Kaon DFs from Empirical Information and all-orders evolution

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J. Rodríguez-Quintero

The complex structure of strong interactions in Euclidean and Minkowski space
Trento, 26-30 May, 2025

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Kaon Distribution Functions from Empirical Information

Zhen-Ni Xu (徐珍妮)[a], Daniele Binosi[b], Chen Chen (陈晨)[c,d], Khépani Raya[a], Craig D. Roberts[e,f], José Rodríguez-Quintero[a,g]

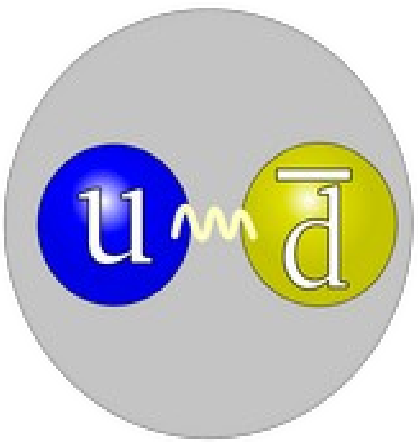
[a]*Department of Integrated Sciences and Center for Advanced Studies in Physics, Mathematics and Computation, University of Huelva, E-21071 Huelva, Spain.*
[b]*European Centre for Theoretical Studies in Nuclear Physics and Related Areas (ECT*), Villa Tambosi, Strada delle Tabarelle 286, I-38123 Villazzano (TN), Italy*
[c]*Interdisciplinary Center for Theoretical Study, University of Science and Technology of China (USTC), Hefei, Anhui 230026, China*
[d]*Peng Huanwu Center for Fundamental Theory (PCFT), Hefei, Anhui 230026, China*
[e]*School of Physics, Nanjing University, Nanjing, Jiangsu 210093, China*
[f]*Institute for Nonperturbative Physics, Nanjing University, Nanjing, Jiangsu 210093, China*
[g]*Irfu, CEA, Université de Paris-Saclay, 91191, Gif-sur-Yvette, France*

chenchen1031@ustc.edu.cn (CC); cdroberts@nju.edu.cn (CDR); jose.rodriguez@dfaie.uhu.es (JRQ)

Date: 2024 November 21

Parton distributions: Hadron scale

ζ_H



- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^\pi(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta = 2+\gamma(\zeta)}$$

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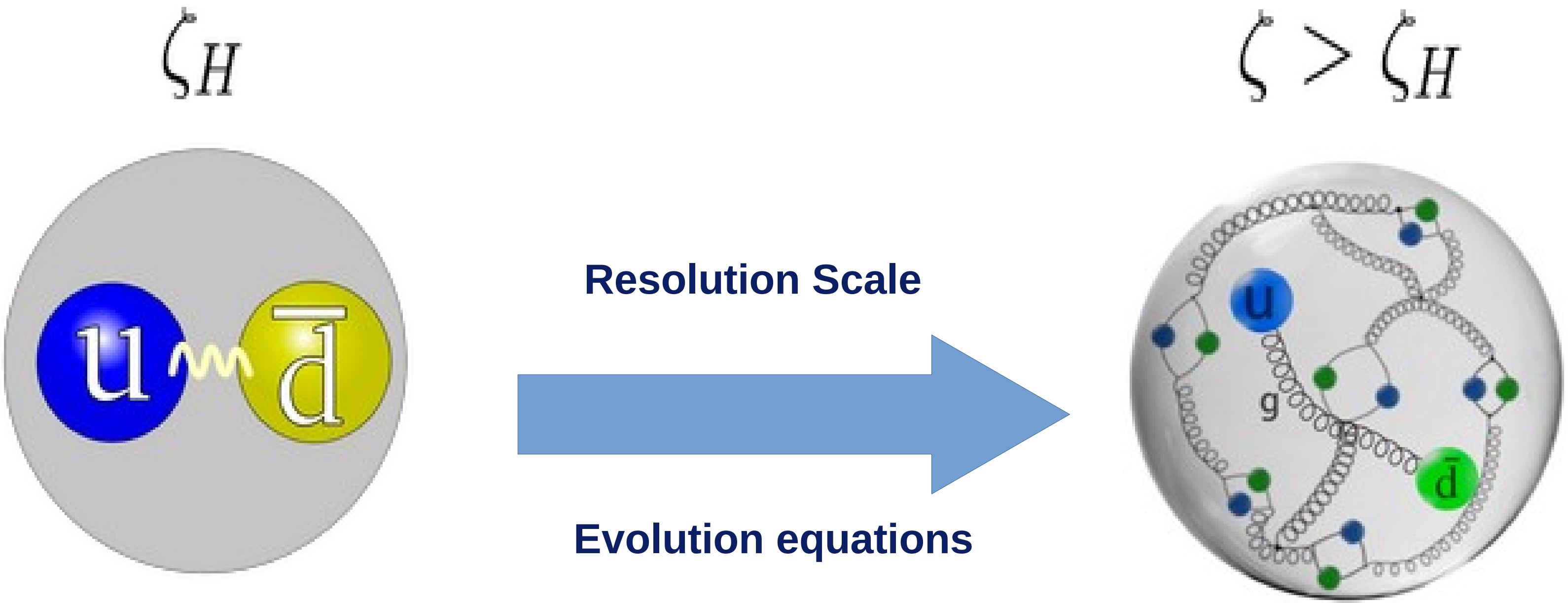

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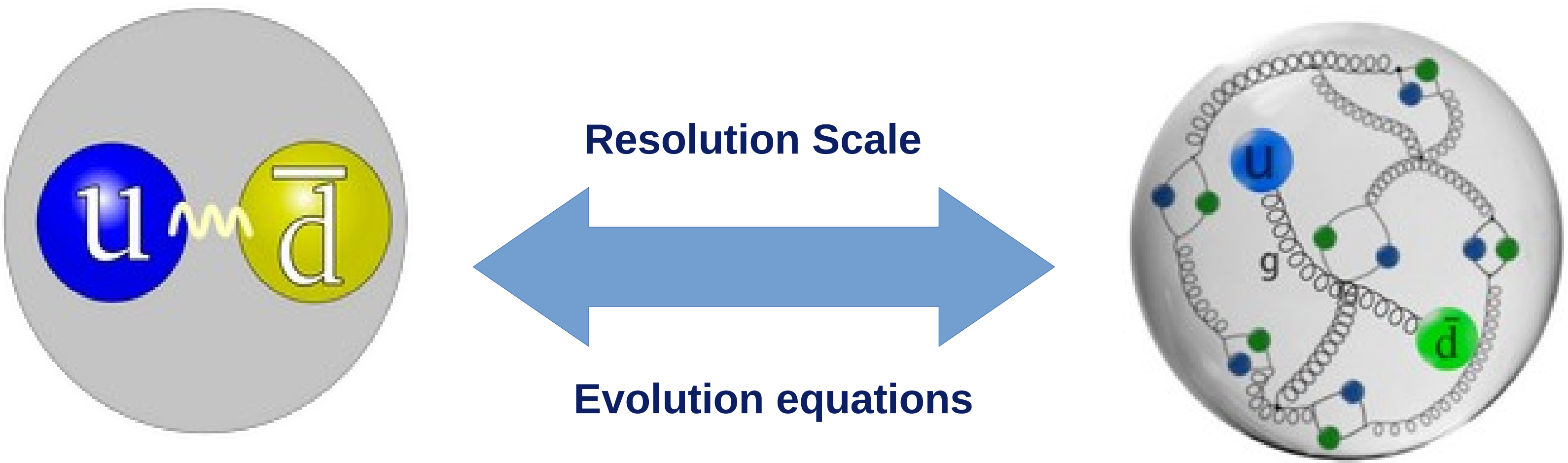
ζ_H $\qquad\qquad\qquad\qquad$ $\zeta > \zeta_H$



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Effective charge:

$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln(\frac{\zeta^2}{\Lambda^2})} + \ldots$$

Which value of Lambda? I

$$t = \ln(\frac{\zeta^2}{\zeta_0^2})$$

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Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t)=\overline{\alpha}(t)\left(1+c\,\overline{\alpha}(t)+\ldots\right)$$

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The evolution will thus depend on the scheme because of the perturbative truncation

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The use of Λ can be interpreted as the choice of new scheme, differing from MS, defined in such a way that one-loop is exact.

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G. Grunberg, Phys.Rev.D 29 (1984) 2315-2338

D. Becirevic et al., Phys.Rev.D 60 (1999) 094509, Phys.Rev.D 61 (2000) 114508

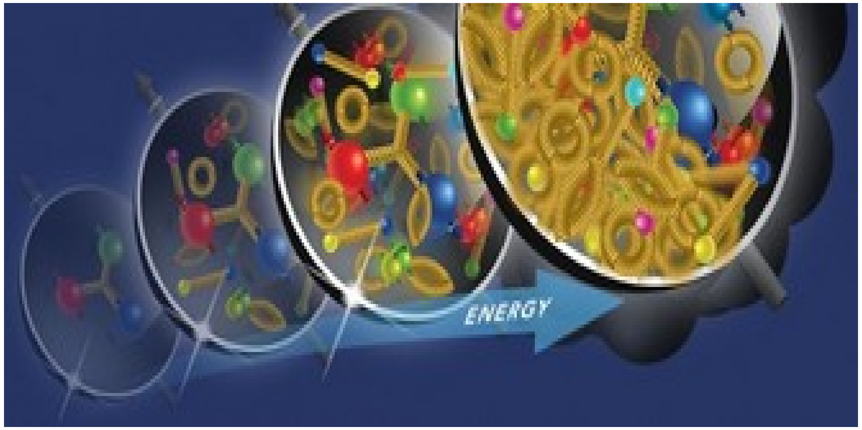
DGLAP: All orders evolution

Raya:2021zrz

Cui:2020tdf

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1\frac{dy}{y}\begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

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Starting from fully-dressed **quasiparticles**, at ζ_H

Sea and **Gluon** content unveils, as prescribed by **QCD**

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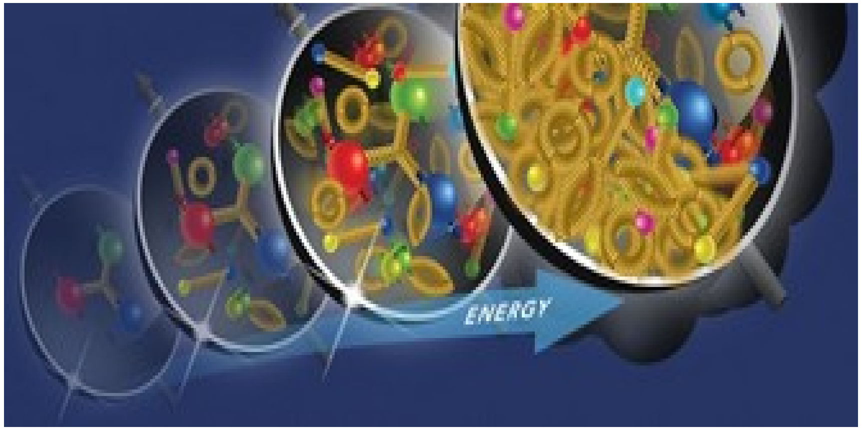

- Not the **LO** QCD coupling but an **effective** one.
- Making this equation exact.
- Connecting with the hadron scale, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

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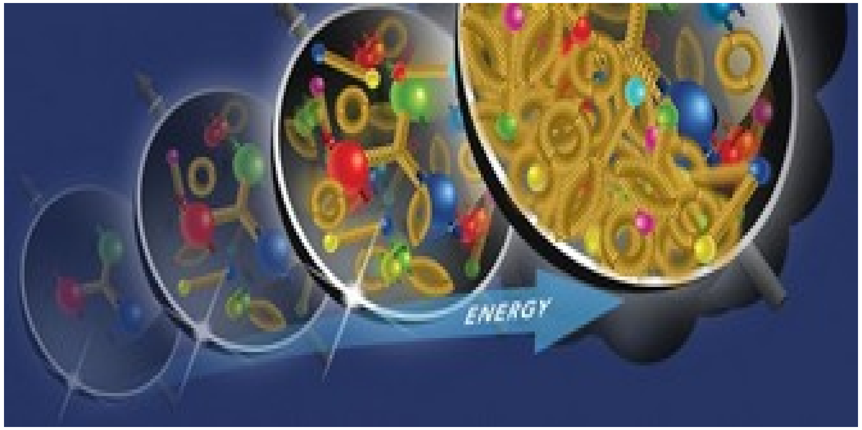
DGLAP ~~leading-order~~ evolution equations

$$\gamma_{AB}^{(n)}=-\int_0^1 dx\, x^n P_{AB}^{C}(x)$$



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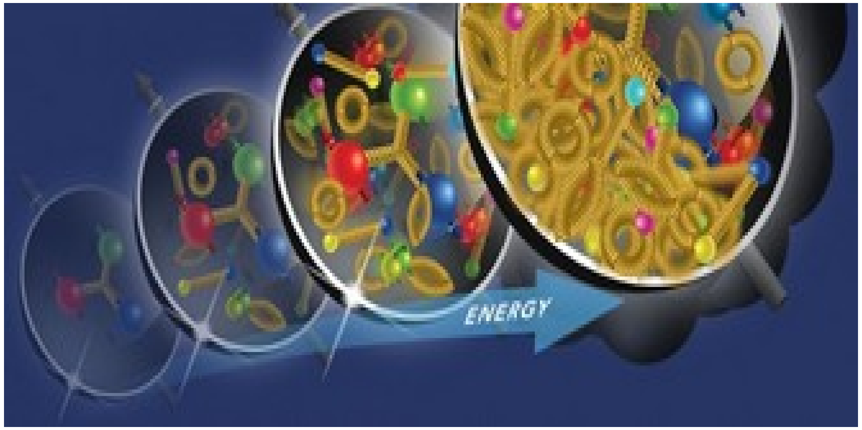

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DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle_{q_H}^{\zeta} = \langle x^n \rangle_{q_H}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n \rangle_{q_H}^{\zeta_H}\left[\frac{\langle x \rangle_{q_H}^{\zeta}}{\langle x \rangle_{q_H}^{\zeta_H}}\right]^{\gamma_{qq}^n/\gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve “all” the Mellin moments up and reconstruct the PDF.

