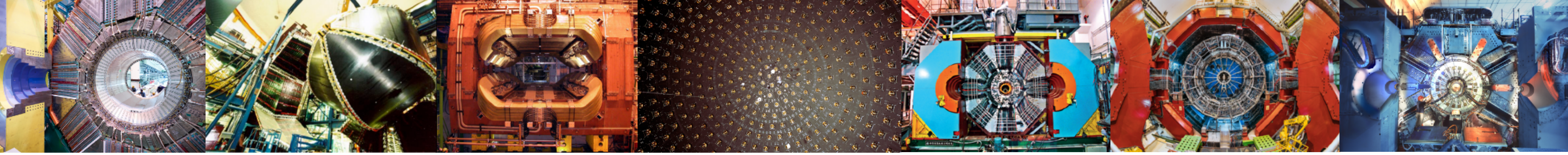


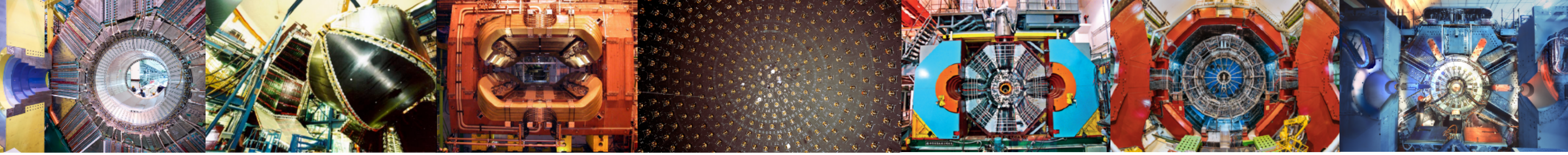
Numerical Analytic Continuation for Lattice Field Theory

William I. Jay

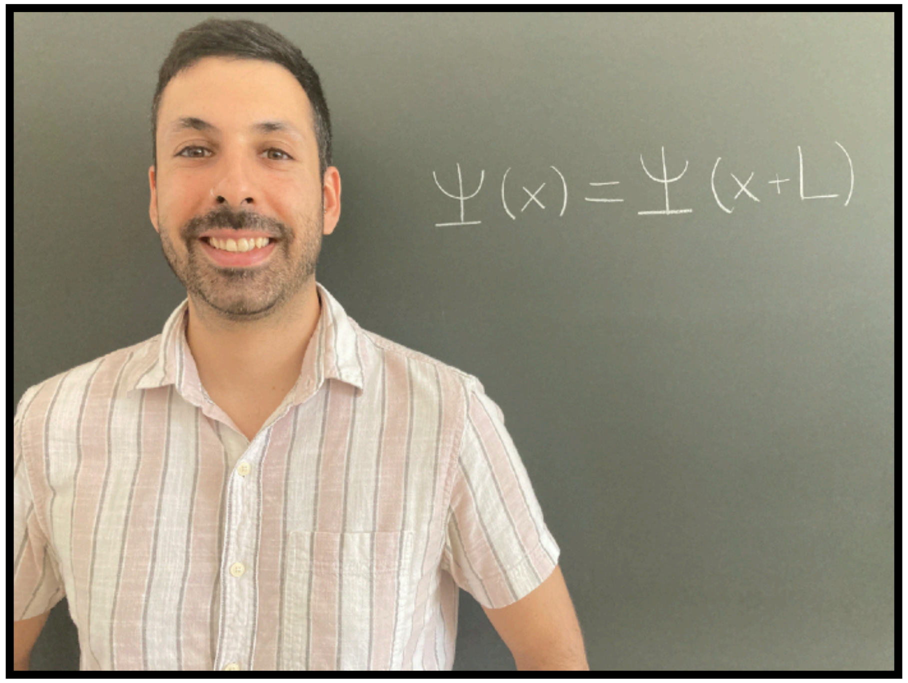
william.jay@colostate.edu

ECT*, Trento, Italy
26-30 May 2025

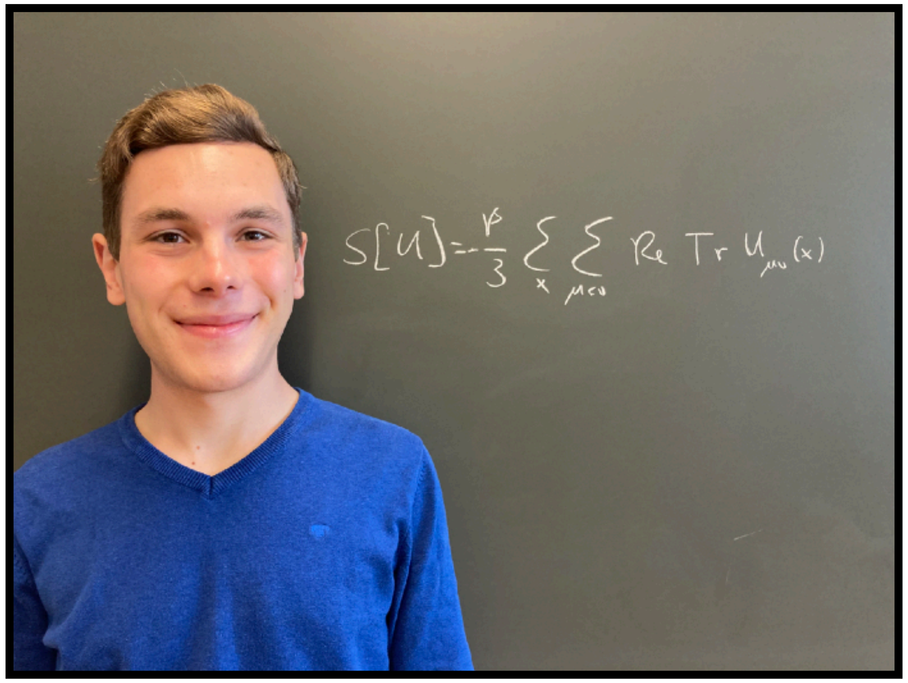
The complex structure of strong interactions in Euclidean and Minkowski space



Work in progress with:



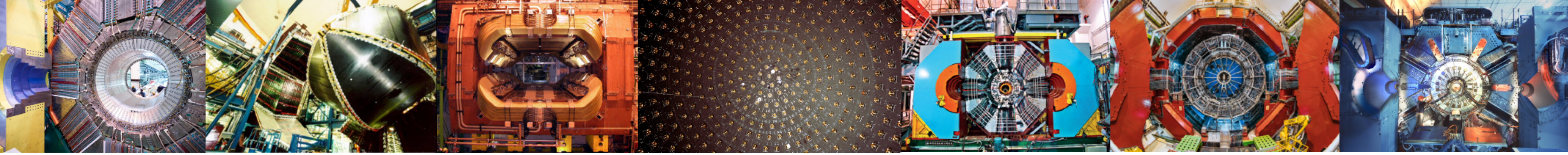
Fernando Romero-López



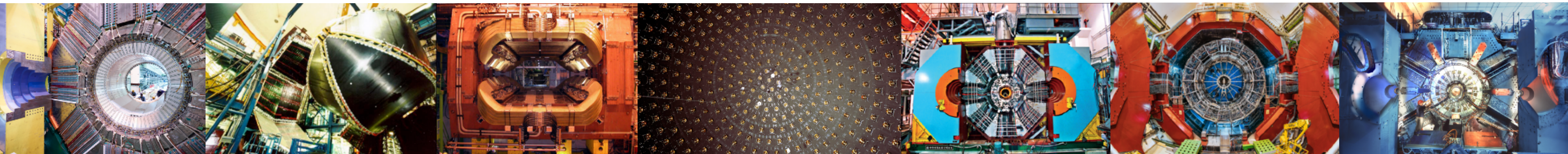
Miguel Salg

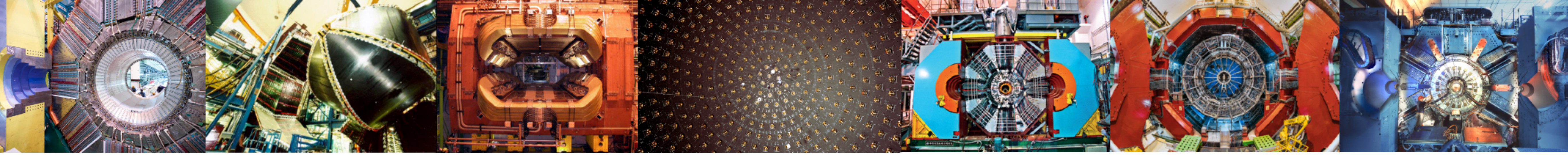
Coming soon to an arXiv near you…

UNIVERSITÄT BERN



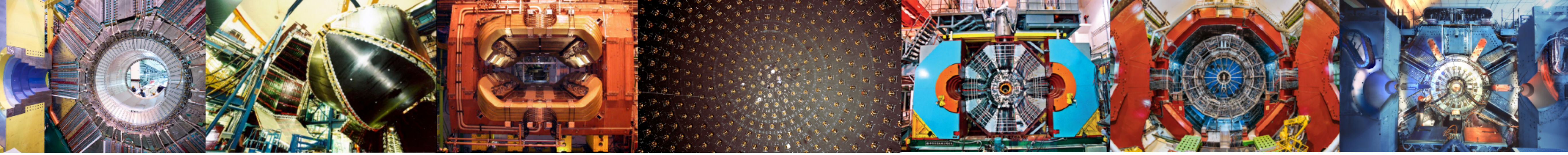
Motivation and Context





Hadron Spectroscopy

- Experimentally, most hadrons appear as resonances, i.e., "bumps" in scattering cross sections.
  - Location of the bump $\Longleftrightarrow$ Mass of the resonance
  - Width of the bump $\Longleftrightarrow$ Lifetime of the resonance

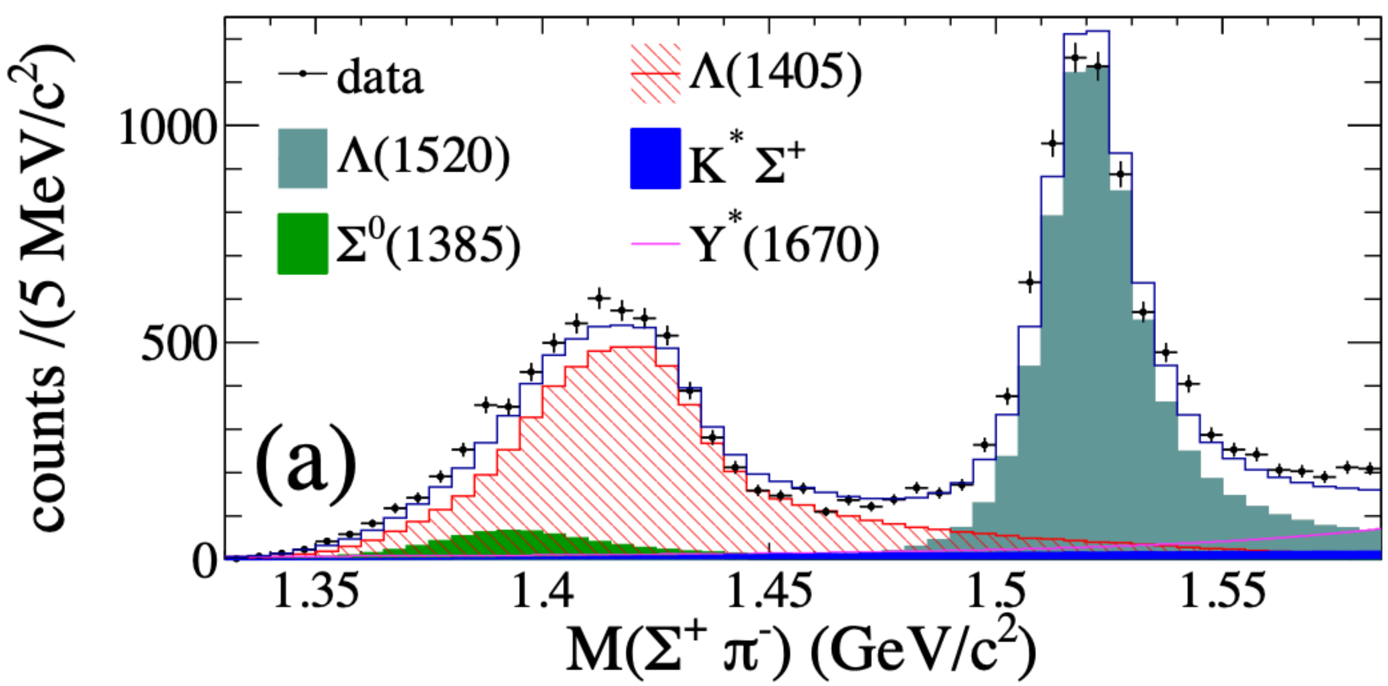


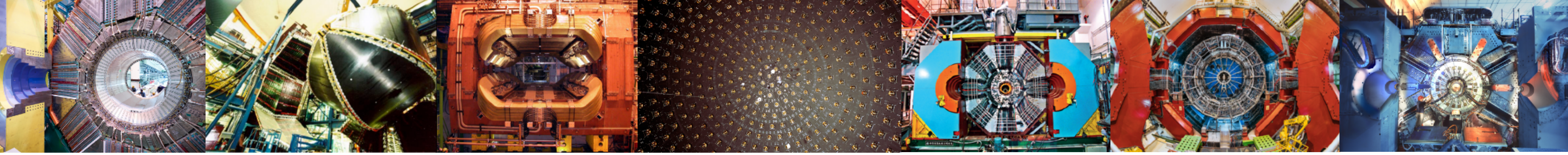
Hadron Spectroscopy

Dalitz & Tuan, PRL 2 (1959) 425-428
Dalitz & Tuan, Annals Phys. 10 (1960) 307-351
Hyodo & Jido, Prog. Part. Nucl. Phys. 67, 55 (2012)
Hyodo & Niiyama, Prog. Part. Nucl. Phys. 120, 103868 (2021)
CLAS Collaboration, PRL 112 (2014) 8, 082004

- Ex: $\pi^-\Sigma^+$ spectrum below K^-p threshold (~ 1442 MeV)
  - More than 60 years of experimental and theoretical study
  - What is the nature of the resonance?
  - Can it be understood from QCD?

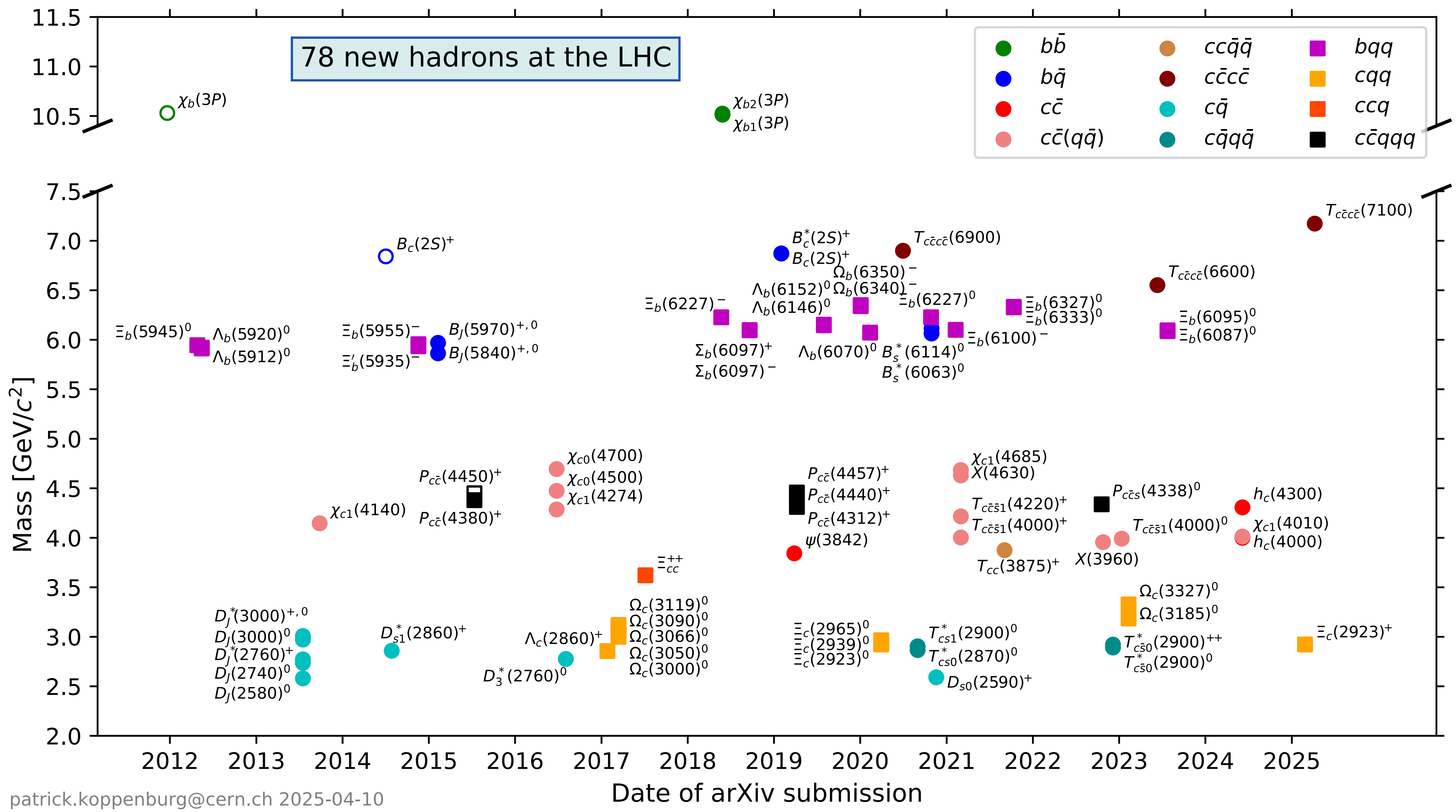
Data - CLAS 2014

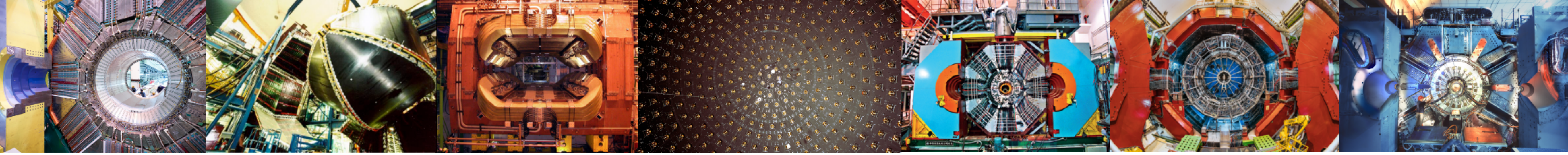




Hadron Spectroscopy

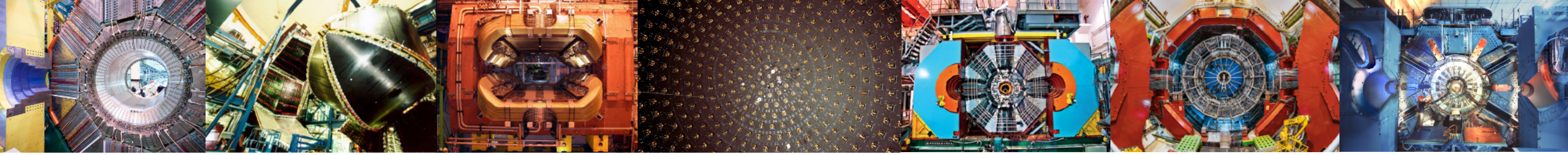
- Ex: Exotic “XYZ” hadrons - a renaissance in hadron spectroscopy





Hadron Spectroscopy

- Theoretically, we want to calculate the properties of resonances directly from the underlying degrees of freedom: QCD
  - Provide predictions/postdictions to test our understanding of low-energy QCD
  - Help characterize the nature of resonances seen in experiment



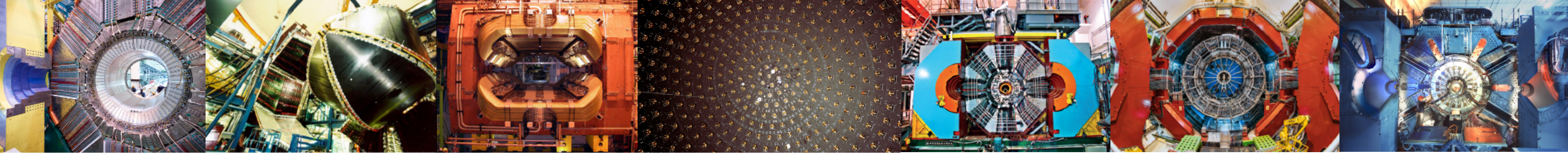
Resonances in field theory

- Resonances are identified with poles of scattering amplitudes $\mathscr{M}$ in the complex plane.

- Ex: Breit Wigner

    - $\mathscr{M}(E) \sim \dfrac{\Gamma/2}{(E_R - E) - i\Gamma/2}$

    - Pole at complex $E = E_R - i\Gamma/2$



Resonances in field theory

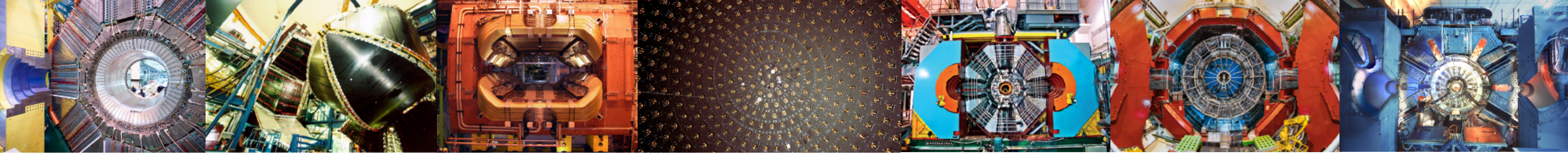
- Resonances are identified with poles of scattering amplitudes $\mathscr{M}$ in the complex plane.

