The setting sun diagram with complex external momenta

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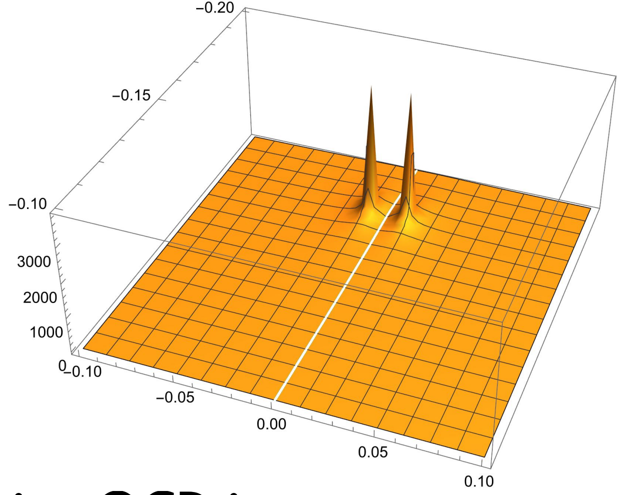
Work in progress in collaboration with D. Dudal and G. Krein

Introduction



- Complex conjugate poles appear in various methods for non-perturbative QCD in Euclidean spacetime: analytical (Refined Gribov-Zwanziger, Curci-Ferrari..), semi-analytical (Dyson-Schwinger, FRG..) and numerical (gauge-fixed lattice)

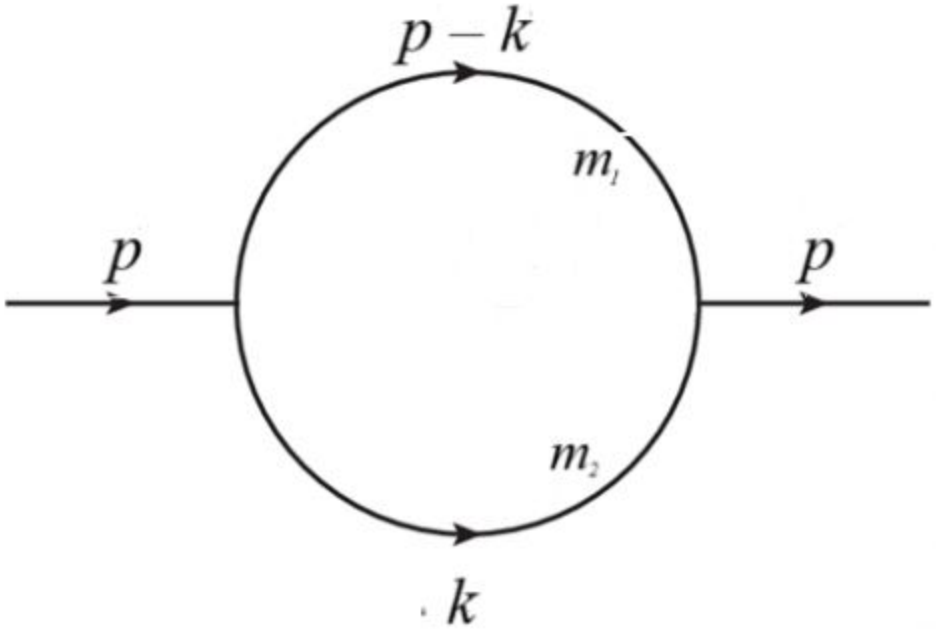
- The Minkowski description of this region is (even) less understood

- Today's talk is just a small demonstration of the subtleties of the analytical continuation between Euclidean and Minkowski space outside of the real axis

- Other people have said important things about this topic, I will briefly discuss this at the end of the talk

A basic question

Are the setting sun diagram defined in Euclidean space and its Minkowskian counterpart connected through analytic continuation $p^2 \to -p^2$ for all $p^2 \in \mathbb{C}$?



