Constructing Local Gauge-Invariant Operators: The Higgs Model as a Case Study

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SUMMARY

Introduction

- In Gauge Field Theory, we have the simple principle:
  "Physical observables are gauge independent."
- The easiest way of fulfilling this requirement would be to work with (correlation functions of) *gauge-invariant operators*.
  "Correlation functions of gauge-invariant operators should be gauge independent."
- However, in general, we deal with correlation functions of gauge variant fields, such as $A_\mu(x)$ (gauge field) and $\psi(x)$ (matter field).
- Gauge independence of S-matrix, physical masses, etc is controlled by the BRST symmetry (Nielsen identities).
- Besides the gauge independence, what can gauge-invariant operators present?

Higgs model as a laboratory

Let us consider the fundamental Euclidian $SU(2)$ Higgs model, defined by the action

$$S = \int d^4x \left[\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + (D_\mu \phi)^\dagger D_\mu \phi + \lambda \left(\phi^\dagger \phi - \frac{v^2}{2} \right)^2 \right], \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu, \tag{2}$$

$$D_\mu \phi = \partial_\mu \phi - ig \frac{\tau^a}{2} A^a_\mu \phi. \tag{3}$$

By expanding ϕ around the minima $\phi_0 = \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix}$, *i.e.*, writing $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h + i\rho^3 \\ i\rho^1 - \rho^2 \end{pmatrix}$

it follows that A^a_μ and h are massive fields (Higgs mechanism),

$$m_A = \frac{vg}{2}, \quad m_h = \sqrt{\lambda} v \tag{4}$$

To quantize this model, we can employ the R_ξ-gauge, namely,

$$S_{gf} = s \int d^4x \bar{c}^a \left(-i\frac{\xi}{2} b^a + \partial_\mu A_\mu^a - \xi m \rho^a\right) \tag{5}$$

where

- ξ is a gauge parameter;
- s is the BRST operator:

$$\begin{aligned} sA_\mu^a = -D_\mu^{ab} c^b, \quad sh = \frac{g}{2} c^a \rho^a, \quad s\rho^a = -\frac{g}{2}(c^a(v+h) - \epsilon^{abc} c^b \rho^c), \\ sc^a = \frac{g}{2}\epsilon^{abc} c^b c^c, \quad s\bar{c}^a = ib^a, \quad sb^a = 0. \end{aligned} \tag{6}$$

REMARKS:

- This model is perturbatively renormalizable and leads to a unitary S-matrix;
- There exists a residual $SU(2)$ global symmetry (Custodial Symmetry);
- This model can be formulated in the *Lattice*, and it shows a confining phase continuously connected to the Higgs phase. (see [Fradkin and Shenker, 1979]);
- In the Lattice, there is no need for a gauge-fixing, thus, due to Elitzur's Theorem [Elitzur, 1975], the VEV of ϕ is zero;
- Frohlich, Morchio and Strocchi [Fröhlich et al., 1981] proposed a mechanism based on gauge-invariant operators. See also the recent work of [Maas, 2019]

Regarding this model, we calculated and analyzed the 2-point functions of $A^a_\mu(x)$ and $h(x)$ up to 1-loop [Dudal et al., 2021], *i.e.*,

$$\langle A^a_\mu(p) A^b_\nu(-p)\rangle = \delta^{ab}\left[A(p^2)T_{\mu\nu}(p) + B(p^2)L_{\mu\nu}(p)\right] \quad \text{and} \quad \langle h(p)h(-p)\rangle. \tag{7}$$

In particular, we investigated the spectral functions,

$$\langle \Phi(p)\Phi(-p)\rangle = \int_0^\infty dt\, \frac{\rho_\Phi(t)}{p^2+t}, \tag{8}$$

where

$$\rho_\Phi(t) = Z\delta(t - M^2_{\text{phys}}) + \tilde{\rho}_\Phi(t). \tag{9}$$

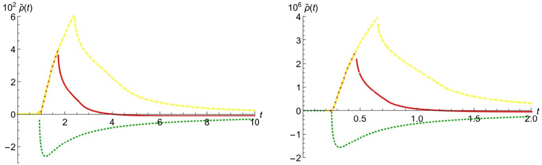

Figure: Spectral function of the transverse component of $\langle A(p)^a_\mu A(-p)^b_\nu \rangle$ for different values of ξ.

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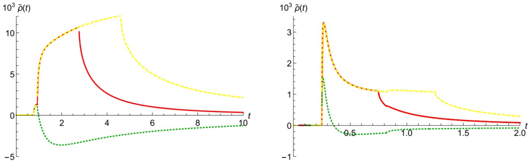

Figure: Spectral function of the Higgs propagator $\langle h(p)h(-p) \rangle$ for different values of ξ.

Notice that both spectral functions are *gauge-dependent* and show *positivity violation*.

