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Finite Volume Thermodynamics in a Uniform Magnetic Field

Perspectives from Chiral Perturbation Theory

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Outline & References

- Chiral Perturbation Theory *[Gasser, Leutwyler, Ann. Phys.* ***158*** *(1984) 142, Nucl. Phys. B* ***250*** *(1985) 465]*
  - Introduction & Discussion of Ingredients
  - Observables in a Magnetic Field
    - Chiral Condensate, Magnetization of the Vacuum, Topological Susceptibility
- Finite Volume Corrections to QCD Observables in Electromagnetic Fields
  - *Case I.* Magnetic Field *[PA, Tiburzi, Phys. Rev. D,* ***107****, 094504 (2023)]*
  - *Case II.* Impact of Parallel (Euclidean) Electric and Magnetic Field *[PA, work in progress]*
  - *Case III.* Minkowski Electric Field at Finite Temperature in Finite Volume *[see Lattice study in Endrodi, Marko, PRD 109 (2024)]*

Chiral Perturbation Theory - Ingredients

- Chiral Perturbation Theory is the low-energy *effective theory* of QCD
- In *massless QCD*, left-and-right-handed quarks do not mix and therefore can be rotated independently $\psi_L \to L\,\psi_L$ and $\psi_R \to R\,\psi_R$. The symmetry of the QCD partition function is
    - $U(1)_V \times SU(2)_V \times SU(2)_A \to U(1)_V \times SU(2)_V$ due to the formation of the chiral condensate, $\langle \overline{\psi}\psi \rangle \neq 0$
        - The $U(1)_A : \psi \to e^{i\theta_5\gamma_5}\,\psi$ subgroup of $U(2)_L \times U(2)_R$ is broken by the partition function.
- The building blocks of the effective theory are
    - *Goldstone Manifold.* $\Sigma \in SU(2)_L \times SU(2)_R / SU(2)_V$
    - *Scalar-Pseudoscalar Source.* $\chi = S + i\,P$
    - *Electromagnetic Gauge Field.* $\partial_\mu \Sigma \to D_\mu \Sigma = \partial_\mu \Sigma + iA_\mu\,[\,Q\,,\Sigma\,]$
- The results are *model-independent* for $p/(4\pi F_\pi) \ll 1$ with $p = -\,i\partial\,,\sqrt{eB}\,,m_\pi$

Chiral Perturbation Theory - Ingredients

- Goldstone manifold, $\Sigma \in SU(2)_L \times SU(2)_R / SU(2)_V$
- The fluctuations are axial since the chiral condensate $\langle \overline{\psi}\psi \rangle$ is only invariant if $L = R \equiv V$.

$$1 \rightarrow \Sigma = 1 + \cdots$$

$$\Sigma \equiv U 1 U \in SU(2)_L \times SU(2)_R / SU(2)_V \qquad U = \exp\left(\frac{i\Phi}{2F}\right) \qquad \Phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \pi^0 \end{pmatrix}$$

- The construction of the Lagrangian involves building a chirally invariant theory under $\Sigma \rightarrow L \Sigma R^\dagger$.

$$\mathscr{L}_2 = \frac{1}{4}F^2 \mathrm{Tr}[\,\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\,] + \frac{1}{4}F^2 \mathrm{Tr}[\,\chi \Sigma^\dagger + \chi^\dagger \Sigma\,]$$

- The mass term of QCD $\Delta\mathscr{L} = m_q \overline{\psi}\,\psi$ also explicitly breaks chiral symmetry $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$. Chiral symmetry can be preserved through the introduction of spurion sources in the QCD Lagrangian that transform accordingly.
  - The leading order mass term is by introducing sources in the QCD Lagrangian that preserves chiral symmetry $\Delta\mathscr{L} = \overline{\psi}_L\,\chi\,\psi_R + \overline{\psi}_R\,\chi^\dagger\,\psi_L$ with $\chi \rightarrow L\chi R^\dagger$.

Chiral Perturbation Theory - Ingredients

- Terms that contribute at higher order terms are generated similarly

$$\mathscr{L}_4 = \frac{\ell_3 + \ell_4}{16}\mathrm{Tr}\left[\chi\Sigma^\dagger + \chi^\dagger\Sigma\right]^2 + \frac{h_1 + h_3 - \ell_4}{4}\mathrm{Tr}\left(\chi\chi^\dagger\right) - \frac{\ell_7}{16}\left[\mathrm{Tr}\left(\chi\Sigma^\dagger - \chi^\dagger\Sigma\right)\right]^2$$
$$+\frac{h_1 - h_3 - \ell_4}{16}\left\{\left[\mathrm{Tr}\left(\chi\Sigma^\dagger + \chi^\dagger\Sigma\right)\right]^2 + \left[\mathrm{Tr}\left(\chi\Sigma^\dagger - \chi^\dagger\Sigma\right)\right]^2 - 2\mathrm{Tr}\left(\Sigma\chi^\dagger\Sigma\chi^\dagger + \chi\Sigma^\dagger\chi\Sigma^\dagger\right)\right\}$$

  - ℓ_i and h_i are low and high energy constants contain divergences and in terms of the MS-bar scale Λ are

$$\ell_i = \ell_i^r(\Lambda) - \frac{\gamma_i\Lambda^{-2\varepsilon}}{2\,(4\pi)^2}\left[\frac{1}{\epsilon} + 1\right] \qquad \Lambda\frac{d}{d\Lambda}\ell_i^r = -\frac{\gamma_i}{(4\pi)^2} \qquad \ell_i^r(\Lambda) = \frac{\gamma_i}{2(4\pi)^2}\left[\bar{\ell}_i + \log\frac{m^2}{\Lambda^2}\right]$$

- In a magnetic field, further counter-terms are required

$$\Delta\mathscr{L}_4 = -\frac{4h_2 + \ell_5}{2}\mathrm{Tr}\left[F^R_{\mu\nu}F^{R\mu\nu} + F^L_{\mu\nu}F^{L\mu\nu}\right] + \ell_5\mathrm{Tr}\left[\Sigma F^L_{\mu\nu}\Sigma^\dagger F^{R\mu\nu}\right] \qquad F^R_{\mu\nu} = F^L_{\mu\nu} = -e\frac{\tau_3}{2}F_{\mu\nu}$$

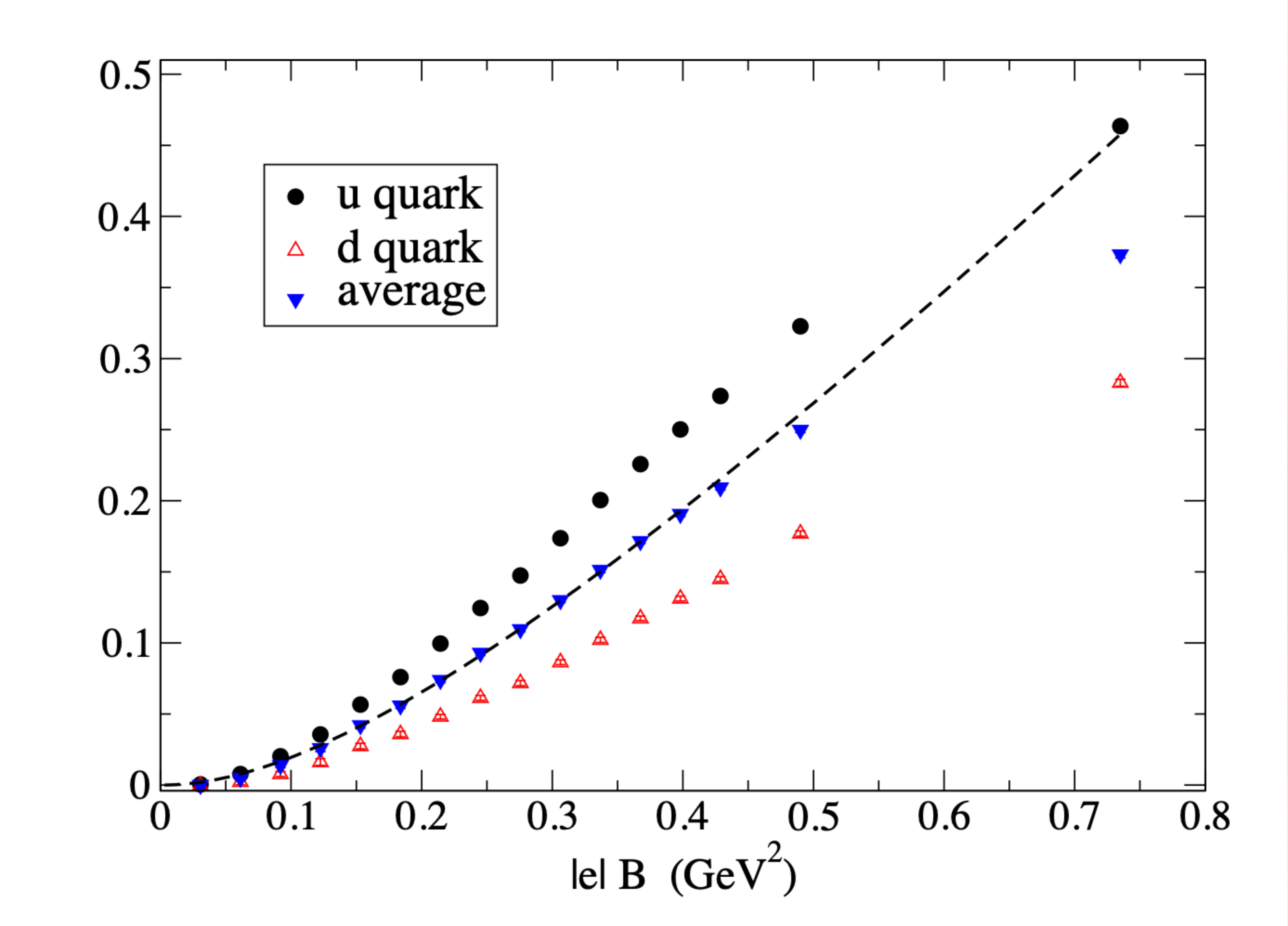
Magnetic Catalysis

- The relative shift in the chiral condensate monotonously increases as a function of eB

$$\frac{\Delta\Sigma(H)}{\Sigma(0)} = -\int_0^\infty \frac{ds}{(4\pi s)^2} e^{-m_\pi^2 s} \left[\frac{eBs}{\sinh eBs} - 1 \right]$$

