Semiclassics for QCD vacuum structure via T^2 compactification

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The complex structure of strong interactions in Euclidean and Minkowski space

ECT*, 26–30 May, 2025

Based on:
JHEP **08** 001 (2024) [arXiv:2402.04320 [hep-th]] with Yuya Tanizaki (Yukawa Institute, Kyoto U.)

What I'll talk about…

- **Aim:** We want to understand confinement/vacuum structure

(We work in Euclidean (sorry!); translation to Minkowski language is just as hard as that of instanton gas/quantum tunneling stuff)

- **Strategy**: **Deformation (preserving confinement)**

As this deformation, we make T^2 compactification (with some trick)

QCD / SU(N) Yang-Mills
(strongly coupled, hard problem)

→

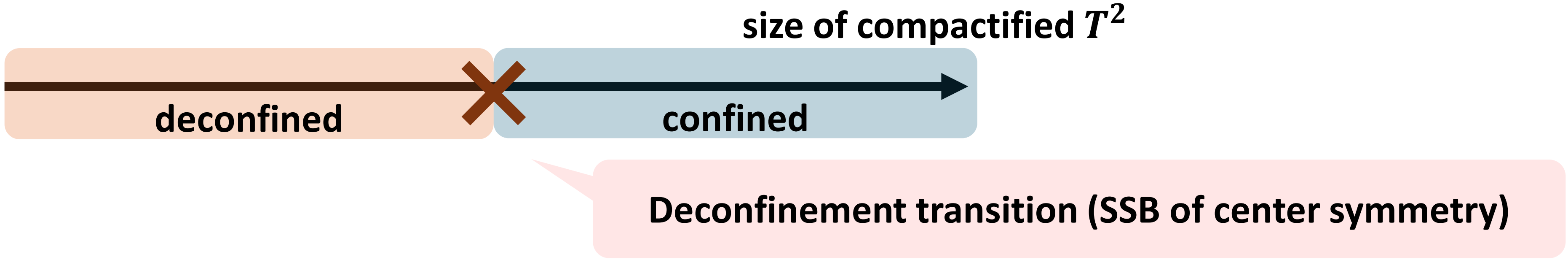
Deformed theory
(weakly coupled, easy problem)

We want to know

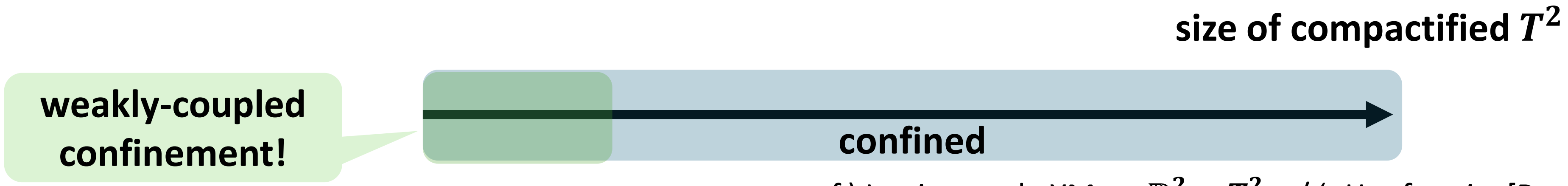
Confinement by gas of fractional instantons

Adiabatic continuity

- In this talk, we consider $\boldsymbol{T^2}$ compactification to have weakly-coupled theory
- With the naïve compactification, there is a deconfinement transition somewhere



- **Nice trick: inserting 't Hooft flux,** which stabilizes the center symmetry / avoids deconfinement ⇒ We expect the **adiabatic continuity** in this setup.
  in a similar spirit as twisted Eguchi-Kawai [González-Arroyo, Okawa '83--]

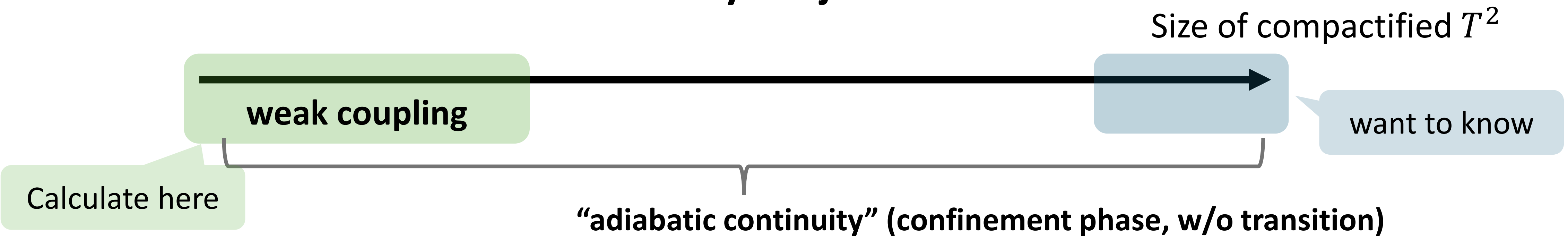


cf.) Lattice work: YM on $\mathbb{R}^2 \times \boldsymbol{T^2}$ w/ 't Hooft twist [Bergner–González-Arroyo–Soler '25]