- ***But***, analytic structure of scattering amplitudes is tightly constrained by causality and unitarity

    - Causality: No poles away from the real line on first (“physical”) Riemann sheet

    - Unitarity: Complex poles come in conjugate pairs on higher Riemann sheets

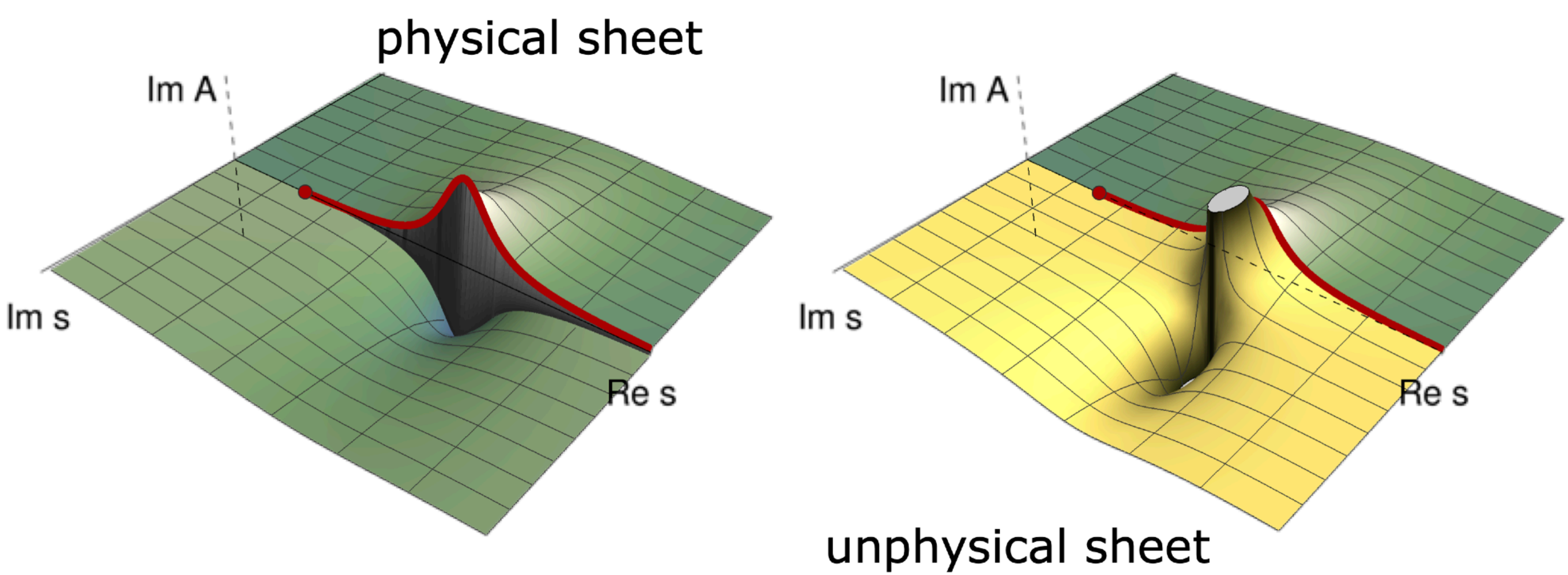
    - Multi-particle thresholds: Branch cuts on the real line

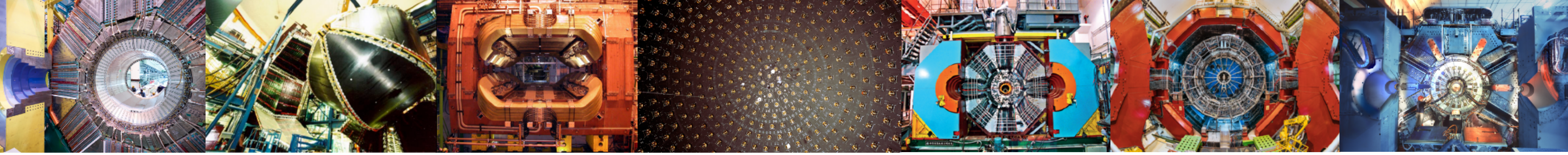


Resonances in field theory

PDG Review: #50 ***Resonances,***
Asner, Hanhart, Mikhasenko

- Resonances are identified with poles of scattering amplitudes $\mathscr{M}$ in the complex plane.





Resonances in field theory

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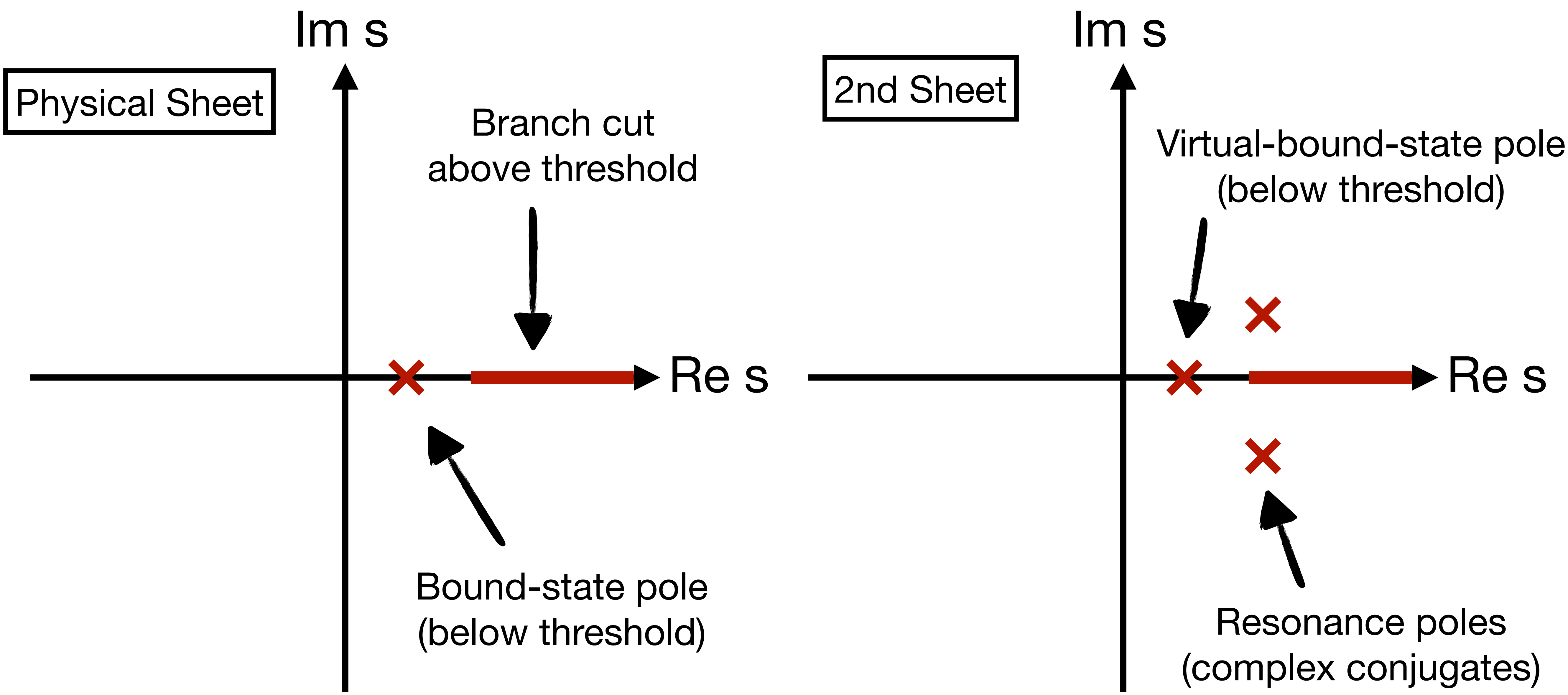
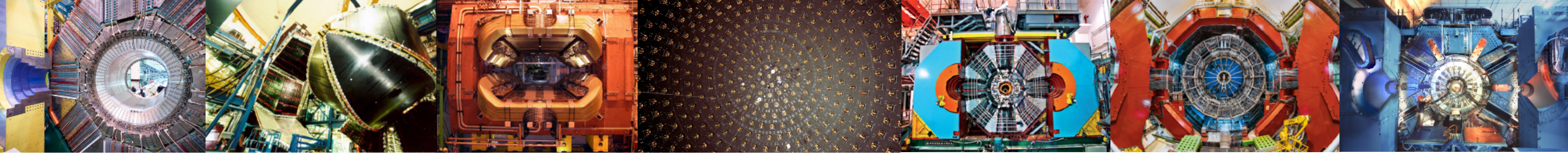
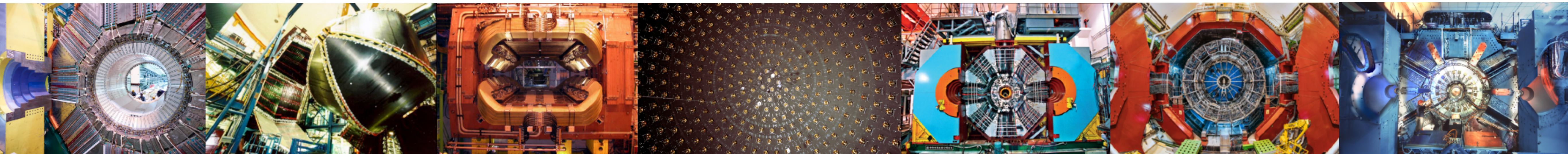
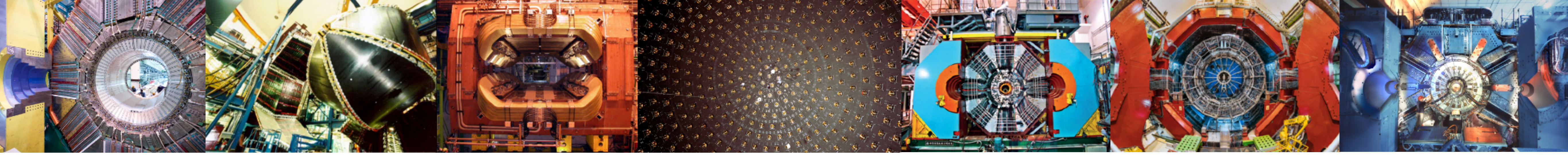




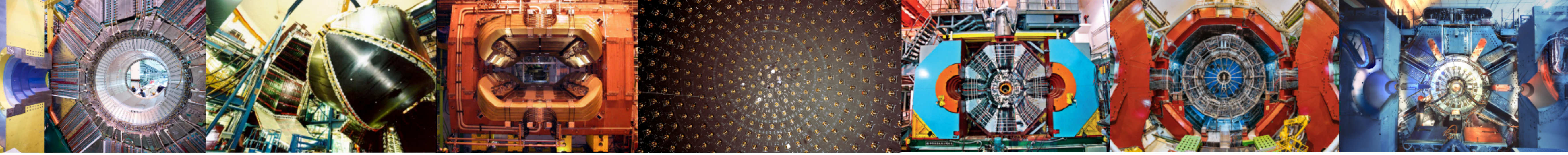
Hadron Spectroscopy in Finite Volume





Hadron spectroscopy in Finite Volume

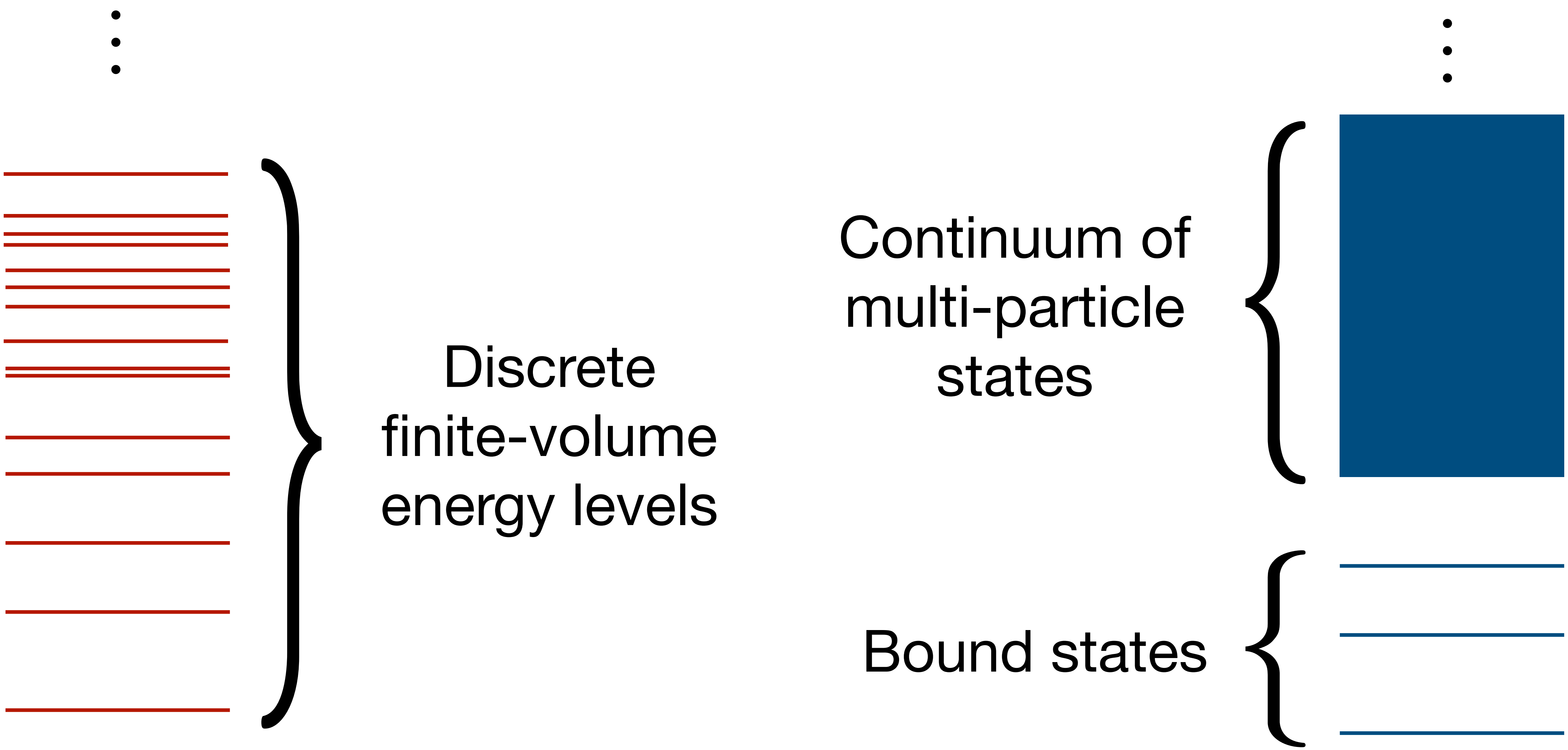
- Scattering amplitudes are defined in infinite volume:
  - Begin with asymptotically separated initial states
  - Particles interact
  - Compute phase shift in outgoing waves at asymptotic separation.
- Conceptual complication in finite-volume:
  - A periodic, few-fm box does *not* have asymptotic states

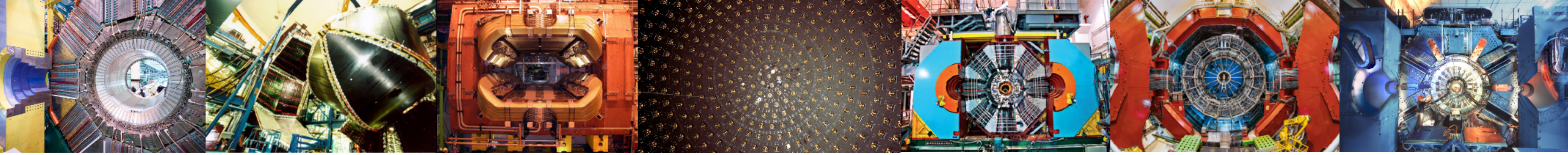


Hadron spectroscopy in Finite Volume

Finite volume

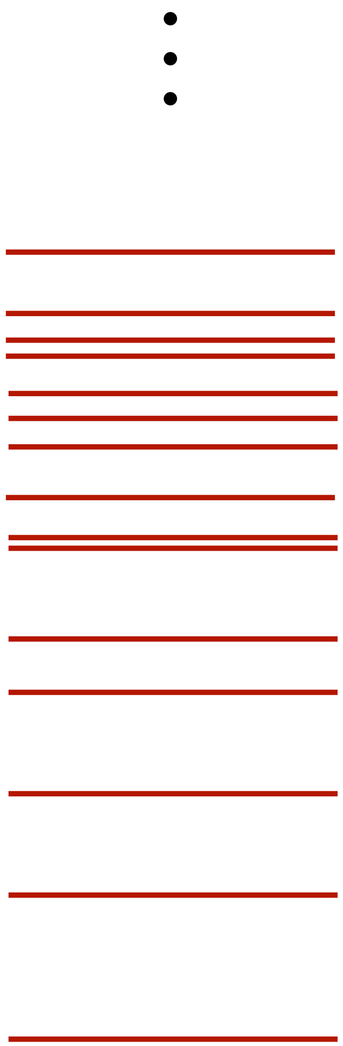
Infinite volume



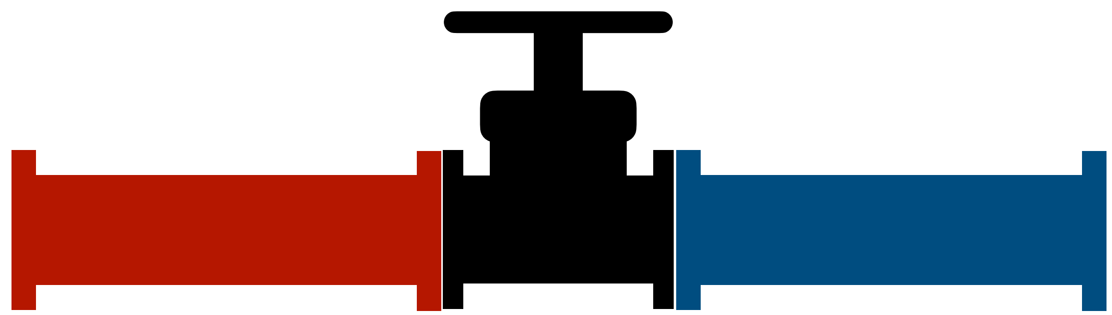


Hadron spectroscopy in Finite Volume

Finite volume



Quantization Condition



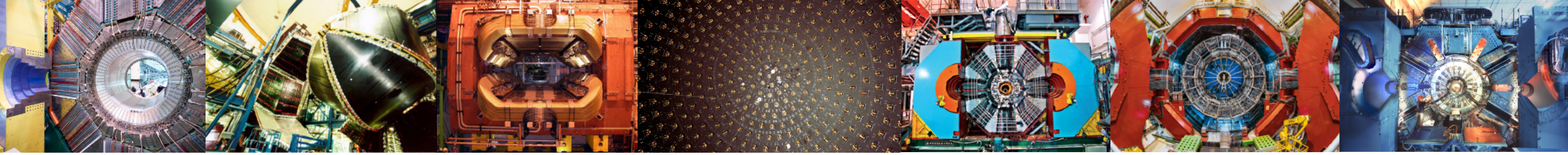
$$\det_{\ell m}\left[F(E,L) + K^{-1}(E)\right]\Big|_{E=E_n} = 0$$

Finite-volume function (known)

Infinite-volume K-matrix (desired)

Infinite volume

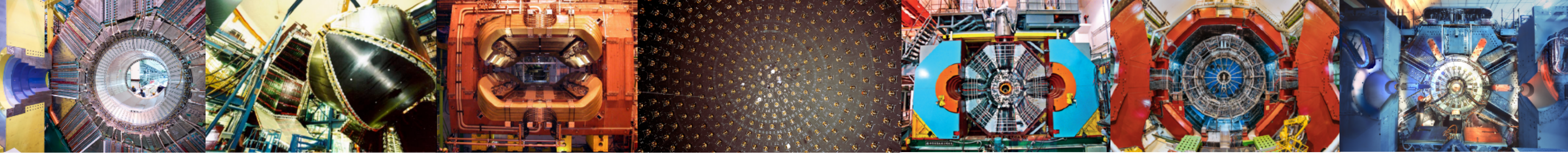
Lüscher, *Commun.Math.Phys.* 104 (1986) 177
Lüscher, *Commun.Math.Phys.* 105 (1986) 153-188
Lellouch and Lüscher, Commun.Math.Phys. 219 (2001) 31-44
+ many, many other contributors!



Aside: Origin of Quantization Condition

Case 1: Infinite volume

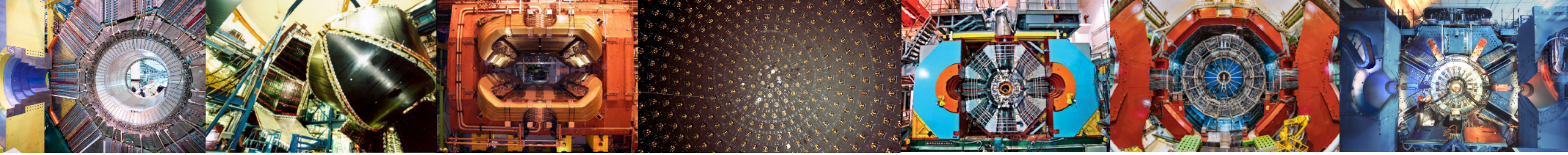
- Consider 2 bosons in non-relativistic QM.
- Consider a finite-range potential $V(r) = 0, r > R$
- Look for scattering ($E > 0$) solutions to the Schrödinger equation
    - $\psi_p(r) \sim \cos\left(pr + \delta(p)\right)$
    - Matching solutions at $r = R$ to find phase shift $\delta(p)$
- Note: momentum p is a continuous



Aside: Origin of Quantization Condition

Case 2: Finite volume

- Look for scattering ($E > 0$) solutions to the Schrödinger equation
  - $\psi_p(r) \sim \cos\left(pr + \delta(p)\right)$
  - Compute phase shift $\delta(p)$ by matching solutions at $r = R$
  - Enforce periodicity at boundary: $\psi_p(r)\Big|_{r=L/2} = 0$
    - $\implies p_n \frac{L}{2} + \frac{2}{L}\delta(p_n) = \pi n$
    - Quantization condition: discrete momenta satisfy this equation



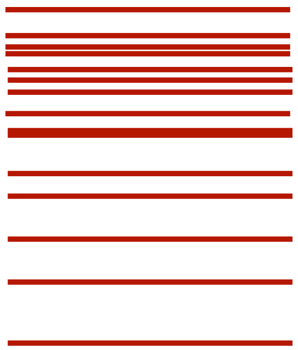
Hadron spectroscopy in Finite Volume

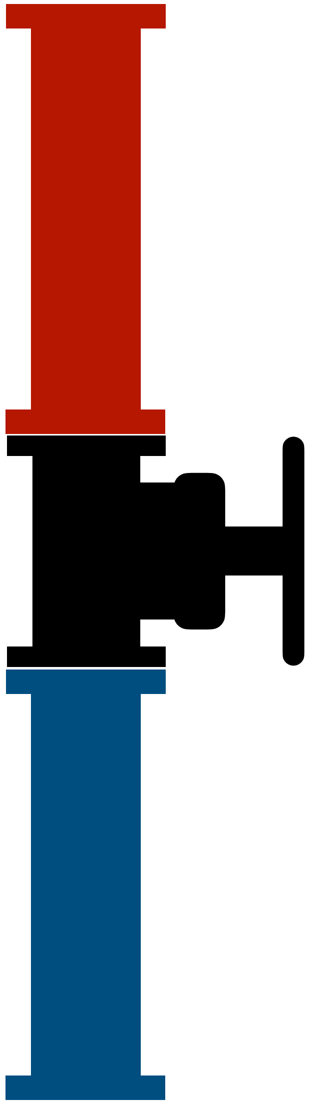
Pipeline of ideas

1.) Compute finite-volume spectrum E_n (lattice QCD)

