on an effective potential

$$V_{LO} = V[\beta(a, a_B, r_e), r_0(a, a_B, r_e)]$$

- We consider this description a LO description

Examples

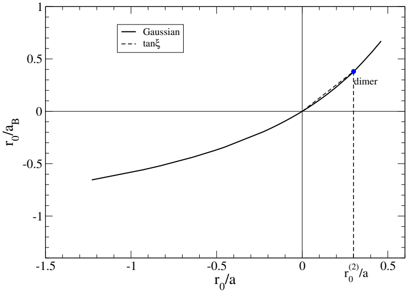
Contact EFT for two bosons

- At LO the potential is $V_\Lambda(r) = C_\Lambda \frac{\Lambda^3}{\pi^{3/2}} e^{-\Lambda^2 r^2/4}$
- The coupling constant C_Λ is fixed to reproduce the scateering length for increasing values of the cutoff $\Lambda \to \infty$

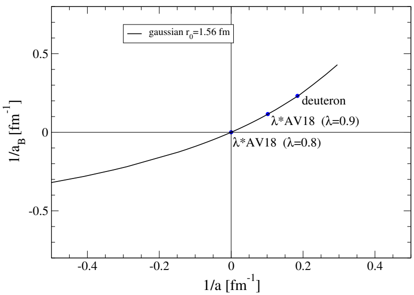
Effective description using a Gaussian potential

- The effective potential is $V_{r_0}(r) = -\frac{\hbar^2}{m r_0^2} \beta e^{-r^2/r_0^2}$
- The coupling constant β and the range r_0 are determined from the scattering length a and the energy length a_B
- For one value of the cutoff $\Lambda = 2/r_0$ the two descriptions are equivalent
- The quantity r_0 defines the two-body scale (to be discussed below)

The two-body scale: assigning dimensions



$\longrightarrow$

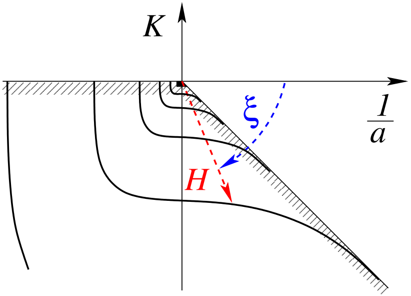


$\tan\xi = \frac{a}{a_B}$ $\qquad r_0^{(2)} = 1.56\,\text{fm}$

The $T = 0$ np plot $(r_0 = 1.56\,\text{fm})$

$$V_{r_0}(r) = -\frac{\hbar^2}{m r_0^2}\beta e^{-r^2/r_0^2}$$

Indications from Efimov Physics: K_*, the three-body parameter



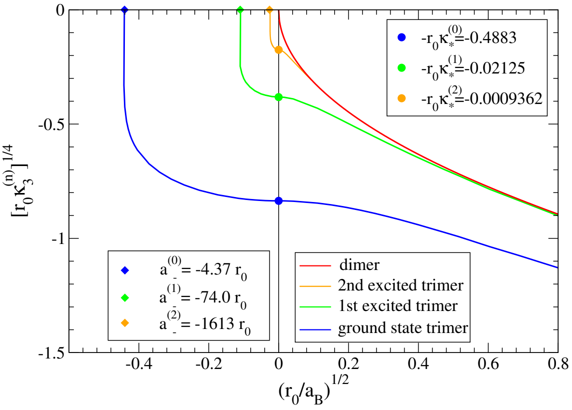
In contact interactions $a = a_B$. The three-body sector is scale invariant and K_*, the binding momentum at the unitary limit, fixes the branch in which the system is located

$Ka = \tan\xi$ and $H = K_* e^{\Delta(\xi)/2s_0}$

with $\Delta(\xi)$ the universal function and s_0 a universal number

The three-body parameter using the gaussian characterization

The case of three bosons: $V = \sum_{ij} V_0 e^{-(r_{ij}/r_0)^2}$



$$a_-^{(0)}\kappa_*^{(0)} = -2.14$$
$$a_-^{(1)}\kappa_*^{(1)} = -1.57$$
$$a_-^{(2)}\kappa_*^{(2)} = -1.51$$