Semiclassics via T^2 compactification

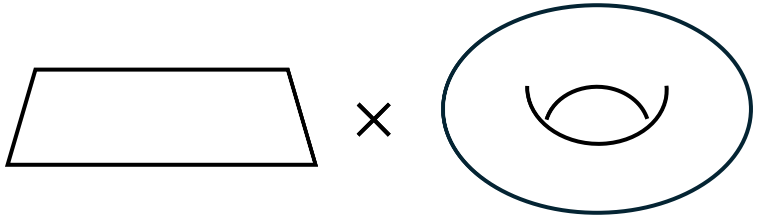
Today's talk: We investigate SU(N) YM/QCD vacuum structure through semiclassical analysis on $\mathbb{R}^2 \times T^2$ with **'t Hooft flux** (+ baryon magnetic flux for QCD)**.**

Main ansatz: adiabatic continuity conjecture



✅ Empirically, this method successfully gives a reasonable picture for confining vacuum in SU(N) YM, SU(N) N=1 SYM, QCD(F), QCD(Sym), QCD(AS), QCD(BF) [Tanizaki-Ünsal '22 '23][Tanizaki-YH-Watanabe '23 '24]. (cf. [Yamazaki-Yonekura '17])
This work: expanding analysis for QCD(F).

Contents

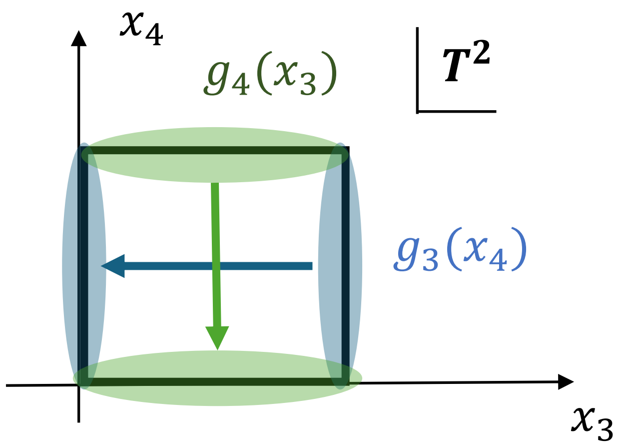


Plan:
Consider SU(N) Yang-Mills theory on $\mathbb{R}^2 \times T^2$ with 't Hooft flux;
Study physics at small T^2, and predict the original theory on $\mathbb{R}^4$ [Tanizaki-Ünsal '22]

SU(N) YM on $\mathbb{R}^2 \times T^2$ with 't Hooft flux (1)

[Tanizaki-Ünsal '22, ……] (cf. [Yamazaki-Yonekura '17])

- **'t Hooft flux for T^2 (or $\mathbb{Z}_N^{[1]}$ background)**



A unit 't Hooft flux ⇔ choose $g_3(0)g_4(L)g_3^\dagger(L)g_4^\dagger(0) = e^{\frac{2\pi \mathrm{i}}{N}}$.

($g_3(x_4), g_4(x_3)$: transition functions on T^2)

$$\begin{cases} a(\vec{x}, x_3 + L, x_4) = g_3^\dagger a g_3 - \mathrm{i}\, g_3^\dagger \mathrm{d} g_3 \\ a(\vec{x}, x_3, x_4 + L) = g_4^\dagger a g_4 - \mathrm{i}\, g_4^\dagger \mathrm{d} g_4 \end{cases}$$

Up to gauge, we can take $g_3 = S,\ g_4 = C$ (shift and clock matrices of $SU(N)$).

⇒ In this gauge, inserting 't Hooft flux ⇔ twisted boundary condition:

$$\begin{cases} a(\vec{x}, x_3 + L, x_4) = S^\dagger\, a\, S \\ a(\vec{x}, x_3, x_4 + L) = C^\dagger\, a\, C \end{cases}$$

e.g.) $N = 3$

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

SU(N) YM on $\mathbb{R}^2 \times T^2$ with ‘t Hooft flux (2)

- **Consequences from ‘t Hooft-twisted compactification**

  ✓**Center symmetry is kept at small T^2**

  Classically, $P_3 = S$ and $P_4 = C \Rightarrow \langle \mathrm{tr}\, P_3 \rangle = \langle \mathrm{tr}\, P_4 \rangle = 0.$

  ✓**Perturbatively gapped gluons:**

$$\begin{cases} a(\vec{x}, x_3 + L, x_4) = S^\dagger a\, S \\ a(\vec{x}, x_3, x_4 + L) = C^\dagger a\, C \end{cases}$$

  ∼**adjoint higgsing by Polyakov loops P_3, P_4:** $SU(N) \to \mathbb{Z}_N$

  ⇒ no zeromode; O(1/NL) KK mass

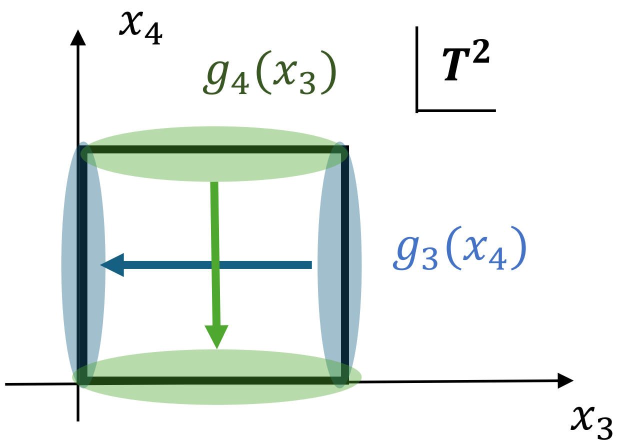
  For confinement on $\mathbb{R}^2$:

  ✓**Numerical evidence for $^\exists$center vortex/fractional instanton with $S = \frac{8\pi^2}{Ng^2}$, $Q_{top} = 1/N$ as a “local solution”** (scale ∼ Size(T^2))