*Cohen, McGady, Werbos, PRC **76**, 2007*

- Lattice QCD observes the same qualitative behavior and has an extended range of validity



D'Elia, Lattice QCD Simulations in External Background Fields (2013)

Free Energy in the θ-vacuum

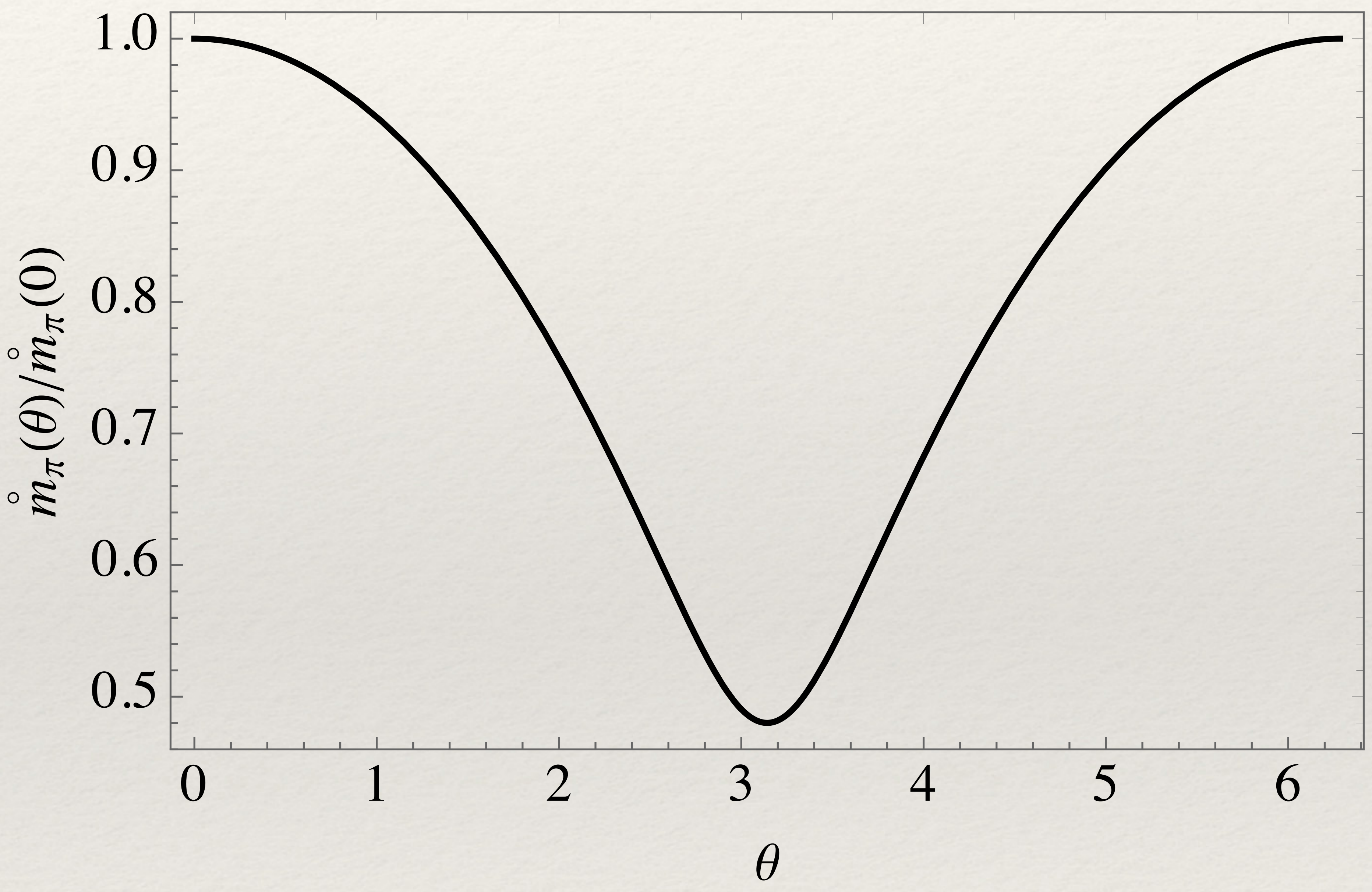
- The sum over Landau levels produces an effective potential (in terms of a proper time integral s)

$$\widetilde{\mathcal{F}}_B(\theta) = \frac{1}{2}B_R^2 - \int_0^\infty \frac{ds}{(4\pi)^2 s^3} e^{-\mathring{m}_\pi^2(\theta)\, s}\left[\frac{eBs}{\sinh eBs} - 1 + \frac{(eBs)^2}{6}\right]$$

Schwinger, PR ***82****, 1951*

- The quadratic contribution renormalizes the external field (with eB *invariant*)

$$B_R = Z_B B \qquad Z_B = 1 + 4e^2 h_2^r + \frac{e^2}{6(4\pi)^2}\left(\log\frac{\Lambda^2}{\mathring{m}_\pi^2(\theta)} - 1\right)$$



$$\mathring{m}_\pi^2(\theta) = \mathring{m}_\pi^2(0)\sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\frac{\theta}{2}}$$

Topological Susceptibility Shift

- The topological susceptibility is monotonous function of eB

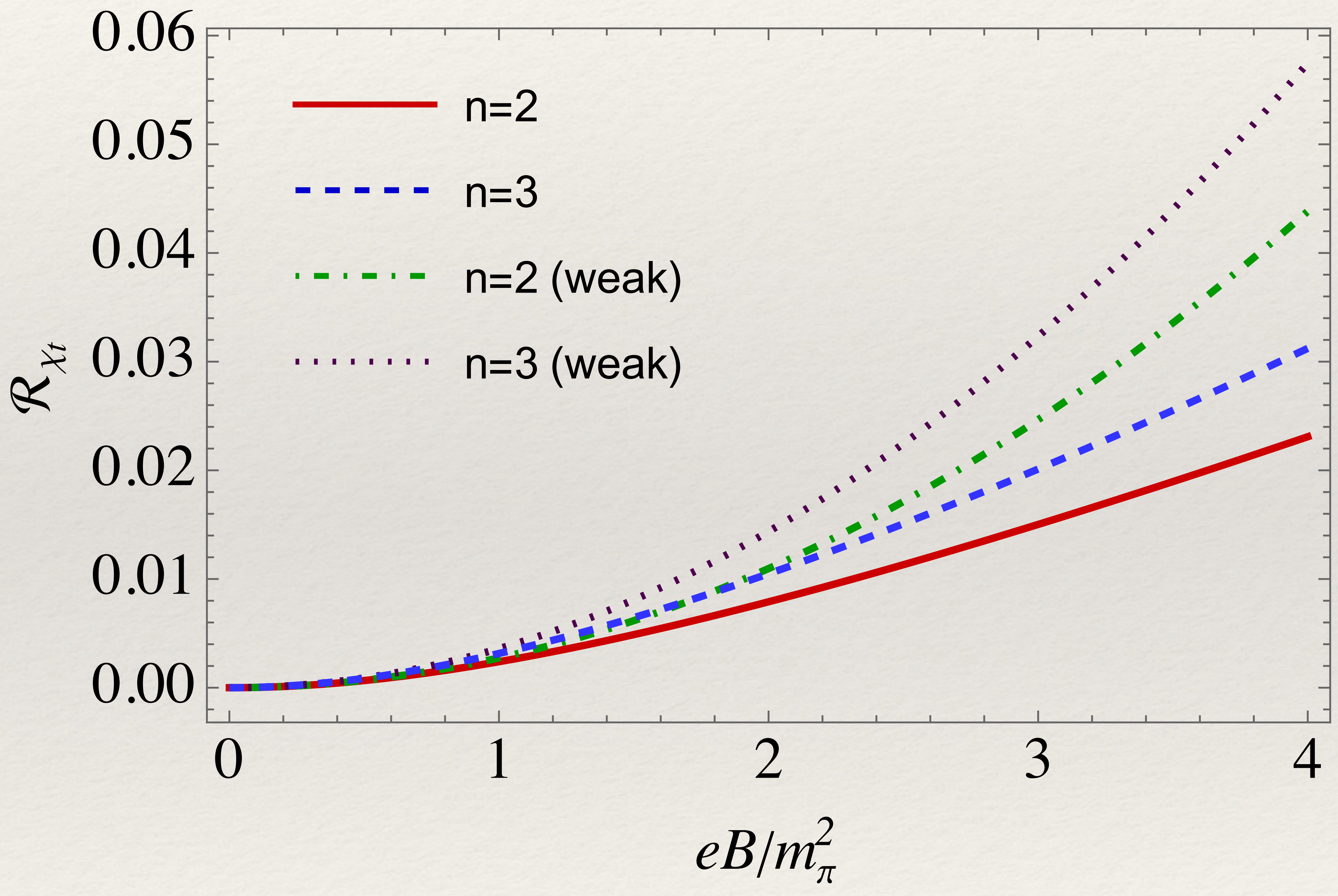
$$\chi_{t,B} = -B_0\,\overline{m}\int_0^\infty \frac{ds}{(4\pi s)^2} e^{-m_\pi^2 s}\left[\frac{eBs}{\sinh eBs} - 1\right]$$
$$= B_0\,\overline{m}\left[\frac{1}{6(4\pi m_\pi)^2}\right](eB)^2 + \mathcal{O}\left\{(eB)^4\right\}$$