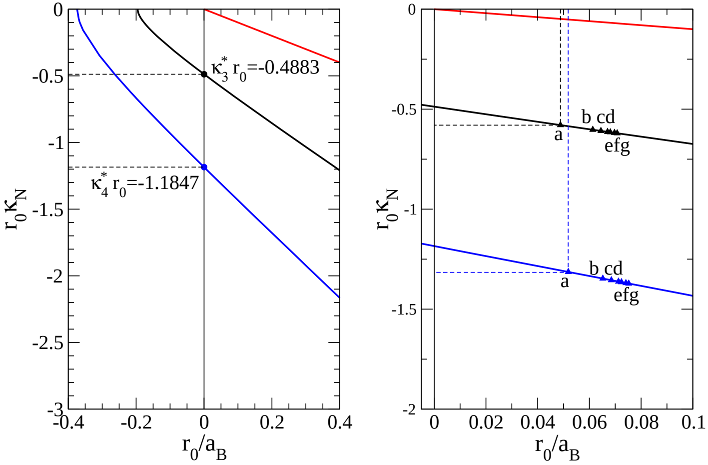
The three-body parameter: the helium case

Potential	E_2	E_3	E_4	$r_0^{(3)}(a_0)$	$r_0^{(4)}(a_0)$
a: HFD-HE2	0.8301	117.2	535.6	11.146	11.840
b: LM2M2	1.3094	126.5	559.2	11.150	11.853
c: HFD-B3-FCH	1.4475	129.0	566.1	11.148	11.853
d: CCSAPT	1.5643	131.0	571.7	11.149	11.851
e: PCKLJS	1.6154	131.8	573.9	11.148	11.852
f: HFD-B	1.6921	133.1	577.3	11.149	11.854
g: SAPT96	1.7443	134.0	580.0	11.147	11.850

Table: Dimer, trimer and tetramer energies (in mK) for the indicated potential

Energies are from E. Hiyama and M. Kamimura, Phys.Rev.A 85, 062595 (2012), A. Kievsky et al, Phys.Rev.A 96, 040501(R) (2017), and P. Barletta and A. Kievsky, Phys.Rev.A 64, 042514 (2001)

The three-body parameter: the helium case



The helium three-body parameter

Scaling each helium-helium potential $V_\lambda = \lambda V_{He-He}(r)$ such that $E_2(\lambda) = 0$

$$E_3^* = -\frac{\hbar^2}{m} K_*^2 = -\frac{\hbar^2}{m} \frac{(0.4883)^2}{[r_0^{(3)}]^2} \approx 83\,\mathrm{mK}$$

$$E_4^* = -\frac{\hbar^2}{m} K_*^2 = -\frac{\hbar^2}{m} \frac{(1.1847)^2}{[r_0^{(4)}]^2} \approx 433\,\mathrm{mK}$$

All the Helium-Helium potentials taken into account share the same three-body parameter and the same four-body parameter!

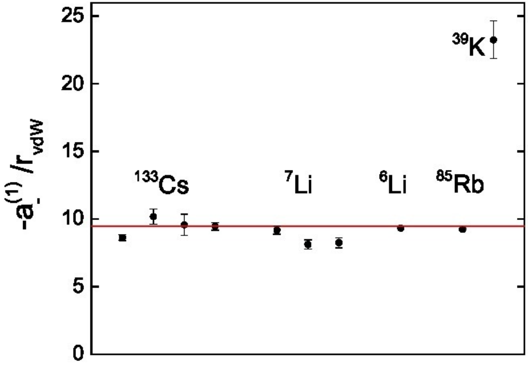
They were not explicitely included in the determination of the potentials!

$r_0^{(3)}$ and $r_0^{(4)}$ are the three-body and the four-body scale.
What information is encoded here?