  [Gonzalez-Arroyo–Montero ’98, Montero ’99 ‘00]

  (cf. [García Pérez–Gonzalez-Arroyo–Soderberg ’90; Itou ‘18] for $\mathbb{R} \times T^3$)

  Note) fractional topological charge: it cannot exist alone if the boundary condition for $\mathbb{R}^2$ is regular



$g_3 = S, \; g_4 = C$

e.g.) $N = 3$

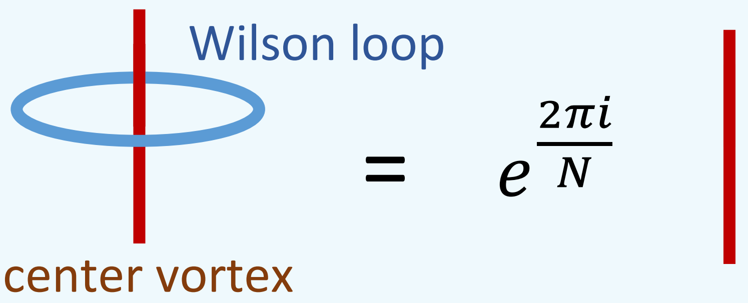
$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

Remark: center vortex?

Center vortex (in general context)
['t Hooft '78, ...: cf. Gastão's talk]

Co-dim-2 object carrying "magnetic flux of center element":

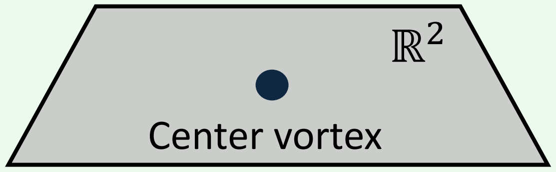


(expected to play an important role in quark confinement)

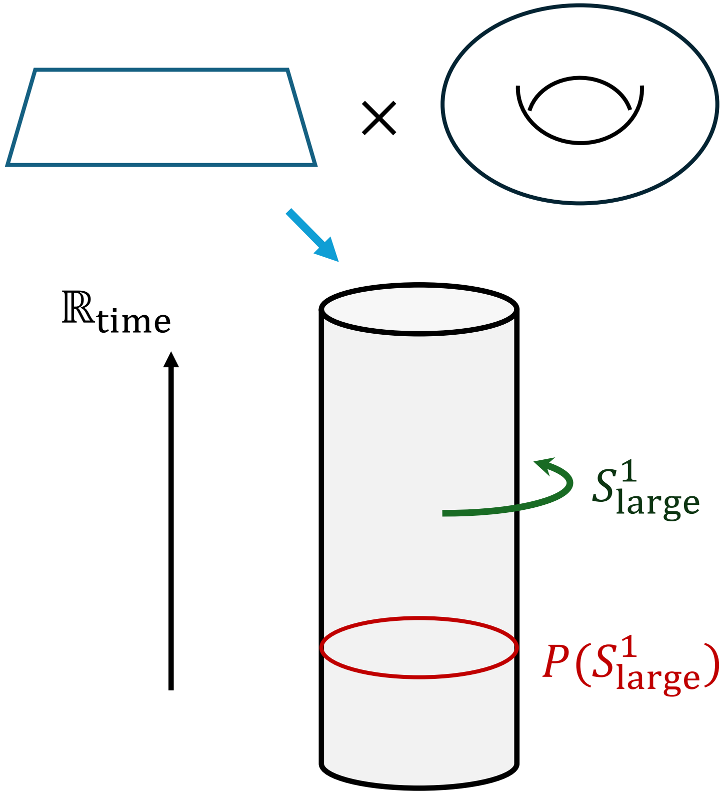
Center vortex (we consider here)

In addition, it is a 1/N fractional instanton satisfying the BPS bound

$$S = \frac{8\pi^2}{Ng^2}, \qquad Q_{top} = 1/N$$



Fractional instanton (in QM)



- <u>Reduction to QM</u> (cf. [Yamazaki-Yonekura ’17])

Further spatial compactification:

$\mathbb{R}^2 \times T^2 \to \mathbb{R}_{\text{time}} \times S^1_{\text{large}} \times T^2$

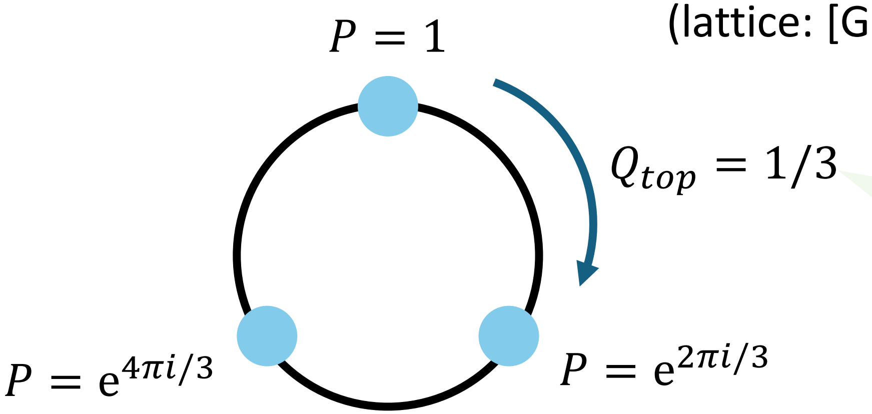
with $\text{Size}(T^2) \ll \text{Size}(S^1_{\text{large}}) \ll \Lambda^{-1}$