- A low energy theorem relates it to the chiral condensate

$$\chi_{t,B} = -\overline{m}\,\langle \bar{q}_f q_f \rangle_B \qquad \overline{m} = \left(\frac{1}{m_u} + \frac{1}{m_d}\right)^{-1}$$

*PA, PLB, **825** 2022*

$$\chi_t(0) = -i\int d^4x \left\langle \mathcal{T}\,\frac{g^2\widetilde{G}G(x)}{32\pi^2}\,\frac{g^2\widetilde{G}G(0)}{32\pi^2}\right\rangle$$

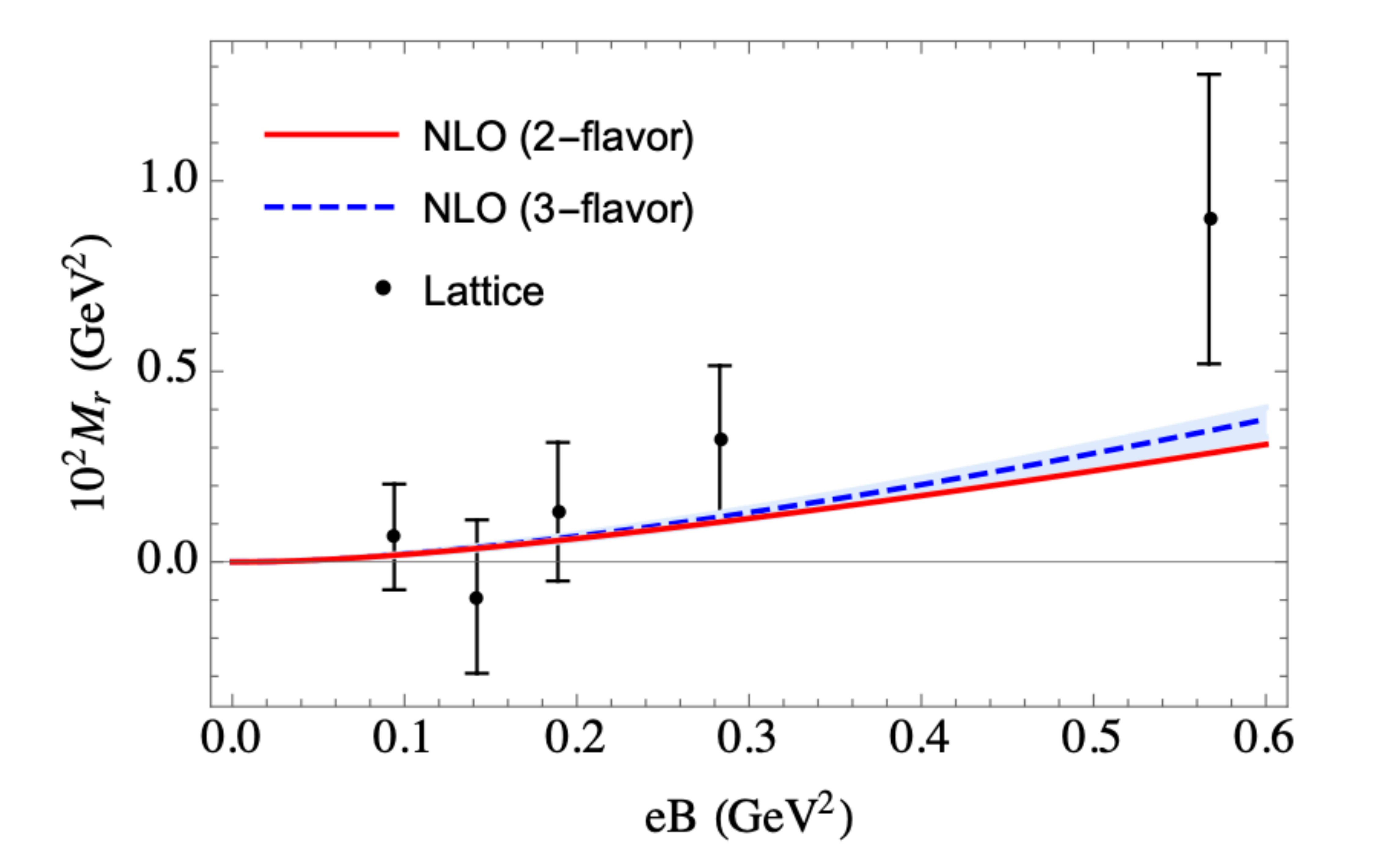


Renormalized Magnetization

- Renormalized magnetization characterizes the response of the QCD vacuum to the external magnetic field. Formally,

$$\mathcal{M}_r = -\frac{\partial \mathcal{F}_B}{\partial (eB)}$$

- Lattice computation of the (renormalized) magnetization, while mostly consistent with chiral perturbation theory, has large uncertainties



Key Questions & Approaches

- Questions
  - What are finite volume effects on observables/condensates in a uniform magnetic field?
    - Chiral condensate, magnetization/pressure anisotropy, neutral pion magnetic polarizability
  - How does generalization to parallel Euclidean electric and magnetic field impact the analysis?
  - Discussion: How does one generalize the approach to real electric fields?
- Approach
  - Chiral Perturbation Theory is the relevant long wavelength description of QCD
  - Periodic boundary conditions is a conventional choice in lattice QCD
  - All analysis will be performed in Euclidean space using indexing convention $w_\mu = (w_0, w_1, w_2, w_3)$
  - Choice of gauge: $A_\mu = (0, -Bx_2, 0, 0)$

Boundary Conditions & Flux Quantization

- Gauge fields are only periodic up to a gauge-transformation

$$A_\mu = (0, -Bx_2, 0, 0) \qquad A_\mu(x + L_2\hat{x}_2) = A_\mu(x) + (0, -BL_2, 0, 0) \equiv A_\mu(x) + \partial_\mu(\theta_2 + \Lambda_2) \qquad \Lambda_2 = -BL_2 x_1$$

- Gauge invariance requires an equivalent gauge transformation of pion fields

$$\phi(x + L_2\hat{x}_2) = W_2(x_1)\phi(x) \qquad W_2(x_1) = e^{-ie(\theta_2 - BL_2 x_1)} \equiv e^{-ie(\theta_2 + \Lambda_2)}$$

- Magnetic flux quantization is a consequence of

$$\phi(x + L_1\hat{x}_1 + L_2\hat{x}_2) = W_2(x_1 + L_1)\phi(x + L_1\hat{x}_1) = e^{-iQ\theta_1} W_2(x_1 + L_1)\phi(x)$$
$$\phi(x + L_2\hat{x}_2 + L_1\hat{x}_1) = e^{-iQ\theta_1}\phi(x_2 + L_2\hat{x}_2) = e^{-iQ\theta_1} W_2(x_1)\phi(x)$$

$$W_2(x_1) = W_2(x_1 + L_1) \implies eB = \frac{2\pi N_\Phi}{L_1 L_2}$$

Green's Function & Coincident Limit

- The infinite volume Green's function is *not* translationally invariant $G_+(x', x) = \langle\, \pi^+(x')\, \pi^-(x)\, \rangle$

$$\left(-D'_\mu D'_\mu + m^2\right) G_+^\infty(x', x) = \delta^{(4)}(x' - x)$$

Schwinger, PR ***82****, 1951*

$$G_+^\infty(x', x) = e^{ieB\Delta x_1 \bar{x}_2} \int_0^\infty \frac{ds}{(4\pi s)^2} \frac{eBs}{\sinh eBs} e^{-\mathring{m}_\pi^2 s} \exp\left[-\frac{eB\left(\Delta x_1^2 + \Delta x_2^2\right)}{4 \tanh eBs} - \frac{\Delta x_0^2 + \Delta x_3^2}{4s}\right] \equiv e^{ieB\Delta x_1 \bar{x}_2} g_\infty(x' - x)$$

