UNIVERSITÄT GRAZ

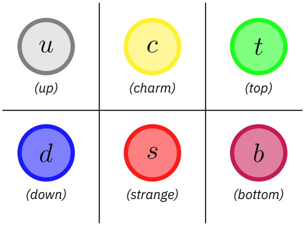
NAWI Graz
Natural Sciences

Hadronic Structure and Contour Deformations

Eduardo Ferreira

ECT*, Trento – May 26-30, 2025

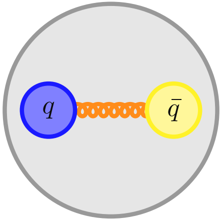
Hadron Physics



Quarks

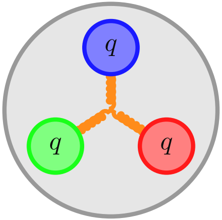
Gluons

Hadron Physics



Mesons

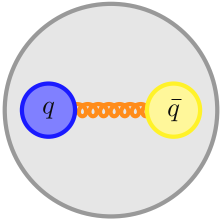
$\pi, \rho, K, \eta, ...$



Baryons

$p, n, \Delta, \Sigma, \Lambda, ...$

Hadron Physics



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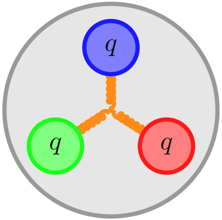
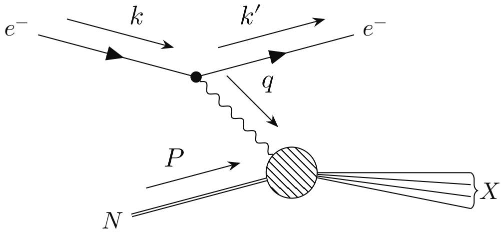


Baryons
$p, n, \Delta, \Sigma, \Lambda, \dots$

- Their dynamics and properties are determined by QCD
  - Atomic nuclei and nuclear stability, EMC effect
  - Radius; spin-, momentum-, charge distributions;
  - Interaction with external currents: $\{e^-, \nu, \dots\}N$ scattering, $N\pi$ scattering, nucleon compton scattering, meson photoproduction

(EIC White Paper), (Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer; 2016), (European Muon Collaboration; 1983), ...

Parton model



(PDG Section 18 (Structure Functions))

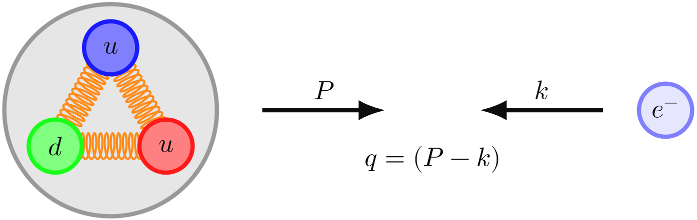
- Common experiment: **Deep inelastic scattering.**
- Electron (or neutrino) probe interacts with nucleon via $\gamma, Z, W^{\pm}$.
- Main idea:
  - Hadrons are *bags* of **partons** .
  - Scattering occurs between the exchanged boson and a parton from the nucleon
- Cross-section separates:

$$\frac{d^2\sigma}{dxdq^2} \propto L_{\mu\nu}W^{\mu\nu}$$

- $W^{\mu\nu}$ described via PDFs.

PDFs/TMDs

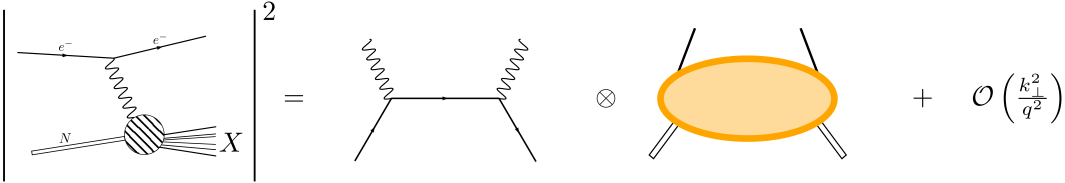
- How to they show up?



$$W_{\mu\nu}(p,q,\sigma) \propto \int d^4z \, \langle p,\sigma| \, J^\dagger_\mu(z) J_\nu(0) \, |p,\sigma\rangle = \sum_i \tau(p,q,\sigma)^i_{\mu\nu} f_i(p,q)$$

PDFs/TMDs

- How to they show up?
- Take the collinear, q^2 very large limit.



$$\sigma \propto \sum_f \int \frac{dx}{x} H(x, Q^2/\Lambda^2) q_f(x, \Lambda^2) + \mathcal{O}\left(\frac{k_\perp^2}{Q^2}\right)$$

$$q_f(x, \Lambda) \propto \int \frac{d^4\ell}{(2\pi)^4} \delta\left(x - \frac{\ell^+}{p^+}\right) \mathrm{Tr}\left[\langle p| \overline{\psi}_f(\ell) \gamma^+ \psi_f(\ell) |p\rangle\right]$$

- Hard scale $Q^2 = -q^2$ introduced allows for factorization – power supression $\mathcal{O}(Q^{-2})$
- Works also on other processes, like Drell-Yan.