2d EFT: $(SU(N) \to)\ \mathbb{Z}_N$ gauge

$\Rightarrow$ **N classical vacua** with $P(S^1_{\text{large}}) = \mathrm{e}^{2\pi k i/N}\mathbf{1}\ (k = 0, \cdots, N-1)$

- “Fractional instanton = tunneling event”

(lattice: [Garcia Perez–Gonzalez-Arroyo–Soderberg '90; Itou ’18])



- does not globally exist if the periodic BC is imposed
- can exist globally under the twisted BC

Semiclassics on $\mathbb{R}^2 \times T^2$ in $SU(N)$ YM [Tanizaki-Ünsal '22]

- **Dilute gas of center vortices**

The center-vortex and anti-center-vortex vertices are:

$$Ke^{-\frac{8\pi^2}{Ng^2}+i\,\theta/N}, \qquad Ke^{-\frac{8\pi^2}{Ng^2}-i\,\theta/N}$$

with a dimensionful constant K.

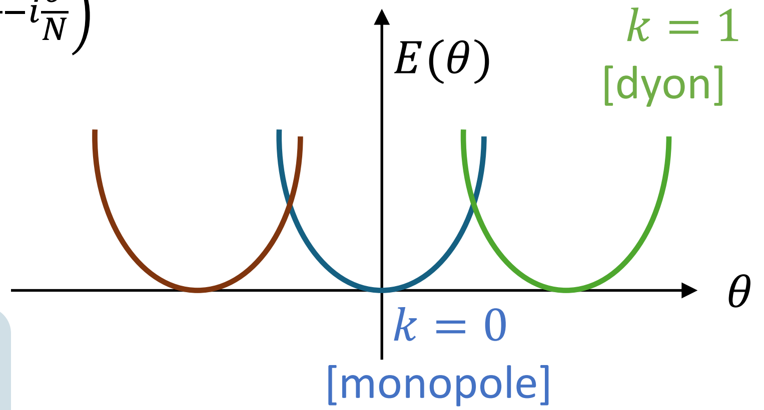
For calculating partition function, we compactify $\mathbb{R}^2$ without 't Hooft flux.
⇒ total topological charge is constrained $Q_{top} \in \mathbb{Z}$

Then, the dilute gas approximation yields, (only configurations with $Q_{top} \in \mathbb{Z}$ are admitted)

$$Z_{YM} = \sum_{n,\bar{n}\geq 0} \frac{1}{n!\,\bar{n}!} \delta_{n-\bar{n}\in N\mathbb{Z}} \left(VKe^{-\frac{8\pi^2}{Ng^2}+i\frac{\theta}{N}}\right)^n \left(VKe^{-\frac{8\pi^2}{Ng^2}-i\frac{\theta}{N}}\right)^{\bar{n}}$$
$$= \sum_{k\in\mathbb{Z}_N} \exp\left[-V\left(-2Ke^{-\frac{8\pi^2}{Ng^2}}\cos\left(\frac{\theta-2\pi k}{N}\right)\right)\right]$$

N semiclassical vacua

Energy density of k-th vacuum
→multibranch structure!



✅ One can also derive area-law falloff of the Wilson loop from the dilute gas of center vortices.

Digression: relation to another method

- **Two deformations** to weak-coupling confined theories

Today's talk

4d Yang-Mills theory

S^1 compactification with center stabilization

T^2 compactification with ‘t Hooft flux

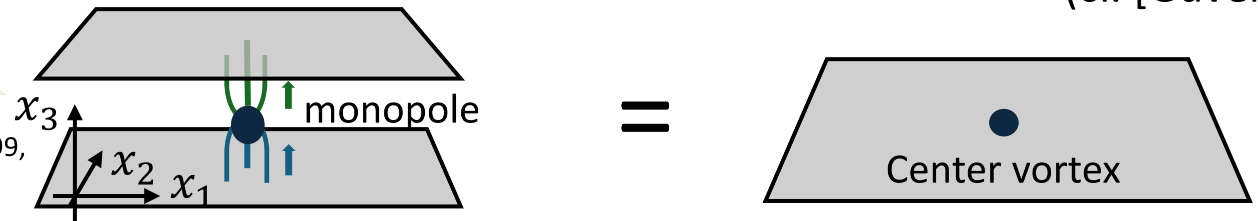
<u>“Monopole semiclassics”</u>
[Ünsal ’07, Ünsal-Yaffe ’08,…]
weakly coupled setup, where BPS/KK monopoles cause confinement

<u>“Center-vortex semiclassics”</u>
[Tanizaki-Ünsal ’22,…]
weakly coupled setup, where center vortices cause confinement

BPS/KK monopole = center vortex [YH-Tanizaki ’24; YH-Misumi-Tanizaki ‘24] (cf. [Güvendik-Schäfer-Ünsal ’24])

“monopole as junction of center vortices”

[Ambjørn-Giedt-Greensite ‘99, Engelhardt-Reinhardt ‘99, Cornwall ’99,…]



Contents

Plan:
Consider similar T^2 compactification for QCD, and study small-T^2 physics
New insights on η’ meson: periodicity extension / η’ mass from fractional instanton
[Tanizaki-Ünsal ’22; YH-Tanizaki ‘24]

Introduction: global structure of η'?