- The finite volume Green's function is *also not* translationally invariant and in both directions of the transverse plane

Tiburzi, Phys. Rev. D 89, 074019 (2014)

$$\left(-D'_\mu D'_\mu + m^2\right) G_+(x', x) = \delta^{(4)}_{L_\mu}(x' - x)$$

$$\delta_{L_1}(x' - x) = \sum_{\nu_1 \in \mathbb{Z}} \delta(x' - x + \nu_1 L_1)$$

$$G_+(x', x) = \sum_{\nu_\mu \in \mathbb{Z}} e^{ie\theta_1 \nu_1} [W_2^\dagger(x_1)]^{\nu_2}\, G_+^\infty(x' + \nu_\mu L_\mu, x)$$

$$G_+(x, x) = \sum_{\nu_\mu \in \mathbb{Z}} (-1)^{N_\Phi \nu_1 \nu_2} [W_1^\dagger(x_2)]^{\nu_1} [W_2^\dagger(x_1)]^{\nu_2} g_\infty(x - x + \nu_\mu L_\mu)$$

$$W_1(x_2) = e^{-ie(\theta_1 + BL_1 x_2)}$$

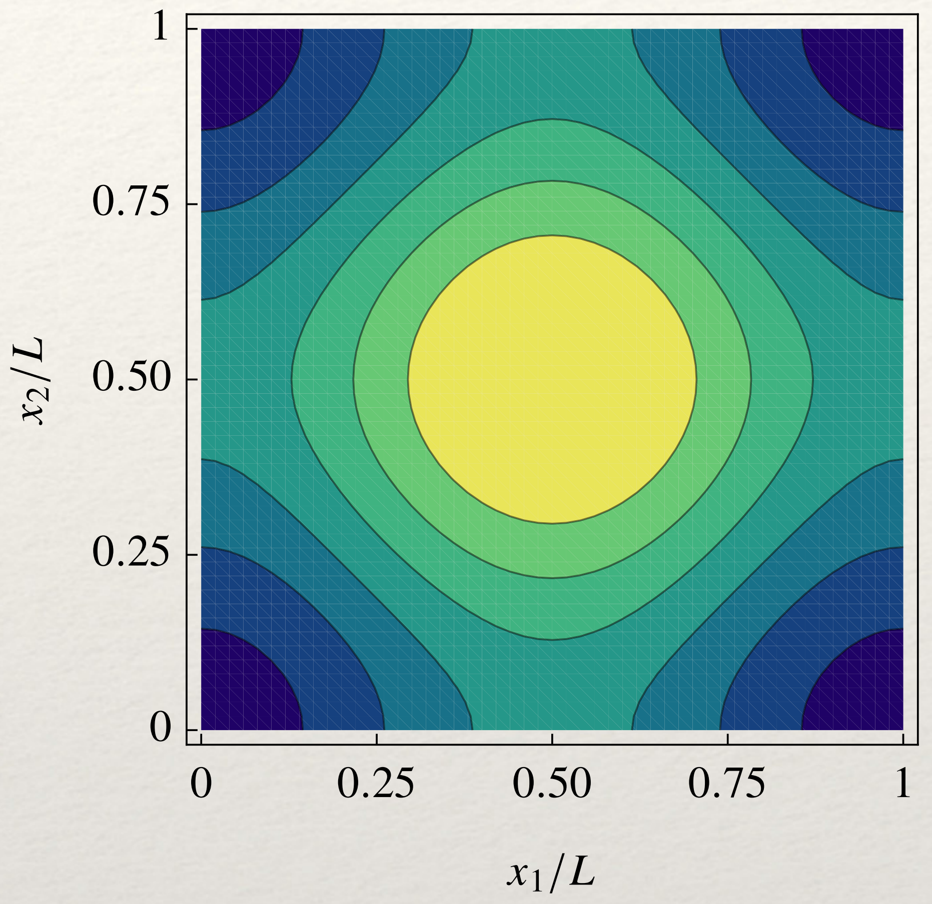
Finite Volume Local Quark Condensate

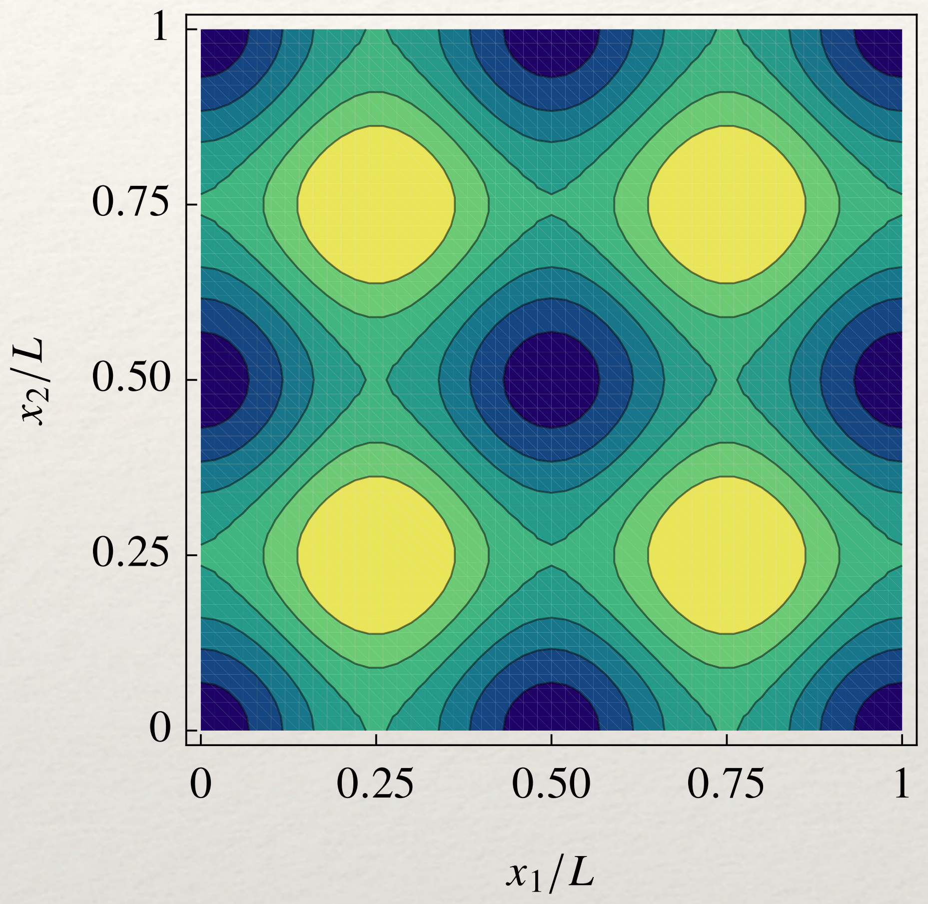
- The quark condensate is therefore spatially variant

$$\mathscr{L}_{\text{mass}} = \frac{F^2}{4}\mathrm{Tr}[\chi^\dagger\Sigma + \chi\Sigma^\dagger] = = 2B_0 m_q \left(F^2 - \tfrac{1}{2}\pi^0\pi^0 - \pi^-\pi^+ + \cdots\right)$$

$$\langle\overline{\psi}(x)\psi(x)\rangle = -2B_0[F^2 - \tfrac{1}{2}G_0(0,0) - G_+(x,x)]$$

$$\langle\overline{\psi}\psi\rangle_0^\infty = -2B_0F^2$$

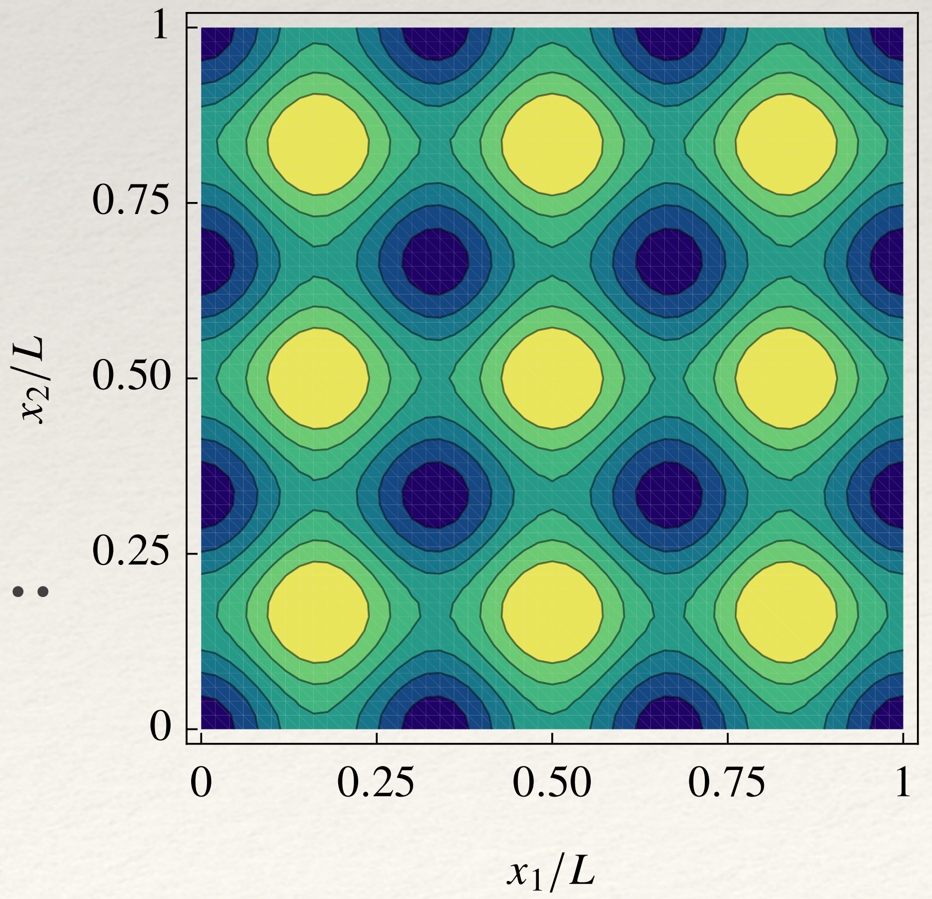




- For asymptotically large boxes

$$R(x_\perp) = \frac{\langle\overline{\psi}(x)\psi(x)\rangle - \langle\overline{\psi}\psi\rangle^\infty}{\langle\overline{\psi}\psi\rangle_0^\infty}$$