A basic question

Are the setting sun diagram defined in Euclidean space

$$S_E\left(p^2\right) = \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2 + m_1^2)} \frac{1}{((k_E + p)^2 + m_2^2)}$$

with $k_E^2 = k_4^2 + \vec{k}^2$ and its Minkowskian counterpart

$$S_M\left(p^2\right) = \int \frac{d^4k_M}{(2\pi)^4} \frac{1}{(k_M^2 - m^2 + i\epsilon)} \frac{1}{((k_M + p)^2 - m^2 + i\epsilon)}$$

with $k_M^2 = k_0^2 - \vec{k}^2$ connected through analytical continuation $S_E(p^2) = -iS_M(-p^2)$ for all *a priori* $p^2 \in \mathbb{C}$?

A very basic question

Are the setting sun diagram defined in Euclidean space in 2D and with two equal real masses *m*

$$S_E\left(p^2\right) = \int \frac{d^2 k_E}{(2\pi)^2} \frac{1}{(k_E^2 + m^2)} \frac{1}{((k_E + p)^2 + m^2)}$$

with $k_E^2 = k_2^2 + k^2$ and its Minkowskian counterpart

$$S_M\left(p^2\right) = \int \frac{d^2 k_M}{(2\pi)^2} \frac{1}{(k_M^2 - m^2 + i\epsilon)} \frac{1}{((k_M + p)^2 - m^2 + i\epsilon)}$$

with $k_M^2 = k_0^2 - k^2$ connected through analytical continuation $S_E(p^2) = -iS_M(-p^2)$ for all *a priori* $p^2 \in \mathbb{C}$?

Methods

- Of course, for $p^2 \in \mathbb{R}_-$ it is well-known that we have $S_E(p^2) = -iS_M(-p^2)$
- The usual way to deal with the sunset diagram is to use Feynman's trick introducing $\int_0^1 dx$ and making the substitution $\ell = k - px$, then set $\int d^d k \to \int d^d \ell$
- However, for complex p^2 this changes the integration limits in a complex way
- We will calculate the integrals by using Cauchy's residue theorem for the k_2-integral and perform the *k*-integral explicitly

$$\oint dk_2 f(k_2) = 2\pi i \sum \mathrm{Res}[f, a]$$

$$S_E\left(p^2\right) = \frac{1}{(2\pi)^2}\int dk \int dk_2 \frac{1}{(k_2^2+k^2+m^2)}\frac{1}{((k_2+p)^2+k^2+m^2)}$$

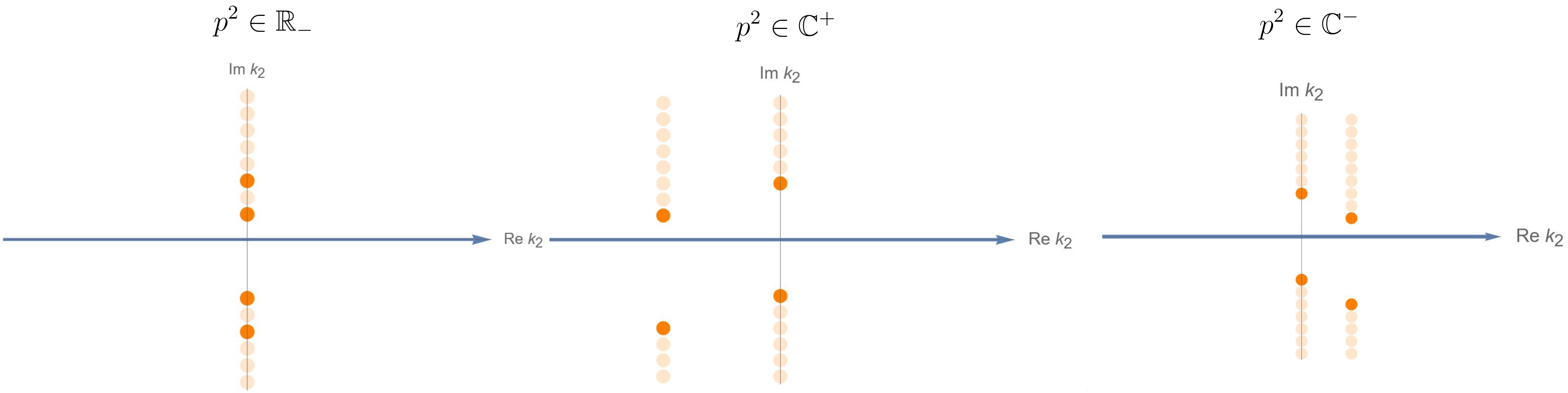
The k_2 integrand has 4 *k*-dependent poles