- **Low-energy effective theory of QCD: $SU(N_f)$ Chiral Lagrangian**

Light pseudoscalar mesons: Nambu-Goldstone bosons of (approximate) $SU(N_f)_{\text{chiral}}$

$$\Rightarrow S[U] = \int f_\pi^2 \, |dU|^2 - \Lambda^3 \mathrm{tr}\,(MU) + c.c.$$

mass matrix from quark mass

- **Chiral Lagrangian with η'?**

Sometimes, one includes η' by considering $U(N_f)$ chiral Lagrangian and adds the instanton-induced η' mass term (Kobayashi-Maskawa-'t Hooft vertex).

$$\Rightarrow S[U] = \int f_\pi^2 \, |dU|^2 - \Lambda^3 \mathrm{tr}\,(MU) - \Delta\, \mathrm{e}^{-\mathrm{i}\theta} \det(U) + c.c.$$

Question: ambiguity in η' mass (or global structure of η')?
It's common to use det-vertex, but it should be log-det(U)-type vertex at large-N…

(Oversimplified) summary

Question: ambiguity in η' mass
Instanton (det-type) vertex? log-det vertex?

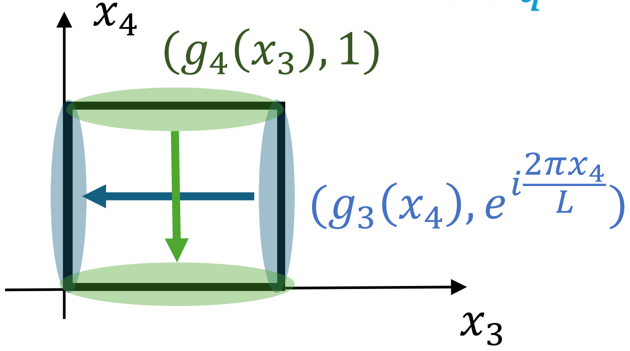
Our suggestion (from 2d semiclassics):
Use fractional instanton vertex $(\det U)^{1/N}$!
(while η' periodicity is extended by N, eating YM vacuum label)

Setup for QCD [Tanizaki-Ünsal '22]

- In the presence of fundamental quarks, it is impossible to insert 't Hooft flux alone ($g_3(0)g_4(L)g_3^\dagger(L)g_4^\dagger(0)=e^{\frac{2\pi \mathrm{i}}{N}}$ leads to an inconsistency).
- To avoid this problem, we also introduce **baryon magnetic flux** simultaneously: $\int_{T^2} dA_B = 2\pi$. (e.g., we can take $A_B=\frac{2\pi}{L^2}x_3 dx_4$; 1/N magnetic flux in terms of $U(1)_q$)

Boundary conditions for quarks (in the gauge $g_3=S,\ g_4=C$):

$$\begin{cases}\psi(\vec{x},x_3+L,x_4)=e^{i\frac{2\pi x_4}{NL}}S^\dagger\psi(\vec{x},x_3,x_4)\\ \psi(\vec{x},x_3,x_4+L)=\ C^\dagger\psi(\vec{x},x_3,x_4)\end{cases}$$



- At small T^2 , there is one 2d Dirac "low-energy mode" (⇔without KK mass) per flavor.

(obtained by solving zeromode equation)

Index theorem "$N\times\int_{T^2} dA_q=1$" $(U(1)_B=U(1)_q/\mathbb{Z}_N)$

Constructing 2d effective theory

$\underline{N_f = 1 \text{ case:}}$

Invariance under
$\theta \to \theta + \alpha, \varphi \to \varphi + \alpha$

- Low-energy mode: one 2d Dirac fermion ($\Leftrightarrow$ compact scalar φ)
- Center-vortex vertex: $Ke^{-\frac{8\pi^2}{Ng^2}+i\,\theta/N}$ "$e^{-i\,\varphi/N}$" from $U(1)_{\mathrm{chiral}}$ spurious symmetry
- Dilute gas approximation

$$\longrightarrow \quad S[\varphi] = \int \frac{1}{8\pi}|d\varphi|^2 - m\mu\cos\varphi \; - 2Ke^{-\frac{8\pi^2}{Ng^2}}\cos\left(\frac{\varphi-\theta-2\,\pi k}{N}\right)$$

$\boldsymbol{\varphi}$ "eats" the vacuum label $\boldsymbol{k} \in \mathbb{Z}_N$ and extends its periodicity to $\boldsymbol{\varphi \sim \varphi + 2\pi N}$.

residual gauge $\boldsymbol{SU(N)} \to \mathbb{Z}_N$

$\underline{N_f \geq 2 \text{ case: the non-abelian bosonization}}$

$\Rightarrow U(N_f)_1$ WZW (+ quark-mass deformation + center-vortex deformation)

$\Rightarrow$ 2d analog of $U(N_f)$ chiral Lagrangian with $\eta' \sim \eta' + 2\pi N$ & $(\det U)^{1/N}$-type η' mass.