The same Higgs model allows us to construct the following local gauge-invariant operators:

$$
\begin{aligned}
O(x) &= \phi^\dagger \phi - \frac{v^2}{2} \\
&= \frac{1}{2}(h^2 + 2vh + \rho^a \rho^a)
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
R^a_\mu(x) &= \frac{1}{2}\Big[-(v+h)\partial_\mu \rho^a + \rho^a \partial_\mu h - \epsilon^{abc}\rho^b \partial_\mu \rho^c + \frac{g}{2}(v+h)^2 A^a_\mu \\
&\quad -g(v+h)\epsilon^{abc}\rho^b A^c_\mu - \frac{g}{2}A^a_\mu \rho^b \rho^b + g\rho^a A^b_\mu \rho^b\Big] .
\end{aligned} \tag{11}
$$

Notice that, $O(x)$ and $R^a(x)$ have linear terms, which implies that

$$
\begin{aligned}
\langle O(p) O(-p) \rangle &= v^2 \langle h(p) h(-p) \rangle + \ldots \\
\langle R^a_\mu(p) R^b_\nu(-p) \rangle &= \frac{g^2 v^4}{16} \langle A^a_\mu(p) A^b_\nu(-p) \rangle + \ldots
\end{aligned} \tag{12}
$$

In [Dudal et al., 2021], we also computed $\langle O(p)O(-p)\rangle$ and $\langle R^a(p)R^b(-p)\rangle$ up to 1-loop and obtained the following spectral functions:

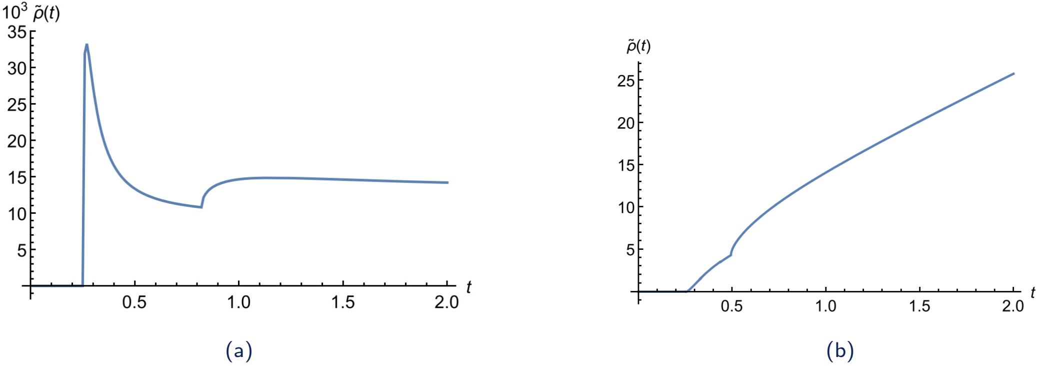

Figure: Spectral functions of $\langle O(p)O(-p)\rangle$ (left) and $\langle R^a_\mu(p)R^b(-p)_\nu\rangle^T$ (right).

Notice that they are both positive!.

Understanding the situation here:

- The perturbative Higgs model has a physical space with semi-positive norm [Kugo and Ojima, 1979]:

$$\mathcal{V}_{phys} = \{|\alpha\rangle; Q_{BRST}|\alpha\rangle = 0\}, \\ \langle\alpha|\alpha\rangle \geq 0 \tag{13}$$

- In this case, the spectral functions of gauge-invariant correlators are indeed positive.

Local Gauge-Invariant Dressings

- Since gauge-invariant operators may possess very desirable properties, it is worthwhile to invest our efforts in studying them.
- In particular, we focus on local gauge-invariant dressings[Stueckelberg, 1938].
- These objects have direct applications in *effective massive models* that explicitly break BRST symmetry, such as infrared-safe models [Tissier and Wschebor, 2011] and the Refined Gribov–Zwanziger models [Gribov, 1978, Zwanziger, 1989, Sorella, 2008].
- Recently, we (Prof. Antônio Pereira and I) investigated the possibility of constructing local gauge-invariant dressings in a very important class of gauges, which we would like to discuss here.

Let us establish the general framework for constructing and quantizing gauge-invariant dressed fields. For simplicity, we consider the $SU(N)$ gauge group.
In addition to the gauge field A^a_μ, we introduce an auxiliary Stueckelberg field ξ^a to construct the local gauge-invariant operator:

$$A^h_\mu(A,h) = h^\dagger A_\mu h + \frac{1}{ig} h^\dagger \partial_\mu h \tag{14}$$

where $h = \exp(-ig\xi^a T^a)$. That is, $A^h_\mu(A^U, h^U) = A^h_\mu(A,h)$ under the gauge transformations:

$$\begin{aligned} A^U_\mu &= U A_\mu U^\dagger + \frac{1}{ig} \partial_\mu U U^\dagger, \\ h^U &= U h \end{aligned} \tag{15}$$

Since ξ^a is an auxiliary field, we can eliminate it by imposing the constraint,

$$\mathcal{F}^a[A^h] = 0, \tag{16}$$

and then solve it for ξ^a as a function of A_μ,

$$\xi^a(A) \Rightarrow h(A). \tag{17}$$

Notice that $\mathcal{F}^a[A^h]$ is gauge-invariant, which implies that

$$A_\mu^h(A, h(A)) \tag{18}$$

is also gauge-invariant.