PDFs/TMDs

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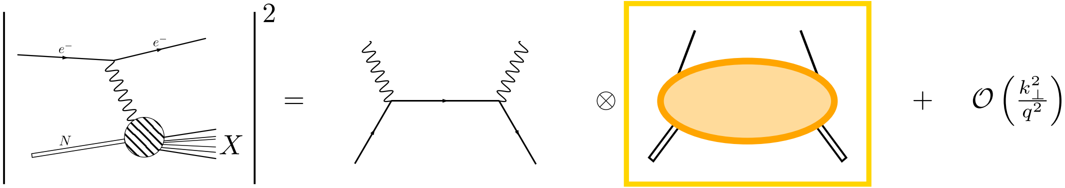

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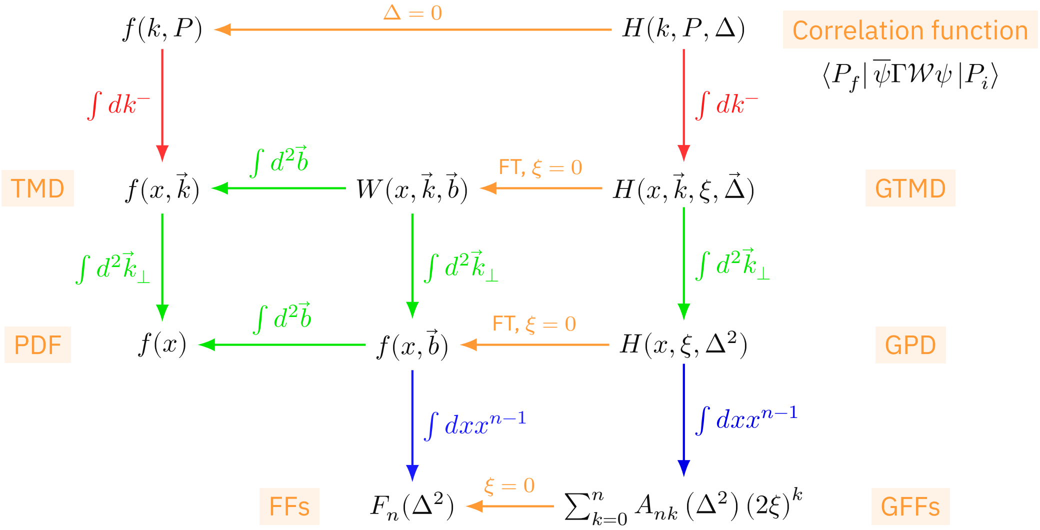
- Focus on the non-perturbative hadronic part.
- Can we build them directly from first-principles **FUN ctional Methods** calculations?

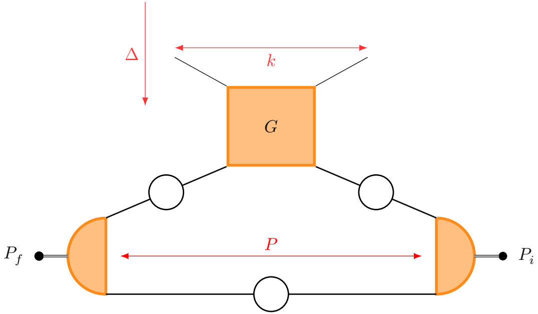
Hadronic quantities



(Lorce, Pasquini, Vanderhaeghen; 2011), (Picture adapted from: Diehl, 2016), (Diehl, 2003), (Meißner, Goeke, Metz, Schlegel; 2008), (Meißner, Metz, Schlegel; 2009)

Our Goal

- **Main Goal:** Get partonic distribution functions from hadron-hadron correlations via **FUN**ctional Methods



- G is the four-point quark correlation function, calculated with scattering equation.
- The BSWF is calculated via the meson BSE.
- First scalar toy model, then QCD.

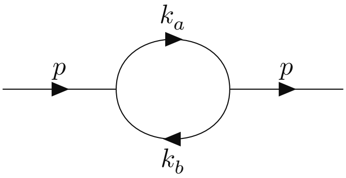
(Mezrag; 2015), (Diehl, Gousset; 1998), (Tiburzi, Miller; 2003), (Mezrag, Chang, Moutarde, Roberts, Rodríguez-Quintero, Sabatié, Schmidt; 2015), many others, ...

$$\mathcal{G}^{[\Gamma]}\left(P,k,\Delta\right) = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \int \frac{d^4z}{2\pi^4} e^{ik\cdot z} \left\langle P_f\right| \overline{\psi}(z)\mathcal{W}\Gamma\psi(0)\left|P_i\right\rangle\right]$$

- Partonic distributions are calculated by integrating the correlator in k^- and taking appropriate traces.