2d version of chiral Lagrangian

- For $N_f > 1$, we use the non-Abelian bosonization: looks like **chiral Lagrangian with η'!**

[$U \in U(N_f)$ with $2\pi N$-periodic (det U)]

$$S[U] = \int \frac{1}{8\pi} |dU|^2 - m\mu \,\mathrm{tr}\,(U) - K \mathrm{e}^{-\frac{8\pi^2}{Ng^2}} \mathrm{e}^{-i\theta/N} (\det U)^{1/N} + c.c. + S_{WZW}^{3d}[U]$$

quark-mass deformation
(if present)

Center-vortex-induced η' mass term
"finite-N version of log-det vertex"

※ "2d version of chiral Lagrangian"

Up to gapped **η'**, this 2d effective theory

= T^2 compactification with $U(1)_B$ flux of 4d $SU(N_f)$ chiral Lagrangian

Coupling to $U(1)_B$ background

$$\int_{M_3 \times T^2} \mathrm{d}A_B \wedge \left(\frac{1}{24\pi^2} \,\mathrm{tr}\,(U^{-1}\mathrm{d}U)^3 \right) \Rightarrow \int_{M_3} \left(\frac{1}{12\pi} \,\mathrm{tr}\,(U^{-1}\mathrm{d}U)^3 \right) = S_{WZW}^{3d}[U]$$

Results

$\eta' \sim \eta' + 2\pi$
$\Rightarrow \eta' \sim \eta' + 2\pi N$

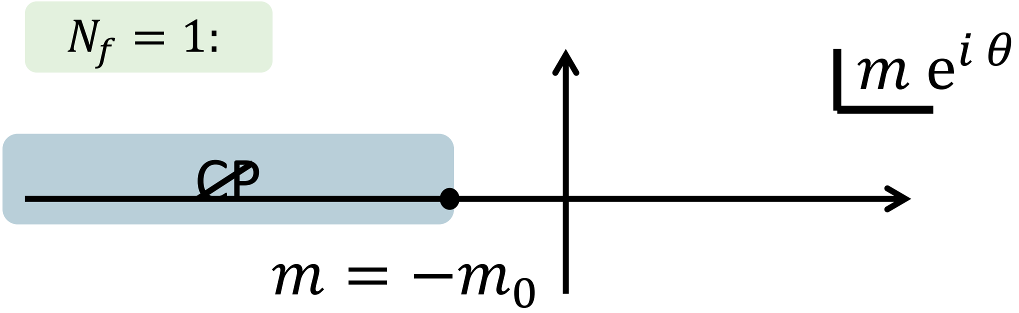
- **2d effective theory on $\mathbb{R}^2$**

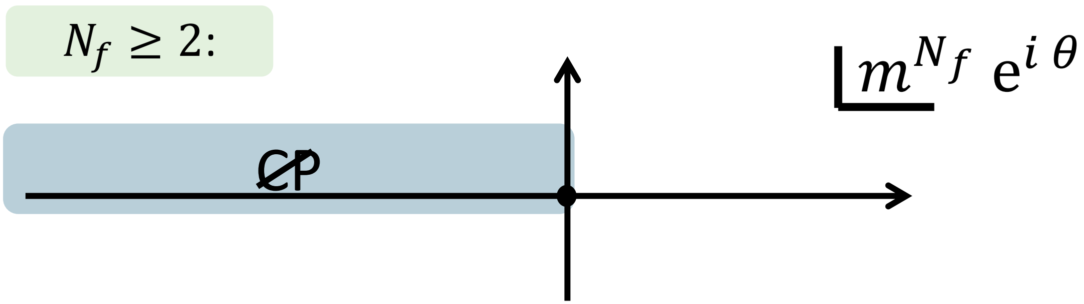
  = 2d analog of chiral Lagrangian + periodicity-extended $\boldsymbol{\eta'}$

  + corresponding $\boldsymbol{\eta'}$ mass term $(\det U)^{1/N}$

finite-N version
of log-det vertex

- This 2d effective theory explains the expected vacuum structure of QCD (phase diagram on $m^{N_f} \mathrm{e}^{i\theta}$):



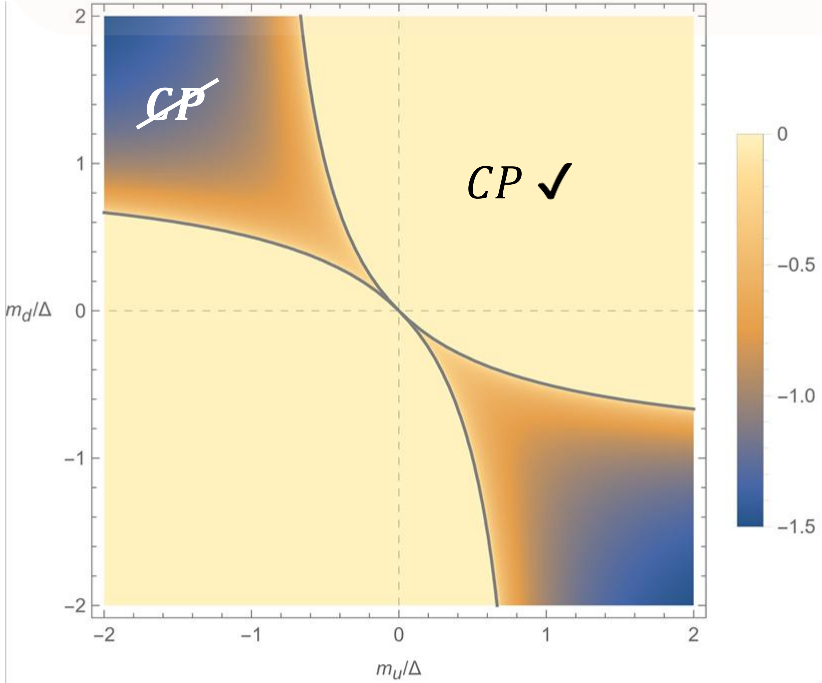


- **$\boldsymbol{\eta'}$ extends its periodicity by absorbing the $\mathbb{Z}_N$ vacuum label;** also for 4d chiral Lagrangian, this prescription improves the global aspects.
  (periodicity extension $\sim$ in the limit where the Chern-Simons DW is infinitely heavy)