DGLAP: All orders evolution

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DGLAP: All orders evolution

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DGLAP: All orders evolution

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Under a sensible assumption at large momentum scale:

$$q(x;\zeta) \underset{x\to 0}{\sim} x^{\alpha(\zeta)} \left(1 + \mathcal{O}(x)\right)$$

$$1 + \alpha(\zeta) = \frac{3}{2} \langle x(\zeta) \rangle \ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle} + \beta(\zeta_H) \langle x(\zeta) \rangle + \mathcal{O}\left(\frac{\langle x(\zeta) \rangle}{\ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle}} \right)$$

DGLAP: All orders evolution

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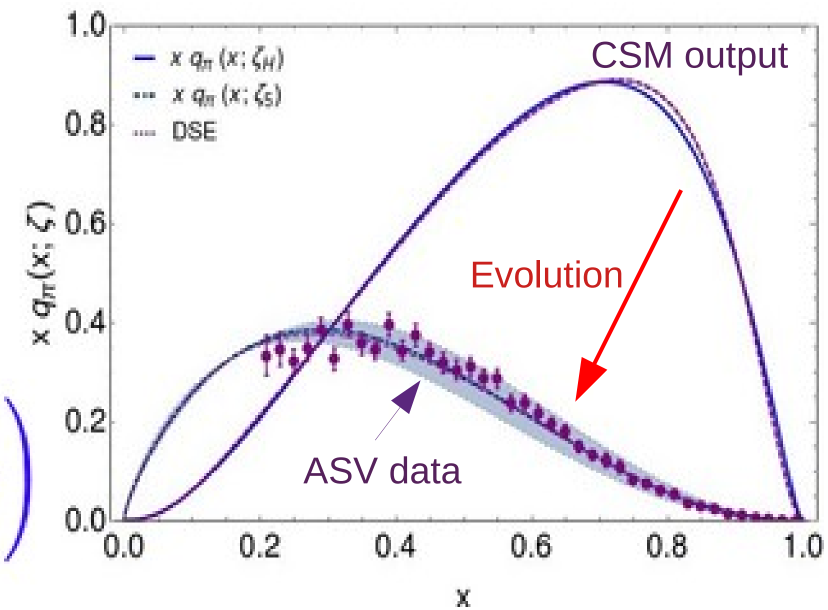
Reconstruction after evolving:



DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle_{u_\pi}^{\zeta_H} = \frac{1}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle_{u_\pi}^{\zeta_H}$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.

DGLAP: All orders evolution

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$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)}$$
$$\times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

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- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

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Reported lattice moments agree very well with the recursion formula

n	$\langle x^n\rangle_{u_\pi}^{\zeta_5}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

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DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

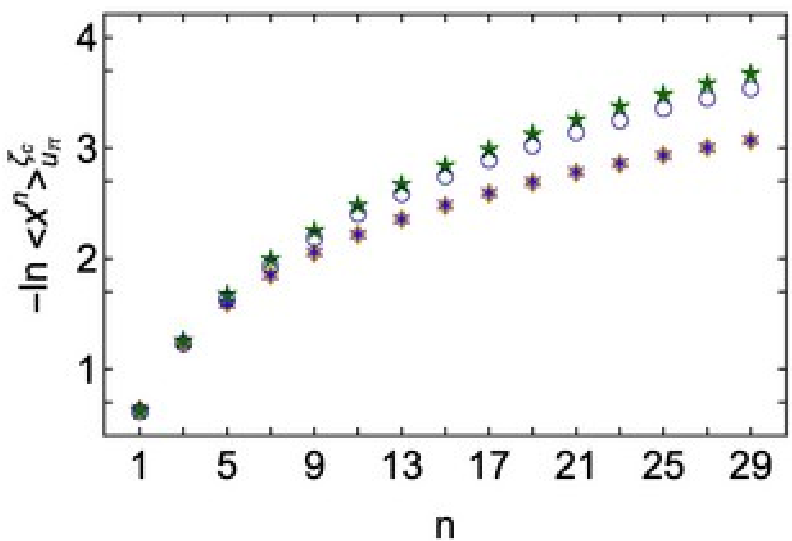
Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

Moments computed from: P. Barry et al., PRL127(2021)232001



DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\langle x^n \rangle^{\zeta}_{u_\pi} (\langle 2x \rangle^{\zeta}_{u_\pi})^{-\gamma_0^n/\gamma_0^1}$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

$$q(x;\zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad\qquad q(x;\zeta_H) = 1$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

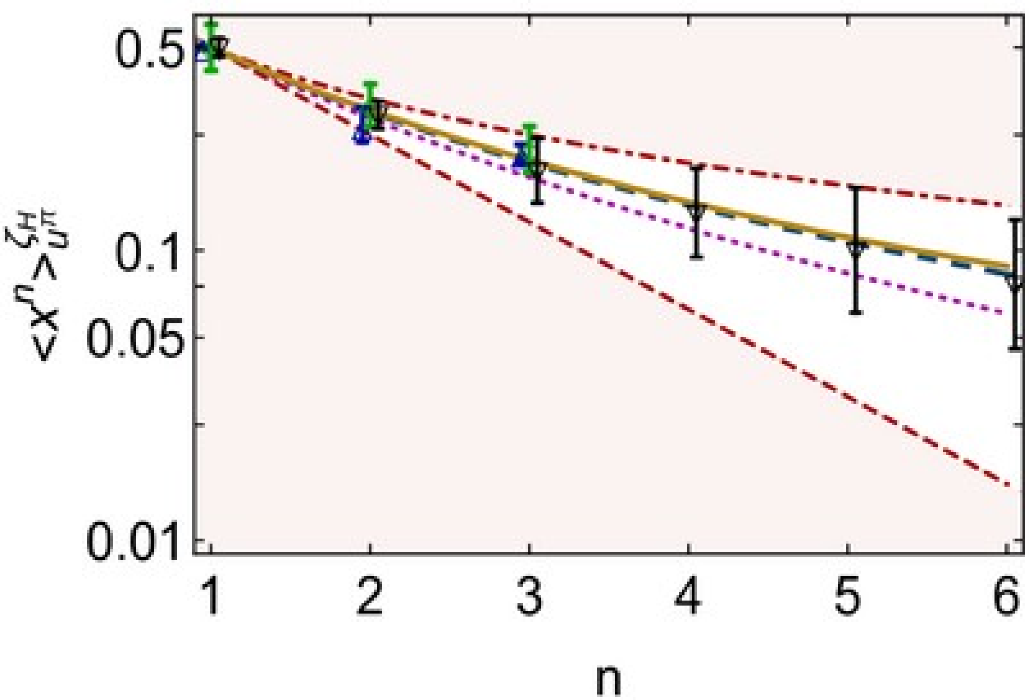
- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} (\langle 2x \rangle^{\zeta}_{u_\pi})^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$q(x;\zeta_H) = \delta(x - 1/2)$ $\qquad$ $q(x;\zeta_H) = 1$



- Keeping **isospin symmetry,** implying:

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	Joo:2019bzr	Sufian:2019bol	Alexandrou:2021mmi
n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

Assumption: define an **effective** charge such that

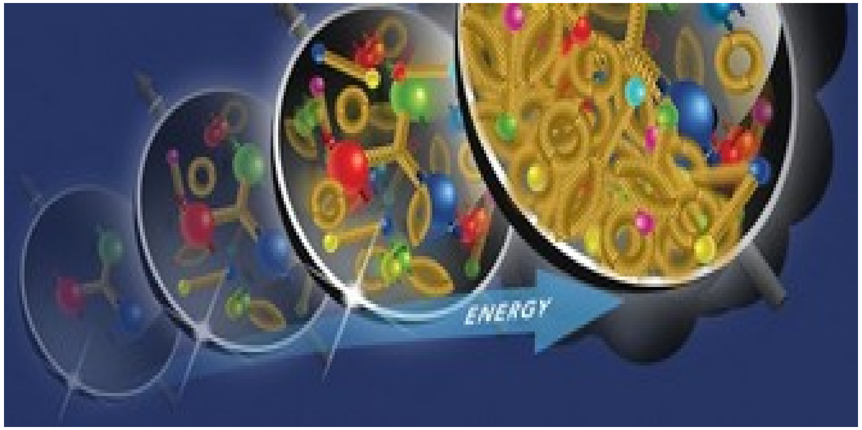
Raya:2021zrz
Cui:2020tdf

Starting from fully-dressed **quasiparticles**, at ζ_H → **Sea** and **Gluon** content unveils, as prescribed by **QCD**

$$\left\{\zeta^2 \frac{d}{d\zeta^2}\mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi}\begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f\gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix}\right\}\begin{pmatrix} \langle x^n\rangle_{\mathrm{NS}}(\zeta) \\ \langle x^n\rangle_{\mathrm{S}}(\zeta) \\ \langle x^n\rangle_g(\zeta) \end{pmatrix} = 0$$

DGLAP ~~leading-order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H \,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right) = \left(\begin{array}{cc} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{array} \right) \left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right)$$

DGLAP: All orders evolution

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$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = [S(\zeta_H, \zeta)]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

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$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

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$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

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DGLAP: All orders evolution

Implication 4: glue and sea from valence

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$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

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DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \boxed{\begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix}} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

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In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

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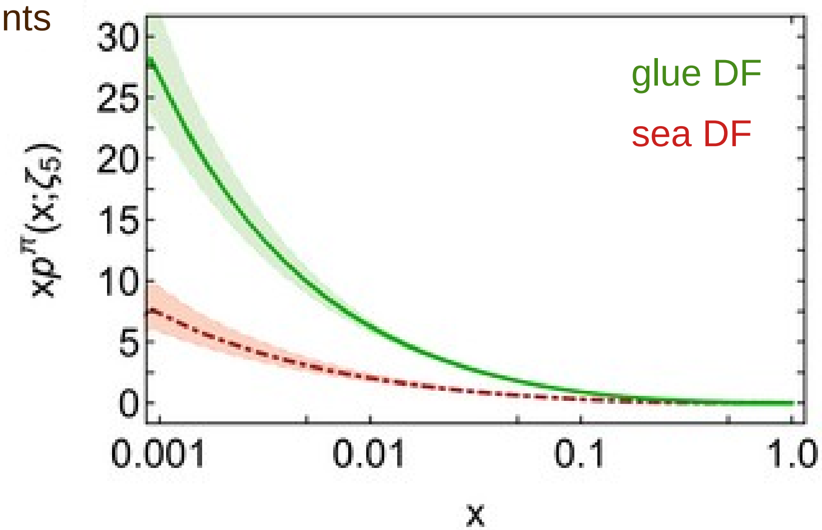
$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \sum_q \langle x^n \rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

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$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

Compute all the moments and reconstruct:

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} \longrightarrow = \left[\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}}\right]^{\lambda^n_\pm/\gamma_{uu}}$$

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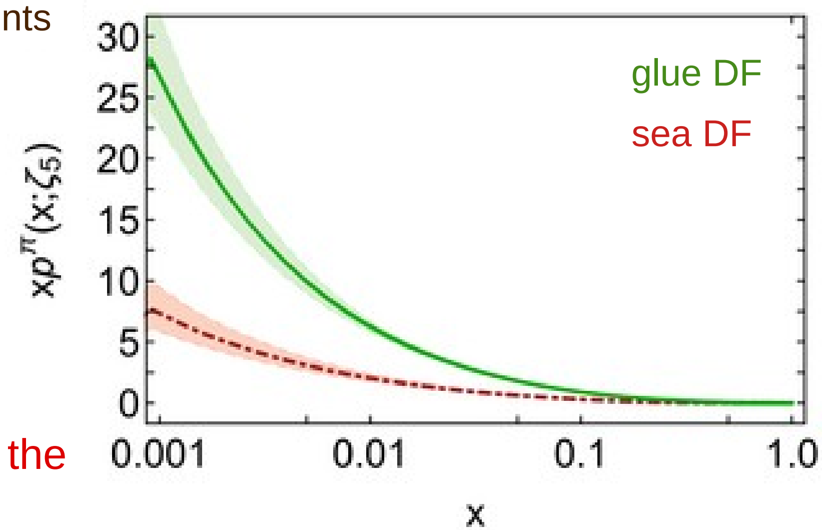

★ The only required input is the the momentum fraction at the probed empirical scale!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H, \ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix} = \begin{pmatrix}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{pmatrix}$$

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$$S^n_\pm = [S(\zeta_H,\zeta)]^{\lambda^n_\pm/\gamma_{uu}} = \left[\frac{\langle x\rangle^{\zeta}_{q_H}}{\langle x\rangle^{\zeta_H}_{q_H}}\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

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n=1 case

$n_f = 4$

$$\langle x\rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x\rangle^{\zeta}_{q_H} + \langle x\rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7}\left[S(\zeta_H,\zeta)\right]^{7/4}$$

$$\langle x\rangle^{\zeta}_{g_H} = \frac{4}{7}\left(1 - \left[S(\zeta_H,\zeta)\right]^{7/4}\right)$$

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DGLAP: All orders evolution

Implication 4: glue and sea from valence

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All quarks active

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n=1 case

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$$\langle x \rangle^{\zeta}_{\Sigma_\pi} = \langle 2x \rangle^{\zeta}_{q_\pi} + \langle x \rangle^{\zeta}_{S_\pi} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

$$\langle x \rangle^{\zeta}_{g_\pi} = \frac{4}{7} \left(1 - \left[S(\zeta_H, \zeta) \right]^{7/4} \right)$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}} = \left[\langle 2x \rangle^{\zeta}_{q_\pi} \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

ζ_5	$\langle 2x \rangle^{\pi}_q$	$\langle x \rangle^{\pi}_g$	$\langle x \rangle^{\pi}_{\mathrm{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

★ The only required input is the the momentum fraction at the probed empirical scale!!