The gaussian scale

- Along the gaussian path:
- The Helium dimer can move along the curve described by a gaussian potential $V_0 e^{-(r/r_0^{(2)})^2}$ with $r_0^{(2)} \approx 10\, a_0$.
- The Helium trimer can move along the curve described by a gaussian potential $V_0 \sum_{i<j} e^{-(r_{ij}/r_0^{(N)})^2}$ with $r_0^{(3)} \approx 11.15\, a_0$.
- The Helium tetramer can move along the curve described by a gaussian potential $V_0 \sum_{i<j} e^{-(r_{ij}/r_0^{(N)})^2}$ with $r_0^{(4)} \approx 11.85\, a_0$.
- The different ranges indicate how the clusters pack: the gaussian range $\rightarrow$ the packing scale

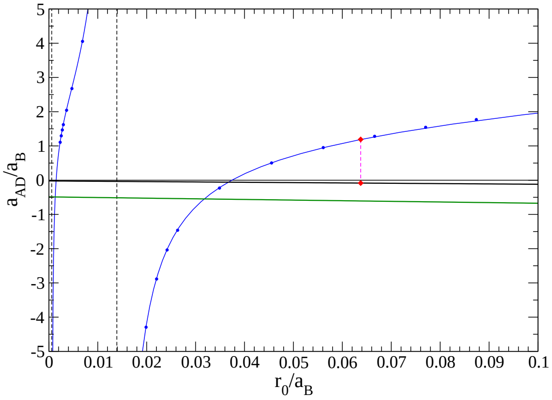
van der Waals universality



For Helium $r_0^{(3)} \approx 11.15\, a_0$ and $r_{vdW} = 5.08\, a_0 \rightarrow a_- = -4.37 r_0 \rightarrow a_-/r_{vdW} = -9.6$

Predictions: the atom-dimer scattering length

$a_{AD}/a_B = d_1 + d_2 \tan[s_0 \ln(K_* r_0 (a_B/r_0) + \Gamma_3) + d_3$ (A. Deltuva et al., PRC 102, 064001 (2020))



$a_{AD}/a_B = 1.19 \rightarrow a_{AD} = 217 a_0$ with the LM2M2 potential the value 218 a_0 is obtained

Unifying the scales: the LO potential

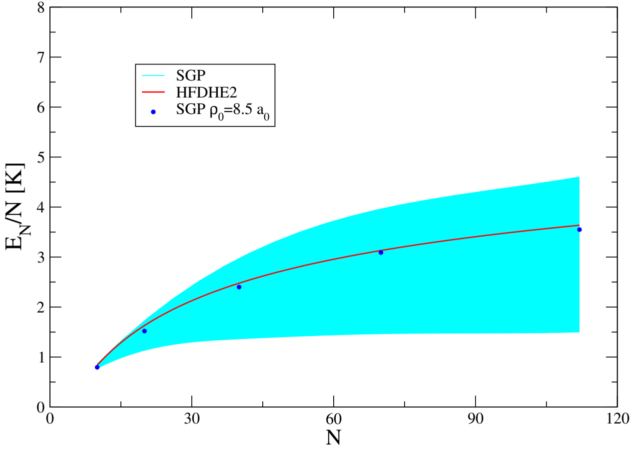
- the two body scale describes the long range physics
- the three- and four body scale describe the packing
- we introduce the following LO potential

$$V_{LO} = V_0 \sum_{i<j} e^{-(r_{ij}/r_0^{(2)})^2} + W_0 \sum_{i<j<k} e^{-(\rho_{ijk}/\rho_0)^2}$$

with W_0, ρ_0 fixing the three- and four-body parameters

- predictions:
- low energy scattering: atom-dimer, atom-trimer, etc
- the E_N/N behavior up to the liquid

Unifying the scales: the LO potential

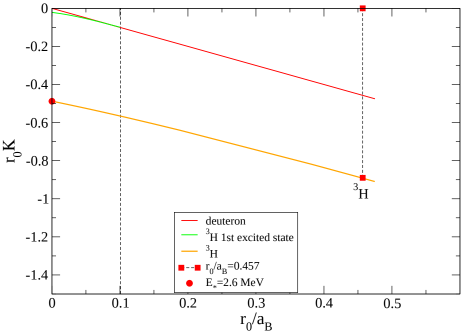


The three-nucleon system: A gaussian characterization

$$V(1,2,3)=\sum_{i<j}V(i,j)=\sum_{i<j}\left(V_0 e^{-(r/r_0)^2}\mathcal{P}_{01}+V_1 e^{-(r/r_0)^2}\mathcal{P}_{10}\right)$$