Lattice QCD

$$\langle \hat{\mathcal{O}}(\tau)\hat{\mathcal{O}}(0)\rangle \Longrightarrow$$





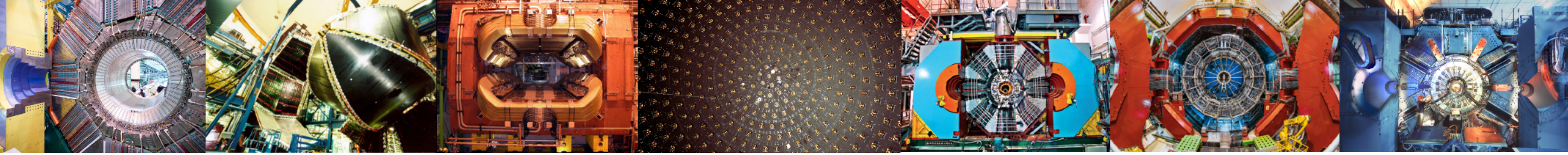
2.) Solve quantization condition for $K^{-1}(E_n)$

$$\det_{\ell m}\left[F(E,L) + K^{-1}(E)\right]\Big|_{E=E_n} = 0$$

Real line: $\mathrm{Re}\mathcal{M}^{-1}(E) = K^{-1}(E)$

3.) Locate poles using analytic continuation

$$\mathcal{M}^{-1}(E) \to \mathcal{M}^{-1}(z)$$



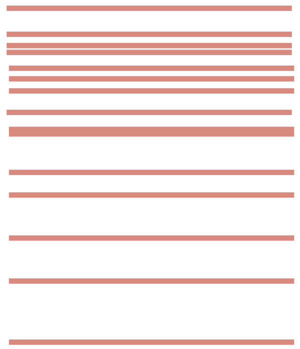
Hadron spectroscopy in Finite Volume

State of the art: Step 1

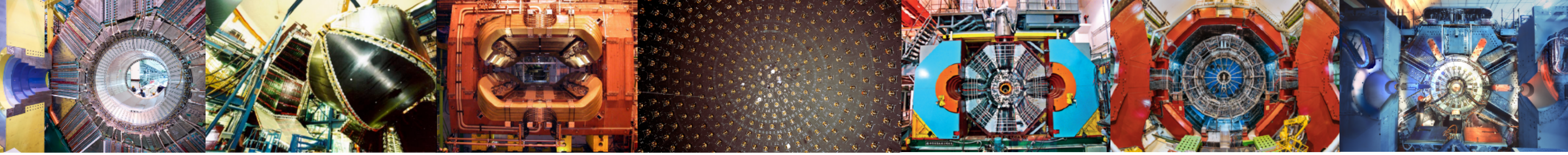
1.) Compute finite-volume spectrum E_n (lattice QCD)

Lattice QCD

$$\langle \hat{\mathcal{O}}(\tau)\hat{\mathcal{O}}(0)\rangle \Longrightarrow$$



- Choose a variational basis of N operators $\hat{\mathcal{O}}_i$ with the quantum numbers of the desired states (e.g., I=1 $\pi\pi$)
- Compute the $N \times N$ matrix of correlation functions $\langle \hat{\mathcal{O}}_i(\tau)\hat{\mathcal{O}}_j(0)\rangle$
- Solve generalized eigenvalue problem to find N energy levels E_n

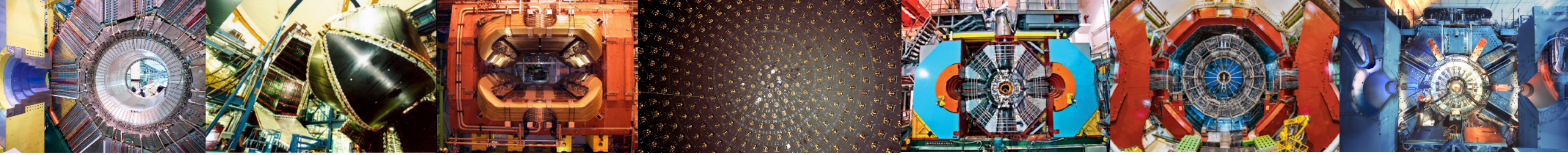


Hadron spectroscopy in Finite Volume

State of the art: Steps 2 and 3

2.) Solve quantization condition for $K^{-1}(E_n)$	$\det_{\ell m}\left[K^{-1}(E)+F(E,\mathbf{P},L)\right]\Big\vert_{E=E_n} = 0$
3.) Locate poles using analytic continuation	$\mathcal{M}^{-1}(E) \to \mathcal{M}^{-1}(z)$

- Assume an explicit parametric model for the K-matrix to relate solutions of the quantization condition at different energies.
- Fix parameters of model with a least-squares fit.
- Solve for poles using the best-fit parameters
- Check parameterization dependence of result to quantify systematic uncertainties



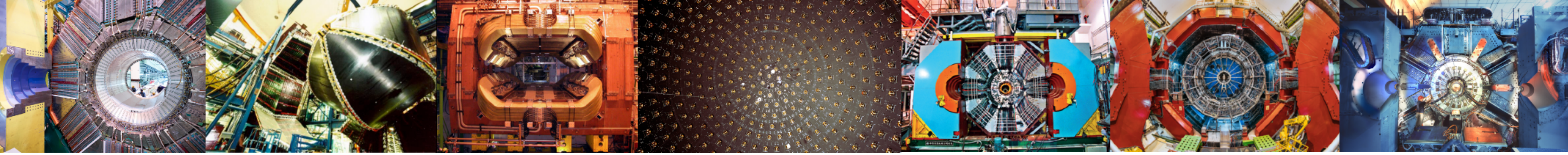
Hadron spectroscopy in Finite Volume

State of the art: Steps 2 and 3

2.) Solve quantization condition for $K^{-1}(E_n)$	$\det_{\ell m}\left[K^{-1}(E)+F(E,\mathbf{P},L)\right]\Big\vert_{E=E_n} = 0$
3.) Locate poles using analytic continuation	$\mathscr{M}^{-1}(E) \to \mathscr{M}^{-1}(z)$

Today's talk:

- New non-parametric approach to solving quantization condition and finding poles via analytic continuation



Hadron spectroscopy in Finite Volume

Solving the quantization condition

Useful to get concrete: Single-channel s-wave ($\ell = 0$) scattering

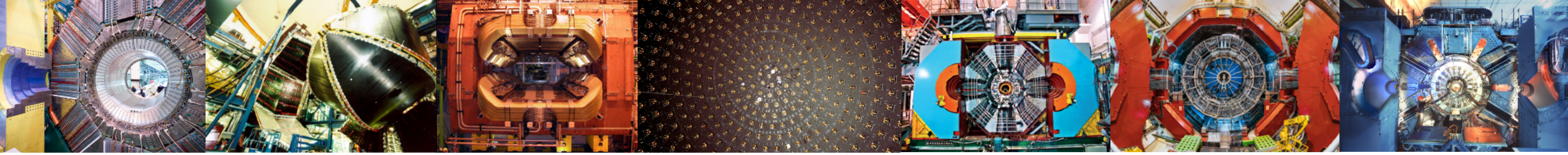
Ex: $\pi\pi \to \sigma \to \pi\pi$ below $K\bar{K}$ threshold.

$$\det\left[K^{-1} + F\right] = 0$$

$$\implies q \cot \delta_0(q^2) = (\text{kinematics}) \times \mathcal{Z}_{00}(q^2, L, \mathbf{P})$$

Determinant simplifies to a single line.

$\mathcal{Z}_{00}$ - Lüscher zeta function (transcendental function for finite volume)



Hadron spectroscopy in Finite Volume

Solving the quantization condition

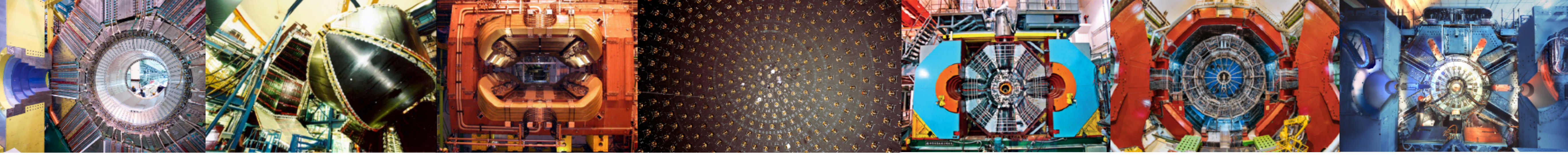
Useful to get concrete: s-wave multi-channel scattering

Ex: $\pi\Sigma \to \Lambda(1405) \to \bar{K}N$

$$\det\left[K^{-1} + F\right] = 0$$

$$\Longrightarrow \det\left[\begin{pmatrix} K^{-1}_{\pi\Sigma;\pi\Sigma}(E) & K^{-1}_{\pi\Sigma;\bar{K}N}(E) \\ K^{-1}_{\bar{K}N;\pi\Sigma}(E) & K^{-1}_{\bar{K}N;\bar{K}N}(E) \end{pmatrix} + \begin{pmatrix} F_{\pi\Sigma}(E) & 0 \\ 0 & F_{\bar{K}N}(E) \end{pmatrix}\right] = 0$$

For each energy E, the quantization condition is a real equation in three unknowns: $K^{-1}_{ij}(E)$

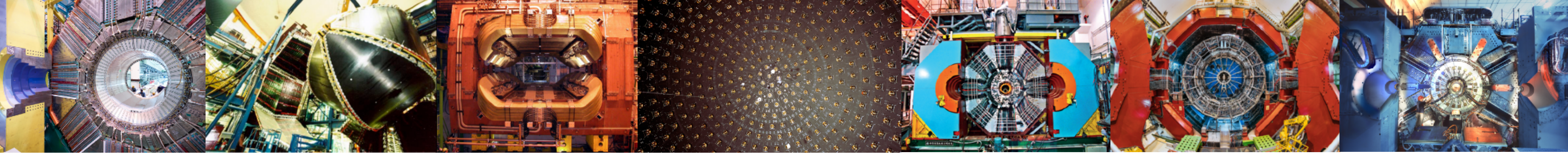


Hadron spectroscopy in Finite Volume

Solving the quantization condition - usual method

- Parameterize the K-matrix — *Ansatz* $K^{-1}(E) = \sum_m \theta_m E^m$
- Solve for discrete energies — Solution $E_{\mathrm{QC}}(\theta)$
- Compare to known energies — E_{LQCD}: $\Delta E = E_{\mathrm{LQCD}} - E_{QC}(\theta)$
- Construct χ^2 function — $\chi^2(\theta) = \Delta E^T \Sigma^{-1}_{\mathrm{LQCD}} \Delta E$
- Define best-fit parameters — $\theta^\star = \mathrm{argmin}_\theta\, \chi^2(\theta)$

A “non-parametric” proposal

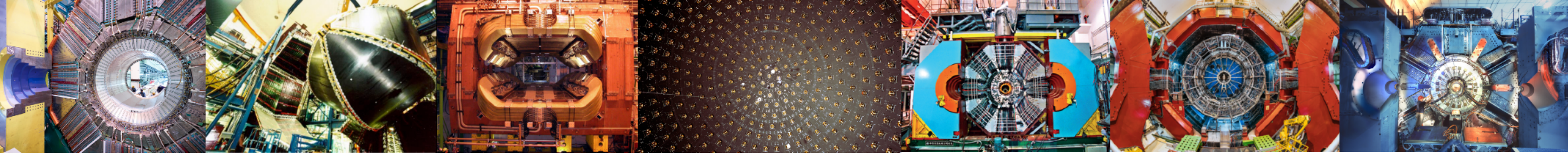


Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

Goal: non-parametric model for K-matrix at nodes on the real line

- Consider interpolating a smooth function $u(E) : \mathbb{R} \to \mathbb{R}$ at a discrete set of "nodes" $\mathbf{E}$ and "values" $\mathbf{u}$
- If typical variations in $u(E)$ occur at scale longer than node spacing, the error from interpolating a discrete set "ought to be small."



Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

Goal: non-parametric model for K-matrix at nodes on the real line

- Bayesian approach $\Longrightarrow$ need "prior" distribution for values $\mathbf{u} = K_\ell^{-1}(\mathbf{E})$:

$$p(\mathbf{u}\,|\,\mathbf{u}_0) \propto \exp\left[-\frac{1}{2}(\mathbf{u}-\mathbf{u}_0)^T \Sigma_p^{-1}(\mathbf{u}-\mathbf{u}_0)\right]$$

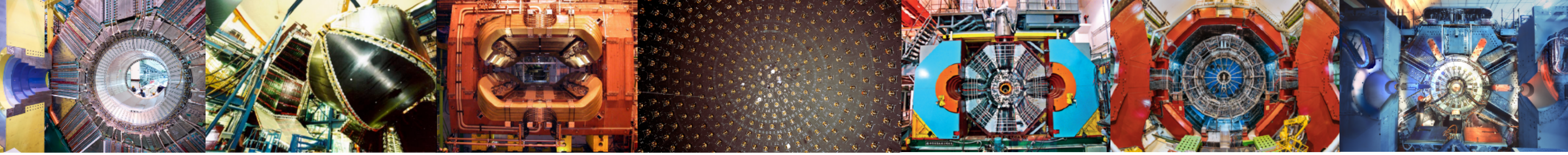
multivariate Gaussian distribution

$$\Sigma_p(E,E') = \sigma(E,E')^2 \exp\left[-\frac{(E-E')^2}{2\ell_c^2}\right] + \epsilon\delta_{E,E'}$$

Radial Basis Function (RBF) kernel

$$p(\mathbf{E}_{\mathrm{LQCD}}\,|\,\mathbf{u}) \propto \exp\left[-\tfrac{1}{2}\chi^2(\mathbf{u})\right]$$

Likelihood - usual χ^2



Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

Goal: non-parametric model for K-matrix at nodes on the real line

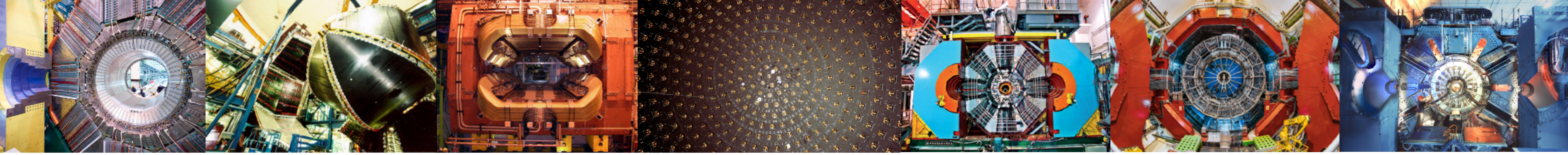
- Bayes Theorem:

$$(\text{posterior}) = (\text{likelihood}) \times (\text{prior})$$

$$p(\mathbf{u} \,|\, \mathbf{E}_{\text{LQCD}}, \mathbf{u}_0) = p(\mathbf{E}_{\text{LQCD}} \,|\, \mathbf{u}) \times p(\mathbf{u} \,|\, \mathbf{u}_0)$$

Note: Similar, but <u>not</u> identical, to a Gaussian process.

$E_{\text{QC}}(\mathbf{u})$ has important nonlinearities in $\mathbf{u}$ from singularities in zeta function $\mathcal{Z}_{00}$



Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

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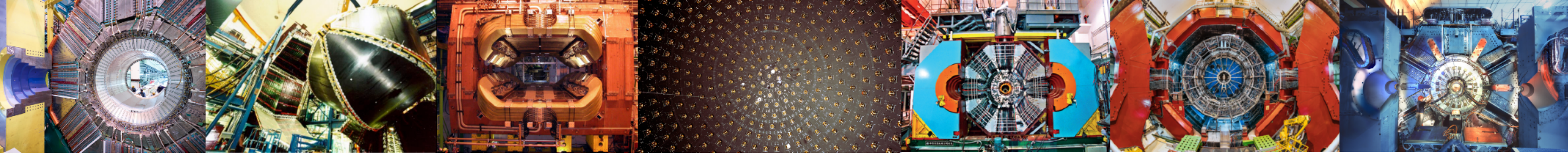
$$(\text{posterior}) = (\text{likelihood}) \times (\text{prior})$$

$$p(\mathbf{u} \,|\, \mathbf{E}_{\text{LQCD}}, \mathbf{u}_0) = p(\mathbf{E}_{\text{LQCD}} \,|\, \mathbf{u}) \times p(\mathbf{u} \,|\, \mathbf{u}_0)$$

- Expectation values (mean, errors) defined with respect to the posterior, as usual:

$$\langle \mathbf{u} \rangle_{\text{post}} = \frac{1}{Z} \int d^d u \, \mathbf{u} \, p(\mathbf{u} \,|\, \mathbf{E}_{\text{LQCD}}, \mathbf{u}_0)$$



Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

Goal: non-parametric model for K-matrix at nodes on the real line

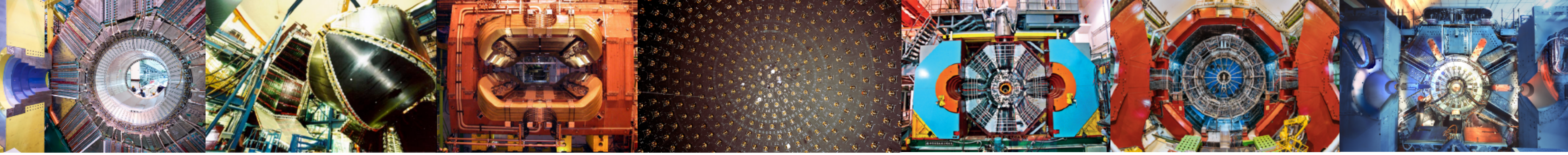
Posterior Sampling: Monte Carlo

Approach #1

1. Sample from Gaussian prior (trivial)

2. Apply reweighting: $w_{\text{post}} = \dfrac{p(\mathbf{u}\,|\,\mathbf{E}_{\text{d}}, \mathbf{u}_0)}{p(\mathbf{u}\,|\,\mathbf{u}_0)}$ $\quad \langle \mathbf{u} \rangle_{\text{post}} = \langle \mathbf{u} w_{\text{post}} \rangle_{\text{prior}}$

- Benefit: no Markov chain, so no autocorrelations.
- Challenge: Sampling can be inefficient when prior and posterior dissimilar



Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

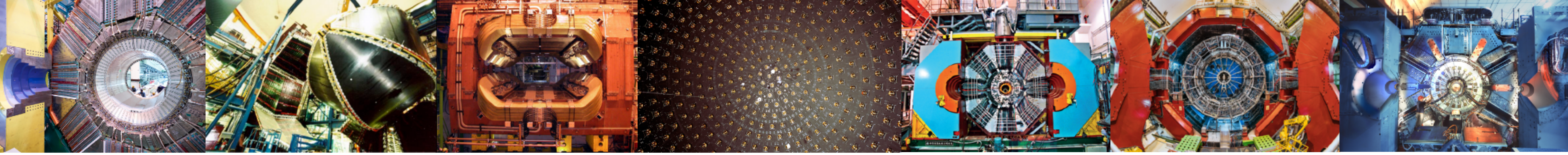
Goal: non-parametric model for K-matrix at nodes on the real line

Posterior Sampling: Monte Carlo

Approach #2

1. Linearize quantization condition
2. Sample from resulting Gaussian approximation to $(\text{Likelihood}) \times (\text{Prior})$
3. Apply reweighting to recover exact result

- Benefit: Easy to compute normalization (Bayesian evidence) for model comparison
- Challenge: Sampling can *still* be inefficient when prior and posterior dissimilar



Hadron spectroscopy in Finite Volume

Solving the quantization condition - a Bayesian approach

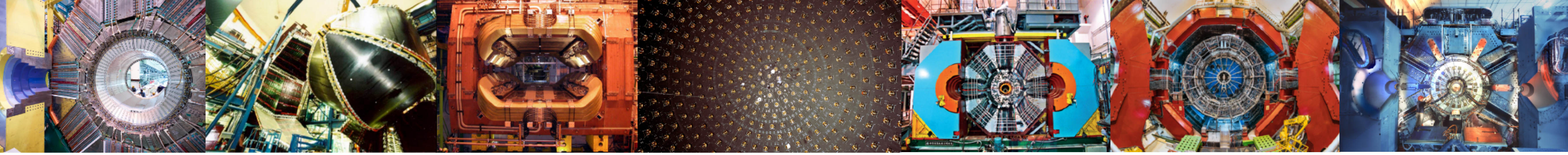
Goal: non-parametric model for K-matrix at nodes on the real line

Posterior Sampling: Monte Carlo

Approach #3

1. Generate samples using Hybrid Monte Carlo algorithm

- Benefit: Robust sampling method
- Challenge: Introduces familiar complications like autocorrelations
- Challenge: Accessing normalization Z for model comparison is more cumbersome

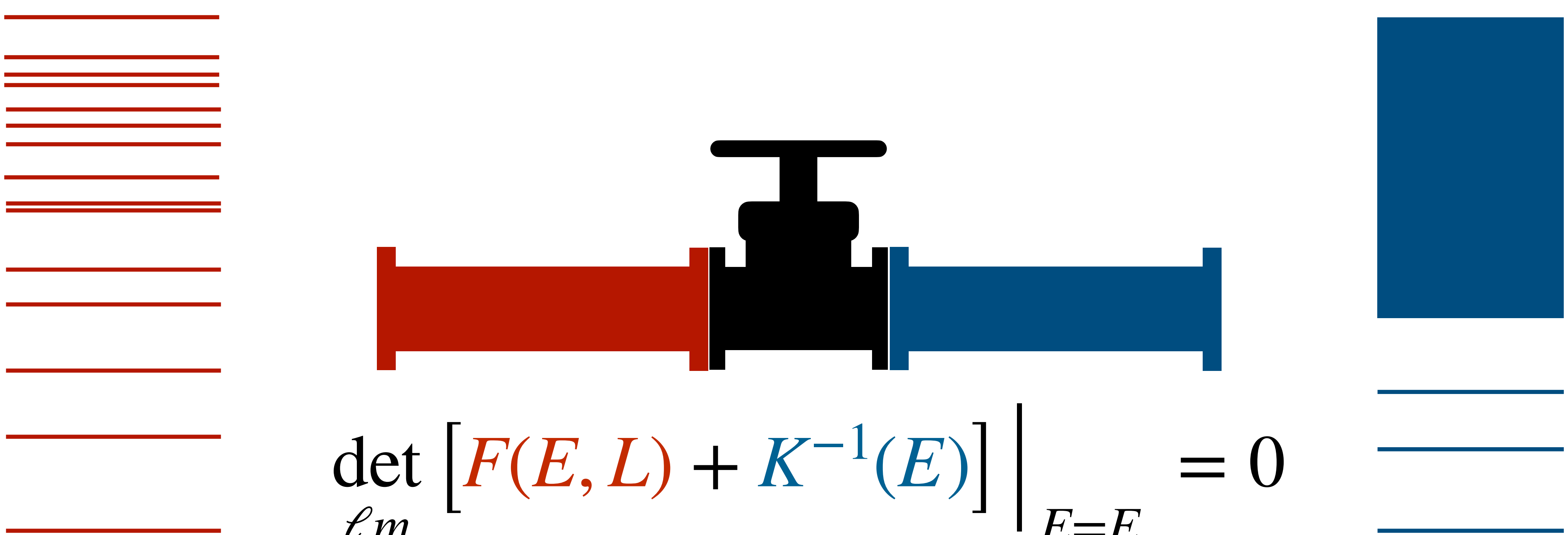


Hadron spectroscopy in Finite Volume

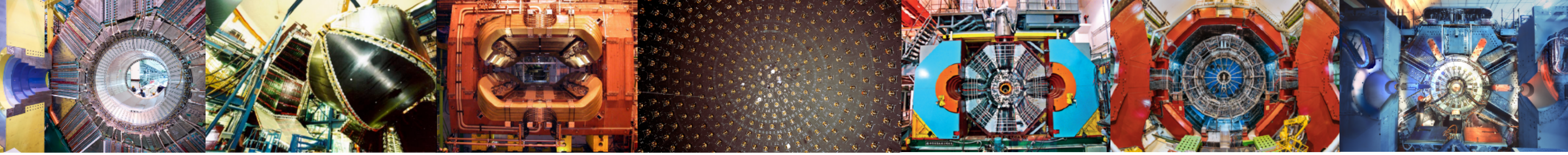
Solving the quantization condition - a Bayesian approach