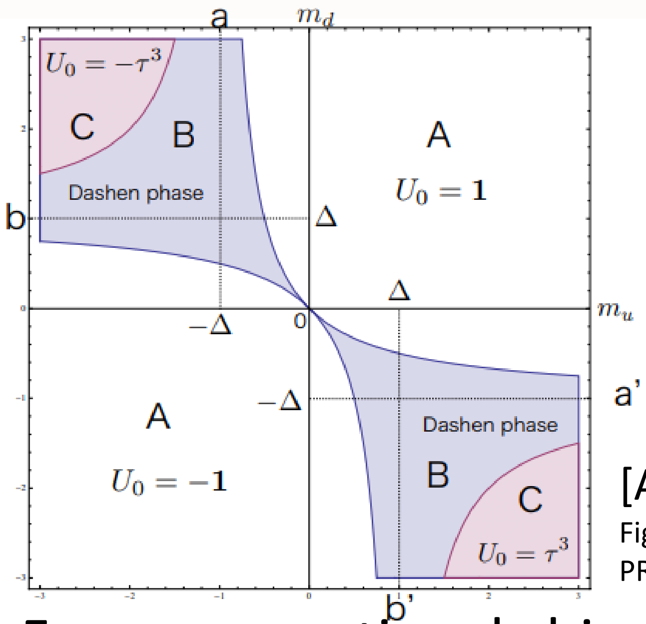
cf.) [Csáki-D'Agnolo-Gupta-Kuflik-Roy-Ruhdorfer ‘23]: η’ in anomaly-mediated SUSY-broken SQCD: N-Nf branches

Application: Dashen phase on (m_u, m_d) plane

Phase diagram of (1+1)-flavor QCD on (m_u, m_d) plane:
The conventional U(2) chiral Lagrangian with det-type η mass has an artificial CP-restored phase (“phase C”).
Using $(\det U)^{1/N}$-vertex eliminates the artificial phase.



with $\eta \sim \eta + 2\pi N$ & $(\det U)^{1/N}$ mass



[Aoki-Creutz ‘14]
Fig. 1 of S. Aoki and M. Creutz, PRL **112** 141603 (2014)

From conventional chiral Lagrangian with η (with det-type mass term)

Summary of this section

We study QCD through semiclassics on $\mathbb{R}^2 \times T^2$ with 't Hooft flux & $U(1)_B$ magnetic flux

Our results:

$\eta' \sim \eta' + 2\pi$
$\Rightarrow \eta' \sim \eta' + 2\pi N$

- **2d effective theory on $\mathbb{R}^2$**

  = 2d analog of chiral Lagrangian + periodicity-extended $\boldsymbol{\eta'}$

  + corresponding $\boldsymbol{\eta'}$ mass term $(\det U)^{1/N}$

  Center-vortex induced mass

- This 2d effective theory explains the expected vacuum structure of QCD (phase diagram on $m^{N_f} \mathrm{e}^{i\theta}$).
- **The periodicity extension of $\boldsymbol{\eta'}$ = inclusion of YM vacuum label**

Also for 4d chiral Lagrangian with $\boldsymbol{\eta'}$, the periodicity extension improves global aspects (particularly, smooth connection to quenched limit).

Contents

4. Summary

Conclusion:
T^2 compactification with ‘t Hooft flux gives **tractable confining theory**
⇒ Study small-T^2 physics tells us some aspects of confinement

- Dilute gas of center vortices (fractional instantons) describes confining vacuum
- New insights on eta prime (on its global structure)
- More developments: other theories (SYM/QCD(adj), QCD(AS/Sym), QCD(BF),……) [Tanizaki-Ünsal ’22 ’22, YH-Tanizaki-Watanabe ‘23 ‘24] , connections to monopole semiclassics [YH-Tanizaki ‘24, YH-Misumi-Tanizaki’24], center stability at large N [YH-Tanizaki-Ünsal ‘25],…

Backup

Technicality: $\mathbb{Z}_N$ gauging and vacuum label

- Problem: Center-vortex vertex: $K e^{-\frac{8\pi^2}{N g^2}+i\,\theta/N}$ "$e^{-i\,\varphi/N}$" looks ill-defined/non-genuine.
- Keypoint: **residual** $\mathbb{Z}_N$ **gauge** after adjoint higgsing by Polyakov loops : $SU(N) \to \mathbb{Z}_N$.
- The residual $\mathbb{Z}_N$ gauge is vector-like to fermion ψ. It couples to φ magnetically

$$\frac{i}{2\,\pi} \int a_{\mathbb{Z}_N} \wedge d\varphi$$

(#fermions) = (#kinks).

Integrating out $a_{\mathbb{Z}_N}$ ⇒ constraint $\int d\varphi \in 2\pi N\,\mathbb{Z}$

⇒ It is possible to regard $\varphi \in \mathbb{R}/2\pi N\mathbb{Z}$.

$e^{-i\,\varphi/N}$ becomes well-defined.