Z-F. Cui et al., arXiv:2006.1465

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \sum_q \langle x^n \rangle^{\zeta_H}_q$$

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$$\langle x \rangle^{\zeta}_{\Sigma_\pi} = \langle 2x \rangle^{\zeta}_{q_\pi} + \langle x \rangle^{\zeta}_{S_\pi} = \frac{3}{7} + \frac{4}{7}\left[S(\zeta_H, \zeta)\right]^{7/4}$$

$$\langle x \rangle^{\zeta}_{g_\pi} = \frac{4}{7}\left(1 - \left[S(\zeta_H, \zeta)\right]^{7/4}\right)$$

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} \longrightarrow = \left[\langle 2x \rangle^{\zeta}_{q_\pi}\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

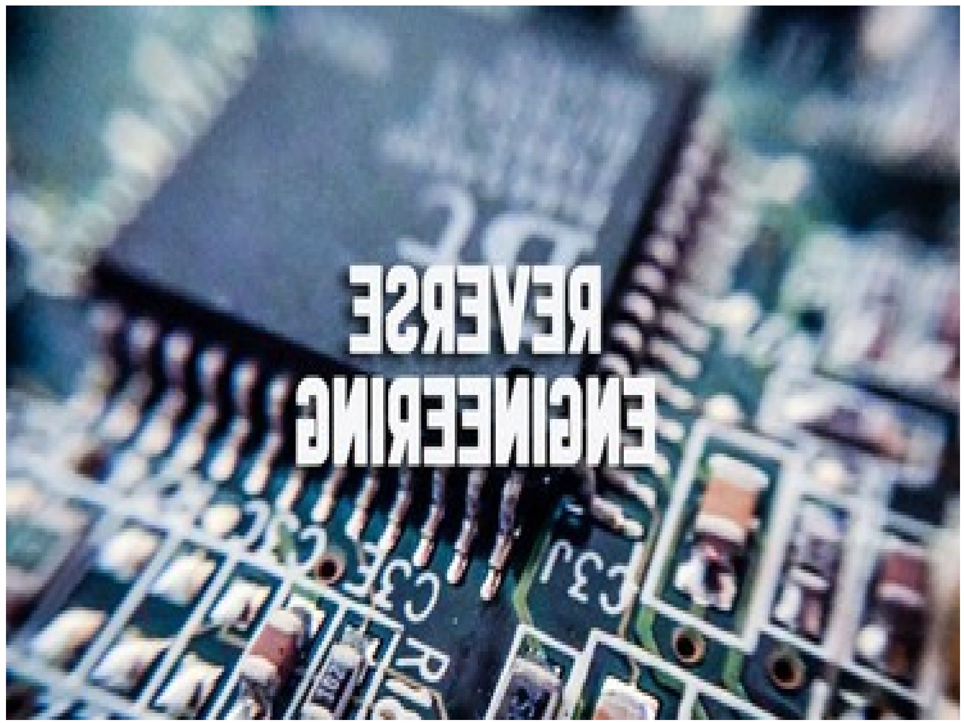
ζ_5	$\langle 2x \rangle^{\pi}_{q}$	$\langle x \rangle^{\pi}_{g}$	$\langle x \rangle^{\pi}_{\text{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

★ The only required input is the the momentum fraction at the probed empirical scale!!

Z-F. Cui et al., arXiv:2006.1465

R.S. Sufian et al., arXiv:2001.04960

Reverse engineering the PDF data



Pion PDF

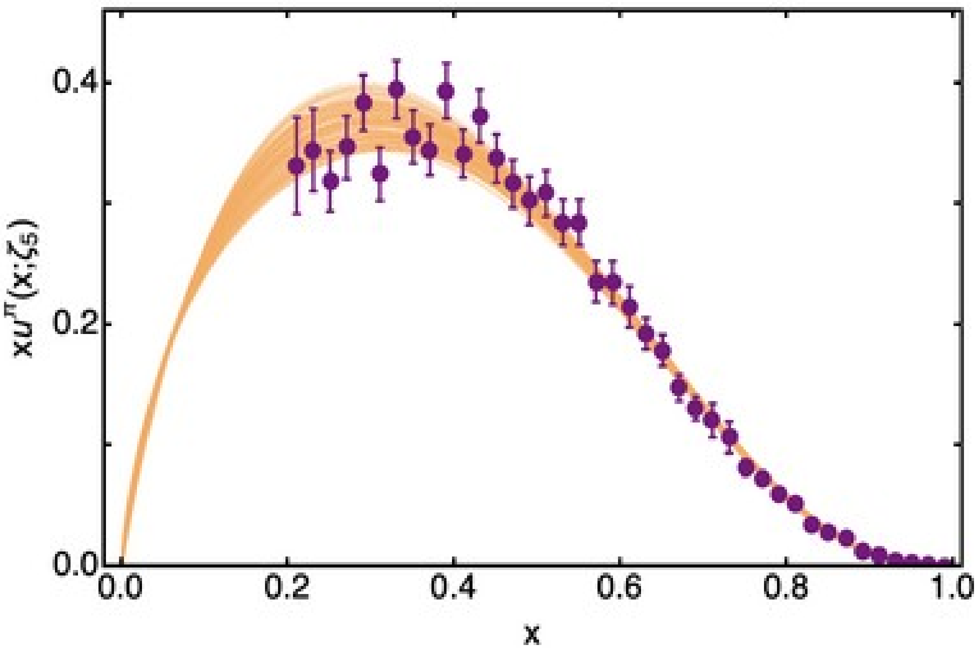
- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{\pi}(x;[\alpha_i];\zeta) = n_u^{\zeta} x^{\alpha_1^{\zeta}} (1-x)^{\alpha_2^{\zeta}} (1+\alpha_3^{\zeta} x^2)$$

Normalization

$$\{\alpha_i^{\zeta} | i = 1,2,3\}$$

Free parameters



Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion **PDF**

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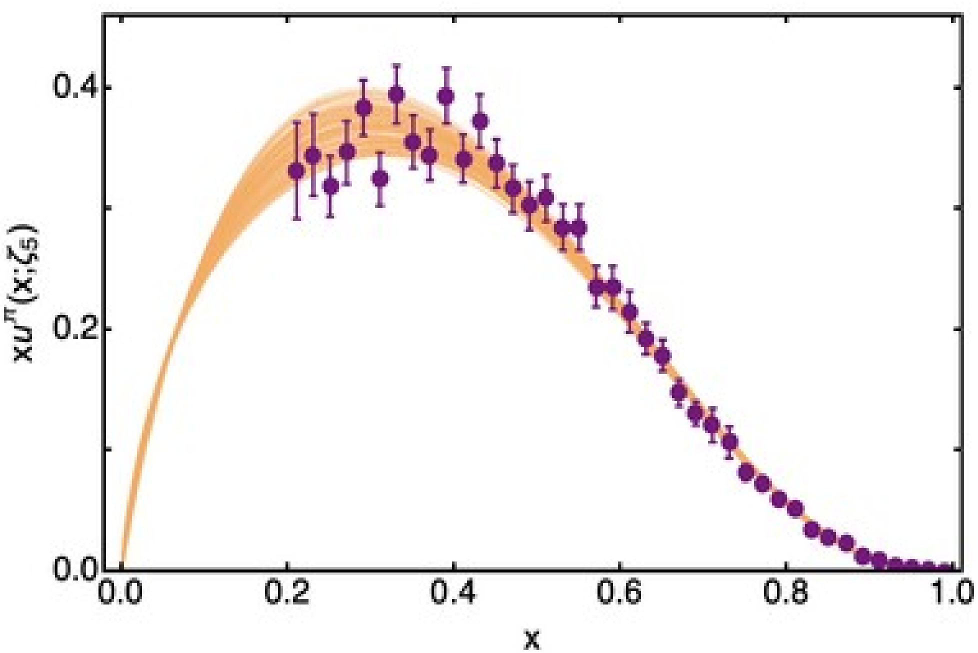

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the data.

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

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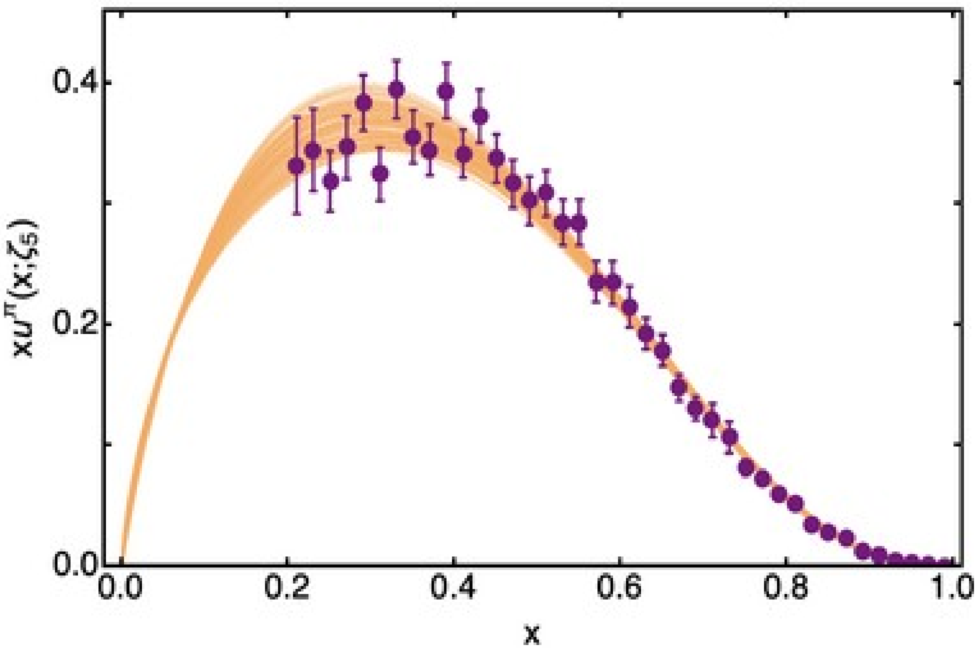

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2) Generate new **values α_i,** distributed randomly around the best fit.

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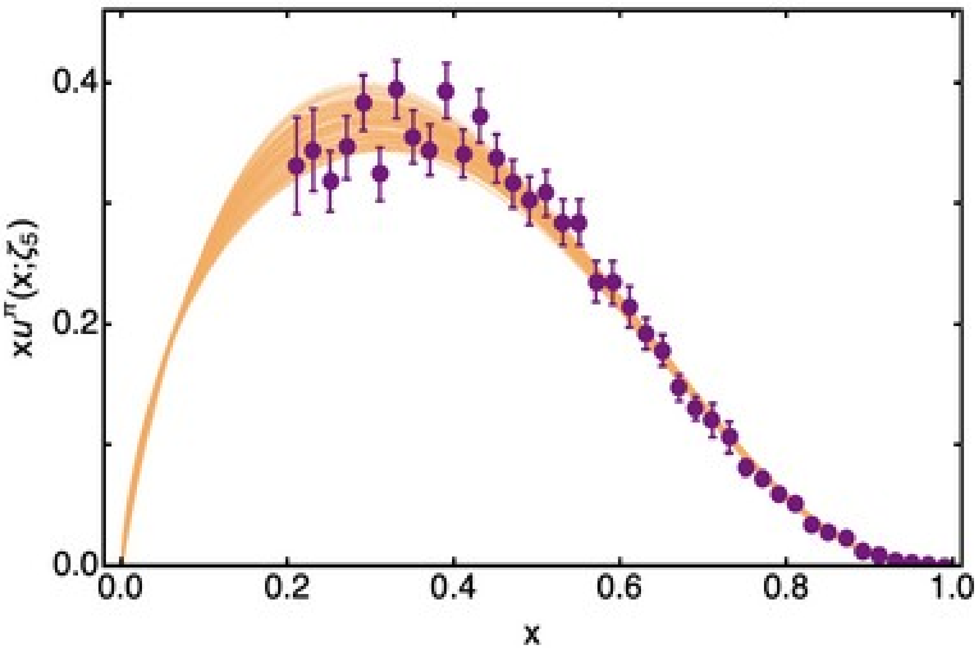

- Then, we proceed as follows:

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3) Using the latter set, evaluate:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

Data point with error

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF

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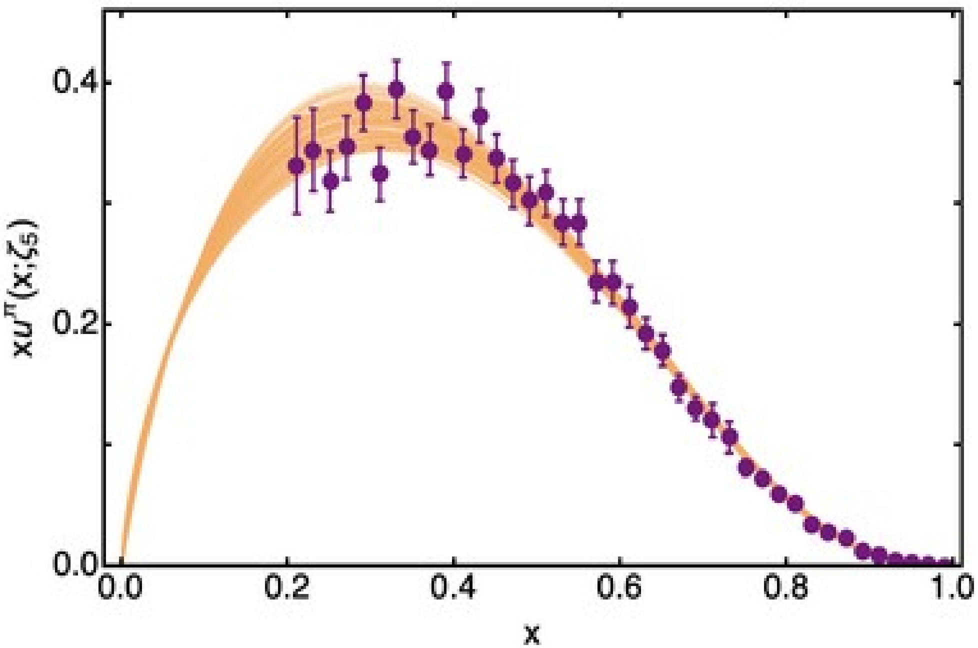

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Data point with error

4) Accept a replica with probability:

$$\mathcal{P} = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,,\quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} \mathrm{e}^{-y/2}$$

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

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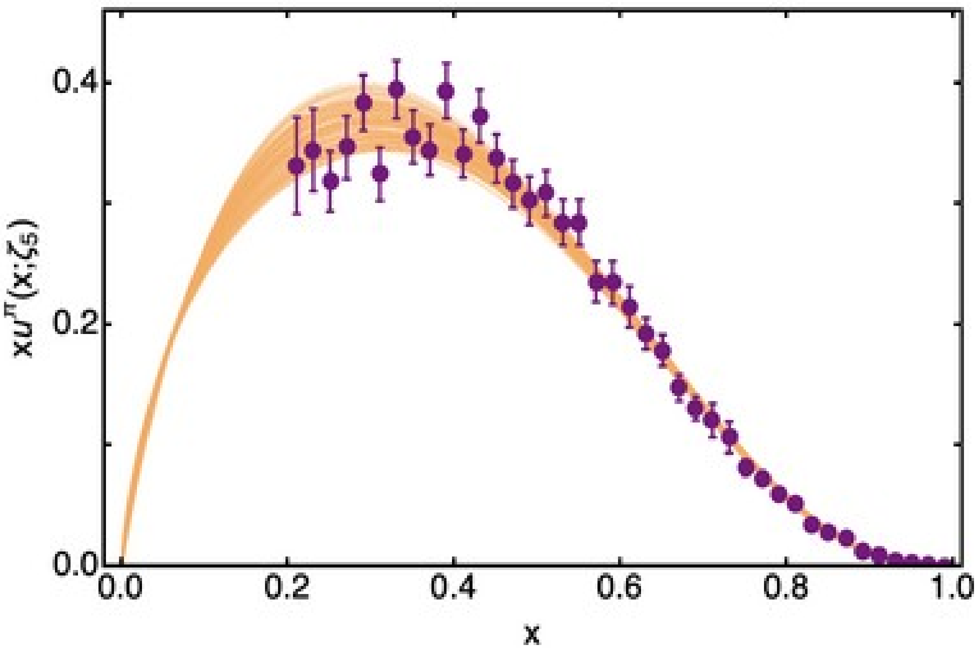

