NANJING UNIVERSITY

# Distribution Functions Using Continuum Schwinger Function Methods


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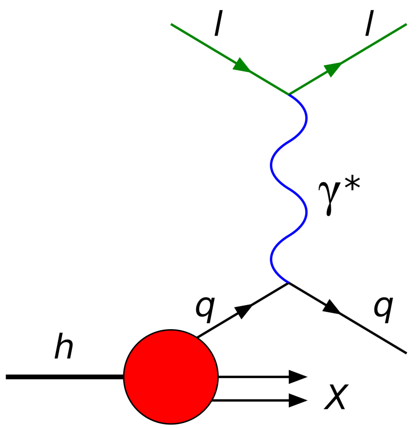
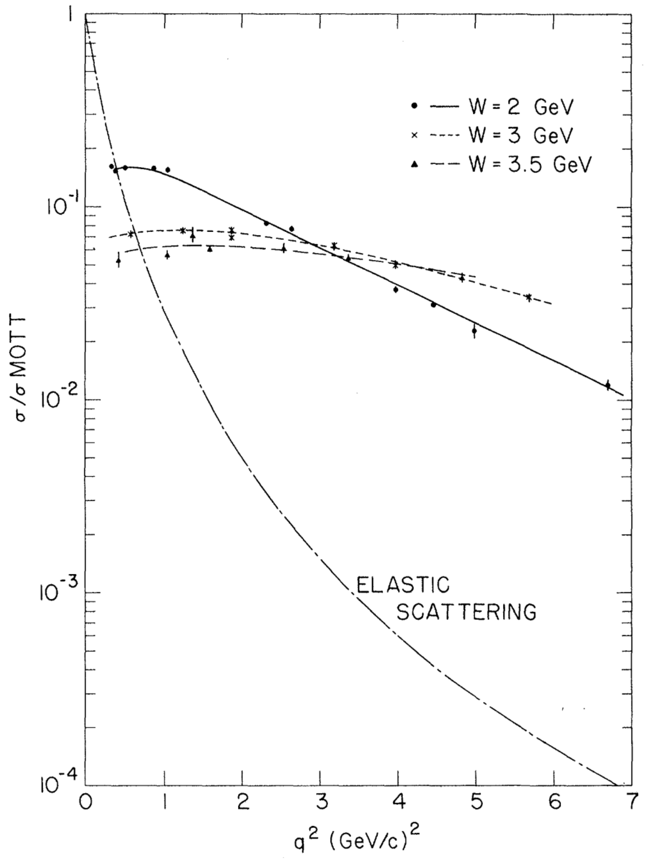
Workshop on the complex structure of strong interactions in Euclidean and Minkowski space, ECT*, Trento, May 26-30 2025

# Deep inelastic scattering (DIS)




➢ DIS of a lepton (l) on a hadron (h), at leading order.
The virtual photon ($\gamma^*$) knocks a quark (q) out of the hadron.




➢ From e–p elastic scattering, at high $q^2$, the differential cross section falls of rapidly with $q^2$ due to the proton not being point-like (and has form factors).

➢ Inelastic scattering cross sections only weakly dependent on $q^2$.

➢ Deep Inelastic scattering cross sections almost independent of $q^2$, i.e. “Form factor” $\rightarrow 1$.

➢ Scattering from point-like objects within the proton!

Phys. Rev. Lett. 23 (1969) 935

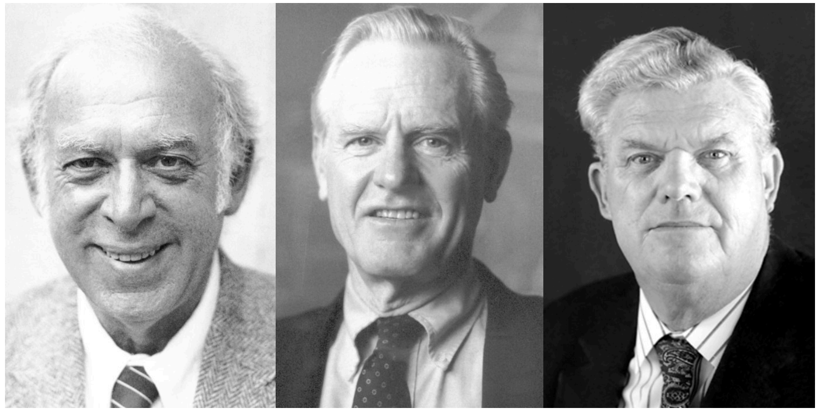
# DIS

➢ Nobel Prize of 1990:

Henry Way Kendall,

Jerome Isaac Friedman

Richard E. Taylor



➢ “for their pioneering investigations concerning deep inelastic scattering of electrons on protons and bound neutrons, which have been of essential importance for the development of the quark model in particle physics.”