$$= -\left[\frac{5}{2} + \cos\left(\frac{2\pi N_\Phi x_1}{L}\right) + \cos\left(\frac{2\pi N_\Phi x_2}{L}\right)\right] \times \frac{m_\pi^2}{F_\pi^2}\frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}} + \cdots$$



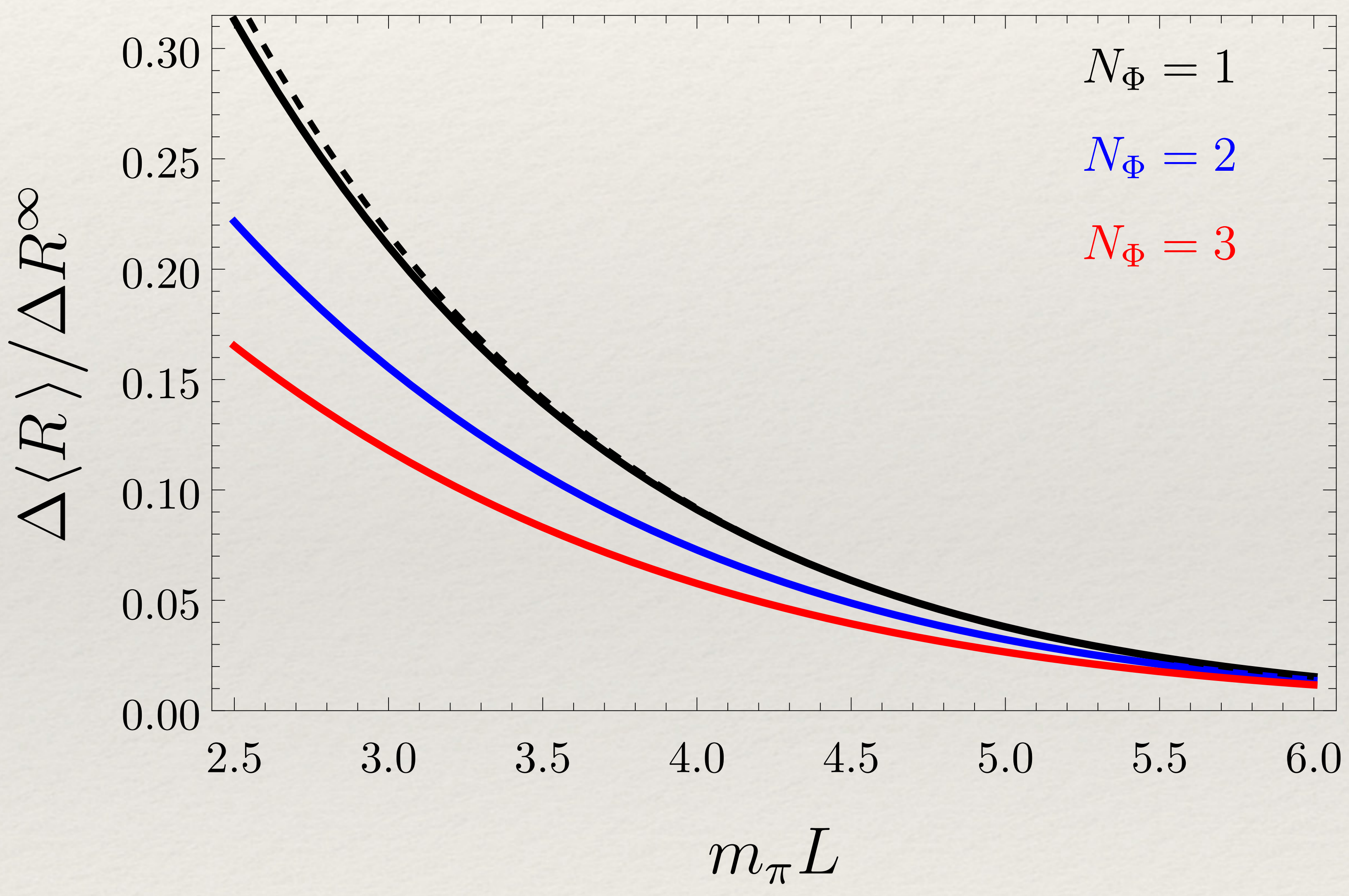
$m_\pi L = 3$

Finite Volume Average Quark Condensate

- The quark condensate is therefore spatially variant

$$\frac{1}{L_1}\int_0^{L_1} dx_1\,[W_2^\dagger(x_1)]^{\nu_1} = \delta_{\nu_1,0}$$
$$\frac{1}{L_2}\int_0^{L_2} dx_2\,[W_1^\dagger(x_2)]^{\nu_2} = \delta_{\nu_2,0}$$

$$\mathscr{L}_{\text{mass}} = \frac{F^2}{4}\mathrm{Tr}[\chi^\dagger\Sigma + \chi\Sigma^\dagger] = = 2B_0 m_q\left(F^2 - \frac{1}{2}\pi^0\pi^0 - \pi^-\pi^+ + \cdots\right)$$
$$\langle\overline{\psi}(x)\psi(x)\rangle = -2B_0[F^2 - \tfrac{1}{2}G_0(0,0) - G_+(x,x)]$$
$$\langle\overline{\psi}\psi\rangle_0^\infty = -2B_0F^2$$



- Finite volume correction of the spatially averaged condensate is characterized by

$$\frac{\Delta\langle R\rangle}{\Delta R^\infty} = \frac{\langle R\rangle - \langle R\rangle_{B=0}}{R^\infty - R^\infty_{B=0}}$$
$$= e^{-m_\pi L}\sqrt{2\pi m_\pi L}\left[1 + \mathcal{O}\left(\frac{1}{m_\pi L}\right) + \mathcal{O}(e^{-m_\pi L}) + \mathcal{O}\left(\frac{N_\Phi^2}{(m_\pi L)^2}\right) + \cdots\right]$$

Pressure Anisotropy & Magnetization

- Pressure can be defined either at fixed field or field flux

$$p_i = -\frac{L_i}{V}\frac{\partial F_B}{\partial L_i}\bigg|_{B,L_{j\neq i}} \qquad \tilde{p}_i = -\frac{L_i}{V}\frac{\partial F_B}{\partial L_i}\bigg|_{\Phi,L_{j\neq i}}$$

- The matter contribution to the free energy $F_B = V\mathcal{F}_B$ admits finite volume corrections

$$\mathcal{F}_B^{\infty} = -\frac{1}{(4\pi)^2}\int_0^{\infty}\frac{ds}{s^3}\, e^{-m_\pi^2 s}\left[\frac{eBs}{\sinh eBs} - 1 - \frac{1}{6}(eBs)^2\right]$$

$$\mathcal{F}_B^{FV} = -\frac{1}{(4\pi)^2}\int_0^{\infty}\frac{ds}{s^3}\, e^{-m_\pi^2 s}\left[\theta_3(\tfrac{L_3^2}{4s}) - 1\right]\left[\frac{eBs}{\sinh eBs} - 1\right]$$

- The longitudinal (transverse) pressures are identical (different)

$$\tilde{p}_3 = p_3 = -\mathcal{F}_B - L_3\frac{\partial \mathcal{F}_B}{\partial L_3} \qquad \tilde{p}_\perp = -\mathcal{F}_B - B\mathcal{M}_r \qquad p_\perp = -\mathcal{F}_B \qquad \mathcal{M}_r = -\frac{\partial \mathcal{F}_B}{\partial B}$$