Goal: non-parametric model for K-matrix at nodes on the real line

- Upshot: sampling furnishes $\langle \mathbf{u} \rangle_{\text{post}} = K^{-1}(\mathbf{E})$

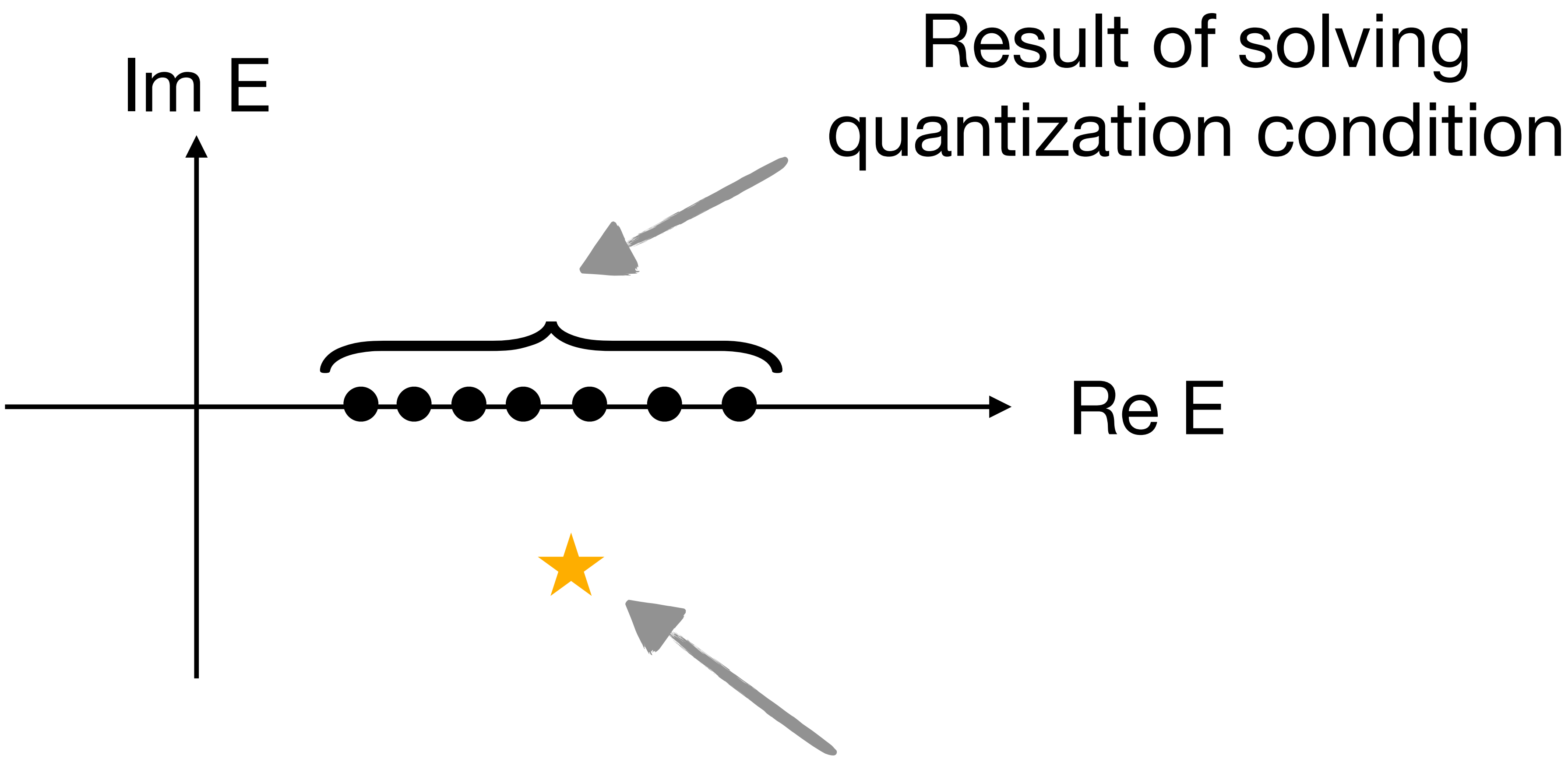


$$\det_{\ell m}\left[F(E,L) + K^{-1}(E)\right]\Big|_{E=E_n} = 0$$

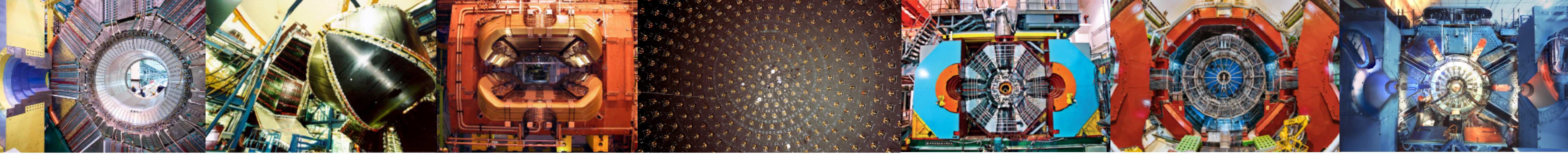


Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane



Goal: Locate pole in $\mathscr{M} \Longleftrightarrow$ Zero in $\mathscr{M}^{-1}$
with robust quantification of uncertainty



Hadron spectroscopy in Finite Volume

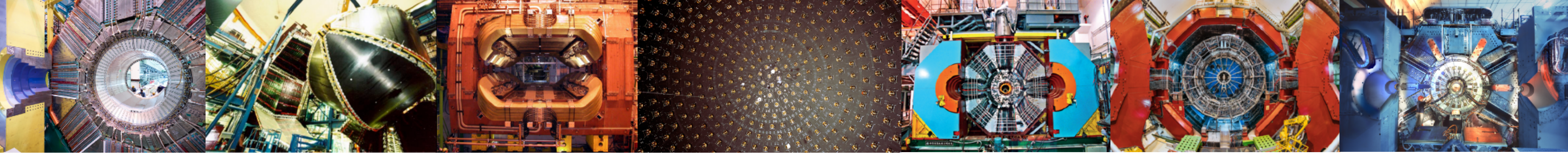
Analytic continuation to locate poles in the complex plane

Im E

Result of solving quantization condition

What does “analytic continuation” mean for a finite set of points?

Goal: Locate pole in $\mathscr{M} \iff$ Zero in $\mathscr{M}^{-1}$ with robust quantification of uncertainty

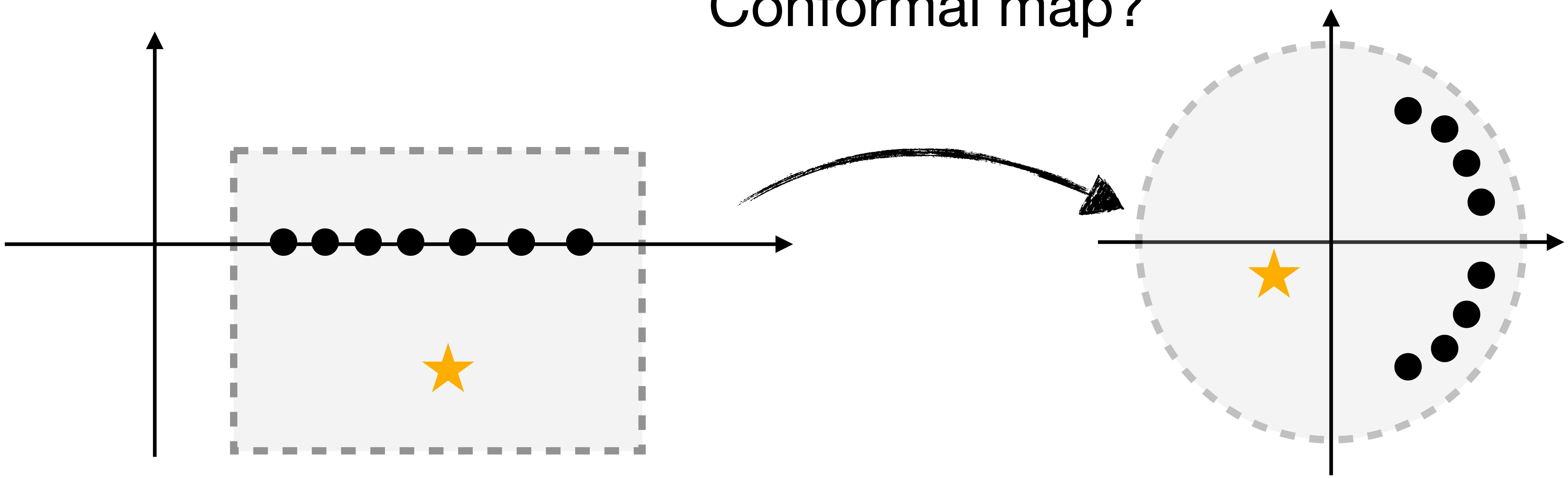


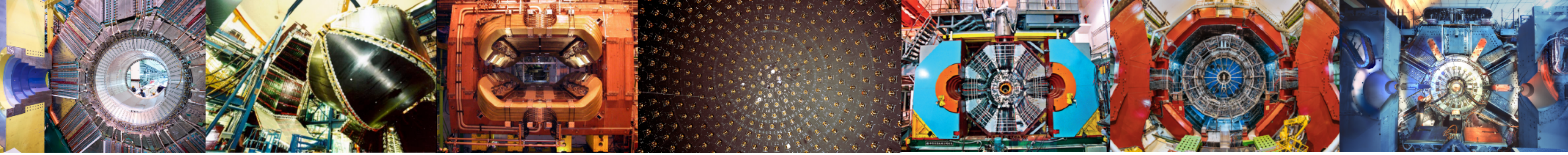
Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane

The natural setting for complex analysis is the unit disk.

Conformal map?

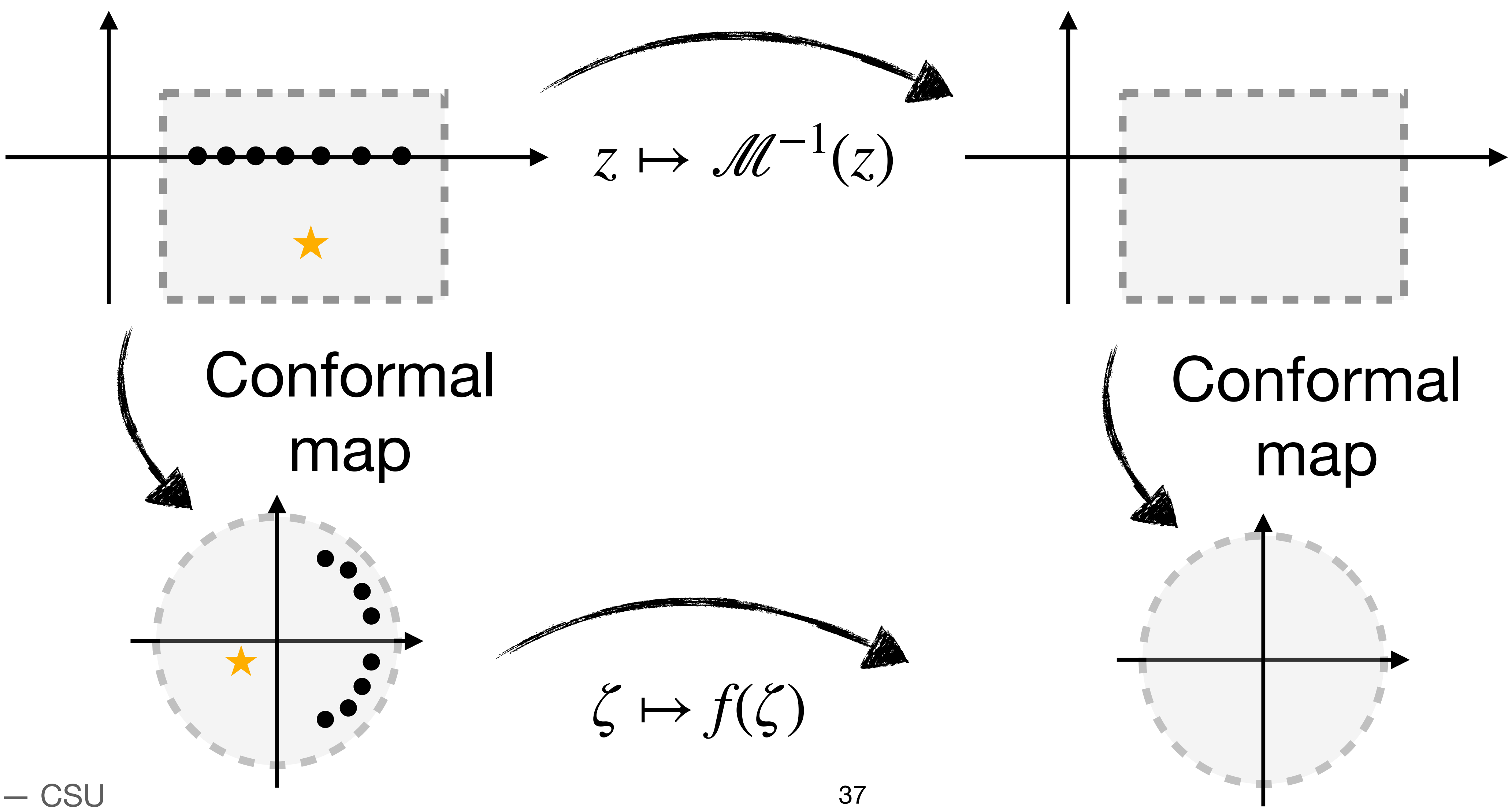


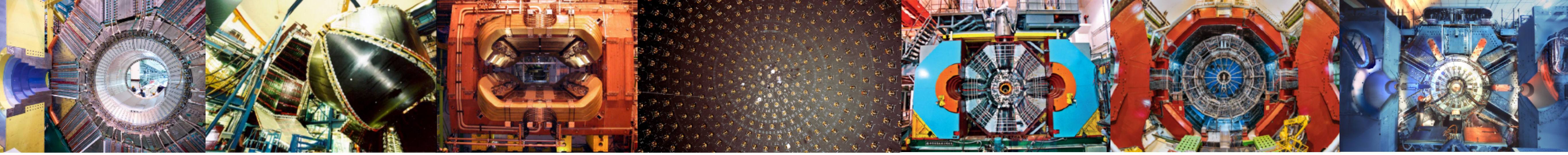


Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane

- *Nota bena*: Such a mapping is required for both domain and codomain.





Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane

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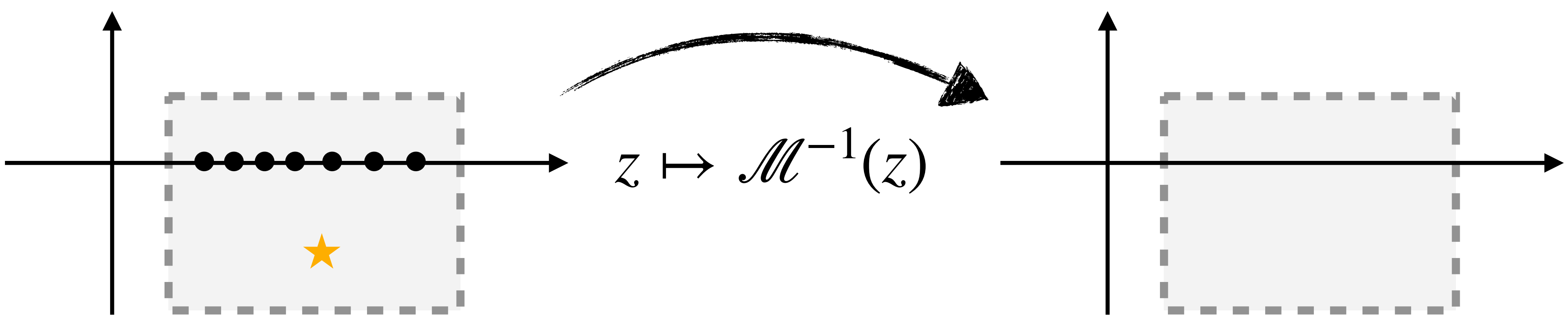
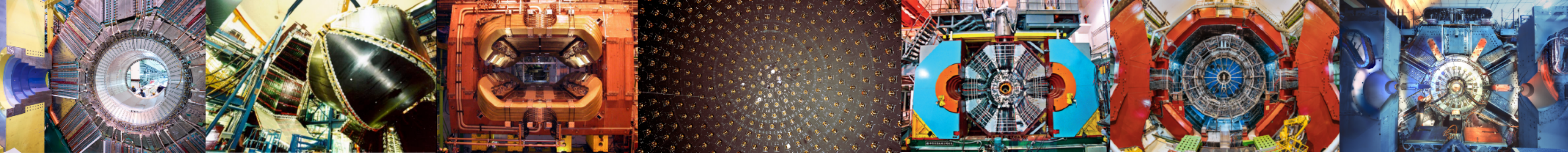


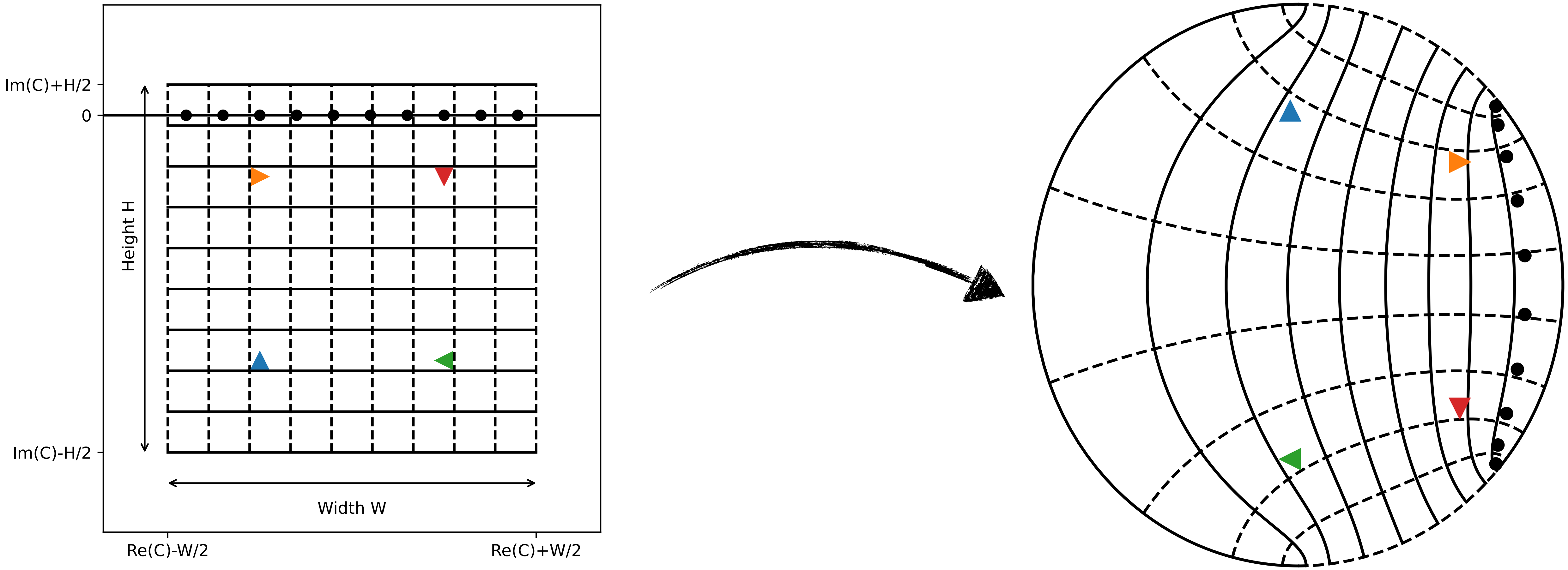
- The Riemann mapping theorem guarantees existence of required conformal maps
  - Use physics knowledge to guide the choice of domain and codomain and maps
  - Ex: Locations of poles and zeros, range of possible values, etc…
- These maps implicitly contain strong theoretical assumptions. Be careful.