# DIS

- The cross-section can be written as a convolution of the leptonic and hadronic tensors: $d\sigma \propto L_{\mu\nu} W^{\mu\nu}$
- Using the general Lorentz decomposition, one can express the e–p DIS hadronic tensor $W_{\mu\nu}$ in terms of the unpolarized structure functions $W_1$ and $W_2$

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(\nu, Q^2) + \frac{1}{m_p^2}\left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P \cdot q}{q^2} q_\nu\right) W_2(\nu, Q^2)$$

- In the deep-inelastic (Bjorken) limit: “scaling” structure functions

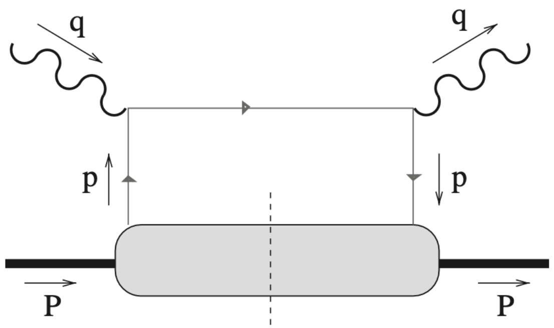
$$\lim_{Bj} W_1(\nu, Q^2) = F_1(x)\,, \quad \lim_{Bj} \frac{\nu}{m_p} W_2(\nu, Q^2) = F_2(x)$$

- Scaling structure functions provide access to the quark parton distribution functions:

$$F_1(x) = \frac{1}{2} \sum_i e_i^2 f_i(x)\,, \quad F_2(x) = 2x F_1(x)\,.$$

# DIS

➢ Another way to obtain the parton model is by considering the quark handbag diagram:

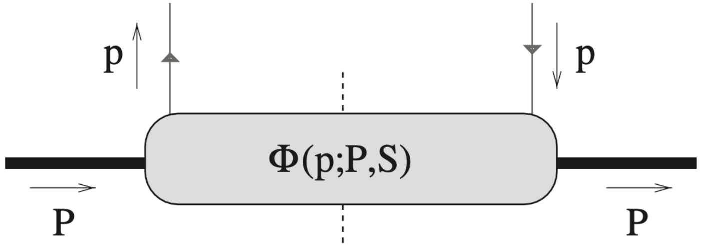



➢ The hadronic tensor:

Electromagnetic Interactions and Hadronic Structure, Edited by Frank Close, Sandy Donnachie, and Graham Shaw, 2007.

$$2m_p W^{\mu\nu}(P,q) = -g_\perp^{\mu\nu}\frac{1}{2}\sum_i e_i^2 \int dp^- d^2p_\perp \,\mathrm{Tr}[\gamma^+\Phi(p)]\Big|_{p^+=xP^+} + \ldots$$

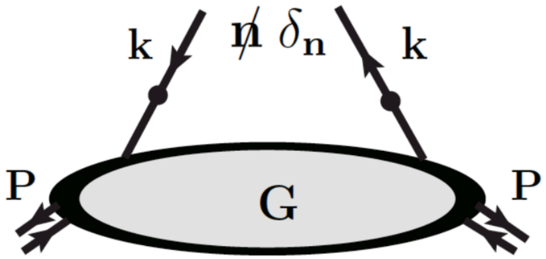
➢ Quark-quark correlation function:




$$f(x) = \int dp^- dp_\perp^2 \mathrm{Tr}\left[\gamma^+\Phi(p)\right]\Big|_{p^+=xP^+} = \int d^4p \,\mathrm{Tr}\left[\gamma^+\Phi(p)\right]\delta(p^+ - xP^+)$$

# Distribution function (DF) of pion

$$f(x) = \int d^4p \mathrm{Tr}\left[\gamma^+ G(p)\right]\delta(p^+ - xP^+)$$




➢ G(p): forward antiquark-target scattering amplitude in rainbow-ladder truncation

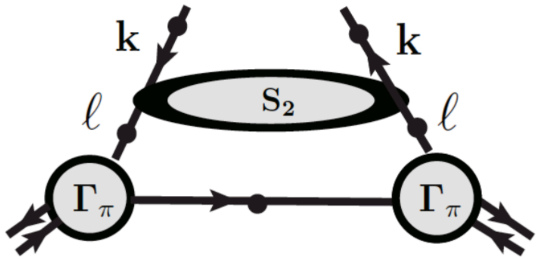
$$G(p) = S_2\Gamma_\pi(l, -P)S(l+P)\Gamma_\pi(l,P)$$

✓ $S_2$: 4-point function, $q\bar{q}$ two-body propagator

✓ $S_l$: dressed quark propagator

✓ $\Gamma_\pi(l,P)$: meson Bethe-Salpeter amplitude

$G =$




*Phys. Rev. C* 83 (2011) 062201

# DF of pion

✓ $S_2$: 4-point function, $q\bar{q}$ two-body propagator

$$S_2\gamma^+\delta(k^+ - xP^+) = S(l)\Gamma^n(l,x)S(l)$$

✓ $\Gamma^n$: generalization of the dressed quark photon vertex, inhomogeneous term is $\gamma^+\delta(l^+ - xP^+)$

$$\Gamma^n(l,x) = n_\mu \frac{\partial S^{-1}(l)}{\partial l_\mu}\delta(n\cdot l - xn\cdot P)$$

where n is a light like vector: $n^2 = 0$ .