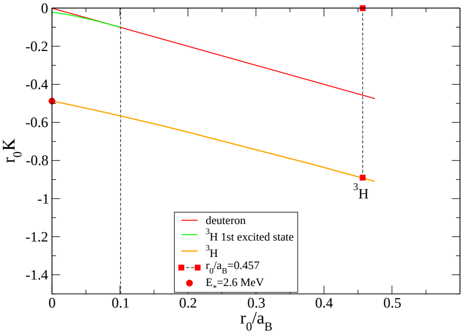
The *np* scattering lengths are: ${}^0a_{np}=-23.74\,\mathrm{fm}$ and ${}^1a_{np}=5.419\,\mathrm{fm}$

To construct the plot we follow the nuclear path defined as ${}^0a_{np}/{}^1a_{np}=-4.38$



The three-nucleon system: A gaussian characterization

$$V(1,2,3) = \sum_{i<j} V(i,j) = \sum_{i<j} \left(V_0 e^{-(r/r_0)^2} \mathcal{P}_{01} + V_1 e^{-(r/r_0)^2} \mathcal{P}_{10} \right)$$



From the plot $r_0/a_B = 0.457 \rightarrow r_0 = 1.97\,\mathrm{fm}$ and $K_* = 0.4883/r_0 = 0.247\,\mathrm{fm}^{-1}$

The nuclear three-body parameter

- The two-nucleon potential alone does not determine the three-body parameter

Potential	E_2(MeV)	E_3(MeV)	E_4(MeV)	$r_0^{(3)}$ (fm)	$r_0^{(4)}$ (fm)
AV18	2.224	7.62	24.23	2.43	2.45
N3LO	2.224	7.85	25.38	2.15	2.20
CD Bonn	2.224	8.00	26.20	2.10	2.15
AV18+UR	2.224	8.48	28.45	1.97	2.07
Exp.	2.224	8.48	28.30	1.97	2.08

- A three-body force is an intrinsic ingredient of the nuclear potential
- Let us scale the NN potential bringing it to the unitary limit

$$V_\lambda = \sum_{i<j} V^\lambda_{AV18}(i,j) = \sum_{i<j} \sum_{ST} \lambda_{ST} V_{ST}(i,j) \qquad \lambda_{0,1} = 1.0616, \lambda_{1,0} = 0.8$$

- with these value $E_d = 0$ and ${}^1a_{np} \to \infty$

The nuclear three-body parameter

	physical point			unitary limit		
Potential	E_2(MeV)	E_3(MeV)	E_4(MeV)	E_2(MeV)	E_3(MeV)	E_4(MeV)
AV18	2.224	7.62	24.9	0	2.27	11.7
AV18+UR	2.224	8.48	29.2	0	2.65	14.1
Gaussian	2.224	8.48	29.1	0	2.53	13.7

- The nuclear potential follows the gaussian path
- the gaussian characterization helps us to determine the important scales:
  - The two-body scale: ${}^1r_0 = 1.83\,\text{fm}$, ${}^3r_0 = 1.55\,\text{fm}$
  - The three-body scale: $r_0^{(3)} = 1.97$
  - The four-body scale: $r_0^{(4)} = 2.08$

- Accordingly we can define the nuclear three- and four-body parameter:
- For $A = 3$: $r_0^{(3)} = 1.97\,\text{fm} \longrightarrow E_*^3 = \frac{\hbar^2}{m}\frac{0.4883}{(r_0^{(3)})^2} = 2.53 \pm 0.1\,\text{MeV}$
- For $A = 4$: $r_0^{(4)} = 2.08\,\text{fm} \longrightarrow E_*^4 = \frac{\hbar^2}{m}\frac{1.1847}{(r_0^{(4)})^2} = 13.7 \pm 0.5\,\text{MeV}$