Pressure Anisotropy & Magnetization

- The pressures (fixed flux) are anisotropic

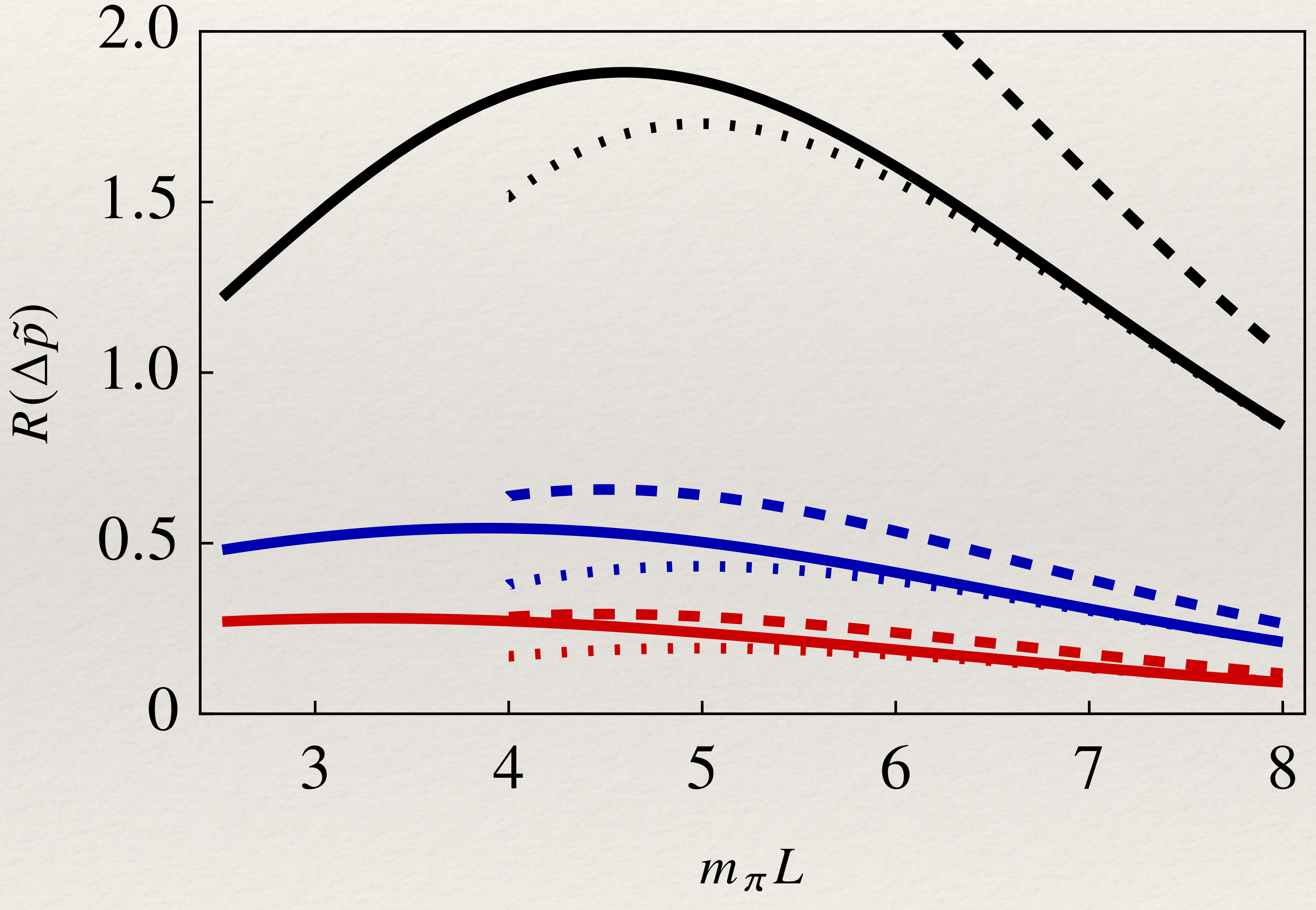
$$\Delta\tilde{p} = \tilde{p}_\perp - \tilde{p}_3 \qquad \Delta\tilde{p}^{\,\infty} = \lim_{L_3\to\infty} \Delta\tilde{p}_\perp = -B\mathscr{M}_r^\infty$$

- The ratio captures finite volume corrections in the infinite volume magnetization

$$R(\Delta\tilde{p}) = \frac{\Delta\tilde{p} - \Delta\tilde{p}^{\,\infty}}{\Delta\tilde{p}^{\,\infty}}$$

$$= \frac{30}{7(2\pi)^{3/2}N_\Phi^2}(m_\pi L)^{9/2}e^{-m_\pi L}\left[1 + \mathcal{O}\left(\frac{1}{m_\pi L}\right) + \mathcal{O}(e^{-m_\pi L})\right]$$

- The finite volume corrections are *extremely acute*



Neutral Pion Magnetic Polarizability

- The ChPT Lagrangian that encodes neutral pion interactions is

$$\mathscr{L}_{\text{int}} = -\frac{m^2}{24F^2}(\pi^0\pi^0)^2 - \frac{m^2}{6F^2}(\pi^+\pi^-\pi^0\pi^0) - \frac{1}{3F^2}D_\mu\pi^+D_\mu\pi^-\pi^0\pi^0 - \frac{1}{3F^2}\pi^+\pi^-\partial_\mu\pi^0\partial_\mu\pi^0 + \frac{1}{3F^2}[\partial_\mu(\pi^+\pi^-)]\pi^0\partial_\mu\pi^0$$

- The neutral pion effective Lagrangian in a magnetic field depends on its energy E with coordinate dependent finite volume dependence encoded in $\mathscr{V}$

$$\mathscr{L}_{\text{eff}} = \tfrac{1}{2}\partial_\mu\tilde{\pi}^0\partial_\mu\tilde{\pi}^0 + \tfrac{1}{2}(E^2 + \mathscr{V})\tilde{\pi}^0\tilde{\pi}^0 \qquad E = m_\pi\left[1 + \frac{eB}{2(4\pi F_\pi)^2}\mathscr{I}\left(\frac{m_\pi^2}{QB}\right)\right] = m_\pi - \frac{1}{2}\beta_M B^2 + \mathscr{O}(B^4)$$

- The renormalized Green's function must be constructed in position space

$$\langle x'|\mathscr{G}_0|x\rangle = \langle x'|G_0|x\rangle + \int_y \langle x'|G_0|y\rangle\mathscr{O}(y)\langle y|G_0|x\rangle$$

$$\mathscr{G}_0^{-1} = G_0^{-1} - \mathscr{O} \qquad G_0^{-1} = -\partial^2 + m^2 \qquad \mathscr{O}(y) = -\left(G_0^{-1}\right)\tfrac{1}{2}V_1(y) - \tfrac{1}{2}V_1(y)\left(G_0^{-1}\right) - V(y)$$

$$\mathscr{G}_0^{-1} = (1 - \tfrac{1}{2}V_1)G_0^{-1}(1 - \tfrac{1}{2}V_1) + V + \cdots$$

Neutral Pion Two-Point Function

- The renormalized Green's function can be projected into zero momentum at the sink

$$\widetilde{G}_0(x',x) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \sum_{\vec{p}} \frac{e^{ip_\mu(x'-x)_\mu}}{p_\mu p_\mu + E^2} \qquad \mathscr{C}_0(T,x_\perp) \equiv \int_{\vec{x}'} \mathscr{G}_0(x',x)$$

$$\mathscr{C}_0(T,x_\perp) = \frac{e^{-ET}}{2E}\left[1 + \mathscr{R}(T,x_\perp,\theta_\perp)\right] = \frac{e^{-(E+\Delta E)T}}{2(E+\Delta E)}\left[1 + \mathscr{R}_\perp(T,x_\perp,\theta_\perp)\right]$$

- As before, averaging over the transverse plane projects to zero winding

$$\langle \mathscr{C}_0(T) \rangle = \frac{e^{-(E+\Delta E)T}}{2(E+\Delta E)} \qquad \Delta E = \frac{m_\pi}{2}\int_0^\infty ds \frac{e^{-m_\pi^2 s}}{(4\pi s F_\pi)^2}\left[\frac{QBs}{\sinh QBs}\left\{\vartheta_3(0,e^{-\frac{L_3^2}{4s}}) - 1\right\} - \frac{1}{2}\left\{\Theta_0(s) - 1\right\}\right]$$