A very important class of gauges has the form

$$\mathcal{F}^a[A] = \partial_\mu A^a_\mu + P^a(A), \tag{19}$$

where $P^a(A)$ is a polynomial of A.
By expanding $h = \exp(-ig\xi^a T^a)$ in power series, and solving $\mathcal{F}^a[A^h] = 0$ iteratively, we obtain:

$$\xi^a = \mathcal{O}^{ab}(A, \partial)\mathcal{F}^a[A] \tag{20}$$

and, consequently,

$$A^{h\,a}_\mu = A^a_\mu + \mathcal{R}^{ab}(A, \partial)\mathcal{F}^a[A] \tag{21}$$

where $\mathcal{O}^{ab}$ and $\mathcal{R}^{ab}(A, \partial)$ are non-local operators.
Notice that if A^a_μ also satisfies the same gauge condition ($\mathcal{F}^a[A] = 0$)

$$\xi^a = 0 \quad \text{and} \quad A^{h\,a}_\mu = A^a_\mu. \tag{22}$$

At the *quantum level*, this construction can be implemented *via* the path integral using the *Faddeev–Popov trick*.
Suppose that $G[(A_\mu^h)_{\text{non-local}}]$ is a functional of the non-local field A_μ^h. It can then be rewritten as:

$$G[(A_\mu^h)_{\text{non-local}}] = \det\left(\mathcal{M}[A^h]\right) \int D\xi^a \, \delta\left(\mathcal{F}[A^h]\right) G[A_\mu^h], \tag{23}$$

where

$$\mathcal{M}^{ab}[A^h] = \frac{\delta \mathcal{F}^a[A^h]}{\delta \xi^b}, \tag{24}$$

and equivalently,

$$G[(A_\mu^h)_{\text{non-local}}] = \int D\xi^a D\tau^a D\eta^a D\bar{\eta}^a \exp\left(-\int d^4x \left[i\tau^a \mathcal{F}^a[A^h] + \bar{\eta}^a \mathcal{M}^{ab}[A^h]\eta^b\right]\right) G[A_\mu^h], \tag{25}$$

where τ^a, η^a, and $\bar{\eta}^a$ are additional auxiliary fields.

Therefore, we end up working with the local BRST invariant action:

$$
\begin{aligned}
S = S_{YM}[A] &+ \int d^4x \left(ib^a F^a[A] + \bar{c}^a M^{ab}[A] c^b \right) \\
&+ \int d^4x \left(i\tau^a \mathcal{F}^a[A^h] + \bar{\eta}^a \mathcal{M}^{ab}[A^h] \eta^b \right)
\end{aligned}
\tag{26}
$$

$$
\begin{gathered}
sA^a_\mu = -D^{ab}_\mu[A]c^b, \qquad sc^a = \frac{gf^{abc}}{2} c^b c^c, \qquad s\bar{c}^a = ib^a, \qquad sb^a = 0, \\
s\xi^a = g^{ab}(\xi)\, c^b, \qquad s\tau^a = 0, \qquad s\eta^a = 0, \qquad s\bar{\eta}^a = 0.
\end{gathered}
\tag{27}
$$

If A_μ and A^h_μ satisfy the same condition, *i.e.*,

$$\mathcal{F}^a = F^a, \tag{28}$$

The entire action is invariant under the dual transformations:

$$\begin{aligned} A_\mu \to A^h_\mu, \quad \xi^a \to -\xi^a, \quad c^a \to \eta^a, \quad \bar{c}^a \to \bar{\eta}^a, \quad b^a \to \tau^a, \\ \eta^a \to c^a, \quad \bar{\eta}^a \to \bar{c}^a, \quad \tau^a \to b^a \end{aligned} \tag{29}$$

Therefore, from these transformations, we have the following relationship for the correlation functions:

$$\langle \Phi(x_1) \ldots \Phi(x_n) \rangle = \langle \Phi^{\text{dual}}(x_1) \ldots \Phi^{\text{dual}}(x_n) \rangle. \tag{30}$$

In particular, we have:

$$\langle A_{\mu_1}(x_1) \ldots A_{\mu_n}(x_n) \rangle = \langle A^h_{\mu_1}(x_1) \ldots A^h_{\mu_n}(x_n) \rangle. \tag{31}$$

Conclusions

- Gauge-invariant operators exhibit very nice spectral properties—at least perturbatively—for theories with a well-defined physical space. (*What happens in the non-perturbative regime?*)
- It is possible to construct a gauge-invariant dressed field for each gauge fixing in the class we discussed, in particular for the *Maximal Abelian Gauge*.
- By using the framework we proposed, the dual symmetry guarantees the validity of the on-shell condition to all orders.
- The result that other gauges also allow a gauge-invariant dressing for the gauge field opens up the possibility of comparisons between different effective massive models formulated in different gauges.

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Thank you!