Unifying the scales: the LO potential

- The two-nucleon LO effective potential should describe the long-range physics. It could have a simply gaussian (or other characteristic form)

$$V_{LO}(2N) = V_0 e^{-(r/r_0)^2} \mathcal{P}_{01} + V_1 e^{-(r/r_0)^2} \mathcal{P}_{10}$$

it could be taken from chiral-EFT fixing $\Lambda = 2/r_0$

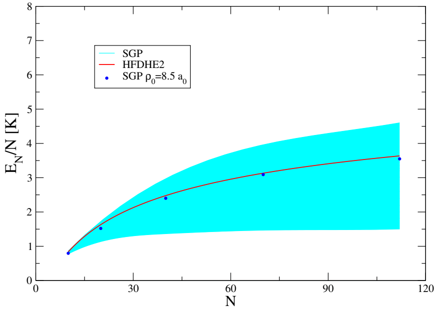
$$V_{LO}(2N) = C_\Lambda^0 e^{-\Lambda^2 r^2/4} \mathcal{P}_{01} + C_\Lambda^1 e^{-\Lambda^2 r^2/4} \mathcal{P}_{10} + V_\pi$$

- At LO a three-nucleon force should be also included. The c_E term, considered in chiral EFT at N2LO, has to be promoted to LO.

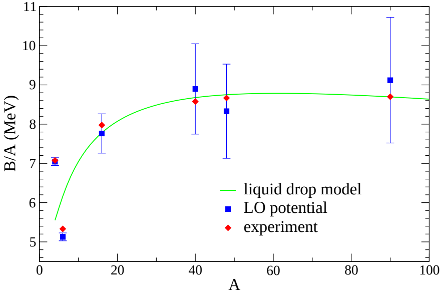
$$V_{LO}(3N) = W_0 \sum_{i<j<k} e^{-(r_{ij}/\rho_0)^2} e^{-(r_{ik}/\rho_0)^2}$$

with W_0, ρ_0 fixed to reproduce $E(^3\mathrm{H})$ and $E(^4\mathrm{He})$

E/A curves in helium and in nuclei at the physical point



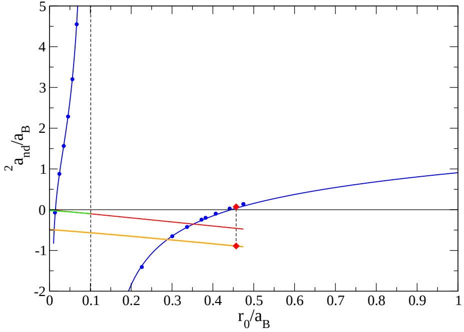
A.K et al, Phys.Rev.A96, 040501(R)(2017)



R. Schiavilla et al. Phys.Rev.C103, 054003 (2021)

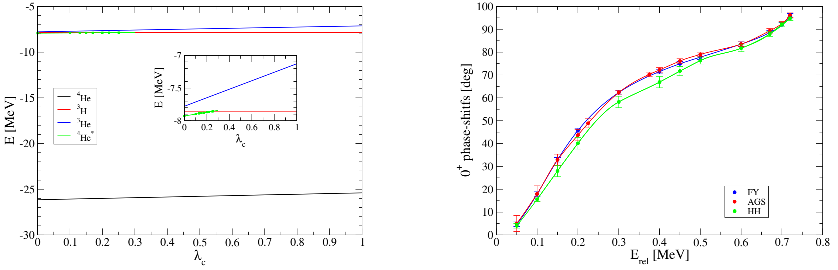
Predictions: the doublet neutron-deuteron scattering length

The doublet ${}^2a_{nd} \approx 0.65$ fm. Its very low value is not easy to understand. However the gaussian characterization of the universal window shows that at the place in which the ^{3}H is located the ${}^2a_{nd}/a_B$ function is going through zero. A detailed analysis predict ${}^2a_{nd} \approx 0.45$ fm.