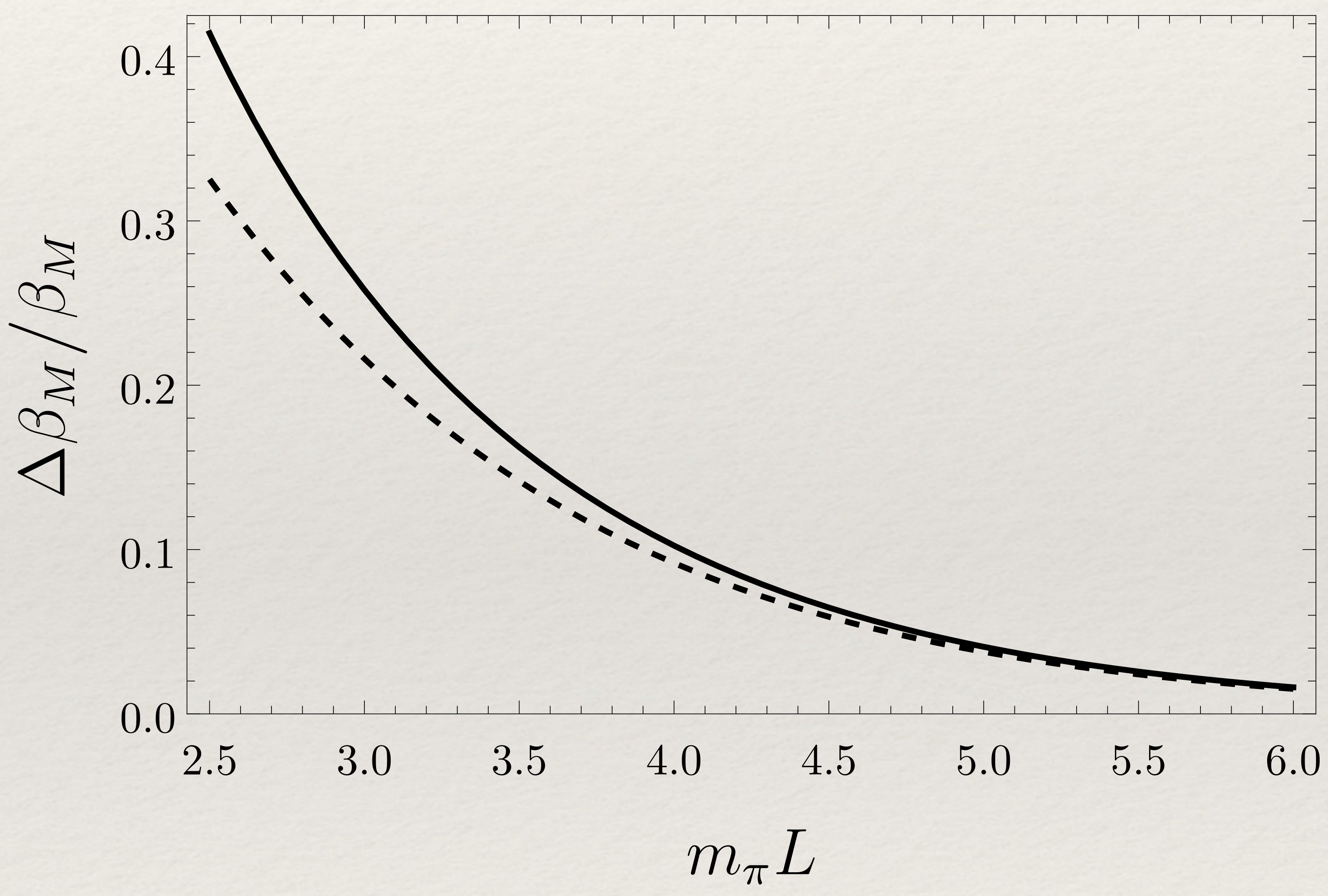
FV Effect on Neutral Pion Polarizability

- The infinite volume polarizability is

$$\beta_M = \frac{e^2}{6m_\pi(4\pi F_\pi)^2}$$

- The relative finite volume correction is

$$\frac{\Delta\beta_M}{\beta_M} = 2\sum_{\nu=1}^{\infty} \nu m_\pi L_3 K_1(\nu m_\pi L_3)$$
$$= \sqrt{2\pi m_\pi L_3}\, e^{-m_\pi L_3} + \cdots$$



Parallel Euclidean Electric and Magnetic Fields

Preliminary result

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Susceptibility of the QCD Vacuum to *CP*-Odd Electromagnetic Background Fields

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We investigate two flavor quantum chromodynamics (QCD) in the presence of *CP*-odd electromagnetic background fields and determine, by means of lattice QCD simulations, the induced effective θ term to first order in $\vec{E}\cdot\vec{B}$. We employ a rooted staggered discretization and study lattice spacings down to 0.1 fm and Goldstone pion masses around 480 MeV. In order to deal with a positive measure, we consider purely imaginary electric fields and real magnetic fields, and then exploit the analytic continuation. Our results are relevant to a description of the effective pseudoscalar quantum electrodynamics—QCD interactions.

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- Generalize the ga
- Gauge invariance

$$A_\mu(x+L_0\hat{x}_0) =$$

$$\phi(x+L_0\hat{x}_0) =$$

- Magnetic and ele

$$\phi(x+L_1\hat{x}_1$$

$$\phi(x+L_0\hat{x}_0$$

- Green's function is constructed easily using separability in infinite volume and method of images in finite volume
- $G_+^\infty(x',x) = \int_0^\infty ds\, e^{-m^2 s} G_\parallel^\infty(x'_\perp, x_\perp \,|\, s) G_\perp^\infty(x'_\perp, x_\perp \,|\, s)$

$$G_\parallel(x'_\parallel, x_\parallel \,|\, s) = \sum_{\nu_\parallel \in \mathbb{Z}} e^{ie\theta_0\nu_0}[W_0^\dagger(x_3)]^{\nu_0} G_\parallel^\infty(x'_\parallel + \nu_\parallel L_\parallel, x_\parallel)$$

$$G_\perp(x'_\perp, x_\perp \,|\, s) = \sum_{\nu_\perp \in \mathbb{Z}} e^{ie\theta_1\nu_1}[W_2^\dagger(x_1)]^{\nu_2} G_\perp^\infty(x'_\perp + \nu_\perp L_\perp, x_\perp)$$

Parallel Euclidean Electric and Magnetic Fields

Preliminary result

- Generalize the gauge background to include electric field $A_\mu(x) = (0, -Bx_2, 0\,, -\mathscr{E}x_0)$
- Gauge invariance requires pion fields to transform as

$$A_\mu(x + L_0\hat{x}_0) = A_\mu(x) + \partial_\mu(\Lambda_0 + \theta_0) \qquad A_\mu(x + L_2\hat{x}_2) = A_\mu(x) + \partial_\mu(\Lambda_2 + \theta_2)$$

$$\phi(x + L_0\hat{x}_0) = e^{-ie\theta_0}e^{-i\Lambda_0}\phi(x) \qquad \phi(x + L_2\hat{x}_2) = e^{-ie\theta_2}e^{-i\Lambda_2}\phi(x)$$

- Magnetic and electric translations must commute for single-valued pion fields

$$\phi(x + L_1\hat{x}_1 + L_2\hat{x}_2) \qquad W_2(x_1) = W_2(x_1 + L_1) \implies eB = 2\pi N_\perp/(L_1L_2)$$

$$\phi(x + L_0\hat{x}_0 + L_3\hat{x}_3) \qquad W_0(x_3) = W_0(x_3 + L_3) \implies e\mathscr{E} = 2\pi N_\parallel/(L_0L_3)$$

- Green's function is constructed easily using separability in infinite volume and method of images in finite volume
- $G_+^\infty(x', x) = \int_0^\infty ds\, e^{-m^2 s} G_\parallel^\infty(x'_\perp, x_\perp\,|\,s) G_\perp^\infty(x'_\perp, x_\perp\,|\,s)$

$$G_\parallel(x'_\parallel, x_\parallel\,|\,s) = \sum_{\nu_\parallel\in\mathbb{Z}} e^{ie\theta_0\nu_0}[W_0^\dagger(x_3)]^{\nu_0} G_\parallel^\infty(x'_\parallel + \nu_\parallel L_\parallel, x_\parallel)$$

$$G_\perp(x'_\perp, x_\perp\,|\,s) = \sum_{\nu_\perp\in\mathbb{Z}} e^{ie\theta_1\nu_1}[W_2^\dagger(x_1)]^{\nu_2} G_\perp^\infty(x'_\perp + \nu_\perp L_\perp, x_\perp)$$

Local Chiral Condensate

Preliminary result

- The chiral condensate is proportional to the coincident Green's functions

$$G_+(x,x) = \sum_{\nu_\mu \in \mathbb{Z}} f_\perp(\nu_\perp, x_\perp) f_\parallel(\nu_\parallel, x_\parallel) g_+(\nu_\mu)$$

$$f_\perp(\nu_\perp, x_\perp) = (-1)^{N_\perp \nu_1 \nu_2} [W_1^\dagger(x_2)]^{\nu_1} [W_2^\dagger(x_1)]^{\nu_2}$$

$$f_\parallel(\nu_\parallel, x_\parallel) = (-1)^{N_\parallel \nu_3 \nu_0} [W_3^\dagger(x_0)]^{\nu_3} [W_0^\dagger(x_3)]^{\nu_0}$$

$$\begin{aligned} g_+(\nu_\mu) &= g_\infty(x + \nu_\mu L_\mu, x) \\ &= \int_0^\infty \frac{ds}{(4\pi s)^2} e^{-m^2 s} \frac{eBs}{\sinh eBs} \frac{e\mathscr{E}s}{\sinh e\mathscr{E}s} e^{\left(-\frac{1}{4s}\left[\frac{eBs}{\sinh eBs}(\nu_\perp L_\perp)^2 + \frac{e\mathscr{E}s}{\sinh e\mathscr{E}s}(\nu_\parallel L_\parallel)^2\right]\right)} \end{aligned}$$