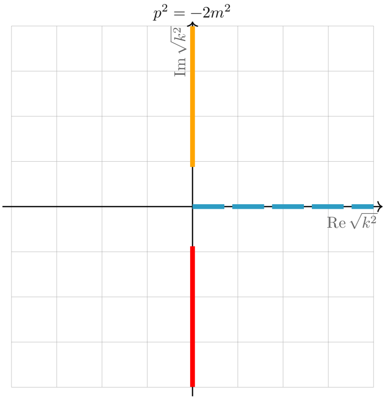
Analytic Structure is Important

- Calculating quantities in time-like momenta is **complicated** !
  - Analytic structure prevents naïve integration.
  - Poles and branch cuts are present.
- Calculate quantities in the P^2 complex plane.
- **Euclidean** $\Leftrightarrow$ Minkowski



$$I(p^2) = \int d^4k \frac{1}{k_a^2 - m^2 + i\epsilon} \frac{1}{k_b^2 - m^2 + i\epsilon}$$

$\Longrightarrow$



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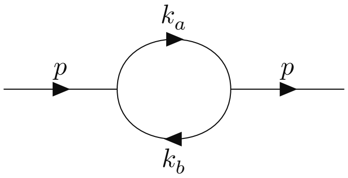

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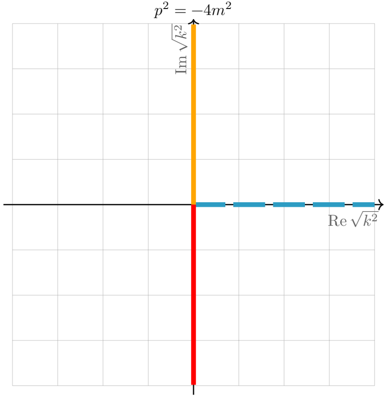

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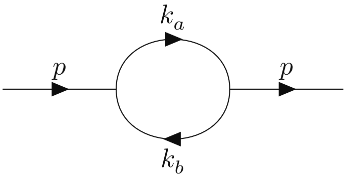

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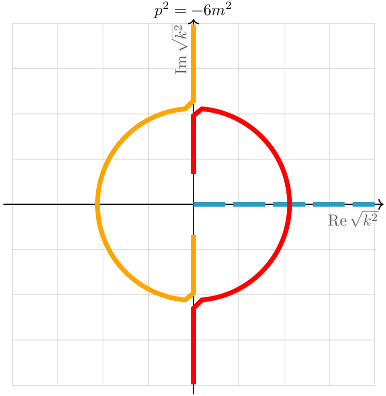

Worked Example

- Simple *tree-level* version of hadronic matrix element.

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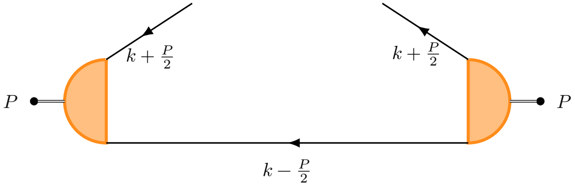
- All elements known – **Residue Integration**



$$\mathcal{G}^{[\Gamma]}\left(P, k, \Delta = 0\right) = \int dq^{-} \, \langle P | \, \overline{\phi}\left(q + \frac{P}{2}\right) \phi\left(q + \frac{P}{2}\right) | P \rangle \Big|_{q_\perp = k_\perp, q^+ = \frac{\alpha}{2} P^+}$$

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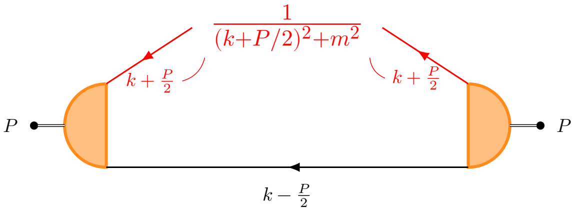

$$\mathcal{G}^{[\Gamma]}\left(\alpha,k_\perp\right) = \int dq^- \left[\frac{1}{\left(q+\frac{P}{2}\right)^2+m^2}\right]^2 \Bigg|_{q^+=\frac{\alpha}{2}P^+,\, q_\perp=k_\perp}$$

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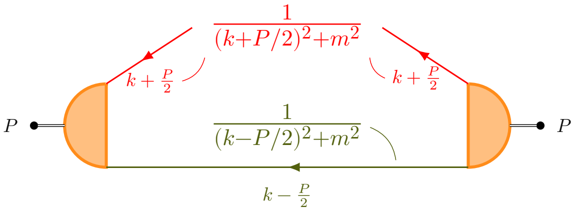

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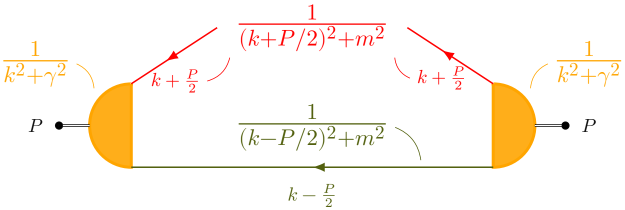

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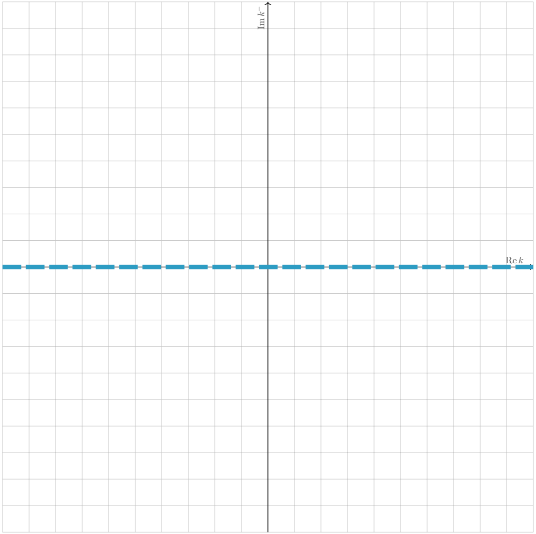
Minkowski/LF