$$\begin{aligned} f_1 &= i\sqrt{k^2+m^2} \\ f_2 &= -i\sqrt{k^2+m^2} \\ f_3 &= i\sqrt{k^2+m^2}-p \\ f_4 &= -i\sqrt{k^2+m^2}-p \end{aligned}$$

with residues

$$R_1(k,p) = -R_2(k,-p) = R_3(k,-p) = -R_4(k,p) = -\frac{1}{2p\left(-ip\sqrt{k^2+m^2}+2k^2+2m^2\right)}$$

$$S_E\left(p^2\right) = \frac{1}{(2\pi)^2}\int dk \int dk_2 \frac{1}{(k_2^2+k^2+m^2)}\frac{1}{((k_2+p)^2+k^2+m^2)}$$



For all cases the contour integral picks up the same poles:

$$S_E\left(p^2\right) = \frac{i}{2\pi}\int dk\left(R_1(k,p)+R_3(k,p)\right) = -\frac{i}{\pi}\frac{\tan^{-1}\left(\frac{ip}{\sqrt{4m^2+p^2}}\right)}{p\sqrt{4m^2+p^2}}$$

$$S_M\left(-p^2\right)=\frac{1}{(2\pi)^2}\int dk_0\int dk\frac{1}{(k_0^2-k^2-m^2+i\epsilon)}\frac{1}{((k_0-ip)^2-k^2-m^2+i\epsilon)}$$

The k_2 integrand has 4 *k*-dependent poles

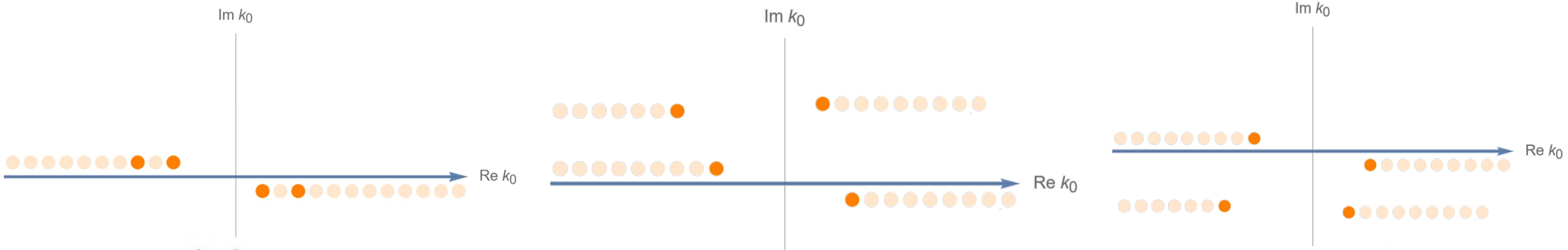
$$\begin{aligned} f_1 &= \sqrt{k^2+m^2}-i\epsilon \\ f_2 &= -\sqrt{k^2+m^2}+i\epsilon \\ f_3 &= \sqrt{k^2+m^2}-i\epsilon+ip \\ f_4 &= -\sqrt{k^2+m^2}+i\epsilon+ip \end{aligned}$$

with residues

$$R_1(k,p) = -R_2(k,-p)=R_3(k,-p)=-R_4(k,p)=-\frac{i}{2p\left(ip\sqrt{k^2+m^2}+2k^2+2m^2\right)}$$

$$S_M\left(-p^2\right) = \frac{1}{(2\pi)^2}\int dk_0 \int dk \frac{1}{(k_0^2 - k^2 - m^2 + i\epsilon)}\frac{1}{((k_0 + ip)^2 - k^2 - m^2 + i\epsilon)}$$