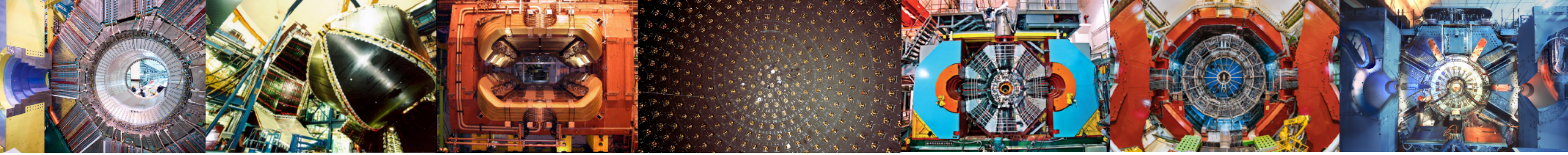


Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane

One appealing choice: Schwarz-Christoffel mapping





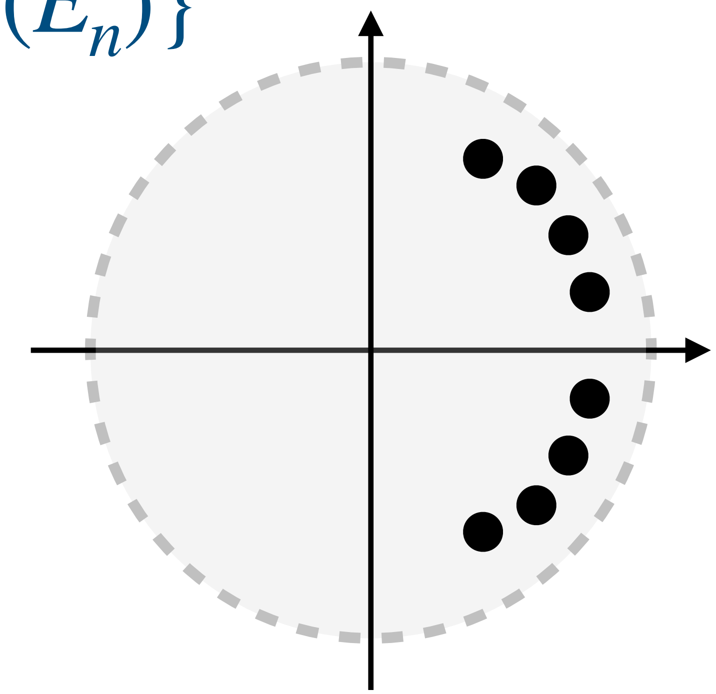
Numerical Analytic Continuation

The sharp technical problem

- **Given data for real energies $\{E_n\}$, $\{\mathscr{M}^{-1}(E_n)\}$**

$$\{E_n\} \to \zeta_n \subset \mathbb{D},$$

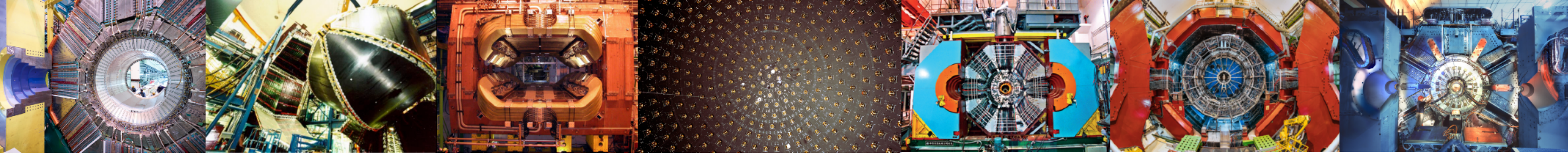
$$\{\mathscr{M}^{-1}(E_n)\} \mapsto w_n \subset \mathbb{D},$$



construct an analytic function $f(\zeta)$

on the disk that interpolates these points: $f(\zeta_n) = w_n$.

- Locate zero by extrapolation to the interior of the disk



Nevanlinna-Pick Interpolation

The big idea: “factor out what you know”

- **Basic fact (maximum modulus principle $\Longrightarrow$):**

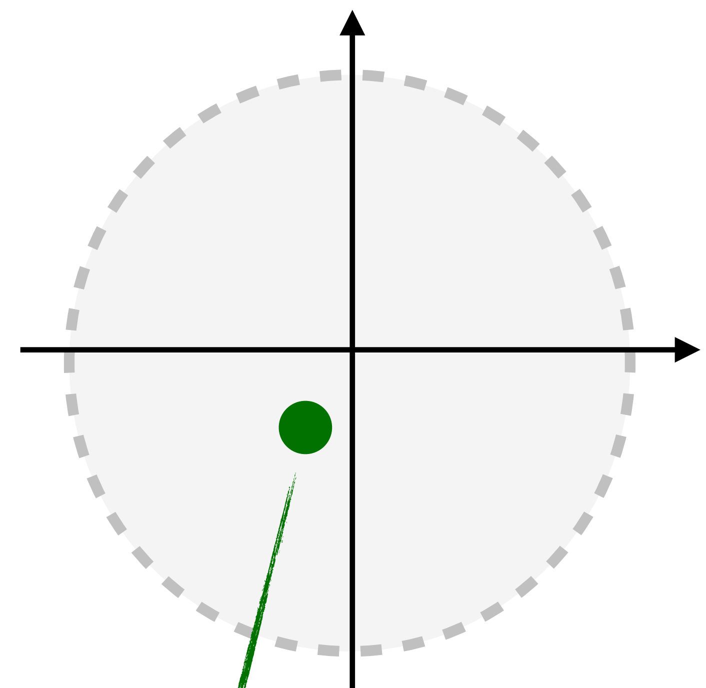
Let $g(\zeta) : \mathbb{D} \to \mathbb{D}$ be an analytic function.

Suppose $g(\zeta)$ has a zero at $\mathbf{a} \in \mathbb{D}$: $g(\mathbf{a}) = 0$.

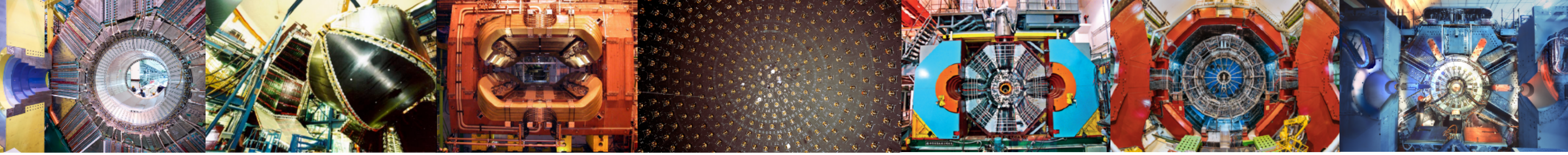
Then $g(\zeta) = b_a(\zeta)\tilde{g}(\zeta)$.

Blaschke factor

“Remainder” (analytic in $\mathbb{D}$)



$g(\mathbf{a}) = 0$



Nevanlinna-Pick Interpolation

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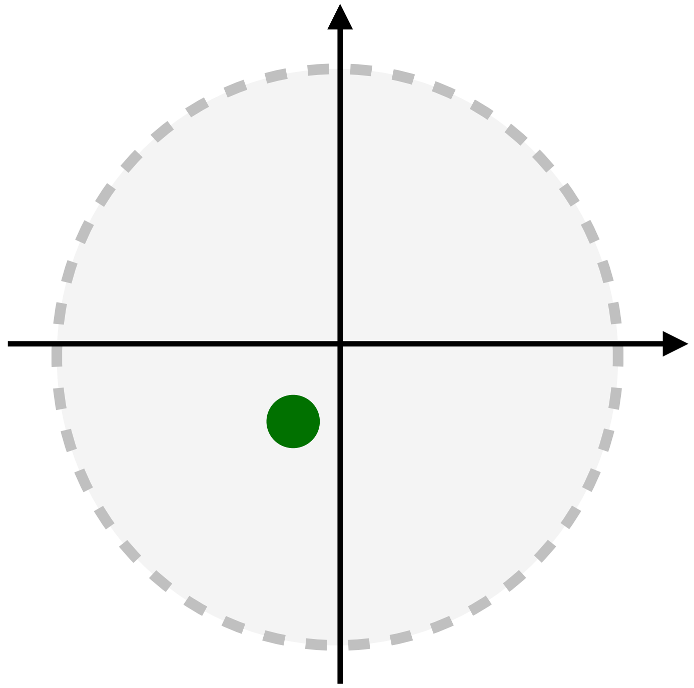
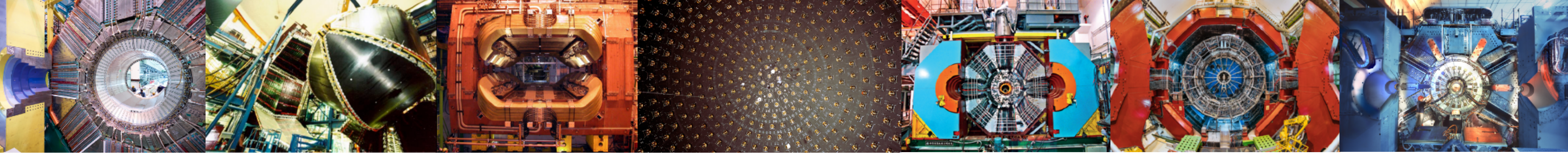


- Note: Setup familiar in quark-flavor physics from *z-expansion* of form factors
  - Blaschke factors "factor out” known analytic structure, e.g., sub-threshold poles.

Boyd, Grinstein, Lebed
Nucl.Phys.B **461 (1996) 493-511**
Phys.Rev.D **56 (1997) 6895-6911**
Caprini, Lellouch, Neubert
Nucl.Phys.B 530 (1998) 153-181



Analytic Continuation

Repeated application of “factoring”

Theorem (Nevanlinna, 1919/1929):

- Any solution to the interpolation problem with N points can be written in the form

$$f(\zeta) = \frac{P_N(\zeta) f_N(\zeta) + Q_N(\zeta)}{R_N(\zeta) f_N(\zeta) + S_N(\zeta)}.$$

- “*Nevanlinna coefficients*” P_N, Q_N, R_N, S_N

  $\Longleftrightarrow$Known / calculable from input data

- Arbitrary function analytic function $f_N(\zeta) : \mathbb{D} \to \mathbb{D}$

  $\Longleftrightarrow$ Freedom to specify further data to constrain the interpolating function

  $\Longleftrightarrow$ Plays role of the “remainder” function on the previous slide

R. Nevanlinna
Ann. Acad. Sci. Fenn. Ser. A 13 (1919)
Ann. Acad. Sci. Fenn. Ser. A 32 (1929)

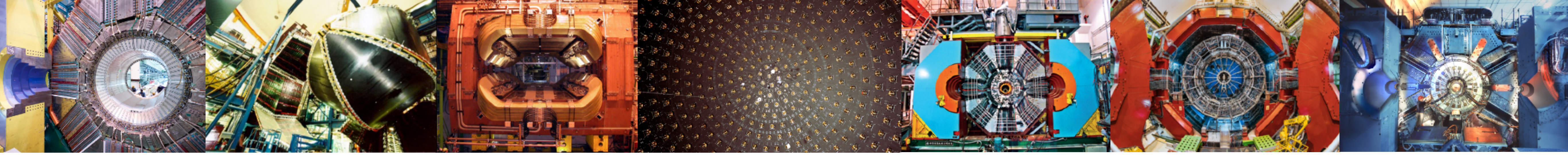
A. Nicolau
Proc. Summer School in Complex and Harmonic analysis… (2016)
[LINK]

First application in QFT (Condensed Matter Physics)
J. Fei, C.-N. Yeh, E. Gull, *PRL* 126, 056402 (2021)
arXiv:2010.04572

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

Patrick Oare

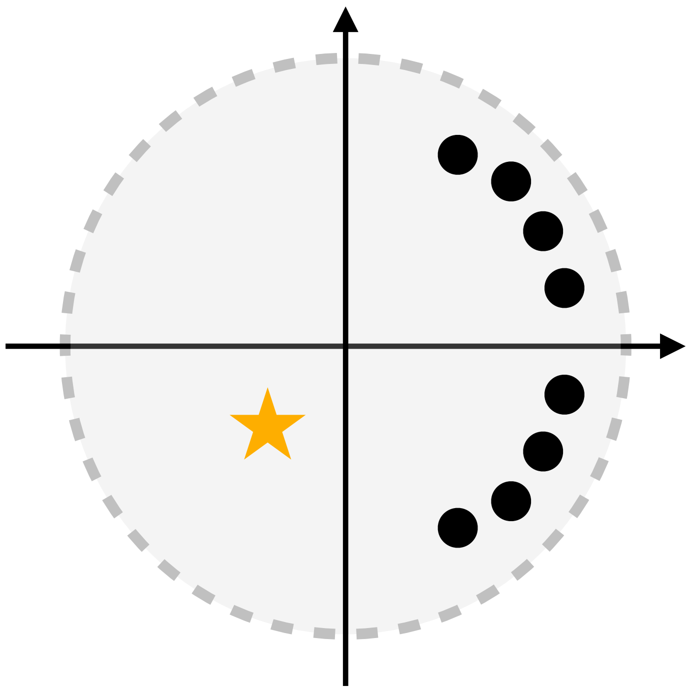
MIT →BNL



Analytic Continuation
The full space of solutions

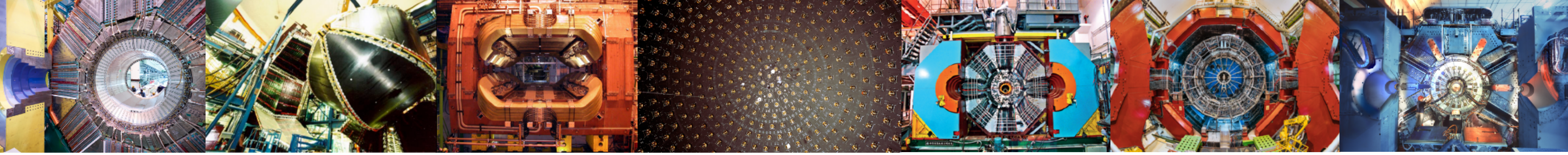
Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

- Key point: The freedom and influence of the "remainder" is constrained, since $f_N(\zeta) \in \mathbb{D}$.
- **Question**: What possible values can the interpolating function $f(\zeta)$ can take when extrapolated to arbitrary points "★"?
    - Remarkably, this set can be parameterized explicitly for each N and each point "★".
    - Size of this set $\Longleftrightarrow$ ambiguity in the analytic continuation



● = given

$f(★) = ?$



Analytic Continuation

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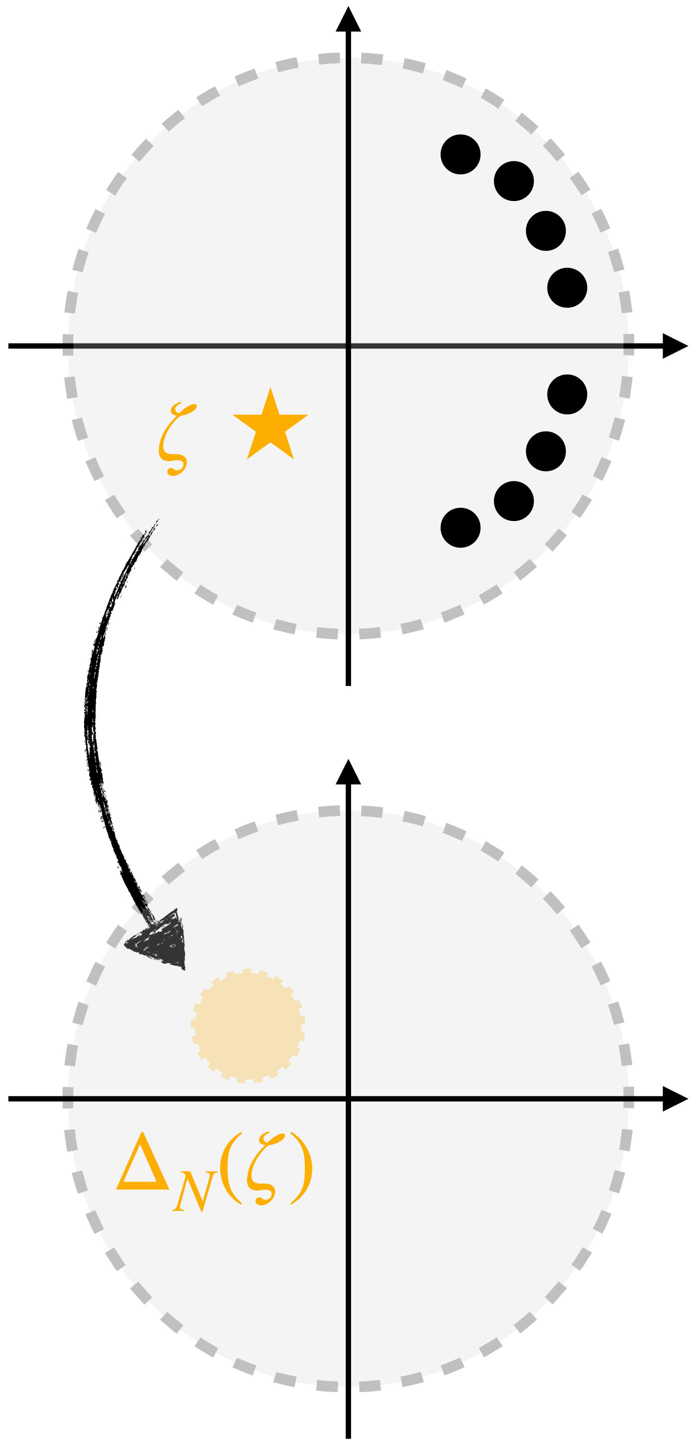
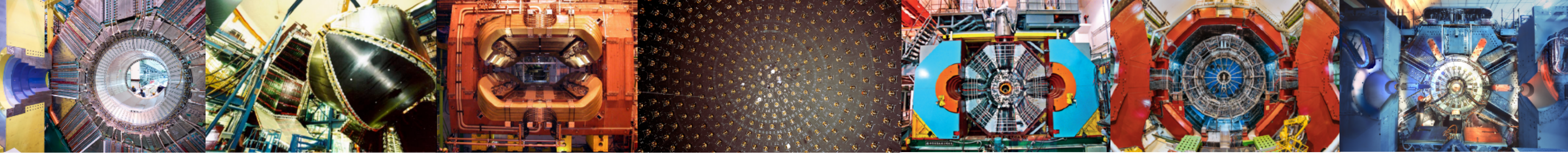
The full space of solutions

- **Answer**: The space of possible values is a disk of radius $r_N(\zeta)$ centered at $c_N(\zeta)$. This disk called the *Wertevorrat* $\Delta_N(\zeta)$.

$$c_N = \frac{P_N\overline{(-R_N/S_N)} + Q_N}{R_N\overline{(-R_N/S_N)} + S_N} \qquad r_N = \frac{|P_N S_N - Q_N R_N|}{|S_N|^2 - |R_N|^2}$$

- Given N **interpolation points**, the *Wertevorrat* $\Delta_N(\zeta)$ rigorously contains all possible analytic continuations at each extrapolation point $\zeta \in \mathbb{D}$.
  - Complete characterization of systematic uncertainty
  - No model assumptions — just analyticity!





Hadron spectroscopy in Finite Volume

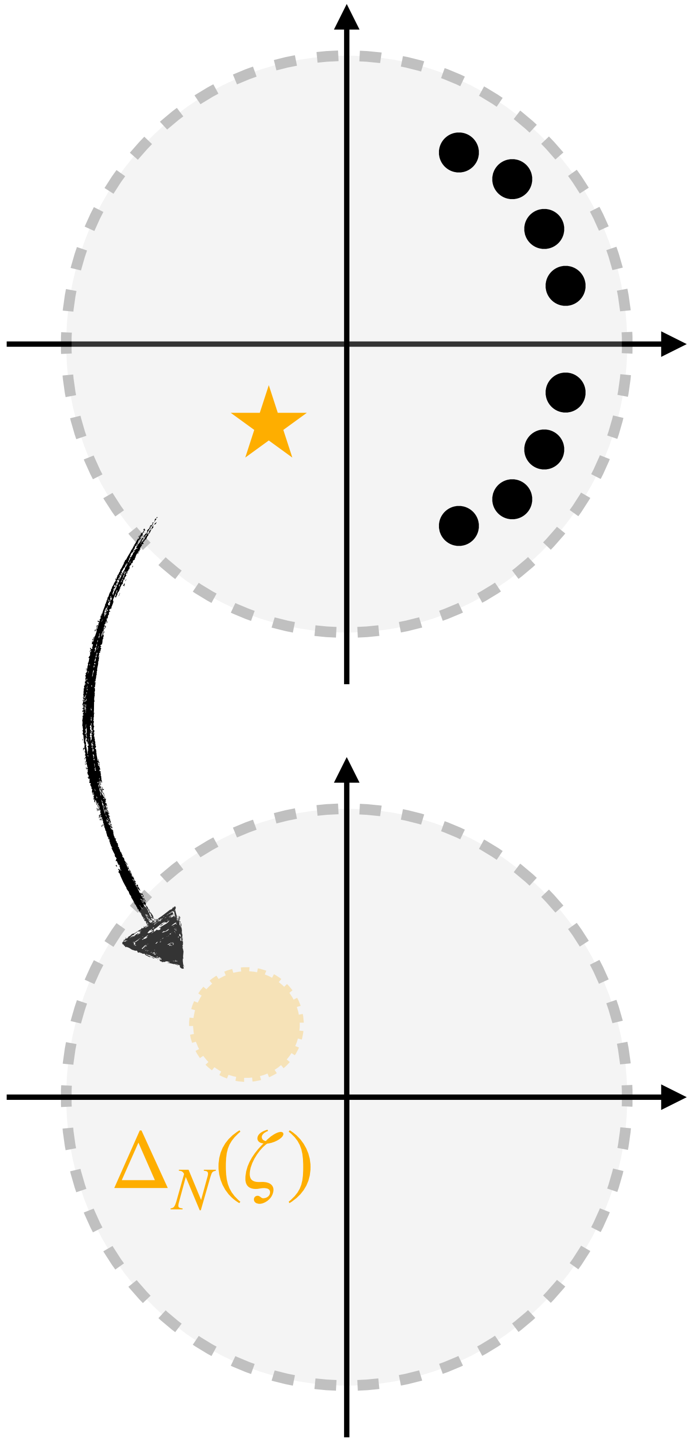
Analytic continuation to locate poles in the complex plane

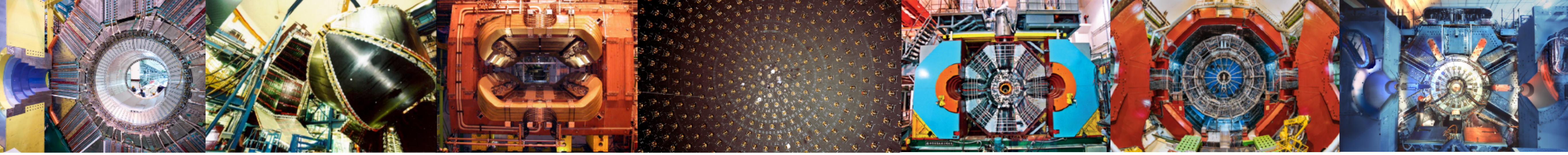
Searching for zeros in $\mathscr{M}^{-1}(E)$ with the Wertevorrat

- Use a *random* point in $\Delta_N(\zeta)$ as the function estimator.
    - Sampling trivial with formulae for radius and center

$$w(\zeta) = c_N(\zeta) + r r_N(\zeta) e^{i\theta}$$

    - Map the $w(\zeta)$ back to original energy coordinates to find zero
- Fix random numbers (r, θ) on each posterior sample
    - Can sample additional values for (r, θ) on each sample for "improved estimator"
    - Similar in spirit to using many point sources on a given gauge-field configuration when solving the Dirac equation in lattice QCD.