✓ DF:

$$f(x) = \int d^4l \mathrm{Tr}\left[n^\mu \frac{\partial S(l)}{\partial l_\mu}\Gamma_\pi(l, -P)S(l+P)\Gamma_\pi(l,P)\right]\delta(n\cdot l - xn\cdot P)\,.$$

# DF of pion

Phys. Lett. B737 (2014) 23-29

- DF: (term A) $$f_A(x) = \int d^4 l \mathrm{Tr}\left[n^\mu \frac{\partial S(l)}{\partial l_\mu}\Gamma_\pi(l,-P)S(l+P)\Gamma_\pi(l,P)\right]\delta(n\cdot l - xn\cdot P)\,.$$

- A second contribution, referred to as "term BC", is introduced to restore symmetry: the pion DF at hadron scale is expected to be a symmetric function if we consider isospin symmetry: $u_{\zeta_H}(x) = u_{\zeta_H}(1-x)\,.$

$$f_{BC}(x) = \int d^4 l \mathrm{Tr}\left[n^\mu S(l)\frac{\partial \Gamma_\pi(l,-P)}{\partial l_\mu}S(l+P)\Gamma_\pi(l,P)\right]\delta(n\cdot l - xn\cdot P)\,.$$

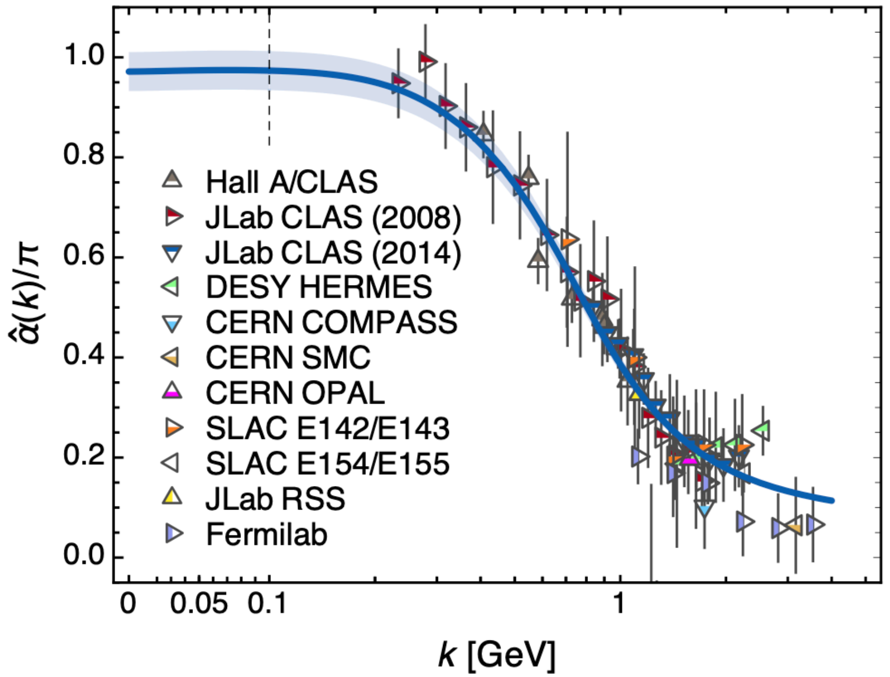
- DF:

$$f(x) = f_A(x) + f_{BC}(x) = \int d^4 l \mathrm{Tr}\left[n^\mu \frac{\partial [S(l)\Gamma_\pi(l,-P)]}{\partial l_\mu}S(l+P)\Gamma_\pi(l,P)\right]\delta(n\cdot l - xn\cdot P)\,.$$

# Process-independent effective charge

➢ Absence of a Landau pole

- ✓ Owing to the appearance of a gluon mass scale
- ✓ $\sqrt{k^2} < \zeta_H$, the running of the coupling slows down, at some point, the running ceases, effectively conformal
- ✓ $\zeta_H \approx 0.331(2)$ GeV
- ✓ Deep infrared, saturates to $\hat{\alpha}(k^2 = 0) = \pi \times 0.97$
- ✓ Gluons are screened, play no dynamical role
- ✓ Valence quasiparticles carry all hadron properties at hadron scale $\zeta_H$.