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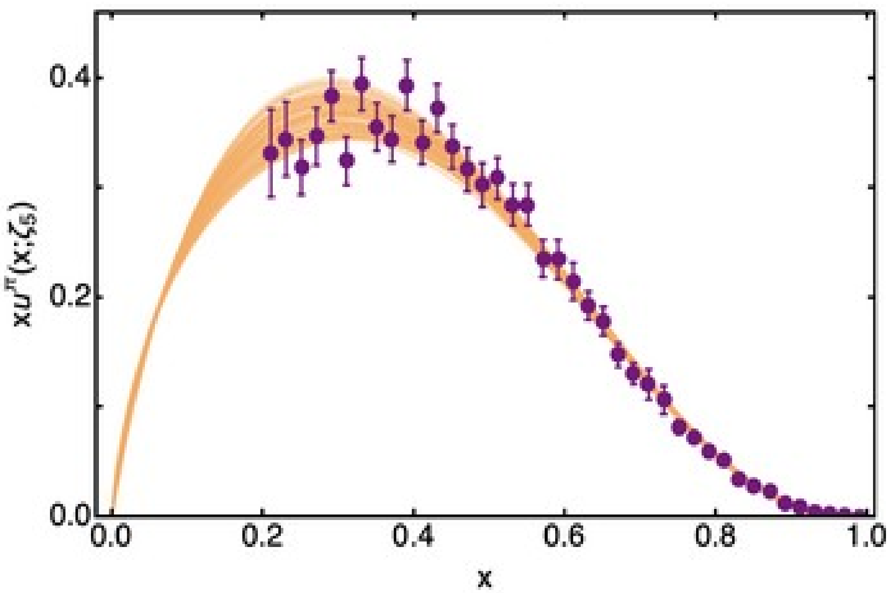
5) Evolve back to ζ_H

Repeat (2-5).

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF: ASV analysis of E615 data

- Applying this algorithm to the **ASV data** yields:



Mean values (of moments) and errors

$\{\{0.5, 2.75144\times10^{-17}\}, \{0.299833, 0.00647045\}, \{0.199987, 0.00735448\}, \{0.142895, 0.0068623\}, \{0.107274, 0.00608759\}, \{0.0835168, 0.00532834\}, \{0.0668711, 0.0046596\}, \{0.0547511, 0.00409028\}, \{0.0456496, 0.00361041\}, \{0.0386394, 0.00320609\}\}$

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF: ASV analysis of E615 data

➢ Applying this algorithm to the **ASV data** yields:

Mean values (of moments) and errors

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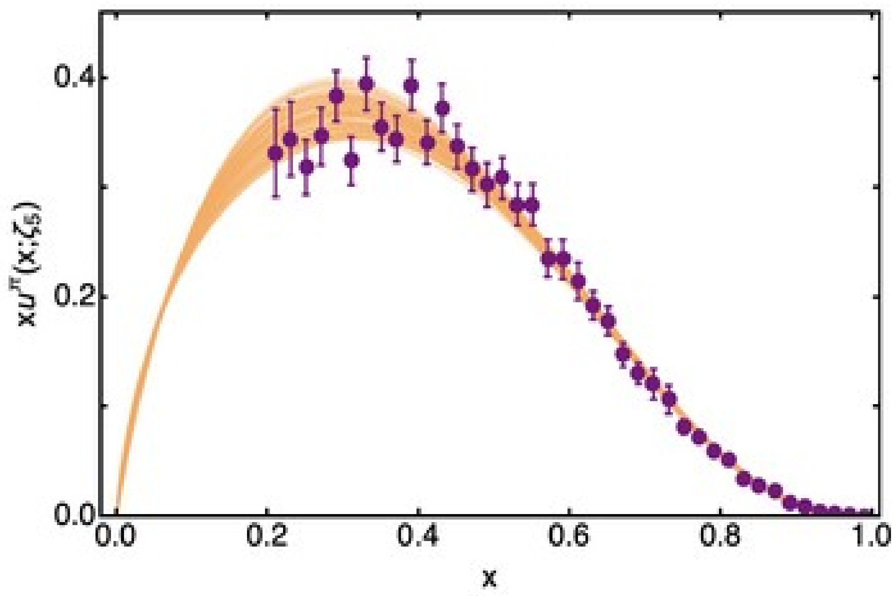
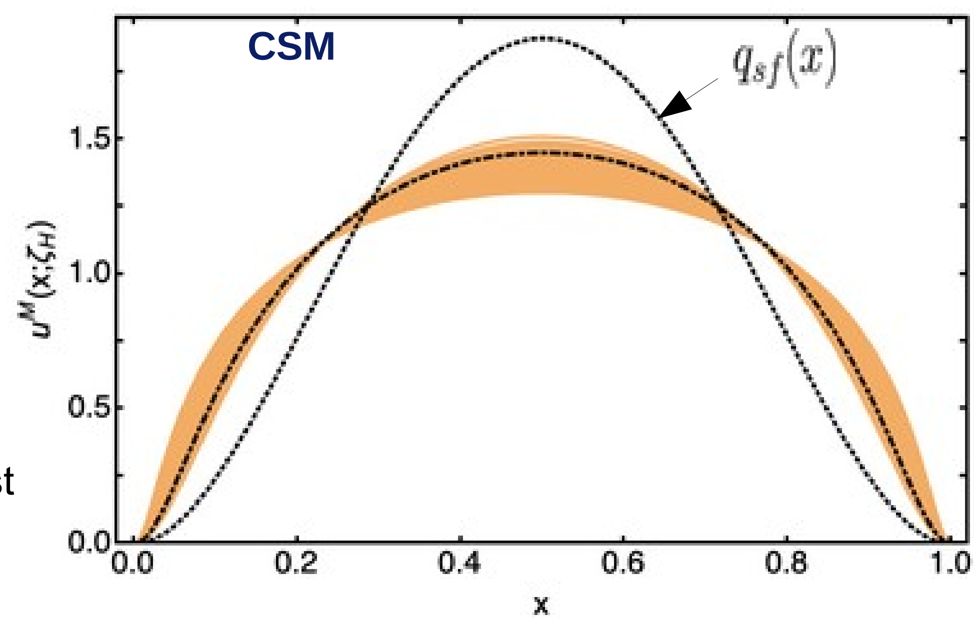


✔ Then, we can **reconstruct** the moments produced by each replica, using the **single-parameter Ansatz**:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$



✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ It seems it favors a **soft end-point** behavior just like the **CSM result**.

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Kaon PDF

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the ASV data for the pion.
2) Use u^K/u^π data to fix the only free parameter for the kaon
3) Generate new **values α_i,** distributed randomly around the best fit parameters
4) With these values, evaluate for the pion:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}, \quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

Kaon PDF

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

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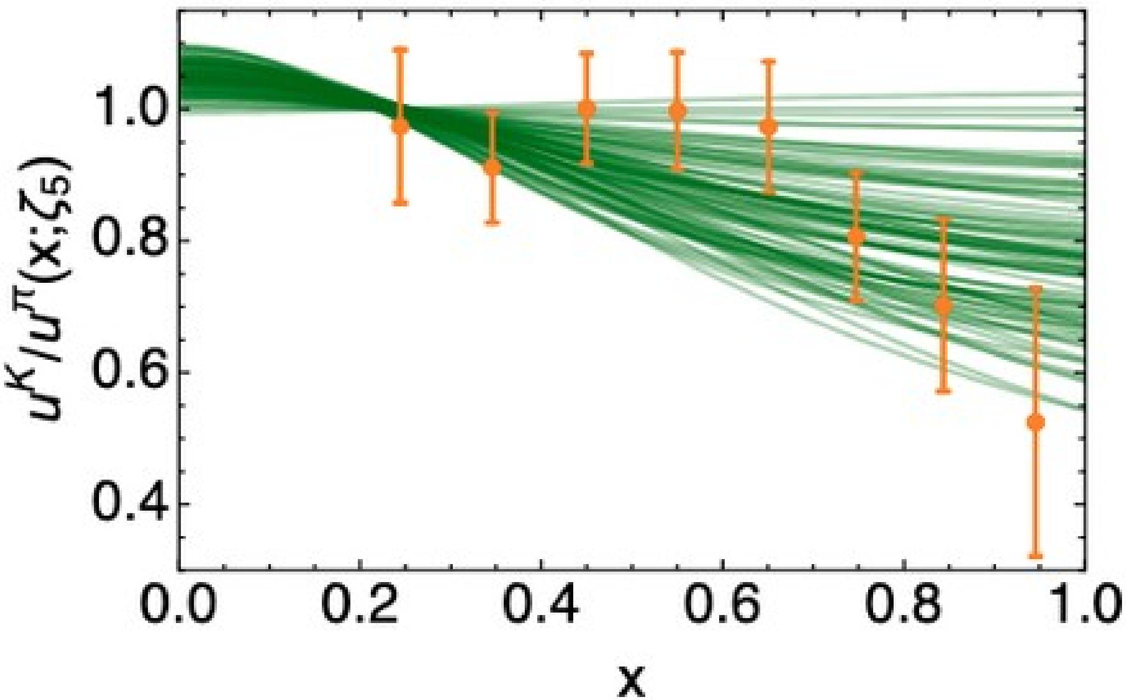

Data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

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$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} \mathrm{e}^{-y/2}$$

5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\ ,\quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}}\mathcal{P}_{u_\pi}$$

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Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

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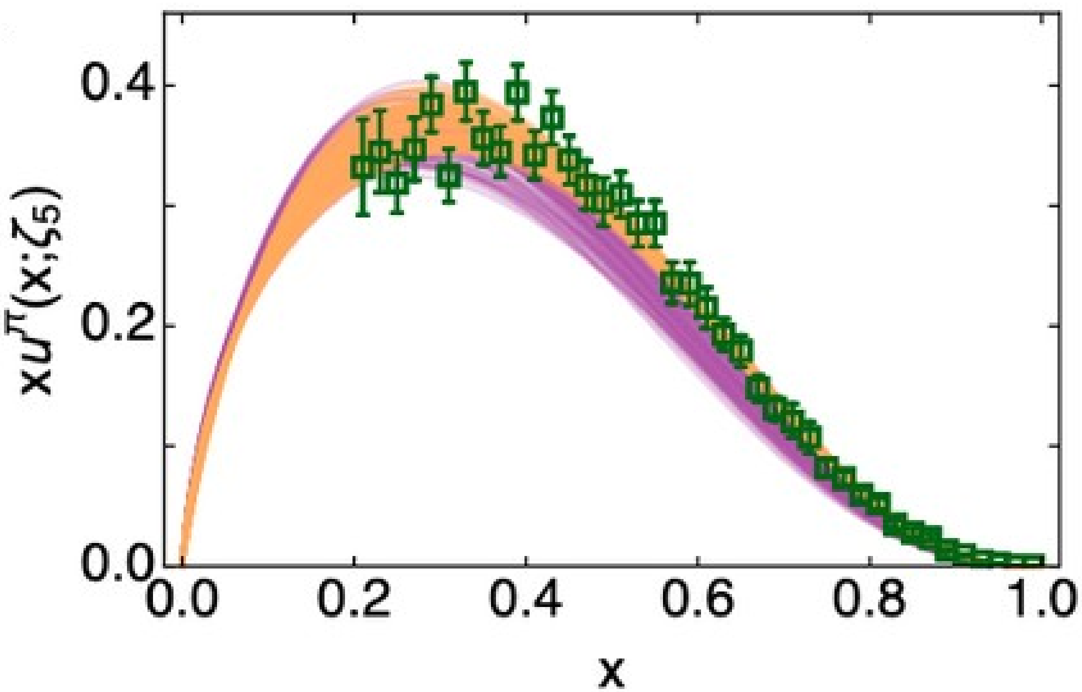

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5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\ , \quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

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- Let us assume the data can be parameterized with a certain functional form, i.e.:

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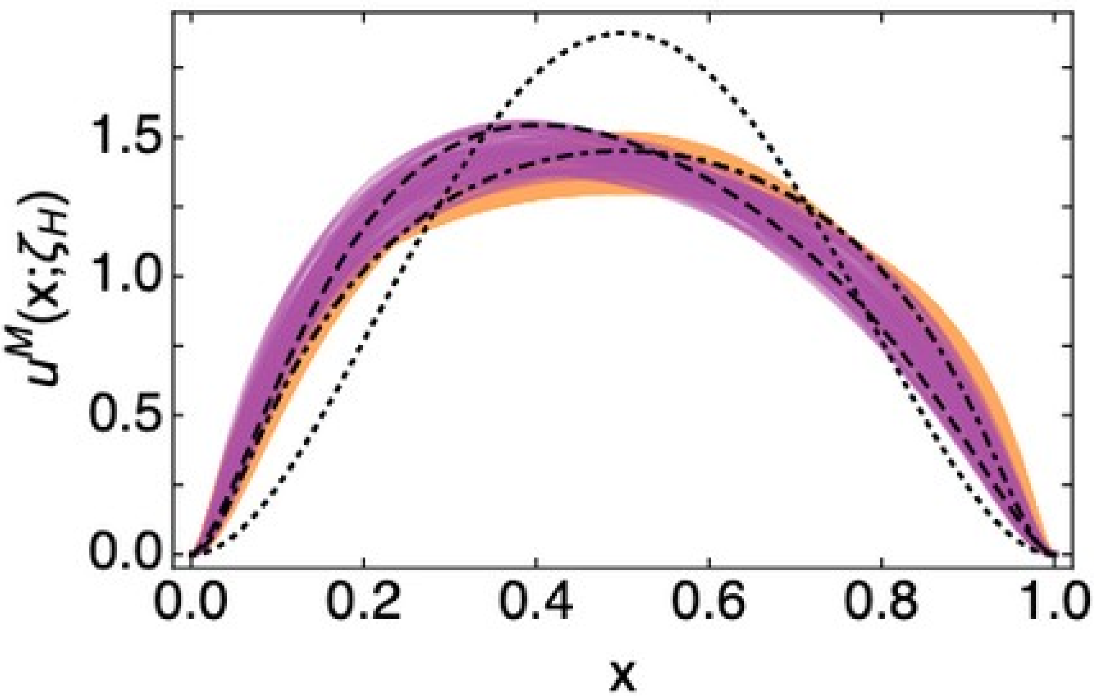

$$q^M(x;\zeta_H) = n_0 \ln\left[1 + x^2(1-x)^2/\rho_M^2\right][1 \pm \gamma_M(1-2x)]$$