Hadron spectroscopy in Finite Volume

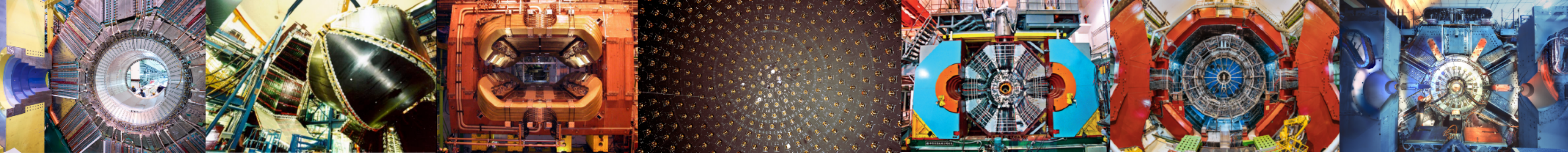
Analytic continuation to locate poles in the complex plane

2.) Solve quantization condition for $K^{-1}(E_n)$	$\det_{\ell m}\left[K^{-1}(E) + F(E, \mathbf{P}, L)\right]\Big\rvert_{E=E_n} = 0$
3.) Locate poles using analytic continuation	$\mathcal{M}^{-1}(E) \rightarrow \mathcal{M}^{-1}(z)$

Our method:

2.) Solve quantization condition with Bayesian analysis

3.) Locate poles using Nevanlinna interpolation



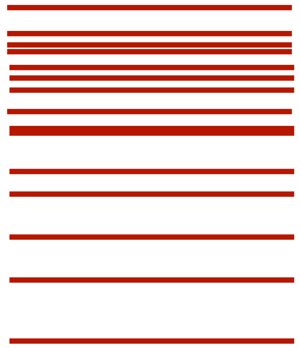
Coupled-channel $\pi\Sigma - \bar{K}N$ Scattering

BaSc Collaboration
PRD 109 (2024) 1, 014511
PRL 132 (2024) 5, 051901

$\Lambda(1405)$ and $\Lambda(1380)$ resonances

Lattice QCD calculation

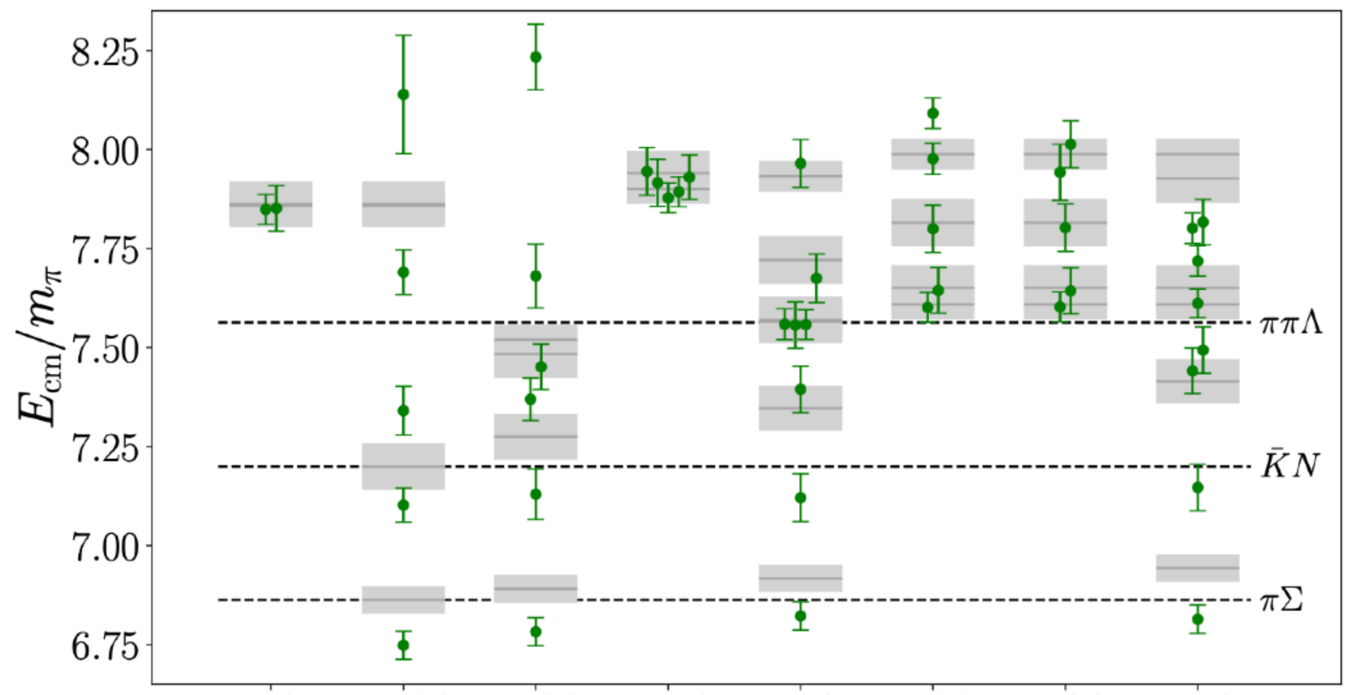
$$\langle \hat{\mathcal{O}}(\tau)\hat{\mathcal{O}}(0)\rangle \implies$$



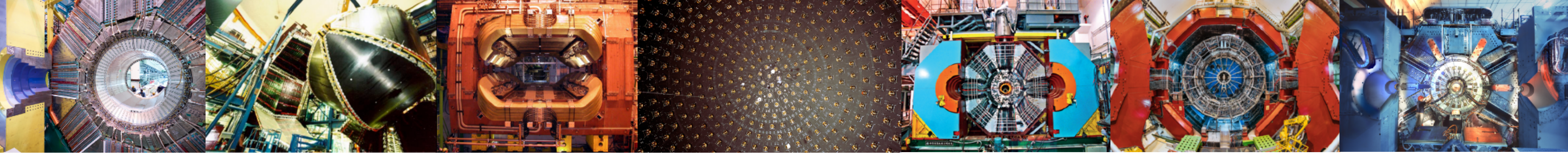
Single ensemble

Energy levels E_n

- $a \approx 0.06$ fm
- $M_\pi \approx 200$ MeV
- $M_K \approx 487$ MeV



Irreducible representations of cubic group



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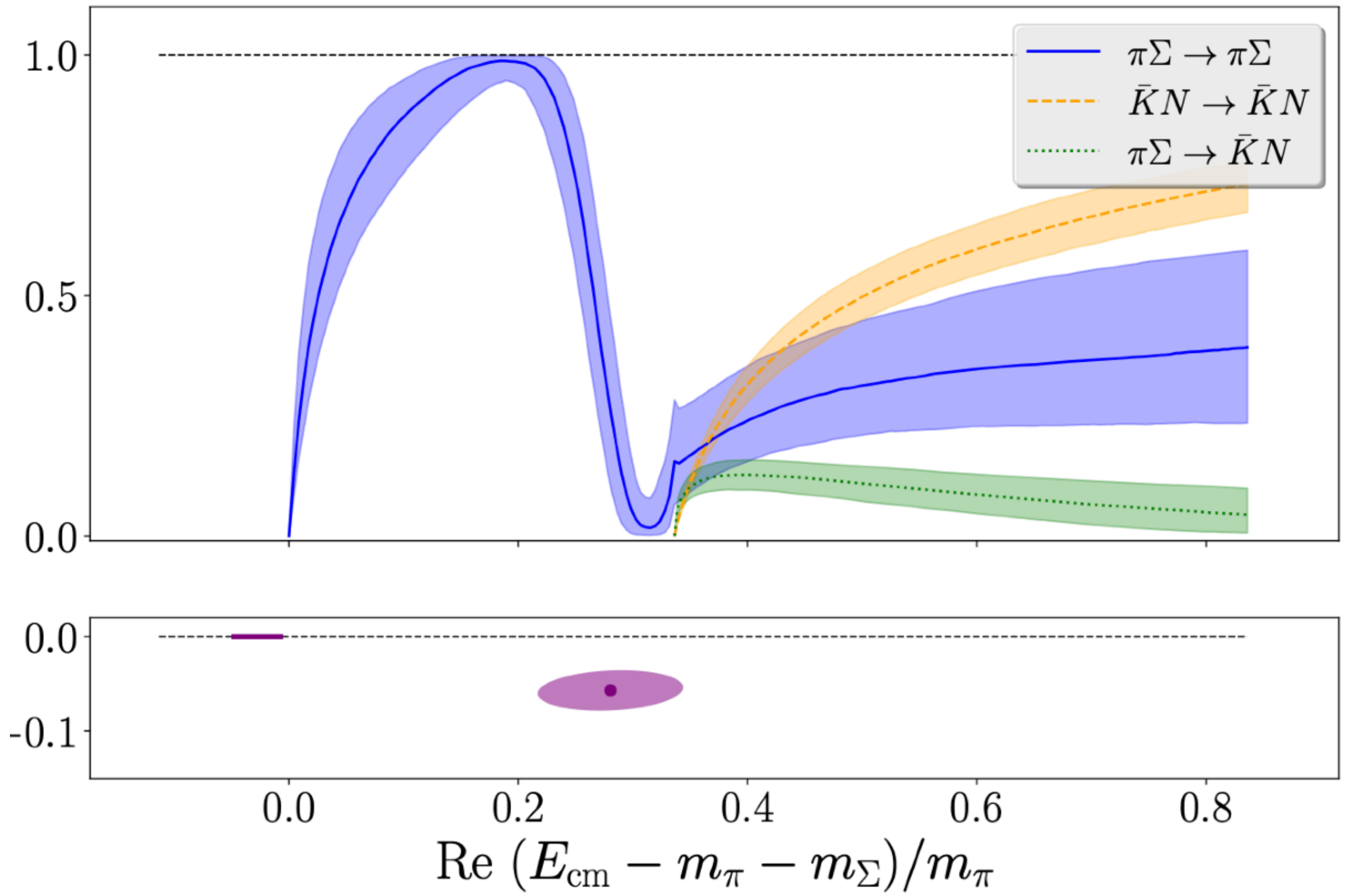
$\Lambda(1405)$ and $\Lambda(1380)$ resonances

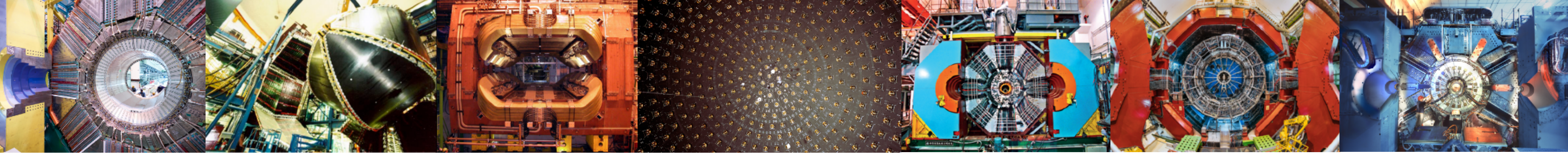
Best fit - Single parameterization

Scattering Amplitudes

Poles (2nd sheet)



Energy w.r.t. $\pi\Sigma$ threshold



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$\Lambda(1405)$ and $\Lambda(1380)$ resonances

Best fit - Single parameterization

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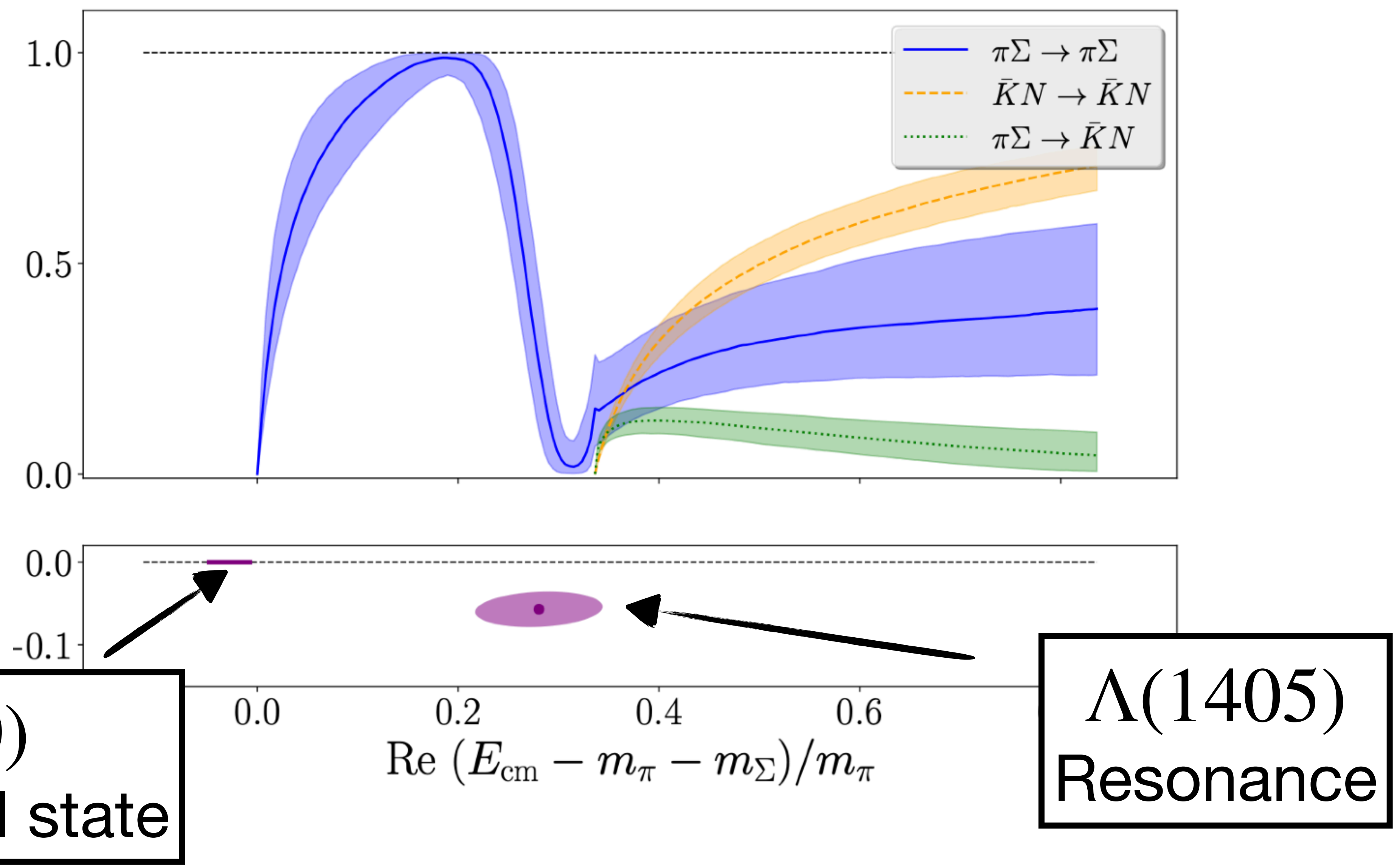

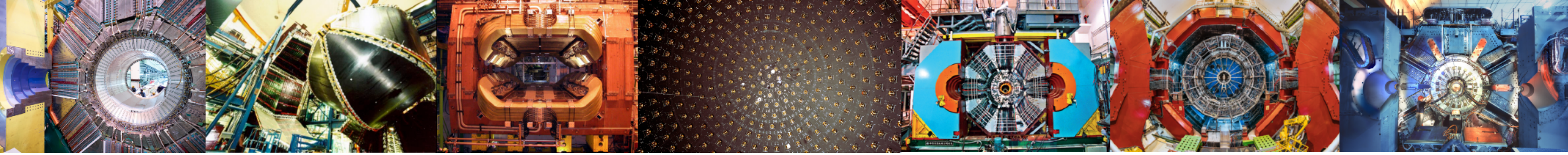

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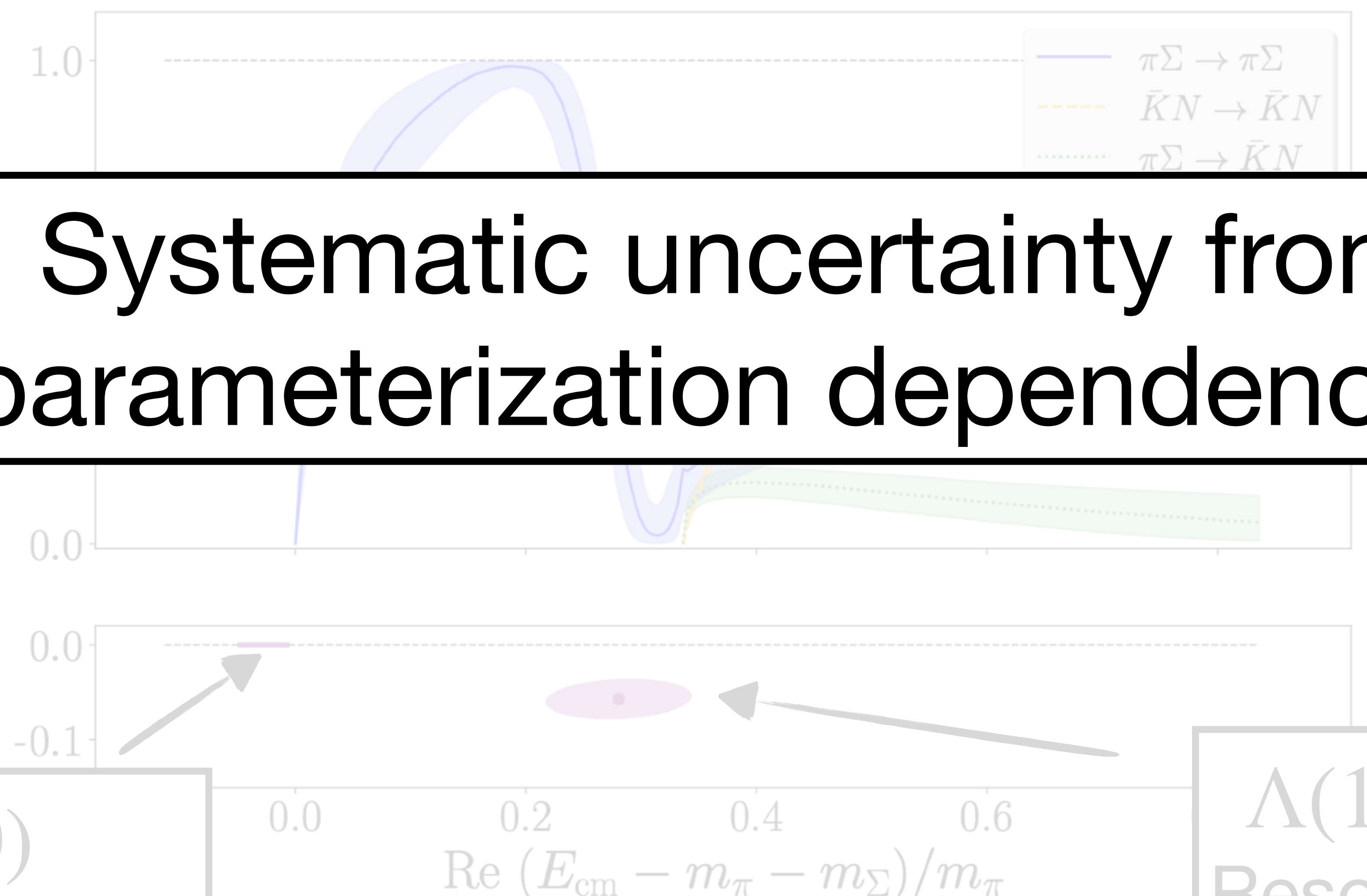
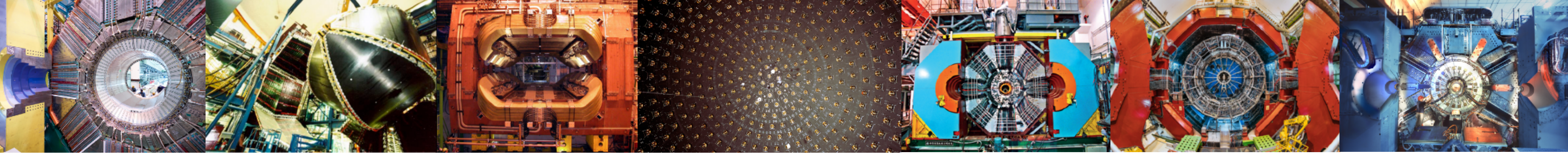
Scattering Amplitudes



Systematic uncertainty from parameterization dependence?

$\Lambda(1380)$
Virtual bound state

$\Lambda(1405)$
Resonance



Coupled-channel $\pi\Sigma - \bar{K}N$ Scattering

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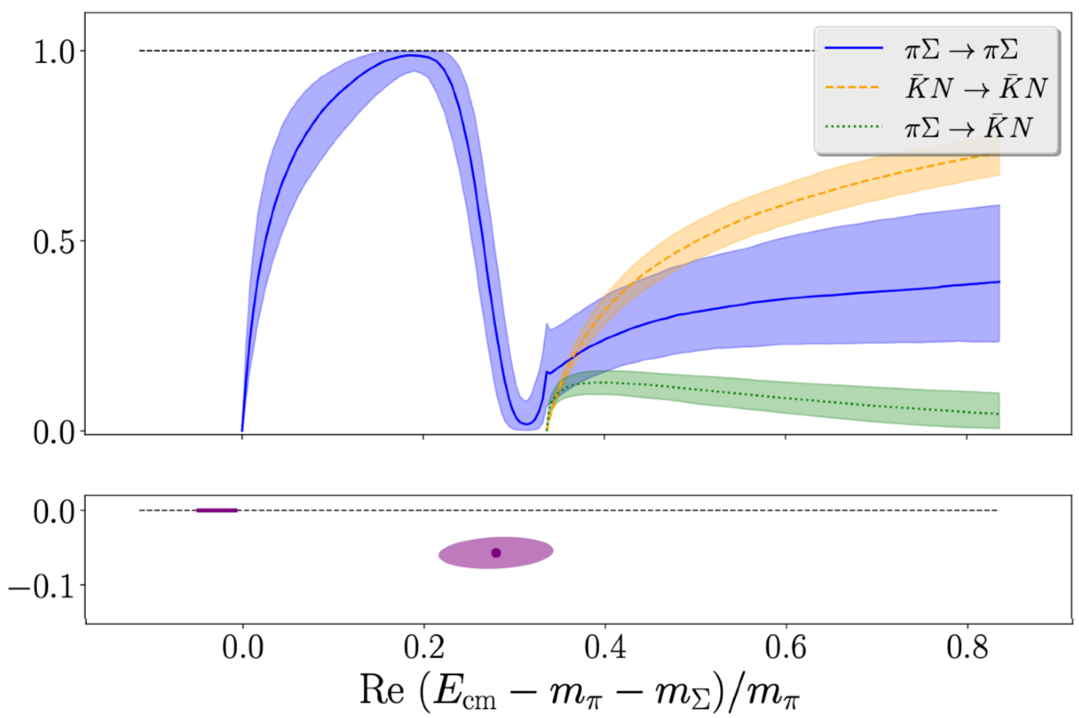
$\Lambda(1405)$ and $\Lambda(1380)$ resonances