$p^2 \in \mathbb{R}_-$ $\quad$ $p^2 \in \mathbb{C}^+$ $\quad$ $p^2 \in \mathbb{C}^-$



$$S_M\left(-p^2\right) = \frac{i}{2\pi}\int dk\,(R_2(k,p) + R_4(k,p))$$

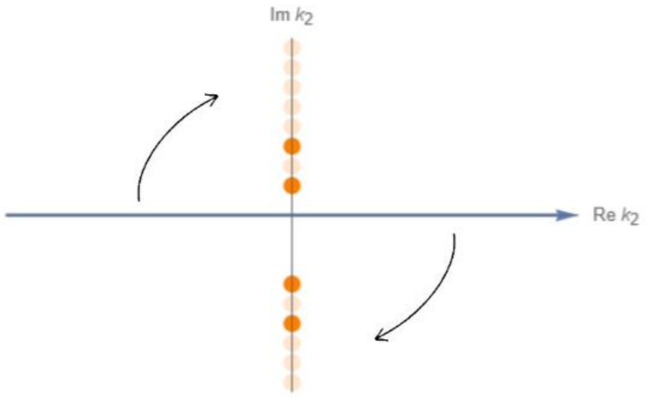
$$S_M\left(-p^2\right) = \frac{i}{2\pi}\int dk\,(R_2(k,p) + R_4(k,p) + R_3(k,p))$$

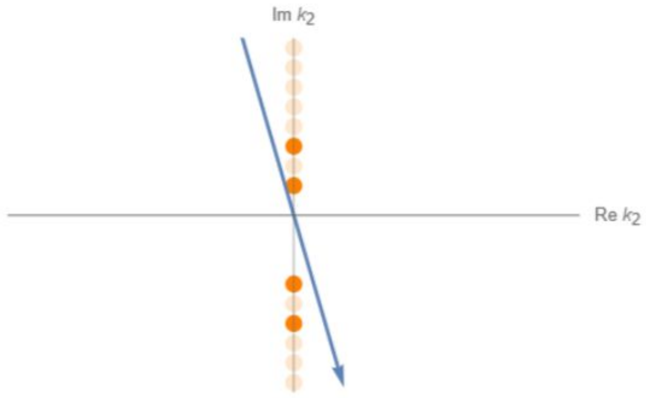
$$S_M\left(-p^2\right) = \frac{i}{2\pi}\int dk\, R_2(k,p)$$

For all cases the contour integral picks up different poles! After integration, we have three different expressions for each of the domains, which are not smoothly connected. Only for $p^2 \in \mathbb{R}_-$do we have $S_E(p^2) = -iS_M(-p^2)$