$\sqrt{t} = 0.8i, \alpha = 0.2$



$$\text{PDF} \propto \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk^- \int d^2k_\perp \langle P| \overline{\phi}\phi |P\rangle$$

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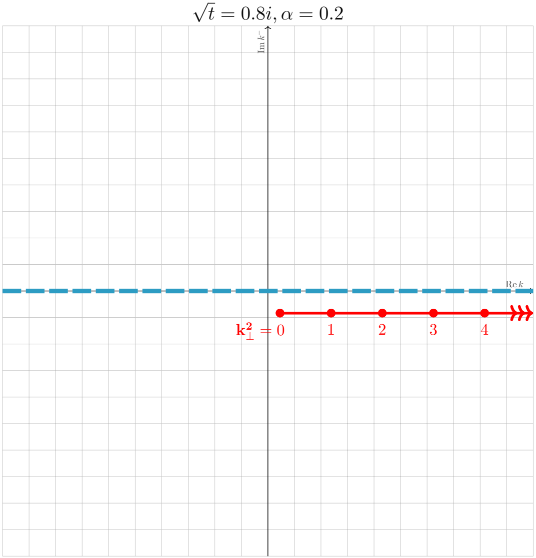

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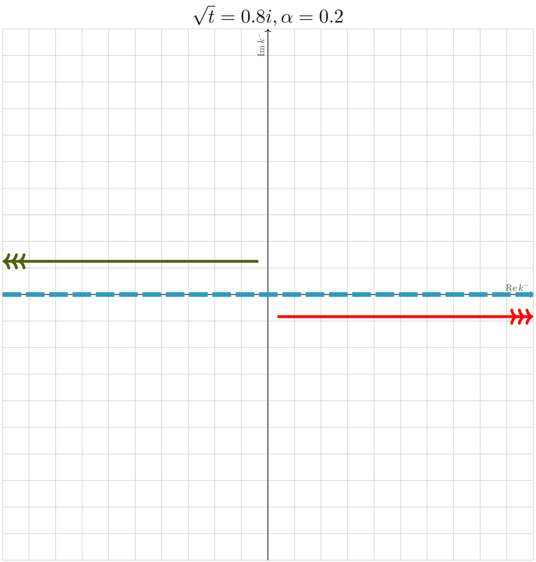

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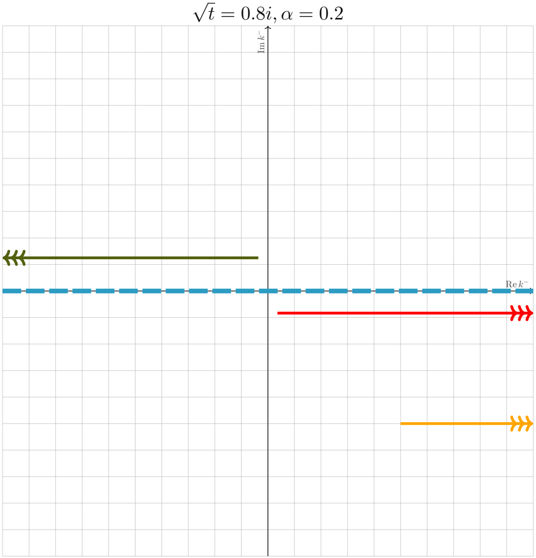

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- Result is given by summing residues for poles above the real axis.

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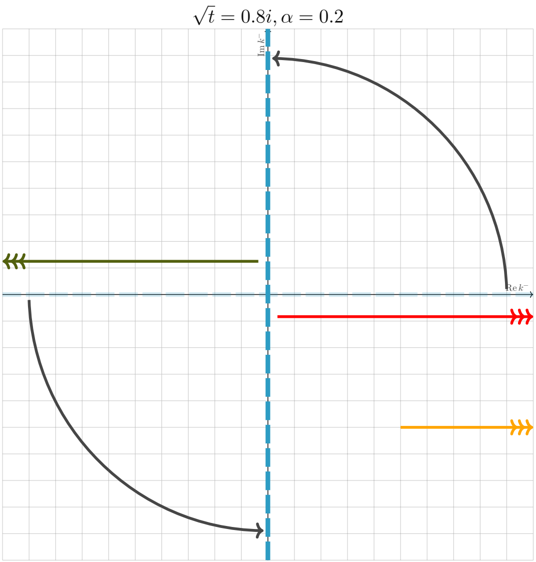

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- Can Wick Rotate (may need contour deformations).

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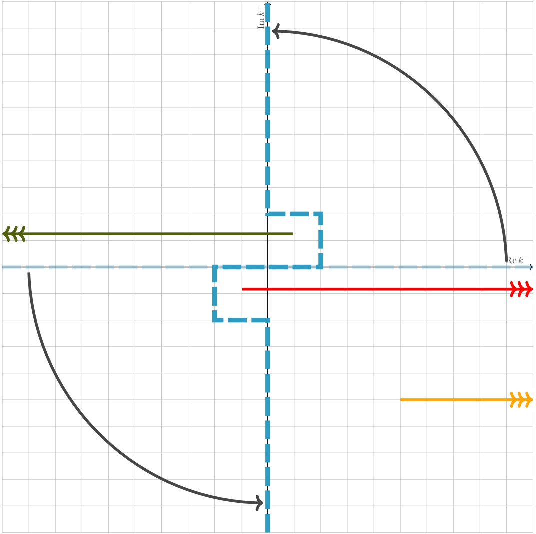