Predictions: the α particle excited state

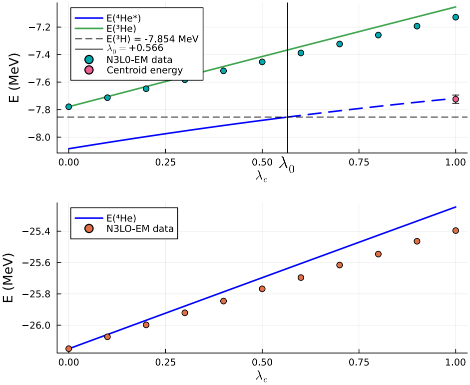
The 0_2^+ state of ^{4}He becomes a true ground state turning off the Coulomb force: $V_C(r) = \epsilon \frac{e^2}{r}$. At $\epsilon = 1$ we have studied the $p - {}^3$H 0^+ phase-shift. The resonance parameters are obtained from the S-matrix complex pole $E_R + i\Gamma/2$.



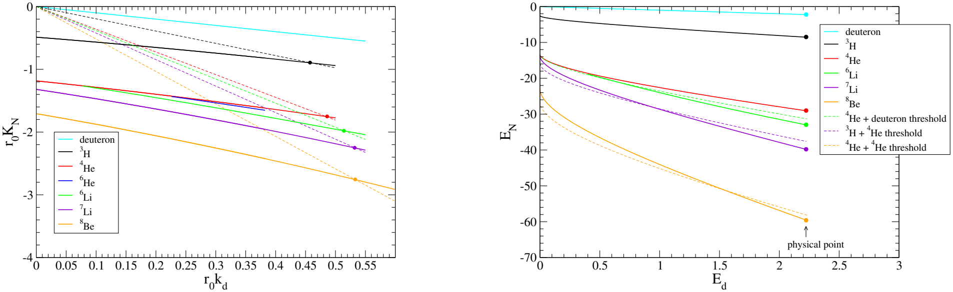
Using the N3LO-EM potential, we have obtained $E_R = 0.13(2)$ MeV and $\Gamma = 0.40(15)$ MeV compared to $E_R^{exp} = 0.39(2)$ MeV and $\Gamma = 0.50(5)$ MeV (P.-Y. Duerinck et al., submitted to PRC)

The α particle excited state: universal concepts

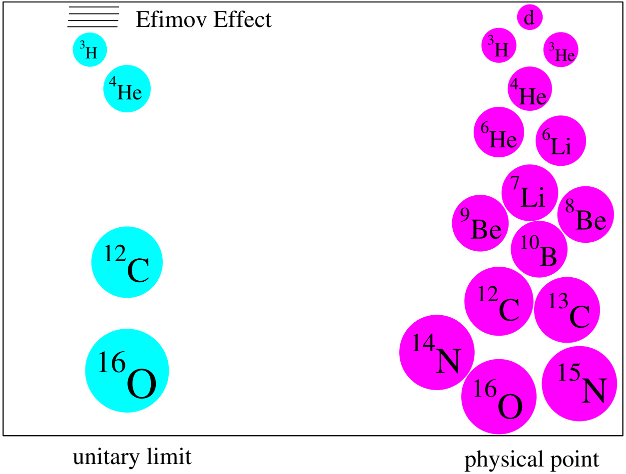
Using the $V_{LO}(2N) + V_{LO}(3N)$ fixed to the N3LO-EM and the ACCC method we obtain



Gaussian characterization of the universal window for $A \leq 8$



Mirroring the nuclear chart at the unitary limit



Conclusions

- Based on the simplet S-matrix having one bound (or virtual) state we have defined the universal window
- Many different systems are located inside the window
- The Eckart (or Gaussian) potential representing the one-state S-matrix can be used to describe these systems. This characterization can be considered a LO description.
- The Gaussian characterization of the universal window allows to assign dimensions to the different system. We have shown the two-, three- and four-body scales.
- The three-body scale is related to the three-body parameter: The energy of the triton at the unitary limit is linked to the physical point following the Gaussian path.
- At the unitary point $E_d = 0$ and ^{3}H shows the Efimov effect
- Using the gaussian path we suggest that the physical point and the unitary point share the same microscopic theory!