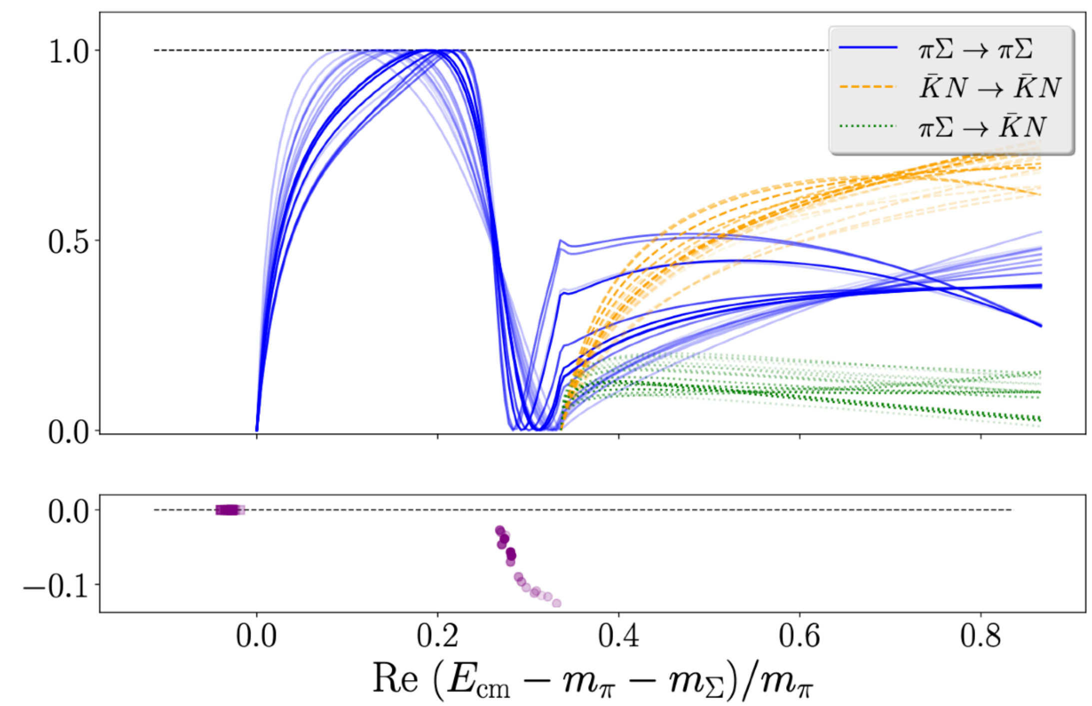
Best fit - Single parameterization

Parameterization dependence

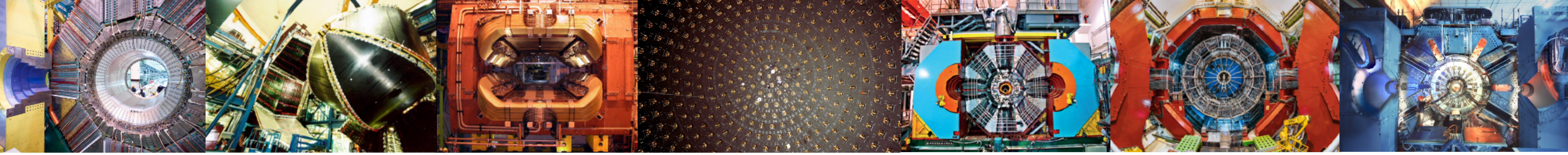
Scattering Amplitudes

Poles (2nd sheet)





Energy w.r.t. $\pi\Sigma$ threshold



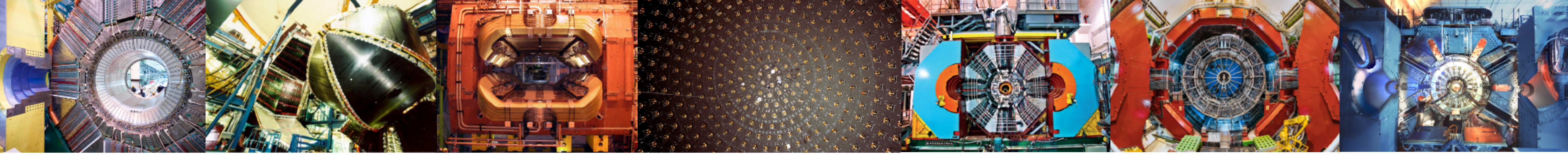
Coupled-channel $\pi\Sigma - \bar{K}N$ Scattering

$\Lambda(1405)$ and $\Lambda(1380)$ resonances

How do results look with the new method?

2.) Solve quantization condition with Bayesian analysis

3.) Locate poles using Nevanlinna interpolation

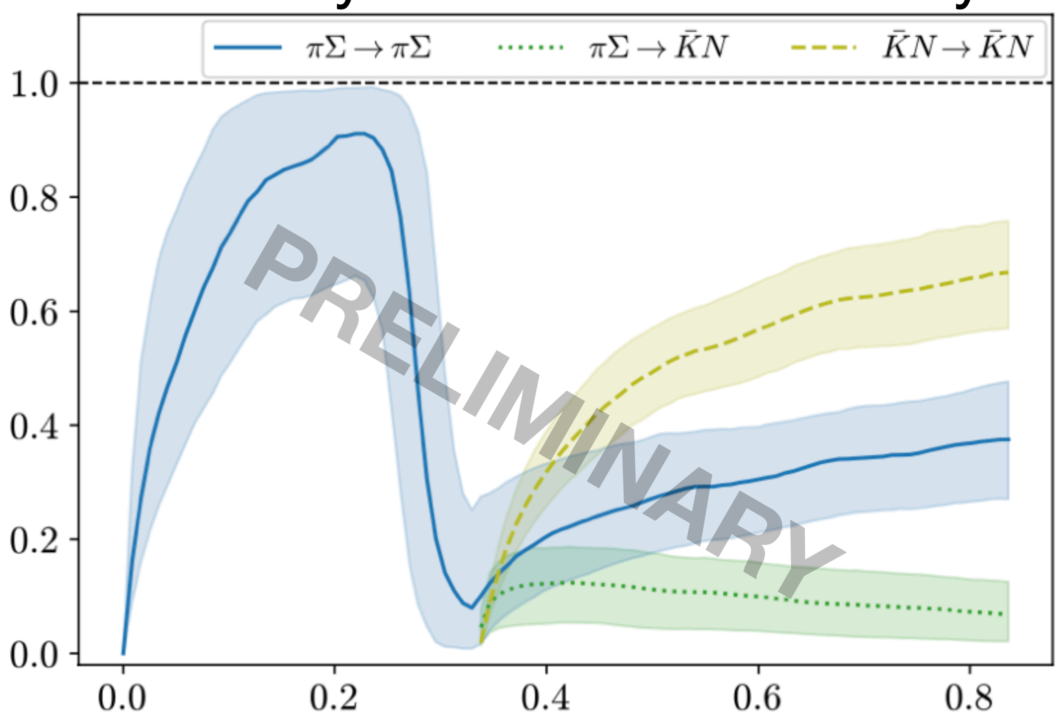


Coupled-channel $\pi\Sigma - \bar{K}N$ Scattering

$\Lambda(1405)$ and $\Lambda(1380)$ resonances

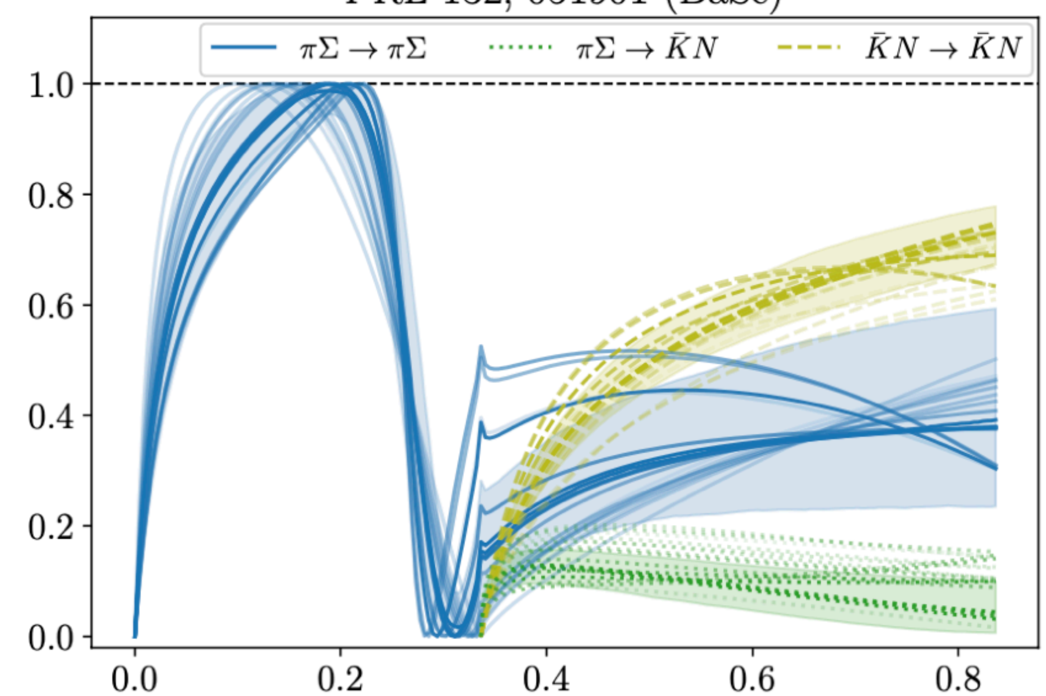
Scattering Amplitudes

New: Bayes+Nevanlinna analysis



Energy

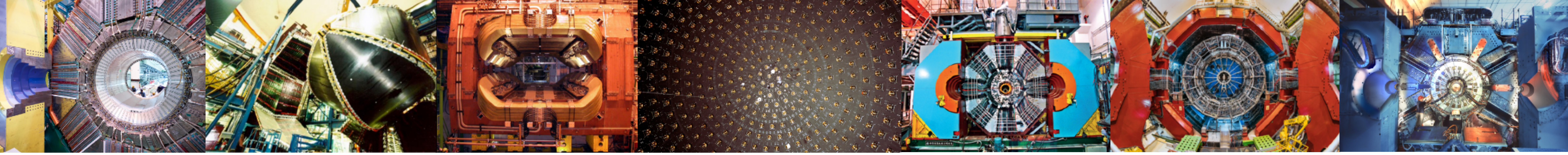
PRL 132, 051901 (BaSc)



Energy

Bayes+Nevanlinna analysis $\Longrightarrow$

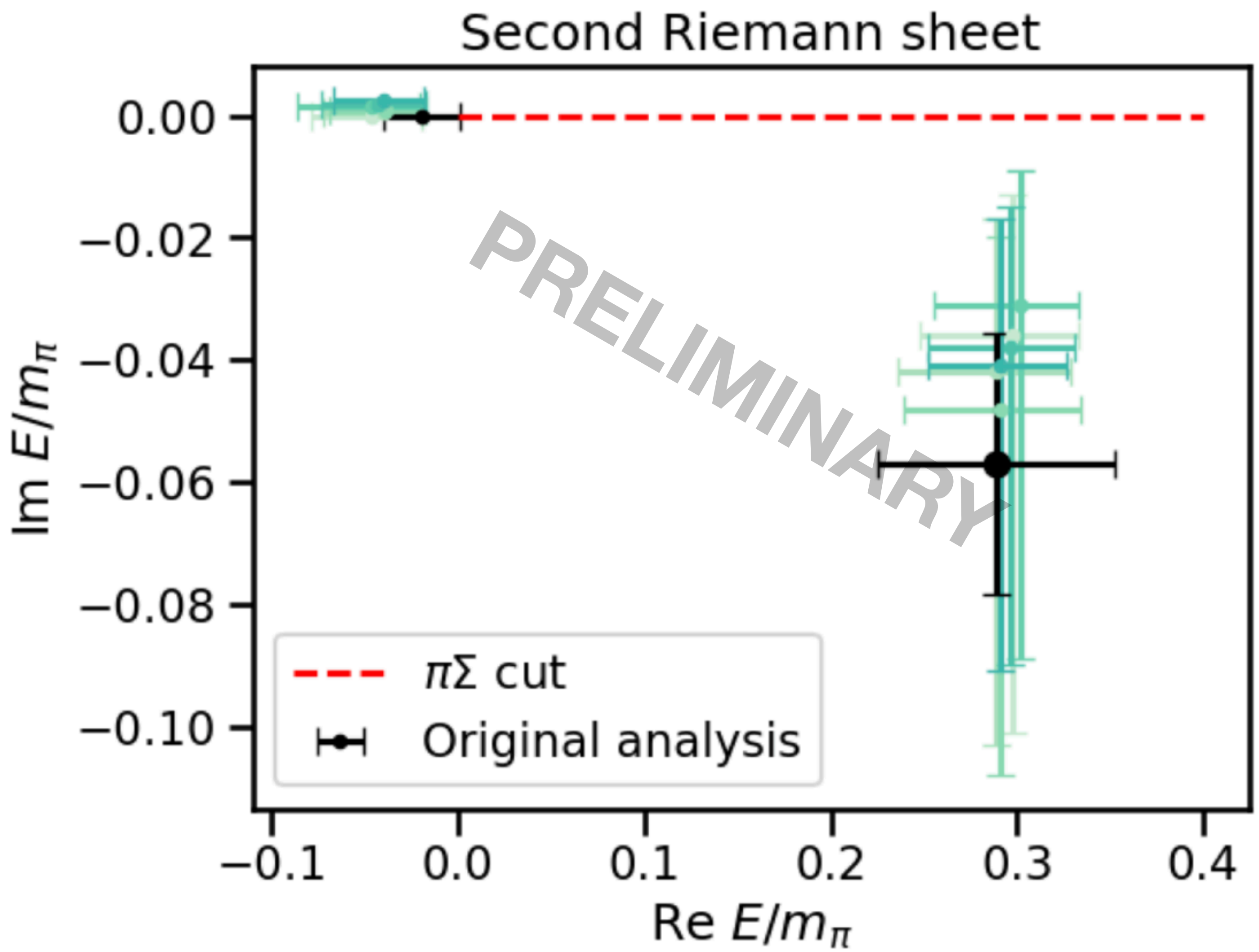
- Results consistent with previous analysis
- Conservative error bands naturally incorporating statistical and systematic uncertainties



Coupled-channel $\pi\Sigma - \bar{K}N$ Scattering

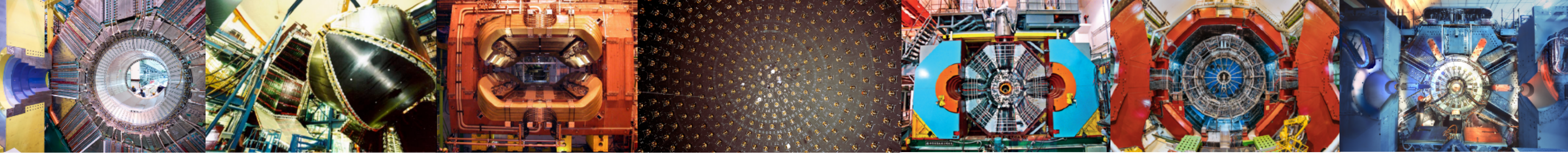
BaSc Collaboration
PRD 109 (2024) 1, 014511
PRL 132 (2024) 5, 051901

$\Lambda(1405)$ and $\Lambda(1380)$ resonances



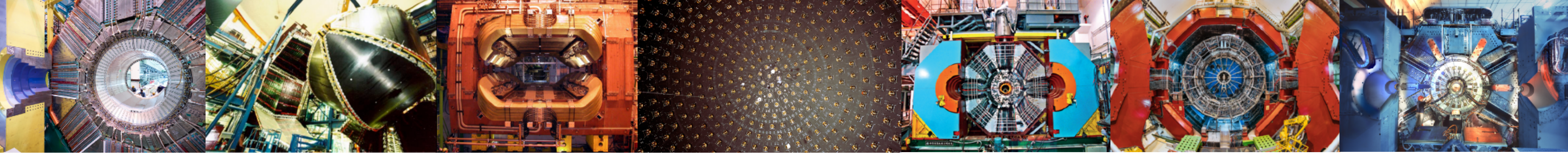
- Two-pole picture consistent with original BaSc analysis
- $\Lambda(1380)$: virtual bound state on second sheet below threshold on real line
- $\Lambda(1405)$: resonance with finite imaginary part
- Results are statistically consistent with previous analysis

✳ Slight vertical offset for $\Lambda(1380)$ to aid readability

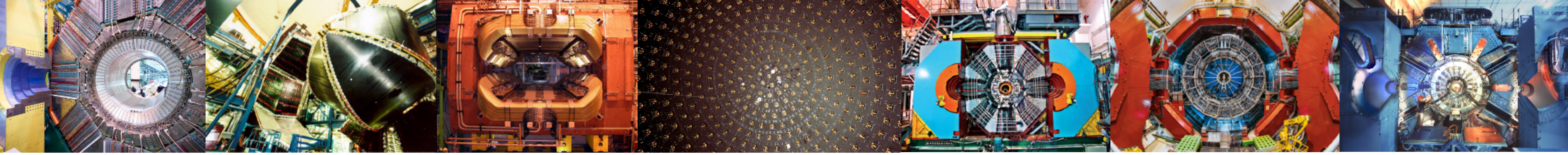


Conclusions

- Lattice QCD calculations of resonance spectroscopy are based on the finite-volume formalism.
- Lattice QCD is reaching the maturity to weigh in on decades-old questions, e.g., the nature of the $\Lambda(1405)$ resonance in $\pi\Sigma - \bar{K}p$ scattering.
- Exotic hadrons ("XYZ") discovered over the past 20 years have caused a renaissance in hadron spectroscopy.
- Analysis methods offering improved systematic control are timely.
- Today's talk: proof-of-concept for a new analysis method based on Bayesian reconstruction and Nevanlinna interpolation.



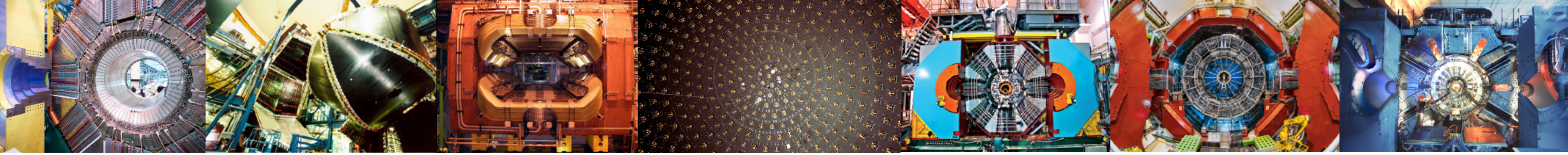
Backup



What about $\sin(iz)$ and friends?

Bergamaschi, WJ, Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

- The context where this is best understood is for Green functions G(z).
- Recall: a green function is a map $G(z) : \mathbb{H} \to \mathbb{H}$ ($\mathbb{H}$=upper half-plane)
    - Functions with this property are called *Nevanlinna functions*
    - Roughly speaking, any Nevanlinna function can be written as an integral of a suitable spectral function.
    - Mapping the problem to the disk to invoke Nevanlinna's theorem invokes these properties in an essential way.
    - In other words, *the interpolating function $f : \mathbb{D} \to \mathbb{D}$ already and automatically has the correct analytic structure*
- The function $\sin(iz)$:
    - Vanishes at infinitely many points, e.g., $z \in i\pi\,\mathbb{N}$
    - Blows up to $\pm\infty \implies$ Not a function $\mathbb{H} \to \mathbb{H}$.
    - Has the wrong singularity structure/asymptotic behavior.
- Constructing an interpolating function $f : \mathbb{D} \to \mathbb{D}$ automatically excludes inconsistent/pathological functions like $\sin(iz)$. This property holds when translated back to $G(z) : \mathbb{H} \to \mathbb{H}$.



What about statistical noise?

The method announces its failure in two ways.

1. The Wertevorrat is expected to decrease monotonically as more information is included. If the radius of the Wertevorrat begins to jitter around some “saturation width,” numerical precision has become a limiting factor.

2. Nevanlinna’s theorem assumes the data satisfy an analytic self-consistency condition: the Pick matrix P_{ij} must be positive semi-definite.

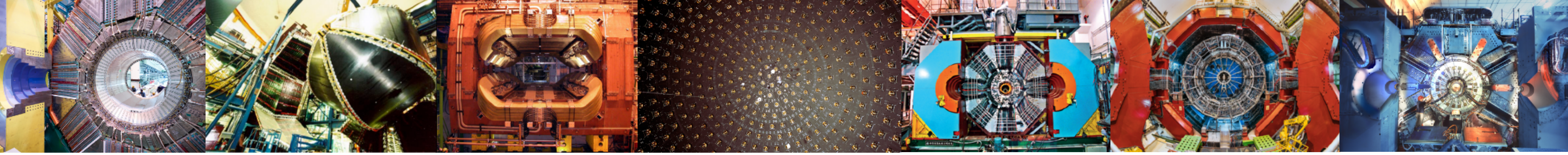
$$P_{ij} = \frac{1 - w_i \bar{w}_j}{1 - \zeta_i \bar{\zeta}_j}$$

Possible Solutions

A. Check this condition and avoid data that violate the hypotheses of the theorem.

B. Rephrase the difficulty as a statistical pre-denoising problem:

Given a statistical sample of $\mathbf{G} \in \mathbb{R}^N$, project to the closest set of points $\mathbf{G}' \in \mathbb{R}^N$ such that P_{ij} is positive semidefinite. “Closest” is determined by the covariance matrix.