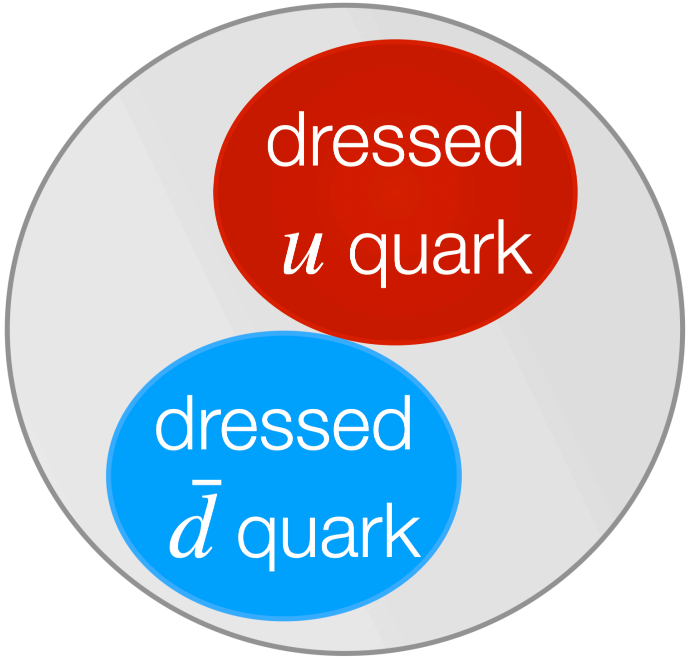



Binosi, Mezrag, Papavassiliou, Roberts, Rodriguez-Quintero, Phys. Rev. D 96, 054026 (2017)

# Pion at hadron scale

- At a hadron scale $\zeta_H$, dressed valence-quarks carry all the pion's light-front momentum and the glue and sea distributions vanish.

Ding, Raya, Binosi, Chang, Roberts, Schmidt, Phys. Rev. D 101 (2020) 5, 054014




- In isospin limit $m_u = m_{\bar{d}}$, pion DF is a symmetric function with respect to x=1/2.

$$\mathbf{u}^{\pi}(1-x,\zeta_H) = \mathbf{u}^{\pi}(x,\zeta_H)$$

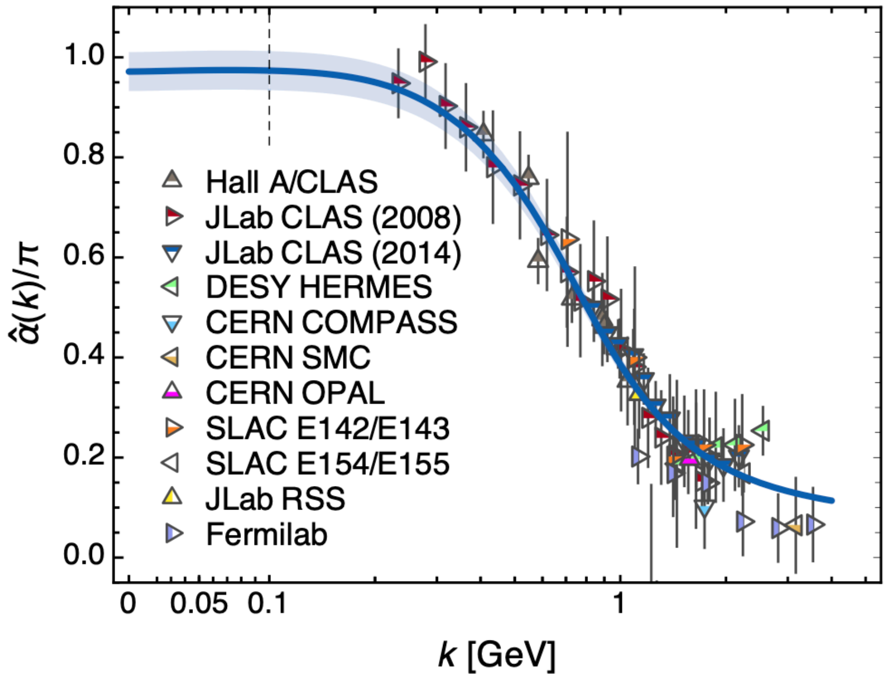
✓ Pion picture at hadron scale: dressed up quark + dressed down antiquark.

# Process-independent effective charge

➢ Infrared completion

- ✓ Effective charge is finite at all accessible momentum scales, from the deep infrared to the far ultraviolet and all scales in between
- ✓ Unique, nonperturbatively well-defined, calculable
- ✓ Large- $k^2$ behavior connects smoothly with one-loop QCD running coupling
- ✓ For practical intents and purposes, indistinguishable from process-dependent charge, determined from the Bjorken sum rule

$$\hat{\alpha}(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{K}^2(k^2)}{\Lambda^2_{QCD}}\right]}, \mathcal{K}^2(y = k^2) = \frac{a_0^2 + a_1 y + y^2}{b_0 + y}$$



# DF evolution: all-orders evolution scheme

- Proposition: there exists an effective charge, $\alpha_{1l}(k^2)$, that, when used to integrate the one-loop pQCD Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations, defines an evolution scheme for parton PDFs that is all-orders exact.

Raya, Cui, Chang, Morgado, Roberts, Rodriguez-Quintero, Chin. Phys. C 46 (2022) 1, 013105.