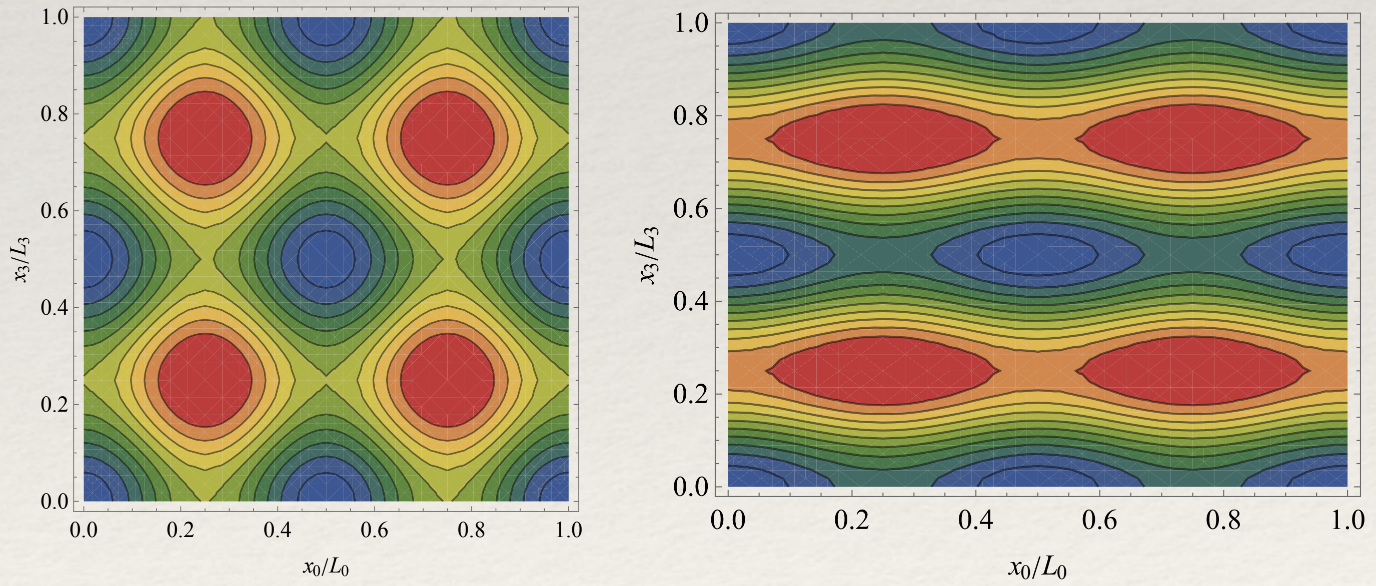
- The ratio $R(x)$ is separable into transverse and longitudinal contributions

$$R(x) = \frac{\langle \overline{\psi}(x)\psi(x)\rangle - \langle \overline{\psi}\psi\rangle^\infty}{\langle \overline{\psi}\psi\rangle_0^\infty} \approx R_\perp(x_\perp) + R_\parallel(x_\parallel)$$

$$R_\perp(x_\perp) = -\left[1 + \cos\left(\frac{2\pi N_\perp x_1}{L}\right) + \cos\left(\frac{2\pi N_\perp x_2}{L}\right)\right] \frac{m_\pi^2}{F_\pi^2} \frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}}$$

$$R_\parallel(x_\parallel) = -\left[\frac{1}{2} + \cos\left(\frac{2\pi N_\parallel x_0}{L_0}\right)\right] \frac{m_\pi^2}{F_\pi^2} \frac{e^{-m_\pi L_0}}{(2\pi m_\pi L_0)^{3/2}} - \left[\frac{1}{2} + \cos\left(\frac{2\pi N_\parallel x_3}{L}\right)\right] \frac{m_\pi^2}{F_\pi^2} \frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}}$$

$m_\pi L = 3$ $m_\pi L_0 = 3{,}4$ $N_\parallel = 2$



Free Energy and Pressure Anisotropy

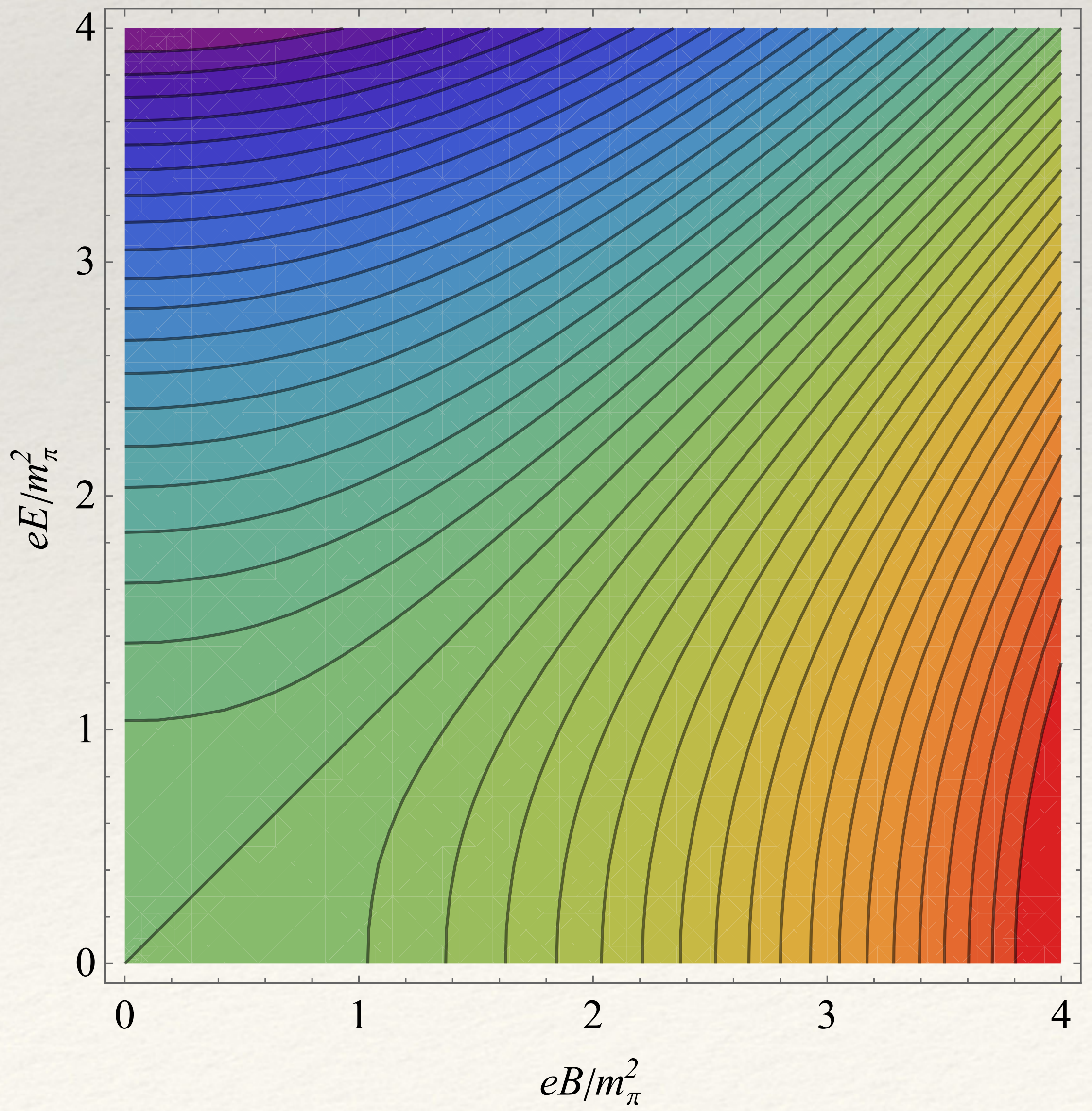
Preliminary result

- Finite volume corrections are absent in the free energy

$$\mathcal{F}_{B,\mathcal{E}} = -\int_0^\infty \frac{ds}{(4\pi)^2 s^3}\, e^{-m_\pi^2 s}\left[\frac{eBs}{\sinh eBs}\frac{e\mathcal{E}s}{\sinh e\mathcal{E}s} - 1 + \frac{1}{6}\left\{(eBs)^2 + (e\mathcal{E}s)^2\right\}\right]$$

- Nevertheless, there is a pressure anisotropy in finite volume due to quantization of fluxes

$$\Delta\tilde{p} \equiv \tilde{p}_\perp - \tilde{p}_3 = -B\mathcal{M}_r + \mathcal{E}\mathcal{P}_r \qquad \mathcal{M}_r = -\frac{\partial\mathcal{F}_{B,\mathcal{E}}}{\partial B} \qquad \mathcal{P}_r = -\frac{\partial\mathcal{F}_{B,\mathcal{E}}}{\partial\mathcal{E}}$$



Conclusion

- Chiral perturbation Theory provides a *model-independent* setting to study the impact of weak fields on the QCD vacuum
- Finite volume lattice corrections to observables/condensates is significant *[PA, Tiburzi, Phys. Rev. D, **107**, 094504 (2023)]*
    - The large size of the correction is due to finite volume effects entering at the same order as finite magnetic field effects
    - The volume effect is worse for observables that are small at weak fields such as magnetization and magnetic susceptibility
    - In parallel (Euclidean) electric and magnetic fields, the local chiral condensate is periodic in all four directions but admits no finite volume corrections upon averaging *[Preliminary result]*
    - While the free energy is free of finite volume corrections, there is a finite pressure anisotropy *[Preliminary result]*

Thank You!

Questions?

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