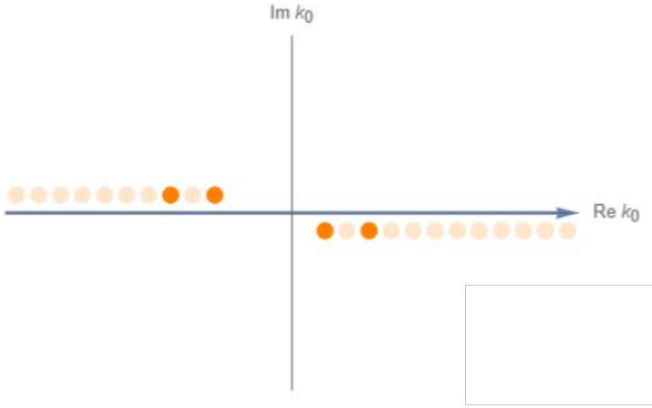
Wick rotation

$$p^2 \in \mathbb{R}_-$$



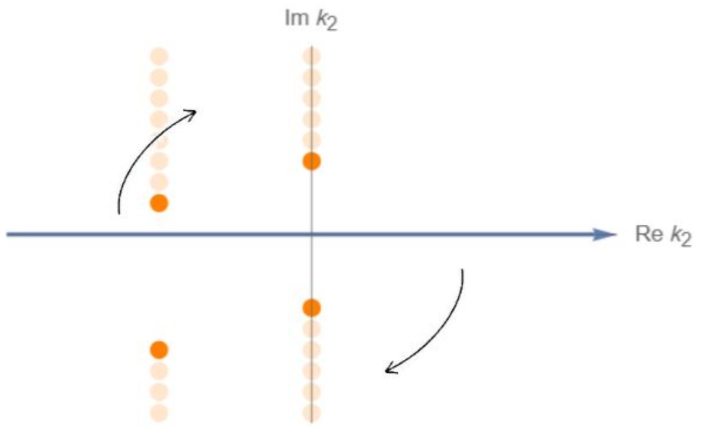


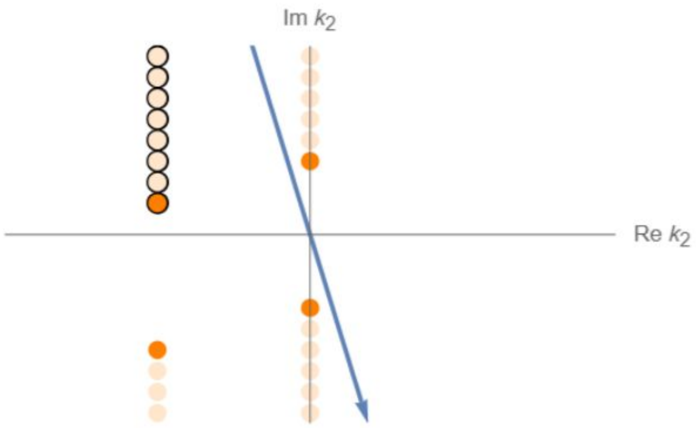
$$\boxed{k_2 = ik_0 \qquad k_0 \to k_0(1+i\epsilon)}$$