- The effective charge scheme was proposed by Grunberg, for example, connecting the nucleon axial charge, to the integral of the isovector part of the nucleon spin structure function

$$\begin{aligned}\Gamma_1^{\mathrm{p-n}}(Q^2) = \frac{g_{\mathrm{A}}}{6}\Bigg[1 &- \frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi} - 3.58\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^2 - 20.21\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^3 \\ &- 175.7\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^4 + \sim -893.38\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^5 \\ &+ \mathcal{O}\left(\left(\alpha_{\mathrm{s}}^{\mathrm{pQCD}}\right)^6\right)\Bigg] + \sum_{n>1}\frac{\mu_{2n}(Q^2)}{Q^{2n-2}}.\end{aligned}$$

$$\Gamma_1^{p-n}(Q^2) =: \frac{g_A}{6}\left(1 - \frac{\alpha_{g_1}(Q^2)}{\pi}\right)$$

✓ Effective charge $\alpha_{g1}(Q^2)$ implicitly incorporates terms of arbitrarily high order, n>1, in the perturbative coupling $[\alpha_s^{pQCD}]^n$.

# DF evolution: all-orders evolution scheme

- We use process-independent effective charge to integrate the one-loop pQCD DGLAP equations, defines an evolution scheme for parton PDFs that is all-orders exact.

Raya, Cui, Chang, Morgado, Roberts, Rodriguez-Quintero, Chin. Phys. C 46 (2022) 1, 013105.

- Mellin moments of DFs: $\langle x^n \rangle = \int_0^1 x^n f(x)\, dx$

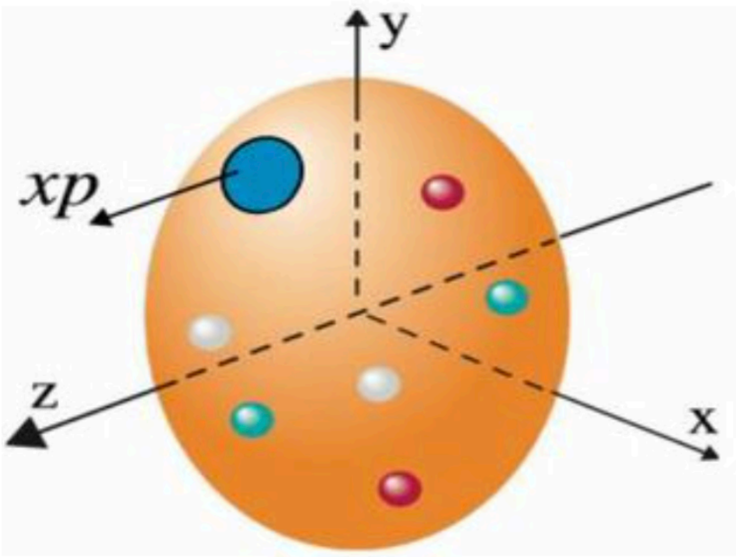
- Valence quark distribution DF Mellin moments:

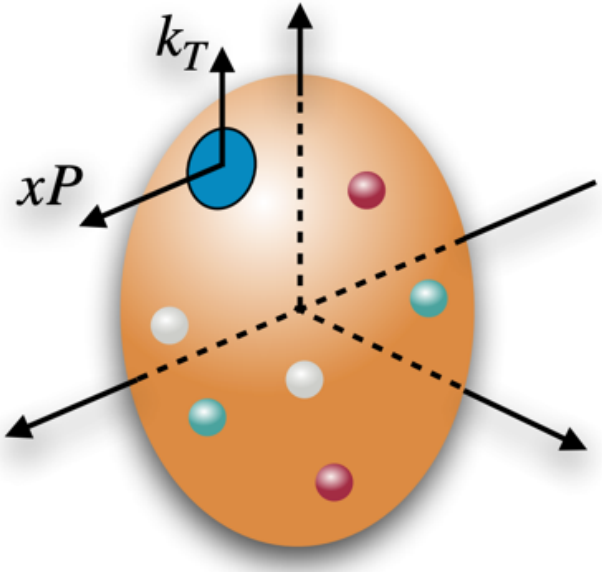
$$\frac{\langle x^n \rangle_V^{\zeta}}{\langle x^n \rangle_V^{\zeta_H}} = \exp\left[\frac{\gamma_0^n}{4\pi}\int_{\ln \zeta^2}^{\ln \zeta_H^2} d(\ln k^2)\, \hat{\alpha}(\ln k^2)\right] ,$$

- Sea quark and gluon DF Mellin moments: (generated by valence DF at hadron scale)

$$\begin{pmatrix} \langle x^n \rangle_\Sigma^{\zeta} \\ \langle x^n \rangle_g^{\zeta} \end{pmatrix} = \begin{pmatrix} \alpha_+^n S_-^n + \alpha_-^n S_+^n \\ \beta_{g\Sigma}^n \left(S_-^n - S_+^n\right) \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_\Sigma^{\zeta_H} \\ 0 \end{pmatrix} .$$

# DFs of pion and kaon




- ➢ 1D distribution function
- ➢ 3D Transverse momentum dependent (TMD) distribution function