- Then, we proceed as follows:

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$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) And for the kaon in terms of data for

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6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\,, \quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

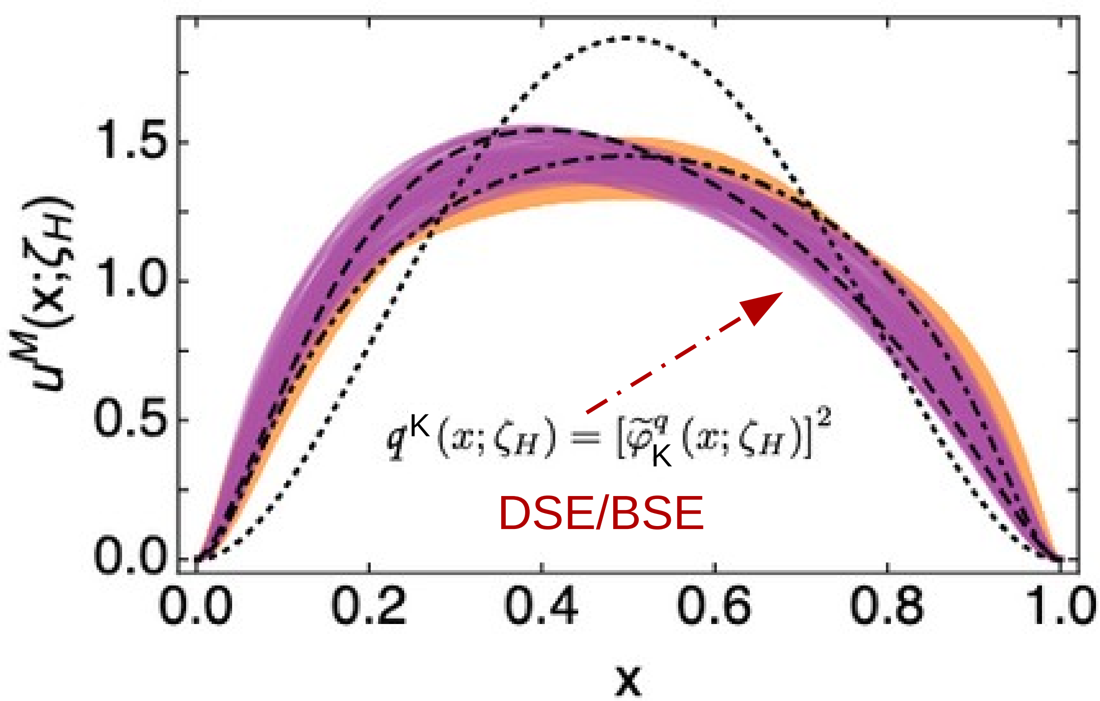
Kaon PDF

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ



$$q^M(x;\zeta_H) = n_0 \ln\left[1 + x^2(1-x)^2/\rho_M^2\right][1 \pm \gamma_M(1-2x)]$$

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$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\ , \quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

Kaon PDF

- Let us assume the data can be parameterized with a certain functional form, i.e.:

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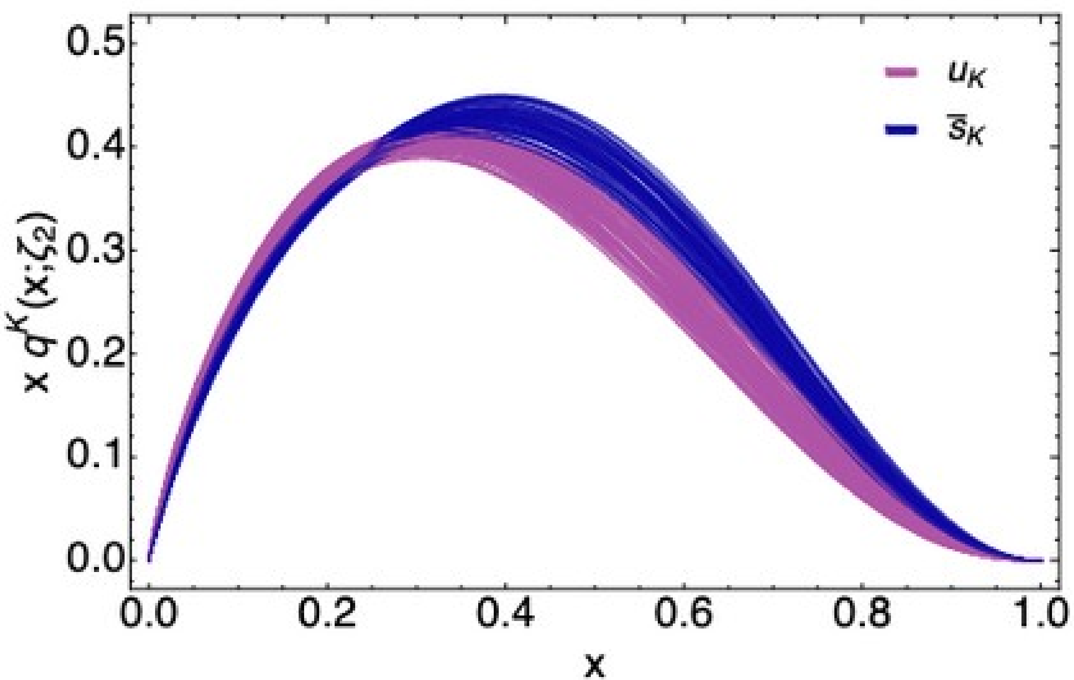
Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ



Capitalizing on $\bar{s}^K(x;\zeta_H) = u^K(1-x;\zeta_H)$, antiquark DF can be derived for each replica and be evolved up again

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the ASV data for the pion.
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$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,,\quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

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$$\mathcal{P}_{u_\pi}\;,\quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

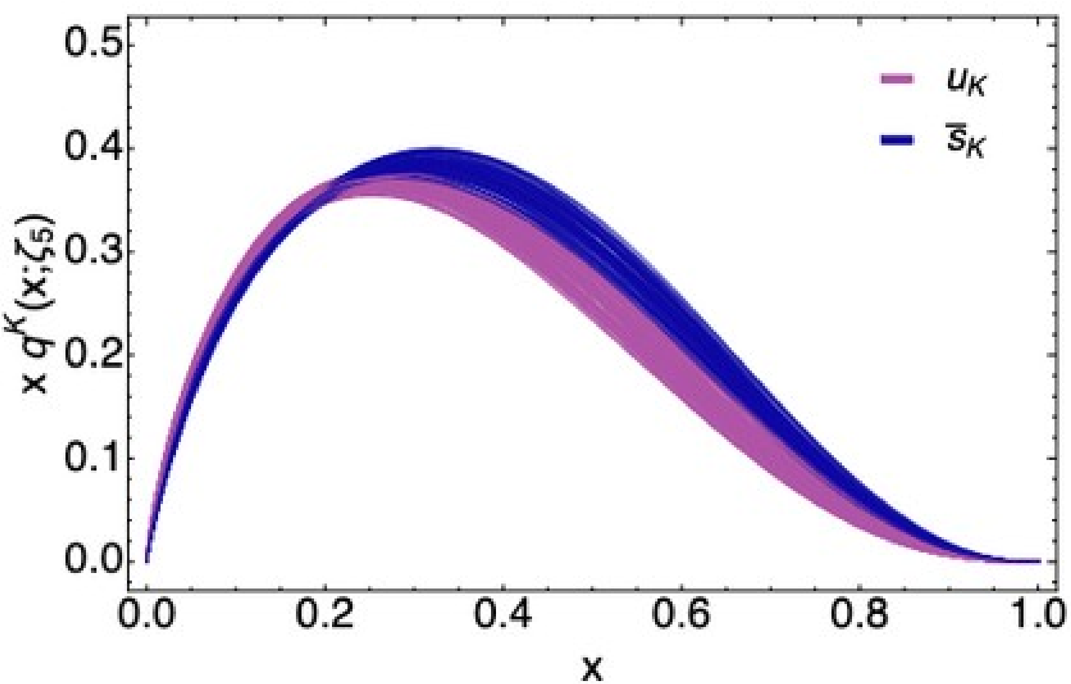
Kaon PDF

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ



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2) Use u^K/u^π data to fix the only free parameter for the kaon

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$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

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5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\ , \quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

Kaon PDF: glue and quark singlet

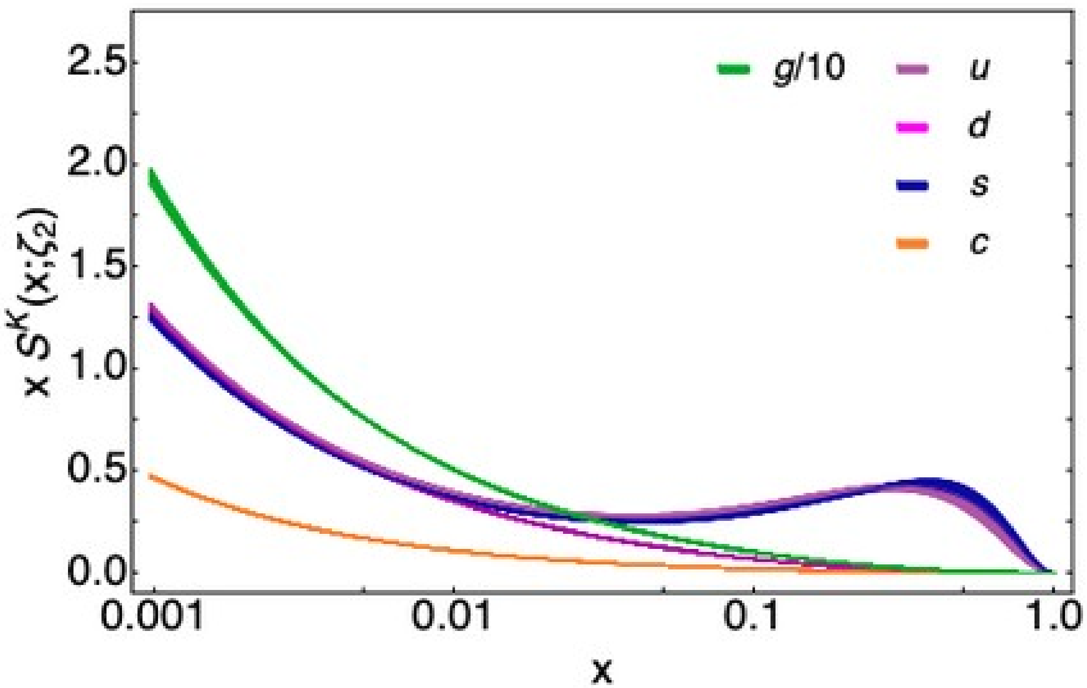
Z-N. Xu et al., arXiv:2411.15376v2

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ



… and, similarly, glue and sea-quark DFs can be also obtained, at different empirical scales!

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the ASV data for the pion.
2) Use u^K/u^π data to fix the only free parameter for the kaon
3) Generate new **values α_i,** distributed randomly around the best fit parameters
4) With these values, evaluate for the pion:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\,, \quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

Kaon PDF: glue and quark singlet

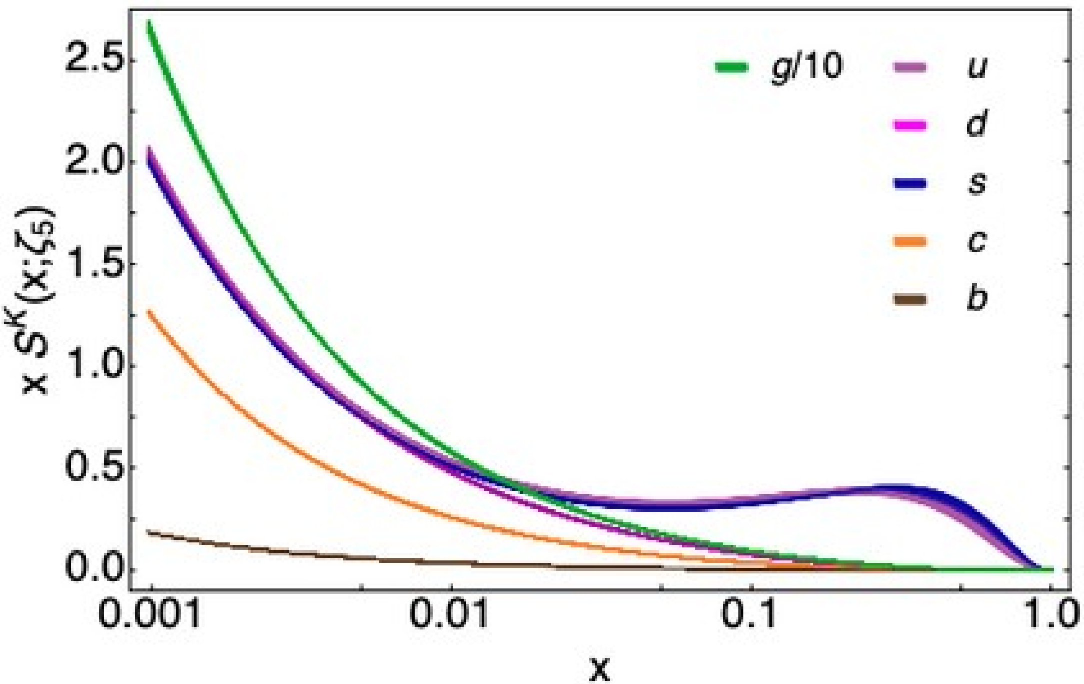
Z-N. Xu et al., arXiv:2411.15376v2

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ



… and, similarly, glue and sea-quark DFs can be also obtained, at different empirical scales!