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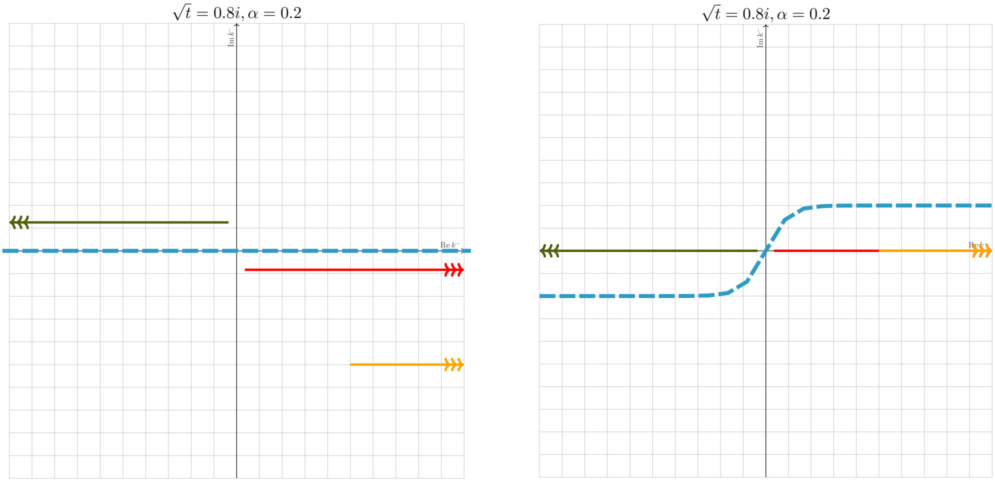
Imaginary time boundary conditions vs $i\epsilon$ prescription



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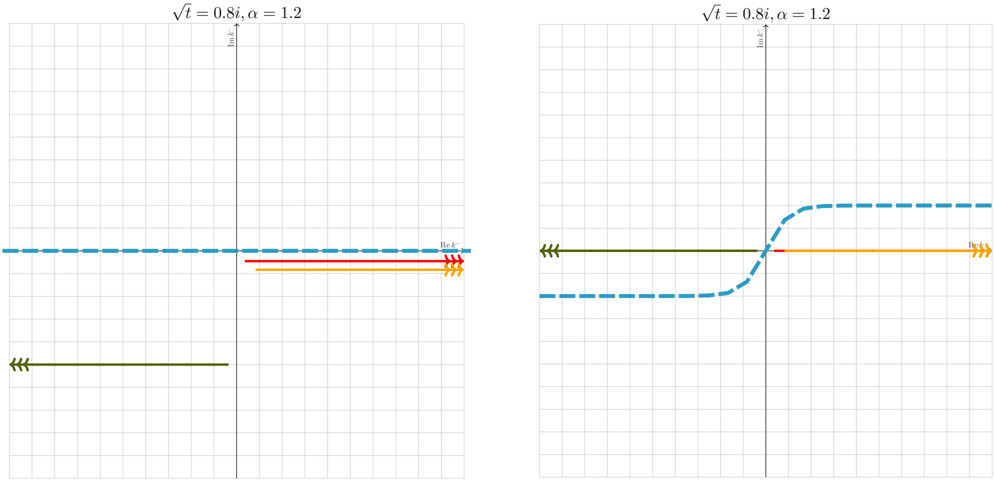

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Euclidean

$$k_\perp^2 = R \quad k^- \propto \hat{k} \cdot \hat{P} = Z \quad P^2 = 4mt^2$$

$$\text{PDF} \propto \int_0^\infty dR \int_{-\infty}^\infty dZ \, \langle P | \, \overline{\phi}\phi \, | P \rangle$$

- Use only Lorentz invariants.
- Hyperspherical coordinates in k.
- We check cuts on $k_\perp^2 = R$.
  - Avoiding cuts on R means selecting the correct residues in Minkowski.

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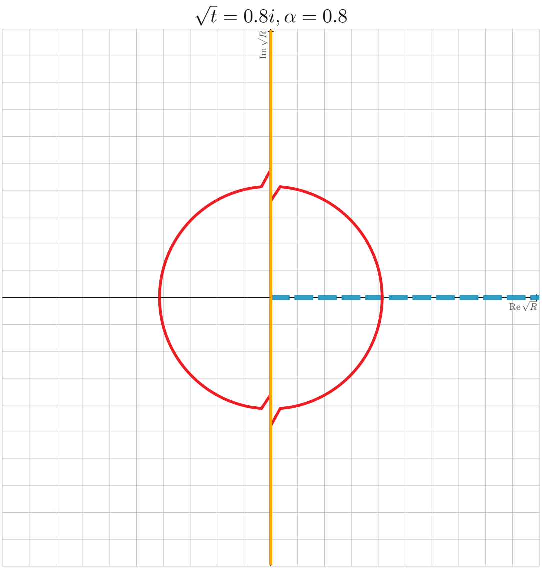