# DF of pion at hadron scale

✓ Solid navy curve:

$$q^{\pi}(x;\zeta_H) = 213.32\,x^2(1-x)^2$$
$$[1 - 2.9342\sqrt{x(1-x)} + 2.2911\,x(1-x)]\,,$$

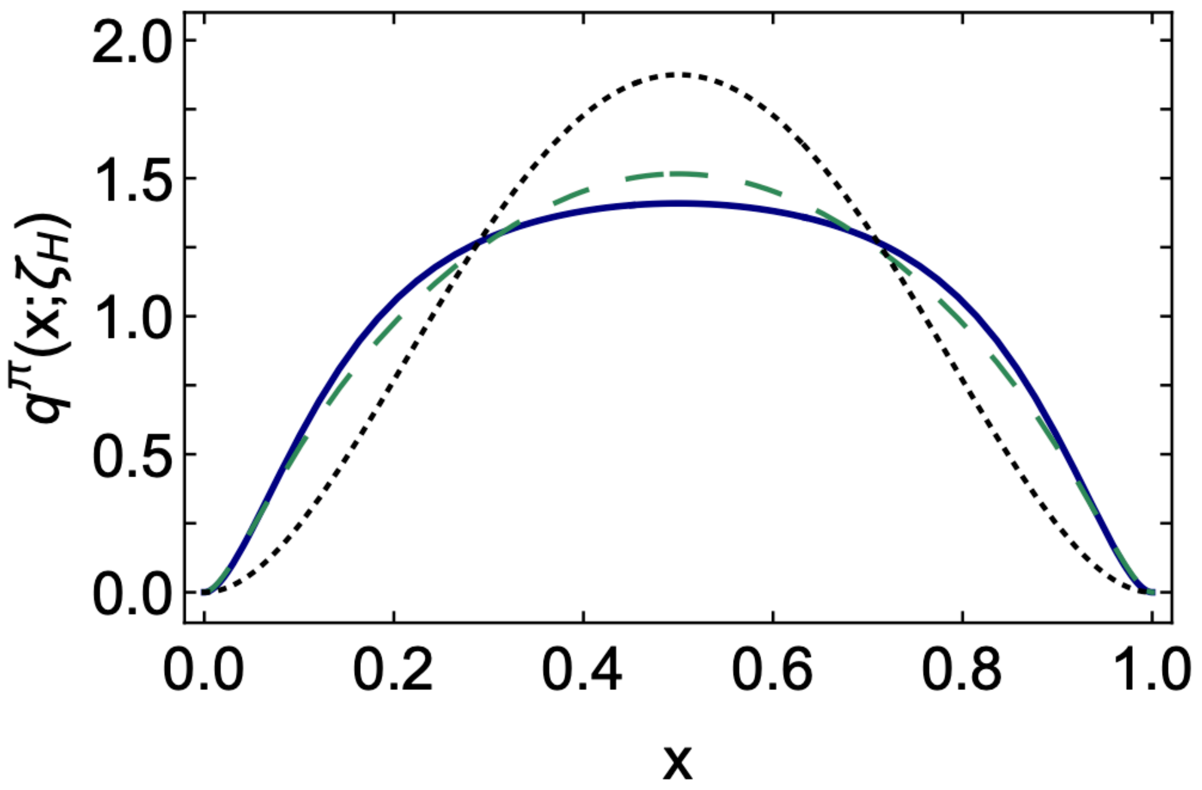
✓ Long-dashed green curve:

$$q^{\pi}_{\tilde{\varphi}^2}(x;\zeta_H) = 301.66x^2(1-x)^2$$
$$[1 - 2.3273\sqrt{x(1-x)} + 1.7889\,x(1-x)]^2$$