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the ASV data for the pion.
2) Use u^K/u^π data to fix the only free parameter for the kaon
3) Generate new **values α_i,** distributed randomly around the best fit parameters
4) With these values, evaluate for the pion:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

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Momentum fractions and comparisons

	$\langle x\rangle^{\zeta}_{q^K}$	$\langle x\rangle^{\zeta}_{S^K_q}$	$\langle x\rangle^{\zeta}_{g^K}$	$\langle x\rangle^{\zeta}_{q^\pi}$
ζ_2				
u	0.230(6)(10)	0.028(2)		0.241(5)(10)
d	0	0.028(2)		0.241(5)(10)
s	0.252(6)(11)	0.026(1)		
c	0	0.008(1)		
b	0	0		
g			0.428(18)	
ζ_5				
u	0.197(5)(9)	0.036(2)		0.207(4)(9)
d	0	0.036(2)		0.207(4)(9)
s	0.216(5)(9)	0.034(2)		
c	0	0.019(1)		
b	0	0.003(1)		
g			0.461(20)	

	empirical		[32, lQCD]	
M	π	K	π	K
l	0.538(15)	0.286(12)	0.499(55)	0.317(19)
s	0.026(01)	0.278(13)	0.036(15)	0.339(11)
c	0.008(01)	0.008(01)	0.013(16)	0.028(21)
q	0.572(15)	0.572(18)	0.575(79)	0.683(50)
g	0.428(18)	0.428(18)	0.402(53)	0.422(67)

[32] Alexandrou, et al., arXiv:2405.08529 [hep-lat]

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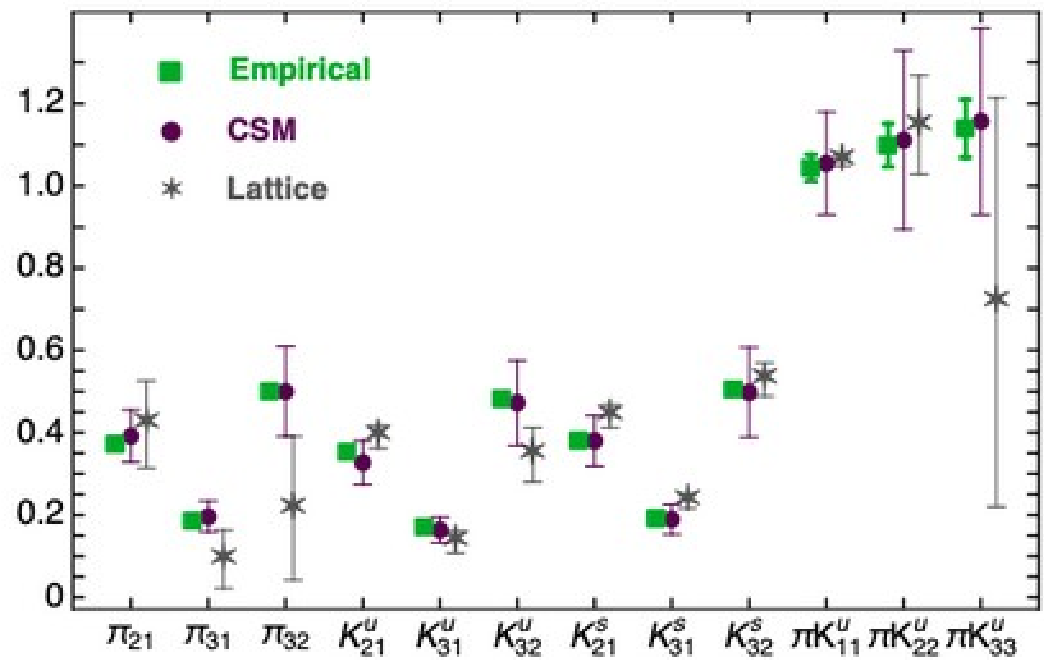

$$\pi_{ij} = \langle x^i\rangle_{u_S^\pi} / \langle x^j\rangle_{u_S^\pi}$$
$$K^q_{ij} = \langle x^i\rangle_{q_S^K} / \langle x^j\rangle_{q_S^K}$$
$$\pi K^u_{ij} = \langle x^i\rangle_{u_S^\pi} / \langle x^j\rangle_{u_S^K}$$

CSM = Z-F Cui, et al., Eur. Phys. J. C80 (2020) 1064.

Lattice = C. Alexandrou, et al., Phys. Rev. D 103 (1) (2021) 014508; Phys. Rev. D 104 (5) (2021) 054504.

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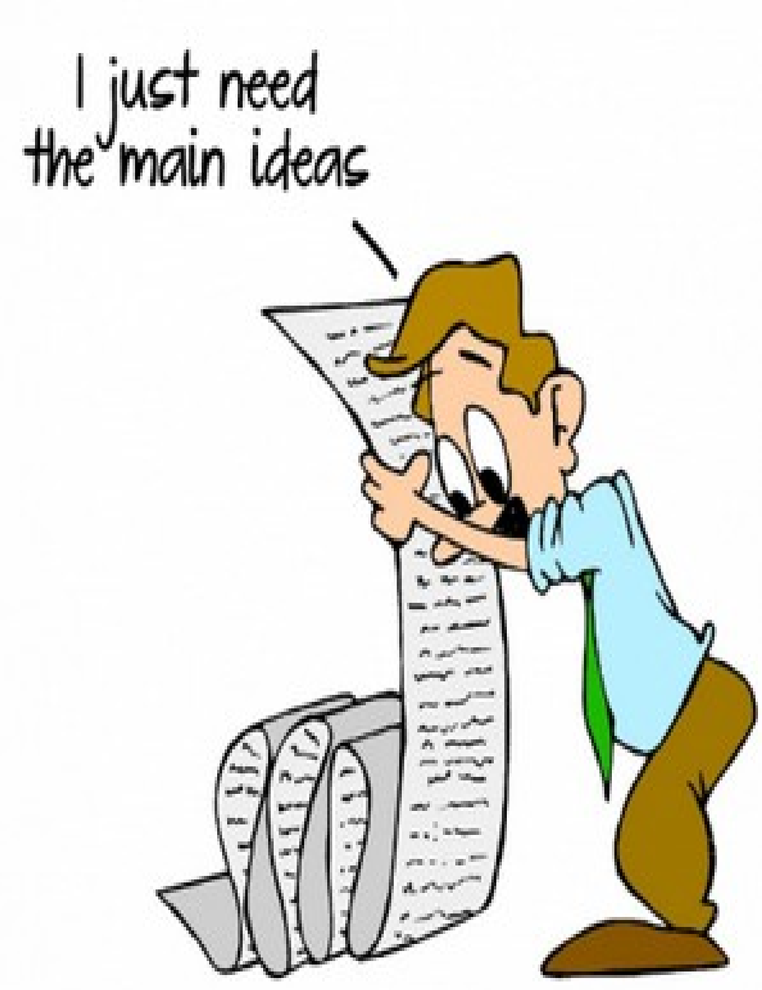
Summary



Summary

- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.
- Capitalizing on the latter, two main ideas emerge: (I) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.
- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and experimental data have been shown to confirm **CSM** results.
- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the pion, kaon and proton cases. A model featuring massless evolution for quark flavors activated after a hard-wall threshold and accounrting for Pauli blocking has been solved analytically, and seen to expose some of the main results implied by the approach.

To be continued...

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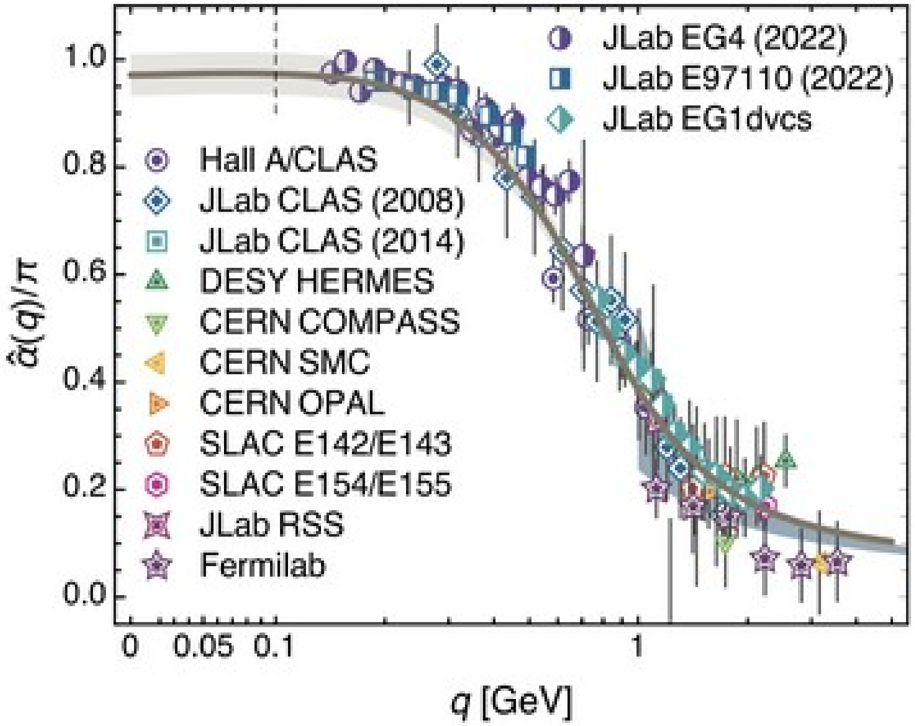
To be continued...

Thank you!

Backslides

QCD effective charge



Modern continuum & lattice QCD analysis in the gauge sector delivers an analogue "Gell-Mann-Low" running charge, from which one obtains a process-independent, parameter-free prediction for the low-momentum saturation

- No landau pole
- Below a given mass scale, the interaction become scale-independent and QCD practically conformal again (as in the lagrangian).

Then, we define:

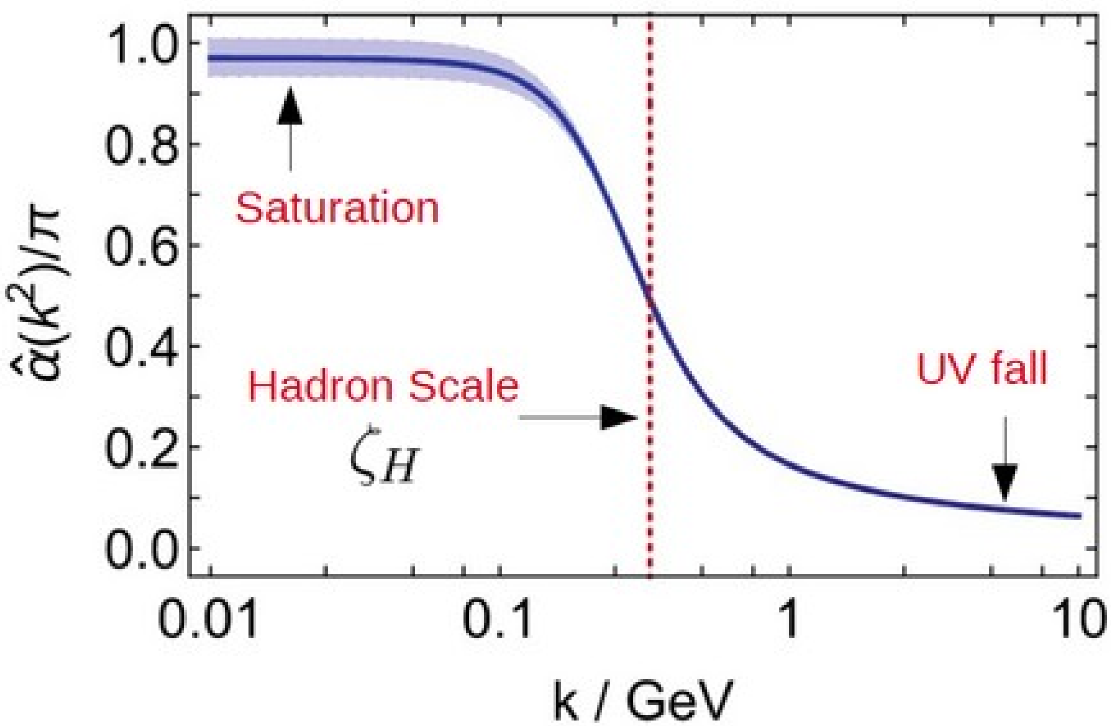
$$\alpha(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda^2_{\rm QCD}}\right]}\,;\ \alpha(0) = 0.97(4)$$

where

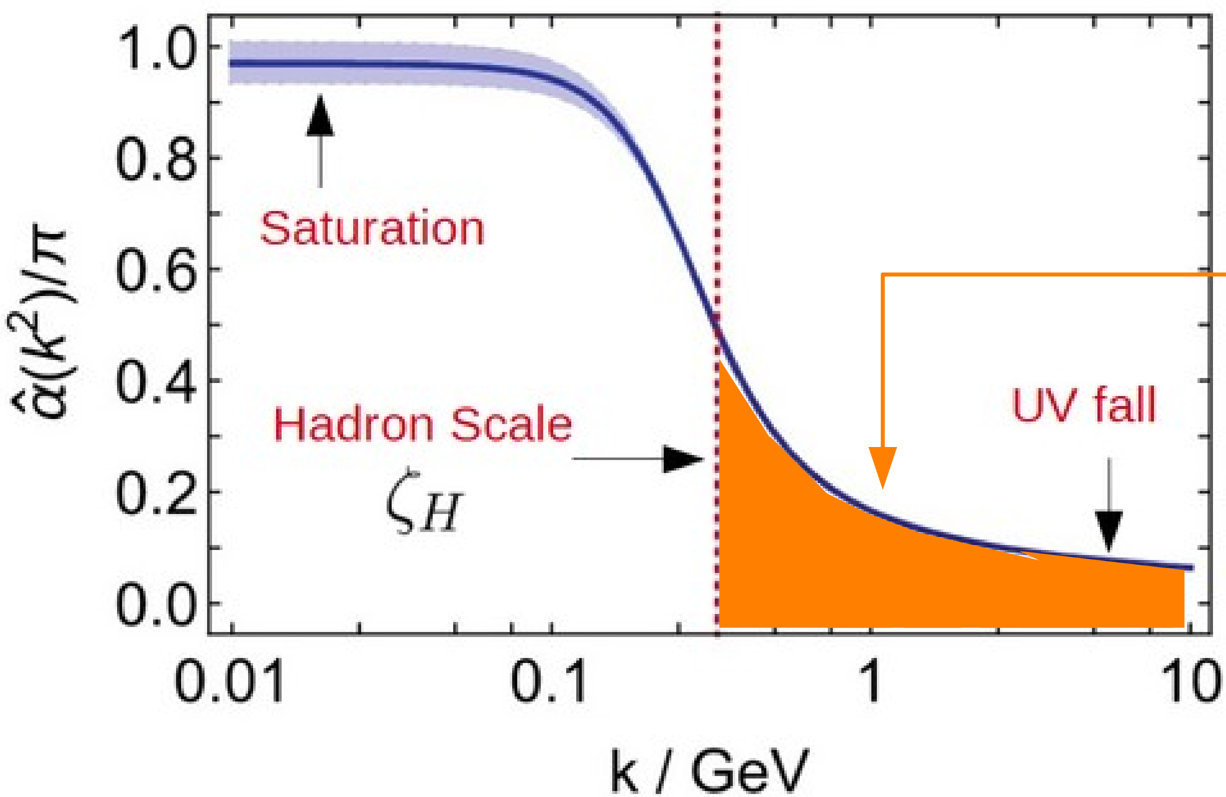
$$\mathcal{M}(k^2 = \Lambda^2_{\rm QCD}) := m_G = 0.331(2)\ \text{GeV}$$

defines the screening mass and an associated wavelength, such that larger gluon modes decouple.