- In the lift from 2π-periodic field to $2\pi N$-periodic field, there is $\mathbb{Z}_N$ ambiguity: $\varphi \to \varphi + 2\pi k$. This 1-to-N correspondence absorbs the vacuum label k. In summary,

$$\int D a_{\mathbb{Z}_N} \sum_{k\in\mathbb{Z}_N} \int_{\varphi\sim\varphi+2\pi} D\varphi\ \ldots \ \Rightarrow\ \int_{\varphi\sim\varphi+2\pi N} D\varphi\ \ldots$$

2d version of chiral Lagrangian

- For $N_f > 1$, we use the non-Abelian bosonization: looks like **chiral Lagrangian with η'!**

[$U \in U(N_f)$ with $2\pi N$-periodic (det U)]

$$S[U] = \int \frac{1}{8\pi}|dU|^2 - m\mu\,\mathrm{tr}\,(U) \; - K\mathrm{e}^{-\frac{8\pi^2}{Ng^2}}\mathrm{e}^{-i\theta/N}(\det U)^{1/N} + c.c. + S_{WZW}^{3d}[U]$$

quark-mass deformation
(if present)

Center-vortex-induced η' mass term
"finite-N version of log-det vertex"

Up to gapped **η'**, this 2d effective theory

= T^2 compactification with $U(1)_B$ flux of 4d $SU(N_f)$ chiral Lagrangian

Coupling to $U(1)_B$ background

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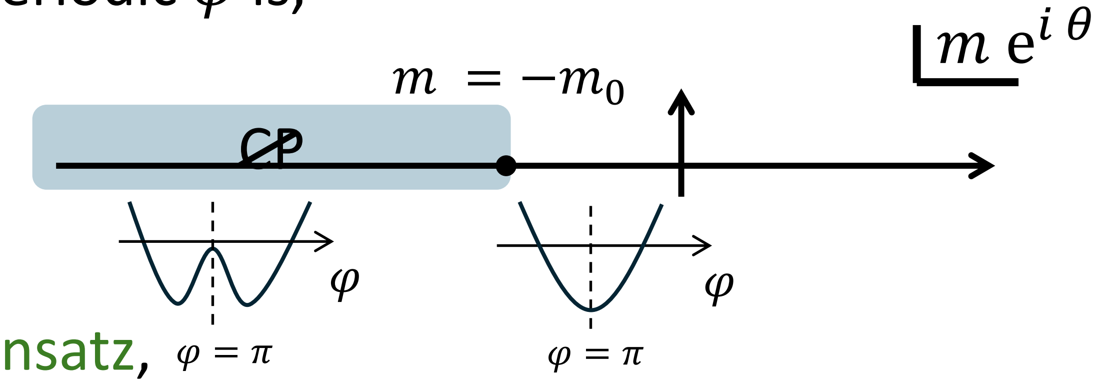
Vacuum structure from 2d effective theory

The 2d effective theory explains the vacuum structure, just by finding potential minima:

- $\boldsymbol{N_f = 1}$ **case**: the effective potential for $2\pi N$-periodic φ is,

$$V[\varphi] = -m\mu\cos\varphi \; - 2Ke^{-\frac{8\pi^2}{Ng^2}}\cos\left(\frac{\varphi-\theta}{N}\right)$$



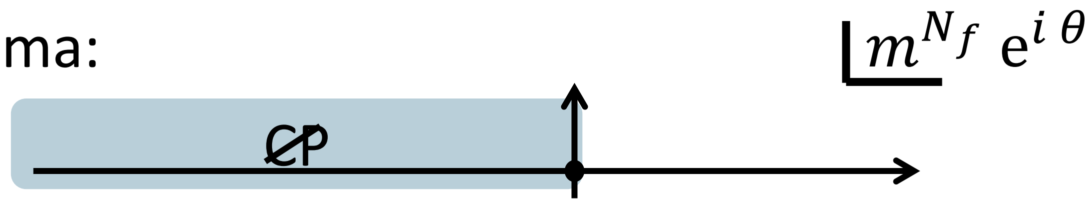
- $\boldsymbol{N_f \geq 2}$ **case**: we take the $SU(N_f)$ symmetric ansatz,

$U = e^{i\,\varphi}1$ with $(\log\det U) = N_f\varphi + 2\pi k$ $(-\pi < \varphi \leq \pi, k \in \mathbb{Z}_N)$

$$\Rightarrow\; V[\varphi] = -N_f m\mu\cos\varphi \; - 2Ke^{-\frac{8\pi^2}{Ng^2}}\cos\left(\frac{N_f\varphi+2\pi k-\theta}{N}\right)$$

At $\theta = \pi$, this potential has two degenerate minima:

$(\varphi = \varphi_*, k = 0)$ and $(\varphi = -\varphi_*, k = 1)$



Discrete anomaly

Baryon-color-flavor anomaly:
Flavor-symmetric QCD with N_f quarks at $\theta = \pi$ has mixed anomaly between $\frac{SU(N_f)\times U(1)_q}{\mathbb{Z}_N}$ and CP if $\gcd(N, N_f) \neq 1$. [Gaiotto-Komargodski-Seiberg '17]

- For $\gcd(N, N_f) = 1$, the variables (k, φ) in the $SU(N_f)$ symmetric ansatz can be combined into single $2\pi N$-periodic one φ: $N_f\varphi + 2\pi k \Rightarrow N_f\varphi \pmod{2\pi N}$. Like the mass deformation in $N_f = 1$ case, a suitable symmetric deformation can single out a unique gapped vacuum (the absence of anomaly).
- For $\gcd(N, N_f) \neq 1$, the $\mathbb{Z}_{\gcd(N,N_f)}$ discrete label cannot be absorbed. (Intuitively, quark fluctuation only bridges k-th vacuum and $(k + N_f)$-th vacuum, so it cannot split the degeneracy of CP-broken vacua: $k = 0$ and $k = 1$.)
- 4d chiral Lagrangian with periodicity-extended η' reproduces this discrete anomaly.

(A more essential point is that the coupling $\int \eta' dA_B \wedge dA_B$ becomes well-defined thanks to the periodicity extension.)