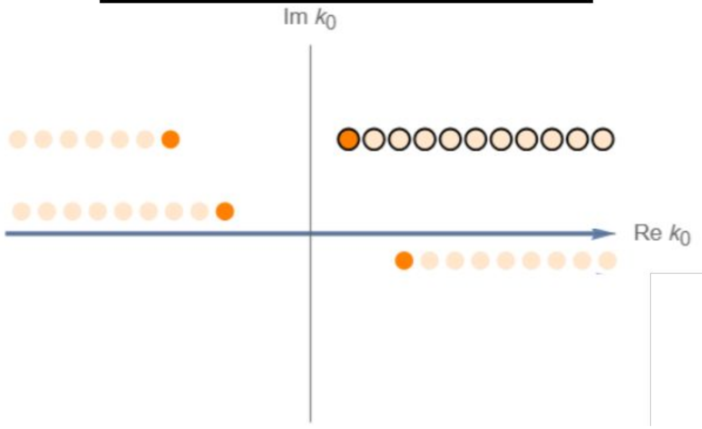
Wick rotation

$$p^2 \in \mathbb{C}^+$$



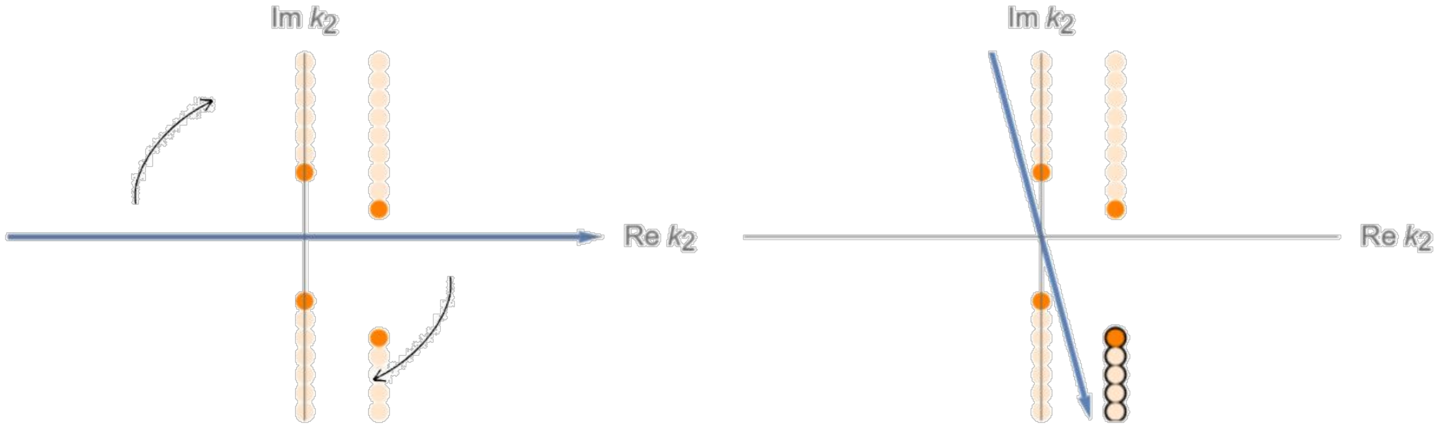


$$\boxed{k_2 = ik_0 \qquad k_0 \to k_0(1 + i\epsilon)}$$

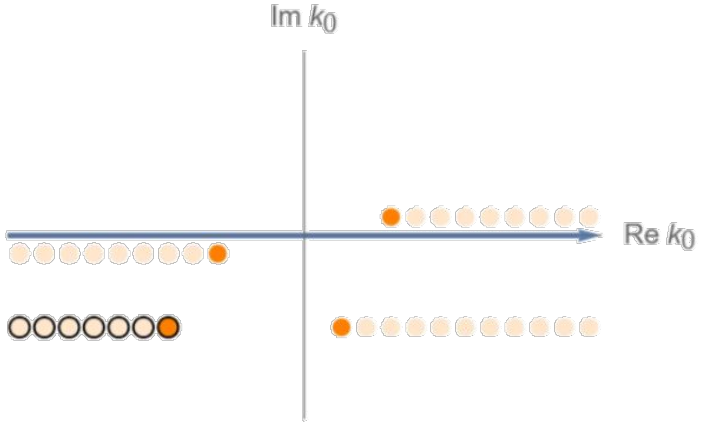


Wick rotation

$$p^2 \in \mathbb{C}^-$$



$$\boxed{k_2 = ik_0 \qquad k_0 \to k_0(1+i\epsilon)}$$



Results?

In summary, what we find from this direct integration method is that

$$\begin{aligned}
\text{for } p^2 \in \mathbb{R}_- : S_E(p^2) &= -iS_M(-p^2) \\
\text{for } p^2 \in \mathbb{C}^+ : S_E(p^2) &= -2iS_M(-p^2) + \frac{i}{2p\sqrt{4m^2+p^2}} \\
\text{for } p^2 \in \mathbb{C}^- : S_E(p^2) &= -2iS_M(-p^2) - \frac{i}{2p\sqrt{4m^2+p^2}}
\end{aligned}$$

This violates the Schwartz reflection principle $iS_M(p^2) = \overline{iS_M(\overline{p}^2)}$
and thus the optical theorem