Then, we identify: $\zeta_H := m_G(1 \pm 0.1)$



QCD effective charge

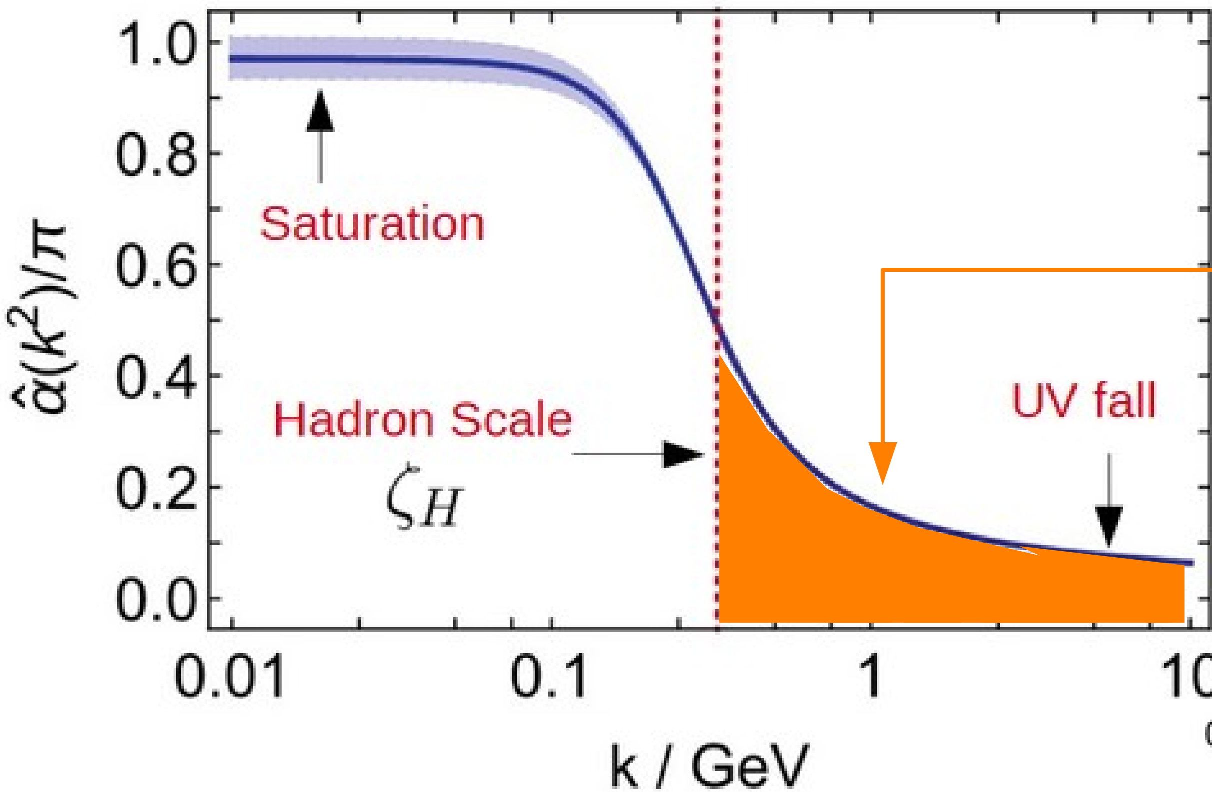


The strength of the charge defines de input for the evolution

$$S(\zeta_H, \zeta_f) = \int_{2\ln(\zeta_H/\Lambda_{QCD})}^{2\ln(\zeta_f/\Lambda_{QCD})} dt\, \widehat{\alpha}(t)$$

$$\langle x(\zeta_5)\rangle_q^\pi = \frac{1}{2}\exp\left(-\frac{8}{9\pi}S(\zeta_H, \zeta_5)\right) = 0.20(2)$$

QCD effective charge



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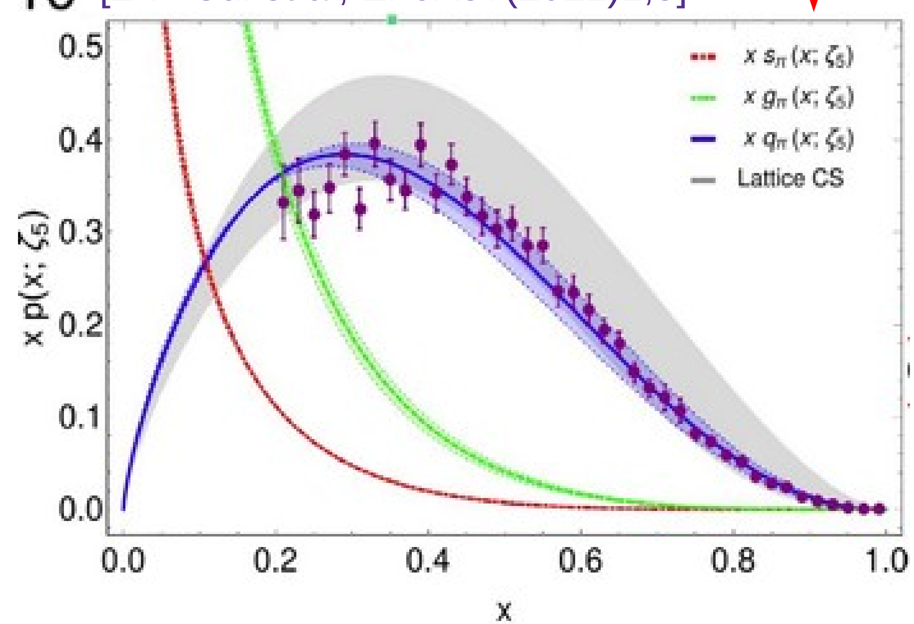
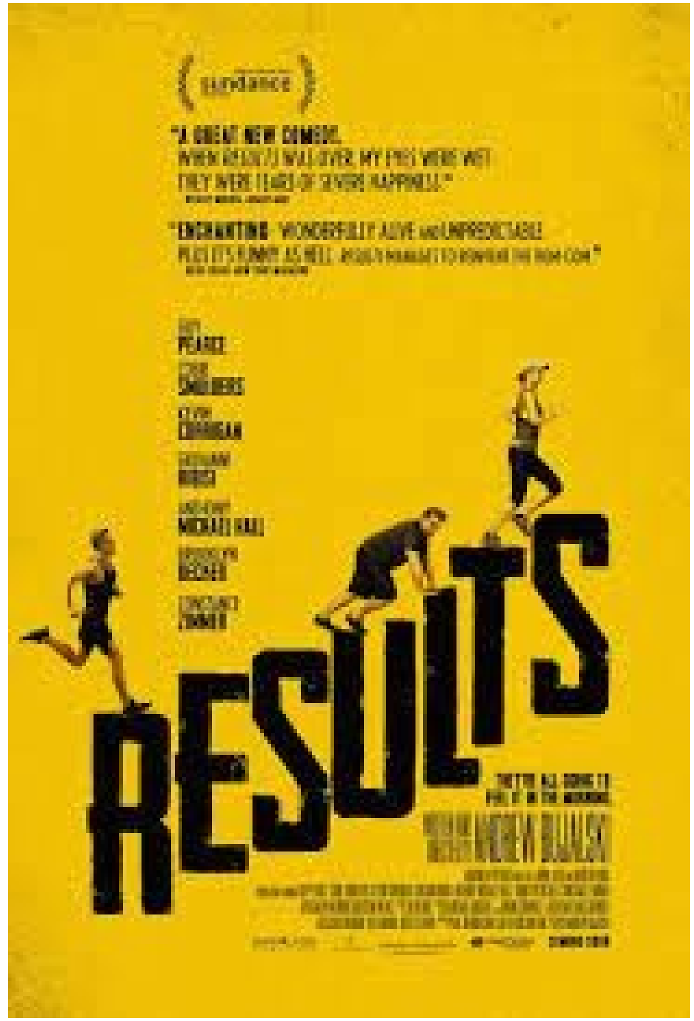
[Z-F. Cui et al, EPJC80(2020)11,1064]
[Z-F. Cui et al, EPJA57(2021)1,5]



Then, the glue, valence- and sea-quark DFs can be predicted, with no tuned parameter, on the ground of the effective charge definition, from the LFWF (or, equivalentely, from a symmetry-preserving DSE/BSE computation of the valence-quarks Mellin moments

[M. Ding et al, CPC44(2020)3,031002]


sundance
"A GREAT NEW COMEDY.
"ENCHANTING
PEARCE
CORRIGAN
RESULTS
ANDREW BUJALSKI

Pion PDF: from CSM (DSEs) to the experiment

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^{\pi}(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar{\eta}\eta};\zeta)\, S(k_{\bar{\eta}};\zeta) \times \{n \cdot \frac{\partial}{\partial k_\eta}\left[\Gamma_\pi^{-P}(k_{\eta\bar{\eta}};\zeta) S(k_\eta;\zeta)\right]\}\,.$$

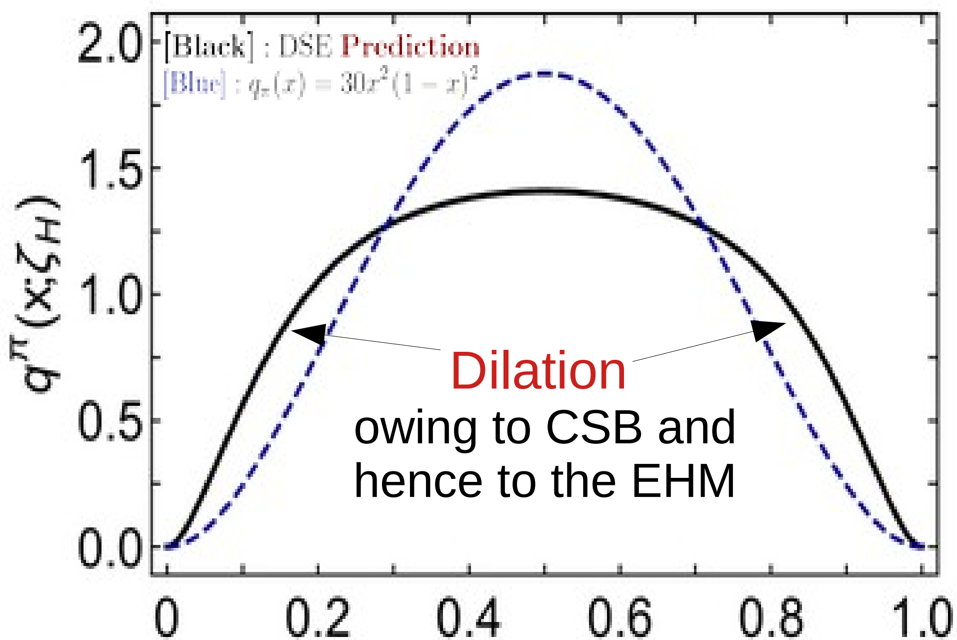
$$q_{\mathrm{O}}^{\pi}(x;\zeta_H) = 213.32\, x^2 (1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$

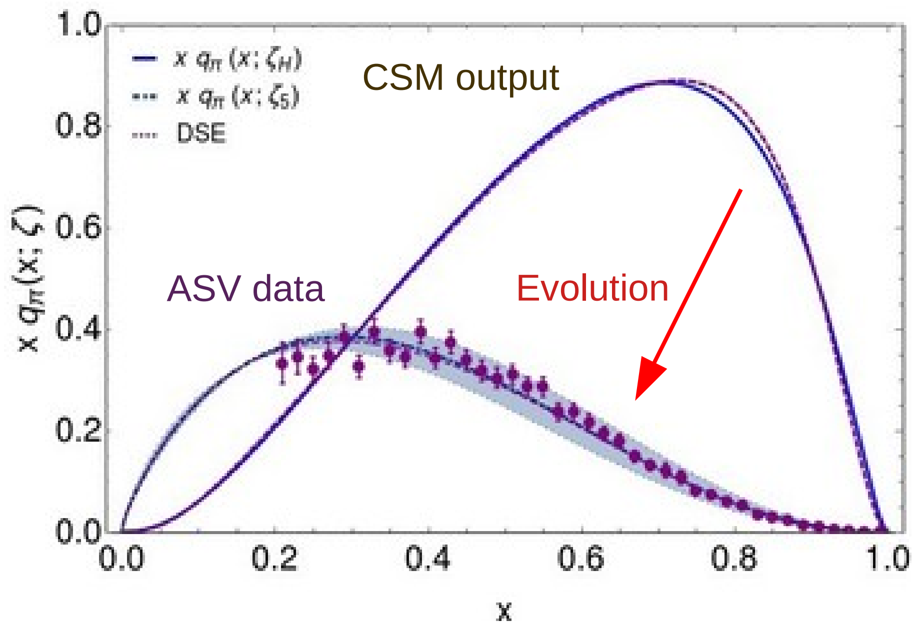
$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\,(1+\mathcal{O}(1-x))$$

$$\beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416
Berger, Brodsky, Phys.Rev.Lett 42(1979)940

- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)