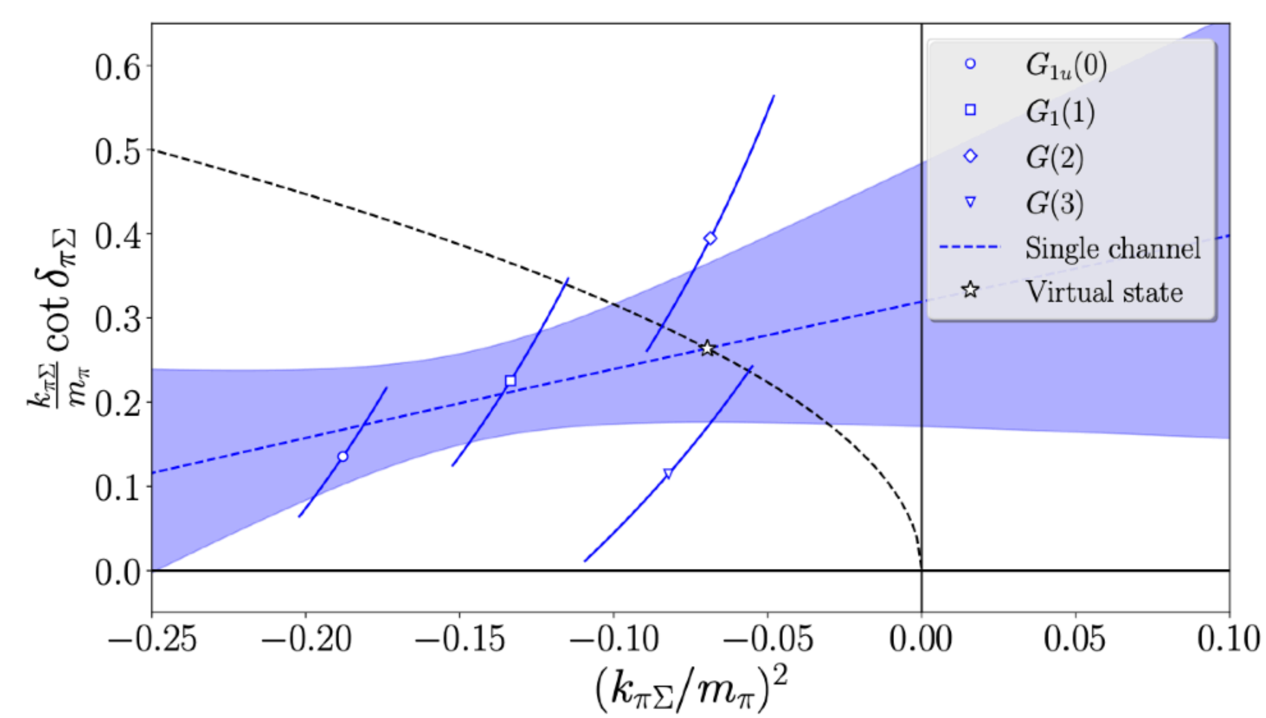


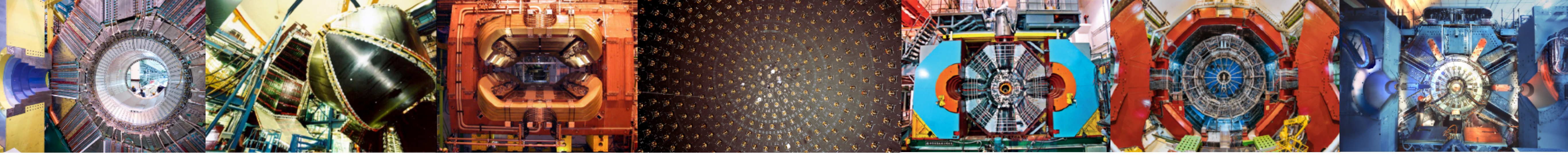
Real vs Virtual Bound State

BaSc Collaboration
PRD 109 (2024) 1, 014511
PRL 132 (2024) 5, 051901

- Bound state:
  - Pole in $\mathscr{M}(E)$ on real axis below threshold of physical sheet
  - Corresponds to a bona fide asymptotic state (e.g., I=0 deuteron)
  - $p = i|p|$ so wavefunction $\psi \sim e^{ipr} \sim e^{-|p|r}$
- Virtual bound state:
  - Pole in $\mathscr{M}(E)$ on real axis below threshold of *unphysical* sheet
  - Gives rise to an observable enhancement in the cross section at threshold. No asymptotic state (e.g., I=1 di-neutron)
  - $p = -i|p|$ so wavefunction $\psi \sim e^{ipr} \sim e^{+|p|r}$

- Look for phase-shift curve to intersect the virtual-bound-state condition:
  - $p \cot \delta(p) - ip = 0$
  - for purely imaginary and negative $p = -i|p|$





Sasa Prelovsek
Talk @ Lattice 2022

Simplest Example: scattering in square-well potential in QM

$\delta = \arctan[\tan(qR)\frac{p}{q}] - pR$

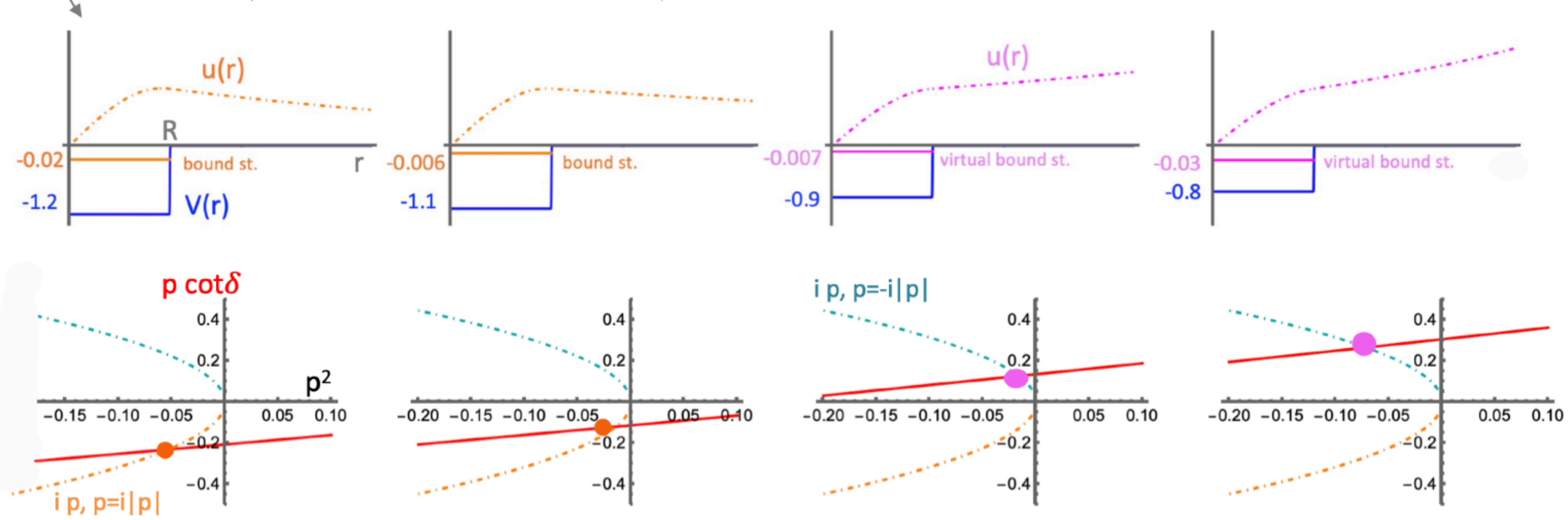
partial wave l=0

$t \propto (p\cot\delta - ip)^{-1}$

$u(r) = A\sin(qr)$ $\quad u(r) = B\sin(pr+\delta)$

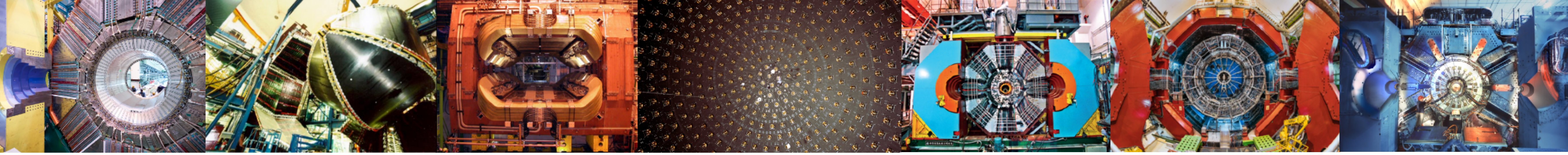
p=i|p| $\quad e^{ipr} = e^{-|p|r}$

p=-i|p| $\quad e^{ipr} = e^{|p|r}$



increasing $m_{u/d}$, decreasing attraction V_0 (or decreasing R)

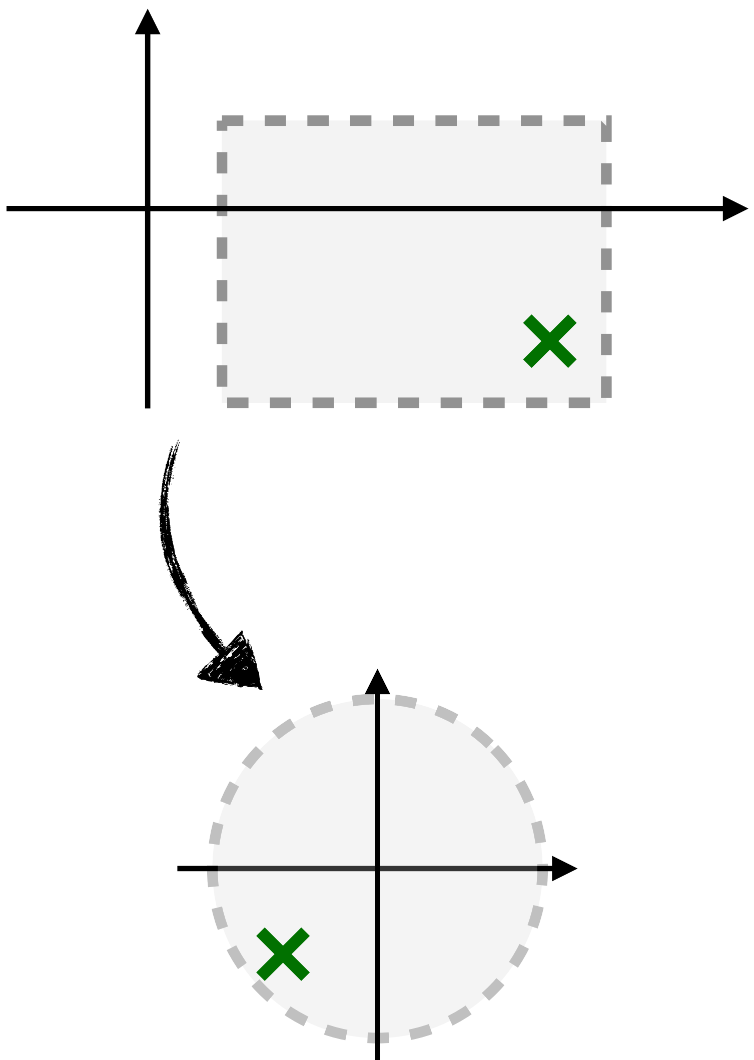
Sasa Prelovsek Doubly charm tetraquark and its quark mass dependence 6



Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane

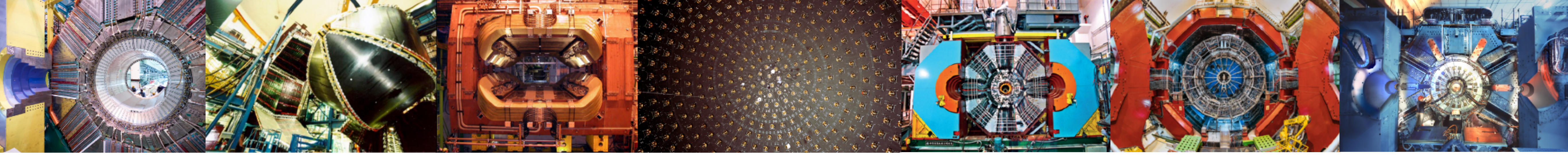
- *What **not** to do:*



Suppose I have point $z \in \mathbb{C}$.

Consider some neighborhood around z.

Map this neighborhood to the unit disk.



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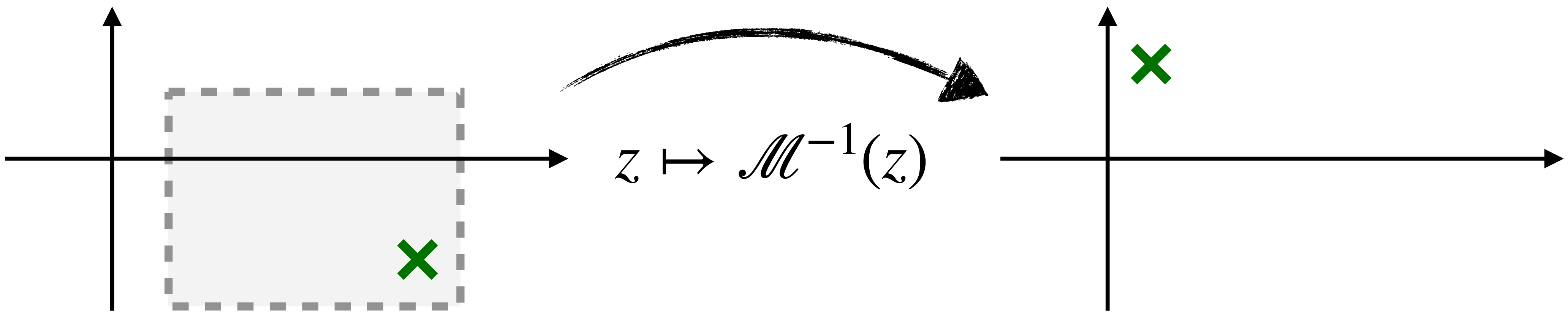
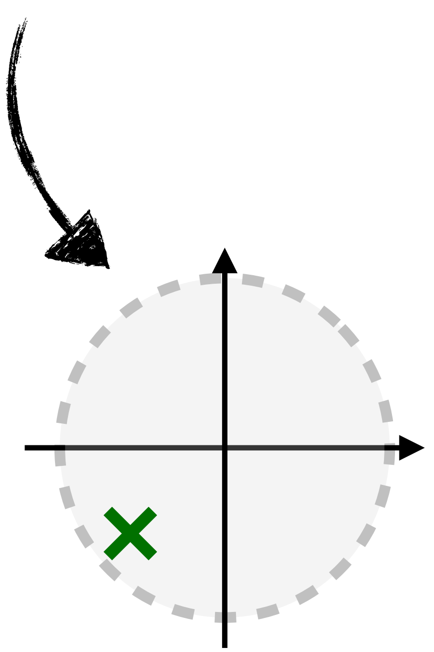
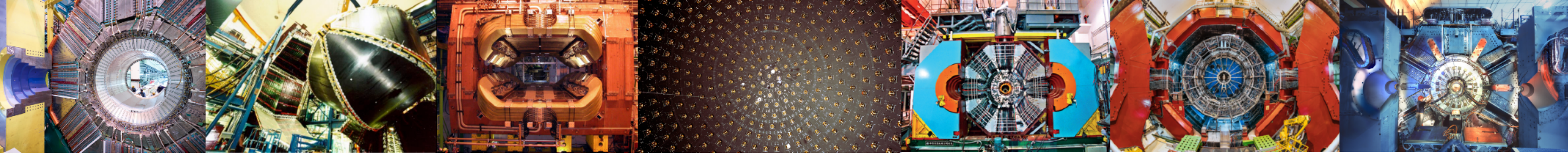




Think carefully about where $\mathscr{M}^{-1}(z)$ could send the point z.



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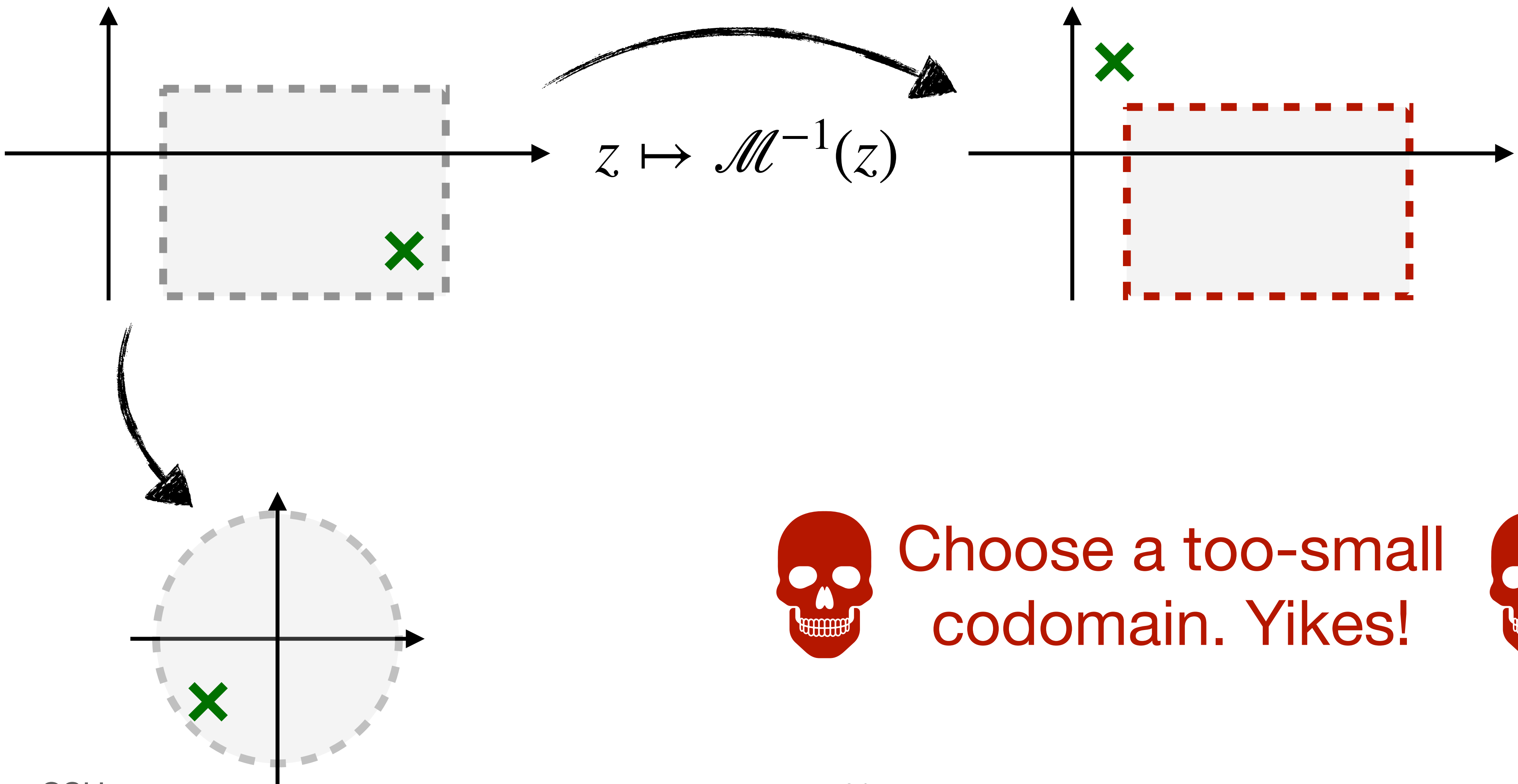
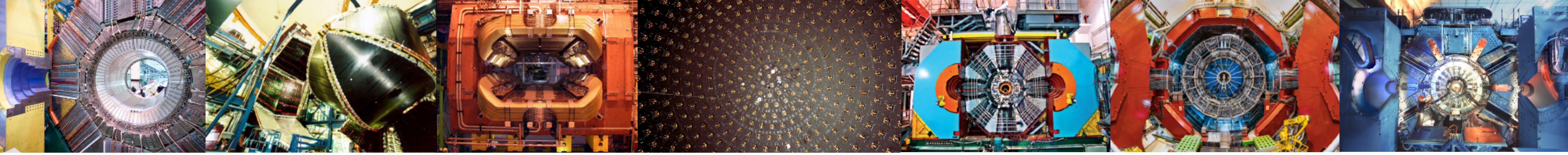


Choose a too-small codomain. Yikes!



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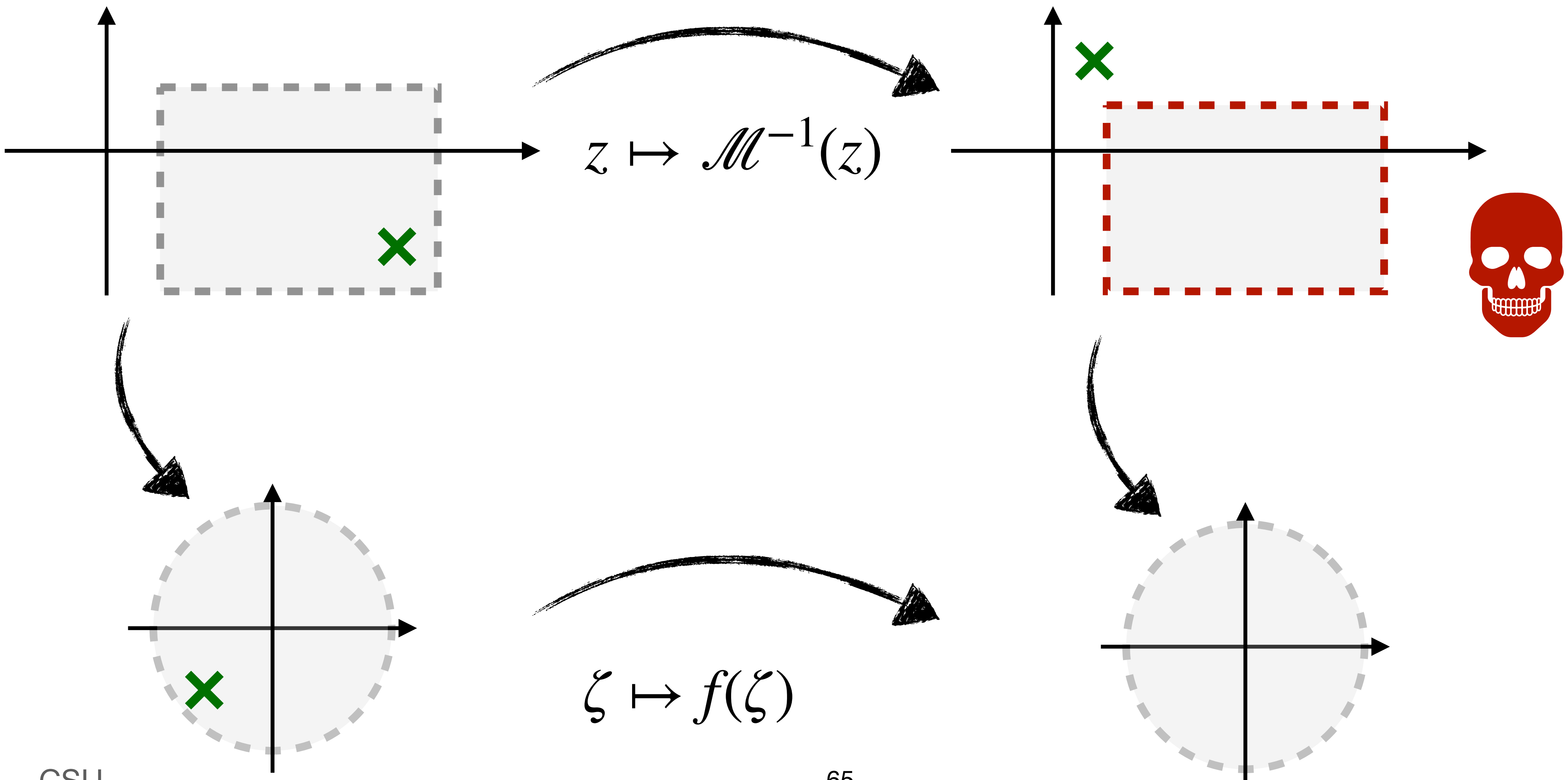

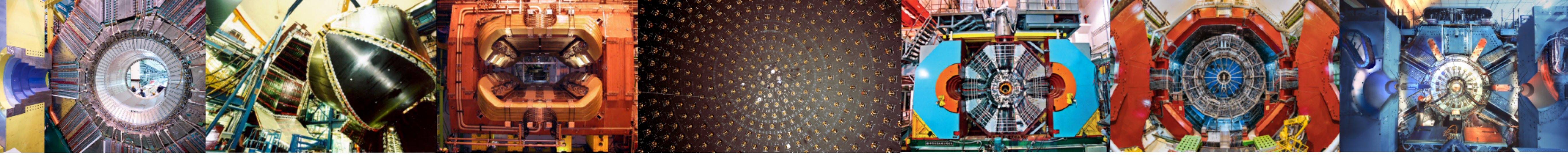

Hadron spectroscopy in Finite Volume

Analytic continuation to locate poles in the complex plane

- *What **not** to do:*

Nevanlinna's *Theorem* still works. By construction, the solution will *not* describe $\mathscr{M}^{-1}(z)$