✓ Dotted black curve: scale free result

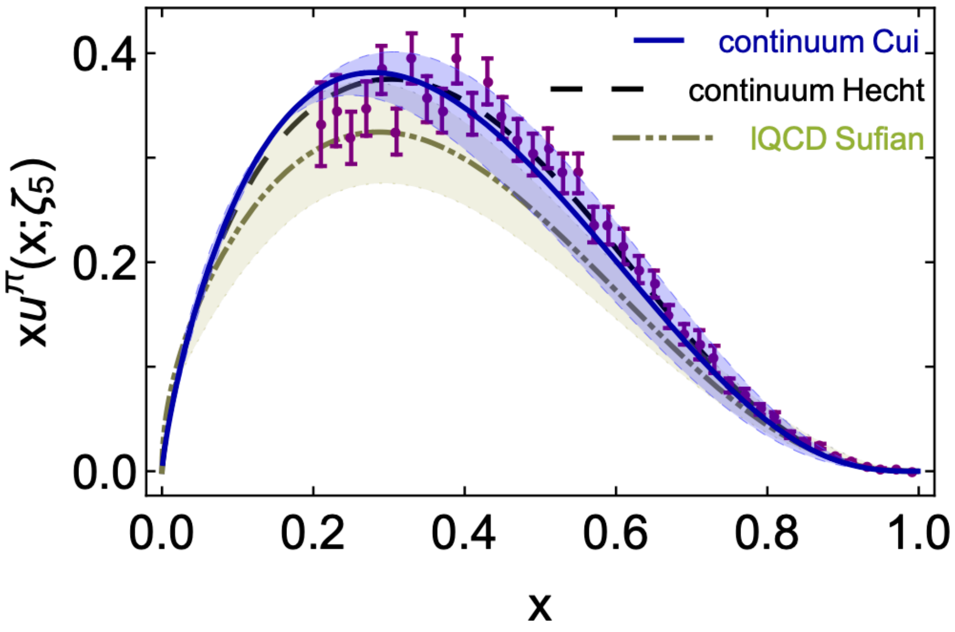
$$q_{\rm sf}(x) = 30x^2(1-x)^2$$

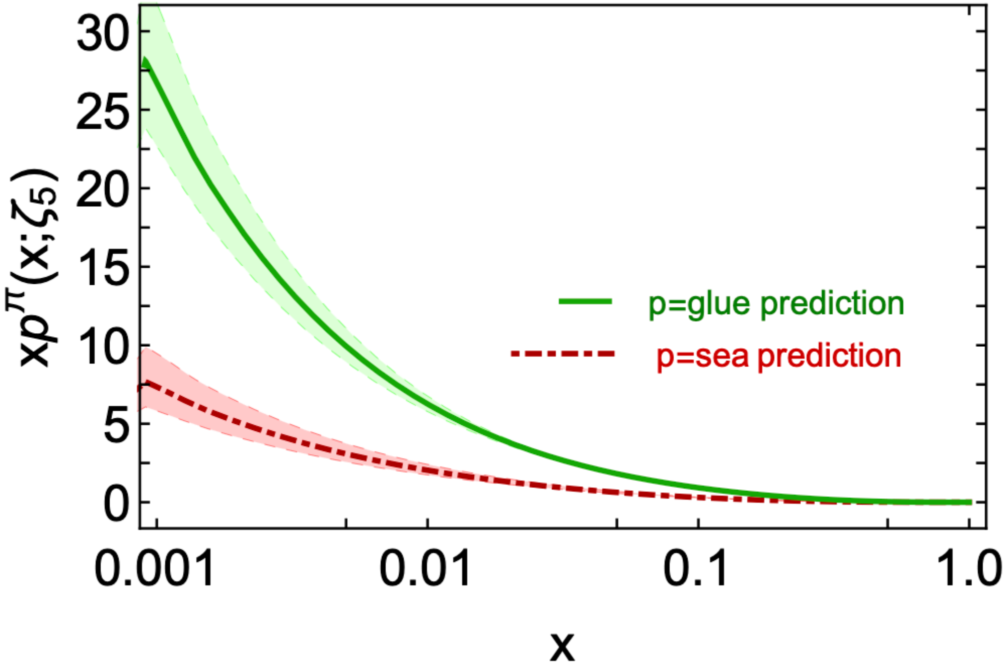
✓ Large x behaviour: $q^{\pi}(x;\zeta_H) \sim (1-x)^2$




✓ Broad function, induced by dynamical chiral symmetry breaking, and emergence of hadron mass (EHM).

# DF of pion at $\zeta_5 = 5.2$ GeV



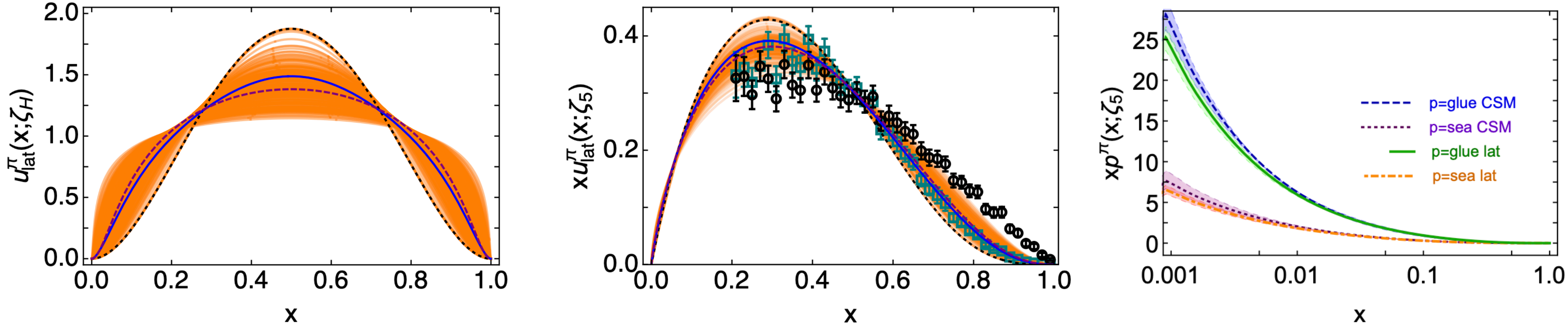




✓ Within uncertainties, continuum valence distribution (Cui) agrees with continuum in 2001 (Hecht), lattice (Sufian), and rescaled E615 experiment data.