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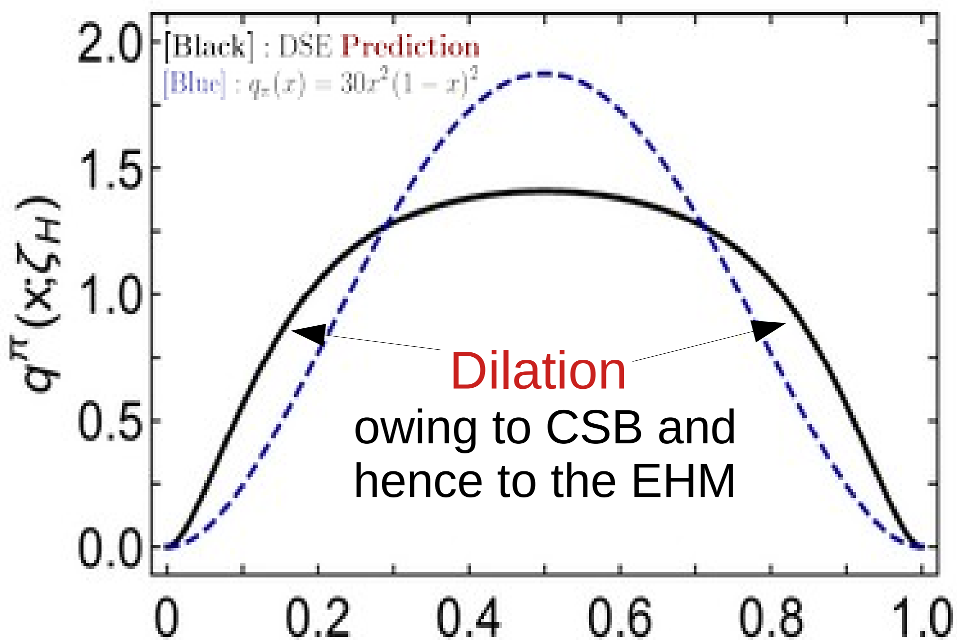
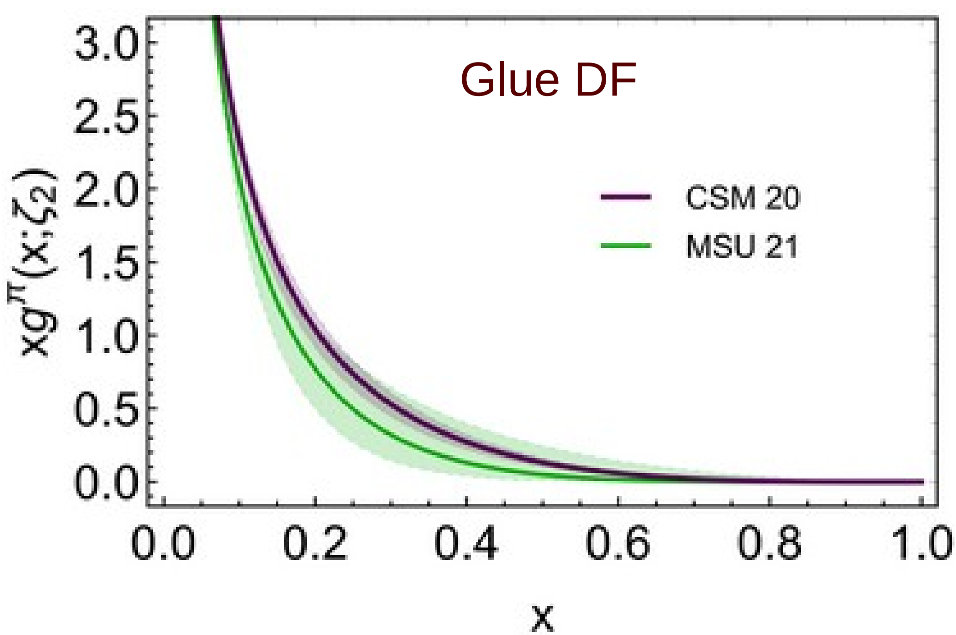
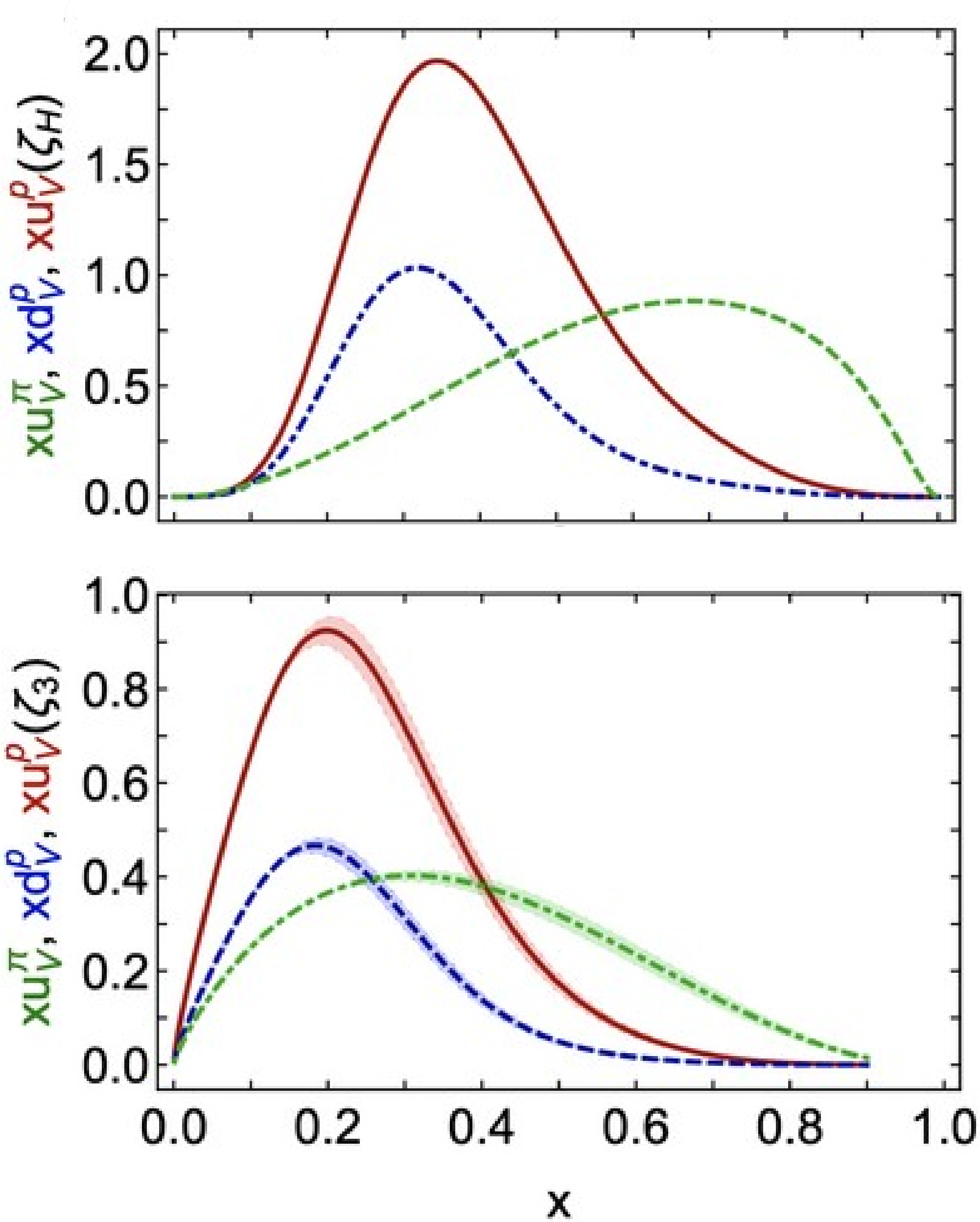




Proton PDF: from CSM (DSEs) to the experiment

An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]



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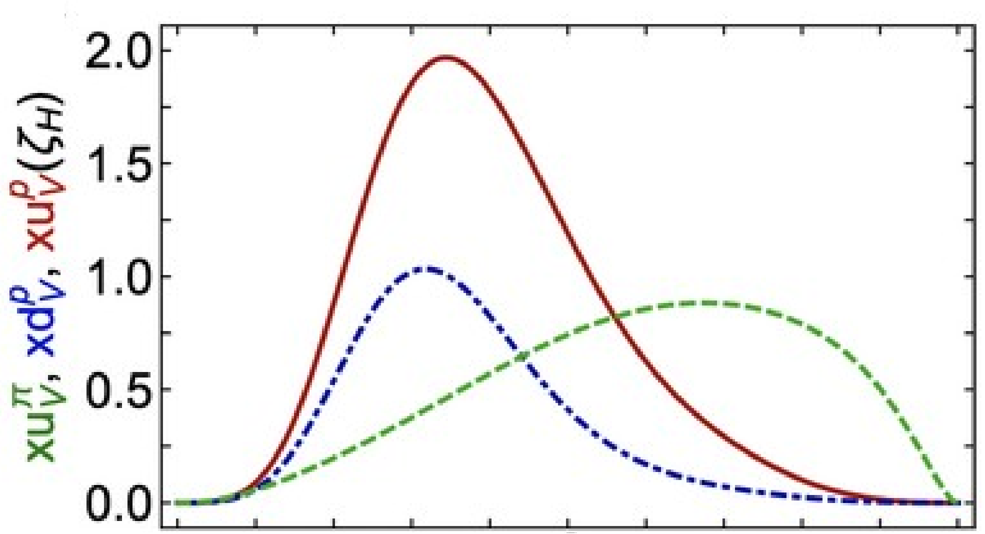
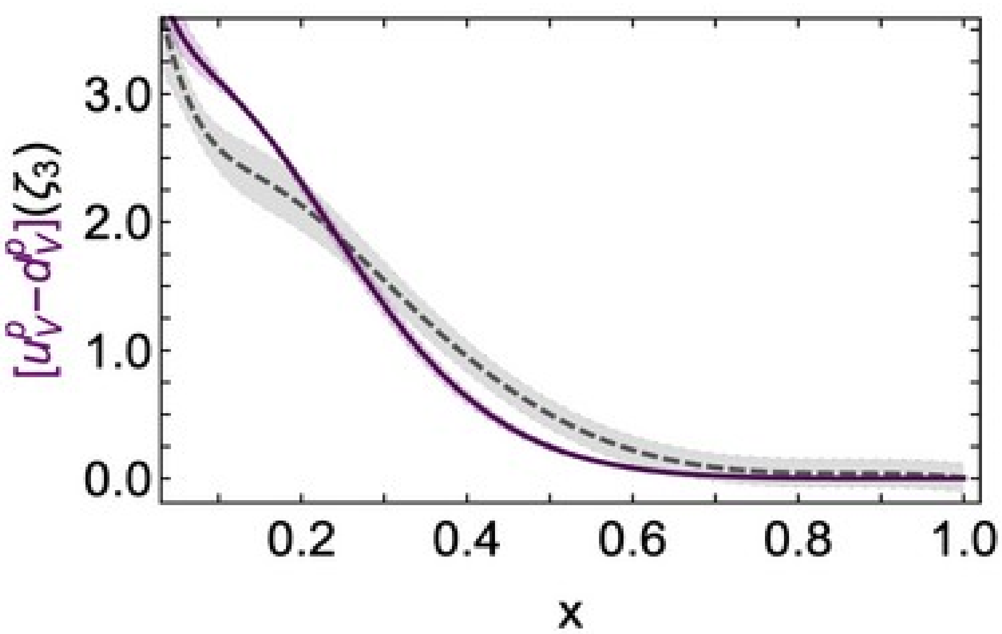
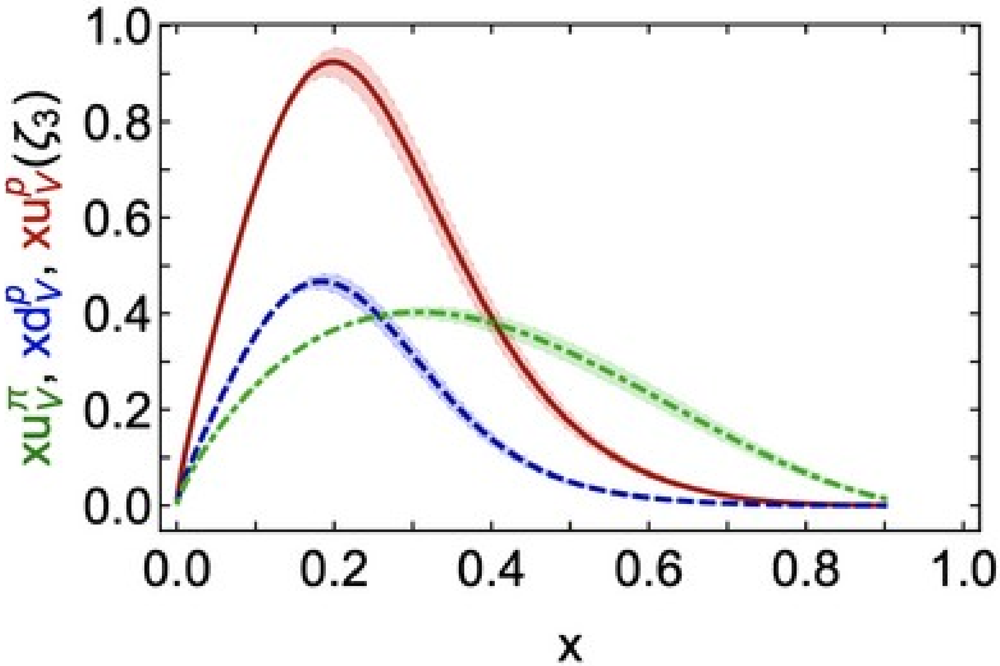
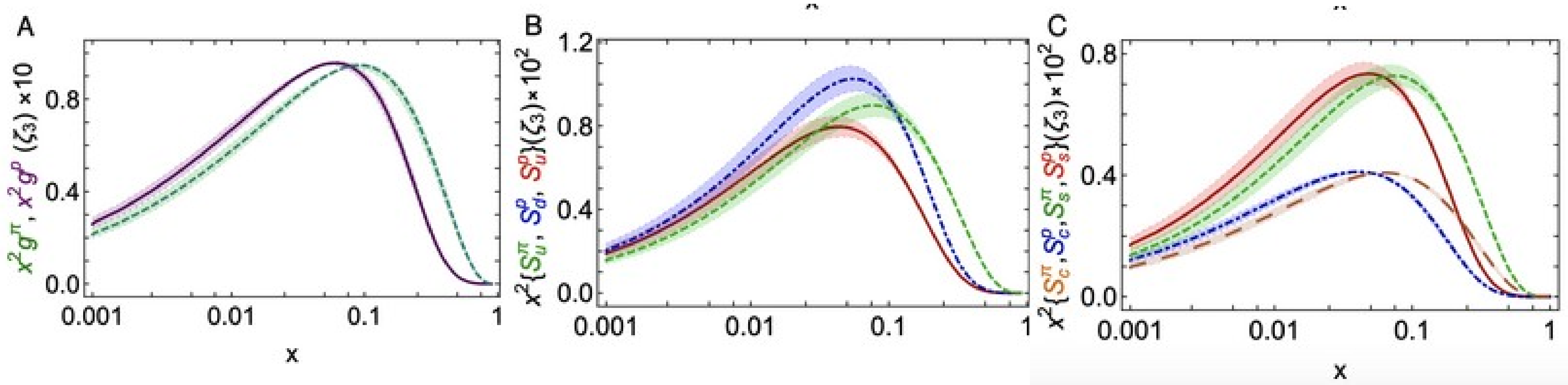


Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]





Proton PDF: pion and proton in counterpoint



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	$S_\pi^{\bar{d}}$	S_π^s	S_π^c
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)