Källén-Lehmann spectral density function

It was established [Dudal & Guimarães, 2011] that the Euclidean sunset diagram in terms of the K-L spectral density function

$$S_E(p^2) = \int_{4m^2}^{\infty} d\tau \frac{\rho(\tau)}{\tau + p^2}$$

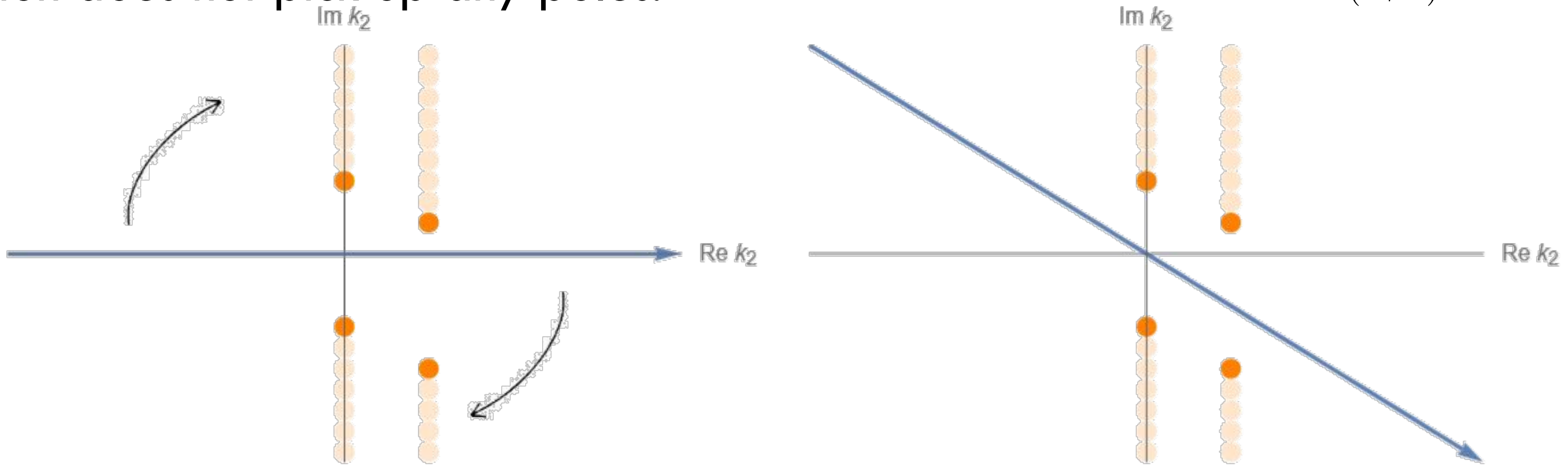
with

$$\rho(\tau) = \frac{1}{2\pi} \frac{1}{\sqrt{\tau^2 - 4m^2\tau}}$$

is analytical in the whole complex p^2-plane except for the interval $[-\infty, -4m^2]$ Therefore if the relation $S_E(p^2) = -iS_M(-p^2)$ holds on the real axis, this relation extends to the complex plane. We have to conclude that we **cannot** perform the integrals with *a priori* complex momenta.

Some suggestions from other works

- Eichmann et al. (2019): the integration limits in Minkowski are $\int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk_0$, so Wick rotation does not pick up any poles.



- Siringo & Comitini (2023): c.c. poles lead to opposite signs in Minkowski space, from this construct general K-L spectral density function
- Oribe et al. (2025): massage the function until the Minkowski function is analytical in $\mathbb{C}/\mathbb{R}_+$

Conclusion

- For the case study of a 2D setting sun diagram, the Euclidean and Minkowski are connected for all $p^2 \in \mathbb{C}$ by analytic continuation

$$S_E(p^2) = -iS_M(-p^2)$$

- The analytic continuation is not established by a Wick rotation from *a priori* complex external momenta
- Instead, one needs to perform the integral for real p^2 and extend the result to complex p^2 to match the spectral representation
- For complex m^2, the K-L integral is not well-defined and we do not have a benchmark to relate the Euclidean and Minkowski integrals