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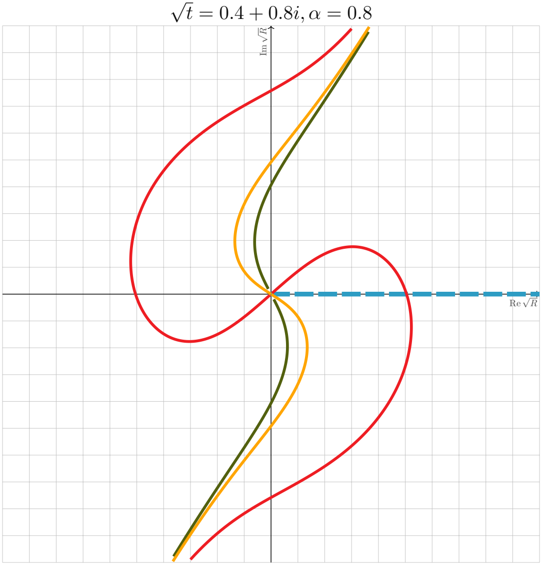

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- Use only Lorentz invariants.
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- We need slighly complex masses.

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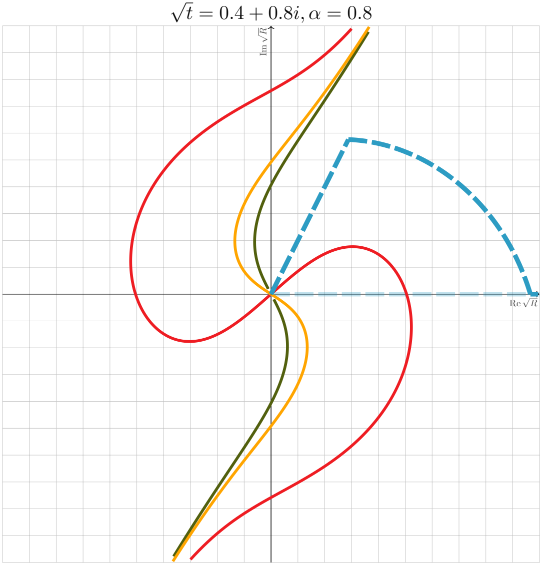

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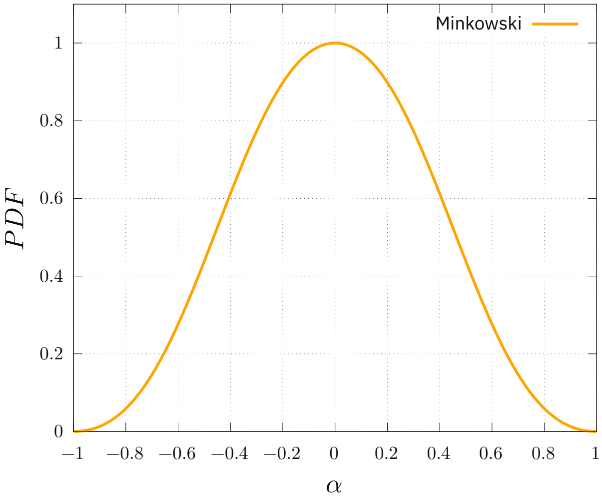

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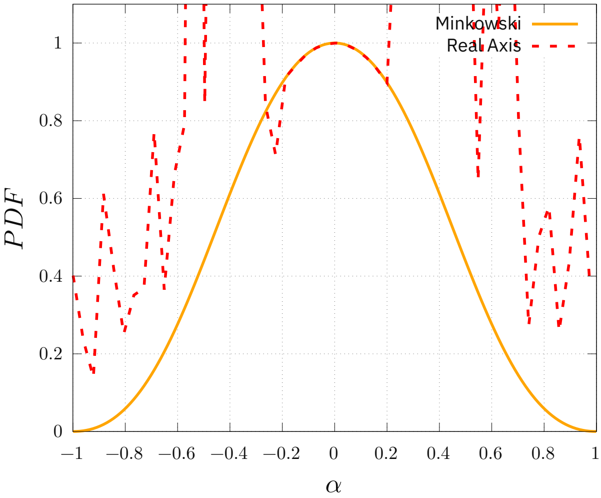

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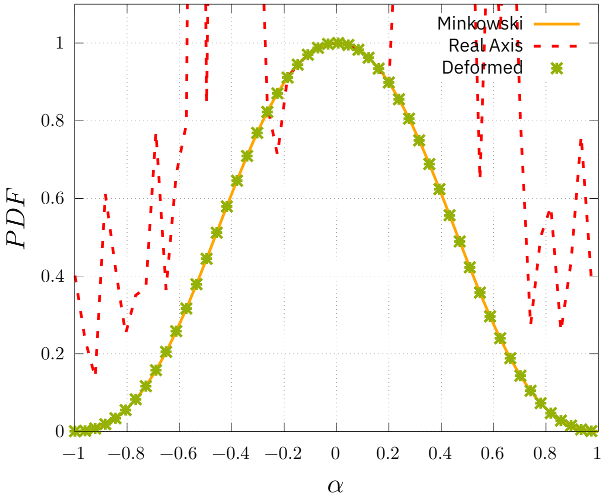

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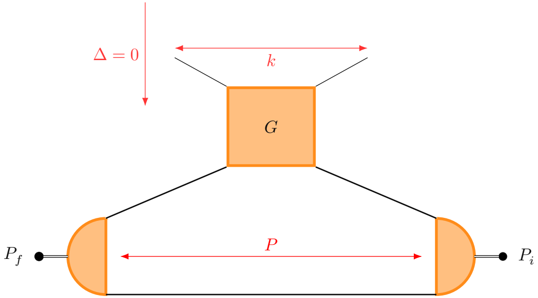
Re-stating

- Get partonic distribution functions from hadron-hadron correlations via **FUN ctional Methods**



- TMDs/PDFs – $\Delta = 0$.
- G is the four-point quark correlation function – added later.
- Tree level propagators.
- First scalar toy model, then QCD.

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- Partonic distributions are calculated by integrating the correlator in k^- and taking appropriate traces.
- This means: integral in $(-\infty, \infty)$ in $\hat{k} \cdot \hat{P}$ – **Analytic Continuation needed**

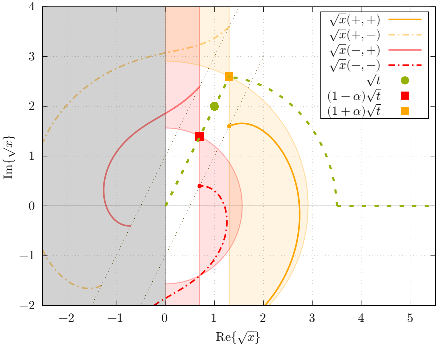
(Eichmann, EF, Stadler; 2022)

Analytic Structure of BSE $\mathrm{Im}\sqrt{t} > \min\left(\frac{1}{1\pm\alpha}\right)$

$$\mathbf{G_0}^{-1} = (l_+^2 + m^2)(l_-^2 + m^2)$$

- Branch cuts in complex $\sqrt{l}$ plane

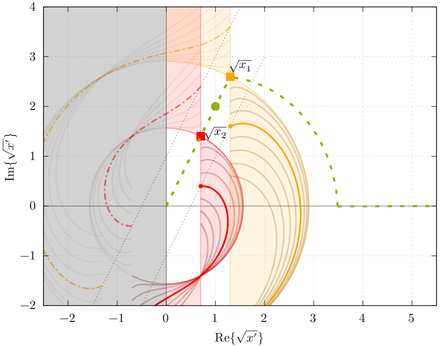
$$\sqrt{l}_\pm^\lambda = \mp(1\pm\alpha)\sqrt{t}\left[z_l + i\lambda\sqrt{1 - z_l^2 + \frac{1}{(1\pm\alpha)^2 t}}\right]$$



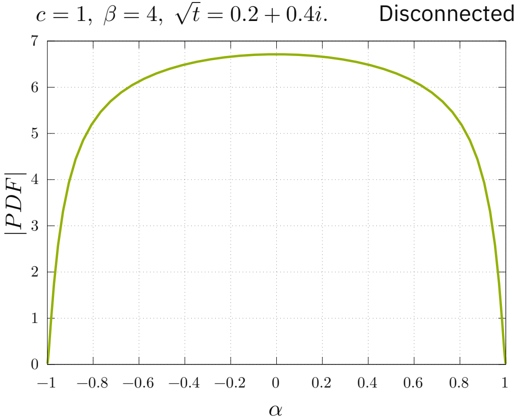
$$\mathbf{K}^{-1} = (q-l)^2 + \mu^2$$

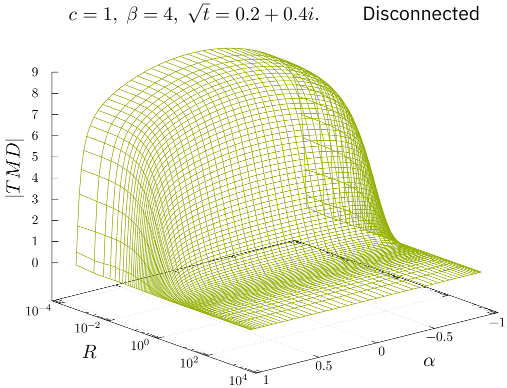
- Branch cuts in complex $\sqrt{l}$ plane depend on path taken:

$$\sqrt{l} = \sqrt{\rho}\left(\Omega \pm i\sqrt{1 - \Omega^2 + \frac{\beta^2}{\rho}}\right)$$



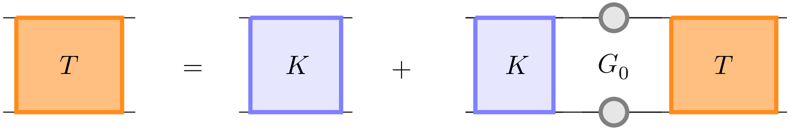
First Results *Without 4-point function*





4-point function

- 4- point function determined from scattering equation:
  $\mathbf{G} = \mathbf{G_0} + \mathbf{G_0}\mathbf{T}\mathbf{G_0} \implies \mathbf{T} = \mathbf{K} + \mathbf{K}\mathbf{G_0}\mathbf{T} \implies \mathbf{T} = (\mathbb{1} - \mathbf{K}\mathbf{G_0})^{-1}\,\mathbf{K}.$



(Eichmann, Duarte, Peña, Stadler; 2019)

4-point function

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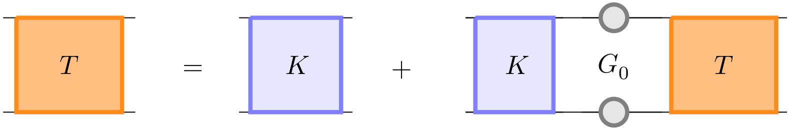
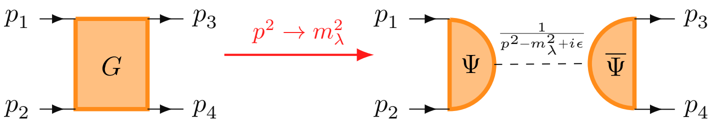


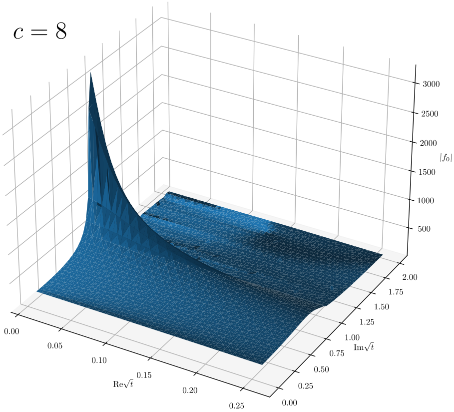


- Same G_0 and K as in the BSE
- *All* dynamics of 2 ϕ particles:
  - **Must produce bound state poles dynamically!**

(Eichmann, Duarte, Peña, Stadler; 2019)

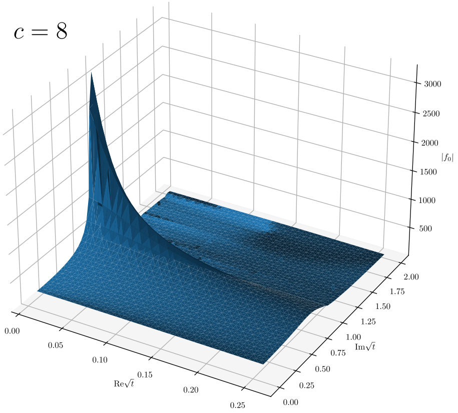
4-point function

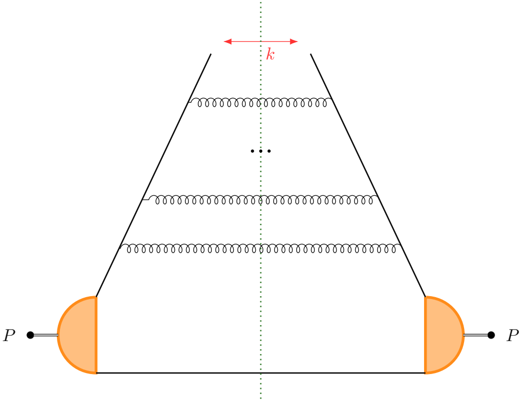
- Partial-wave expansion shows bound-state pole in the first Riemann sheet



4-point function

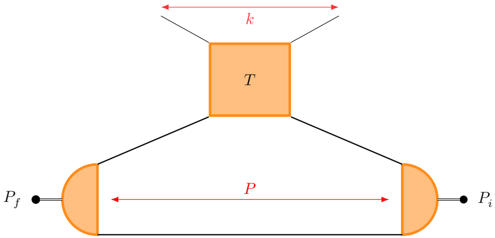
- Partial-wave expansion shows bound-state pole in the first Riemann sheet





- Describes both long-range and short-range qq dynamics.
- Further Fock states?

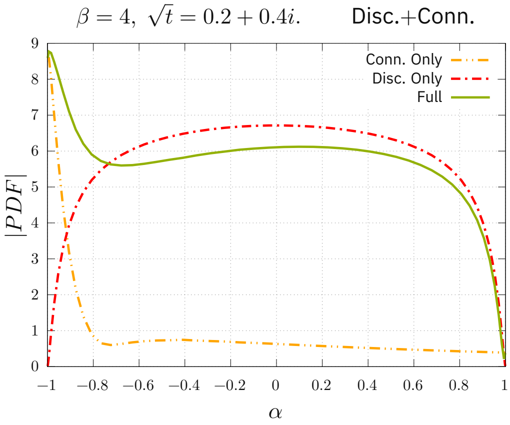
Put everything together

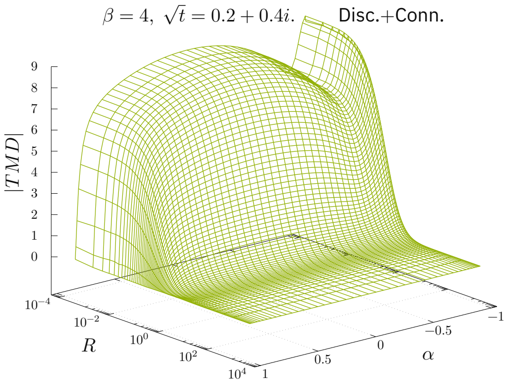


1. Calculate 4-point function
2. Calculate BSE
3. Do the loop integration
4. Project to LF

- We solve one triangle for each point in the R, α grid – **HPC Needed!**
  - $N_\alpha \times N_R \times N_Z$ 4-point functions!
- Obtained PDFs/TMDs for a system of two bound scalars ϕ.

Results *With 4-point function*





- Publication coming soon!

Conclusions

- Framework for calculation of PDFs/TMDs.
- Applications to hadronic structure calculations starting from self-consist first principles calculations.
- Calculation of $q\bar{q}$ scattering equation – rich dynamics possible.
- Future applications to realistic QCD models soon.