Conway et al.. Phys. Rev. D 39 (1989) 92-122. Hecht et al.. Phys. Rev. C 63, 025213 (2001). Aicher et al.. PRL 105, 252003 (2010). Sufian et al.. Phys. Rev. D 99, 074507 (2019). Cui et al., Eur. Phys. J. C 80, 1064 (2020).

✓ Mellin moments: $\langle 2xu^{\pi}(x;\zeta_5)\rangle = 0.40(2)\,, \langle x\rangle_g^{\pi} = 0.45(2)\,, \langle x\rangle_{\rm sea}^{\pi} = 0.14(2)$

# DF of pion at $\zeta_5 = 5.2$ GeV




➢ Exploiting results from numerical simulations of lattice-regularised QCD, parameter-free predictions for pion valence, glue, and sea PDFs are obtained. Cui et al. Phys. Rev. D 105 (2022) 9, L091502

# Future facilities and experiments on pion DFs

- High Intensity, High Luminosity Facilities

  (1) Jefferson Lab at 22 GeV;

  (2) CERN: COMPASS++/AMBER;

  (3) Electron Ion Collider (EIC) in USA;

  (4) Electron-ion collider in China (EicC).

Jefferson Lab
Thomas Jefferson National Accelerator Facility

AMBER
A new QCD facility at the M2 beam line of the CERN SPS

# Future facilities and experiments on pion DFs: COMPASS++/AMBER at CERN

➢ AMBER at CERN - Apparatus for Meson and Baryon Experimental Research. Three phase-1 experiments:

✓ Proton charge-radius measurement using muon-proton elastic scattering

✓ Drell-Yan and J/ψ production experiments using the conventional M2 hadron beam

- The structure of the pion: determination of the pion valence and sea-quark distributions

✓ Measurement of proton-induced antiproton production cross sections for dark matter searches

| Program | Physics Goals | Beam Energy [GeV] | Beam Intensity [$s^{-1}$] | Trigger Rate [kHz] | Beam Type | Target | Earliest start time, duration | Hardware additions |
|---|---|---|---|---|---|---|---|---|
| muon-proton elastic scattering | Precision proton-radius measurement | 100 | $4 \cdot 10^6$ | 100 | $\mu^\pm$ | high-pressure H2 | 2022 1 year | active TPC, SciFi trigger, silicon veto, |
| Hard exclusive reactions | GPD $E$ | 160 | $2 \cdot 10^7$ | 10 | $\mu^\pm$ | $NH_3^\uparrow$ | 2022 2 years | recoil silicon, modified polarised target magnet |
| Input for Dark Matter Search | $\bar{p}$ production cross section | 20-280 | $5 \cdot 10^5$ | 25 | $p$ | LH2, LHe | 2022 1 month | liquid helium target |
| $\bar{p}$-induced spectroscopy | Heavy quark exotics | 12, 20 | $5 \cdot 10^7$ | 25 | $\bar{p}$ | LH2 | 2022 2 years | target spectrometer: tracking, calorimetry |
| Drell-Yan | Pion PDFs | 190 | $7 \cdot 10^7$ | 25 | $\pi^\pm$ | C/W | 2022 1-2 years | |
| Drell-Yan (RF) | Kaon PDFs & Nucleon TMDs | $\sim$100 | $10^8$ | 25-50 | $K^\pm, \bar{p}$ | $NH_3^\uparrow$, C/W | 2026 2-3 years | "active absorber", vertex detector |
| Primakoff (RF) | Kaon polarisa-bility & pion life time | $\sim$100 | $5 \cdot 10^6$ | $> 10$ | $K^-$ | Ni | non-exclusive 2026 1 year | |
| Prompt Photons (RF) | Meson gluon PDFs | $\geq 100$ | $5 \cdot 10^6$ | 10-100 | $K^\pm$ $\pi^\pm$ | LH2, Ni | non-exclusive 2026 1-2 years | hodoscope |
| $K$-induced Spectroscopy (RF) | High-precision strange-meson spectrum | 50-100 | $5 \cdot 10^6$ | 25 | $K^-$ | LH2 | 2026 1 year | recoil TOF, forward PID |
| Vector mesons (RF) | Spin Density Matrix Elements | 50-100 | $5 \cdot 10^6$ | 10-100 | $K^\pm, \pi^\pm$ | from H to Pb | 2026 1 year | |

Table 2: Requirements for future programmes at the M2 beam line after 2021. Muon beams are in blue, conventional hadron beams in green, and RF-separated hadron beams in red.

Letter of Intent: A New QCD facility at the M2 beam line of the CERN SPS (COMPASS++/ AMBER), arXiv:1808.00848 [hep-ex]. Proposal for Phase-1: COMPASS++/AMBER: Proposal for Measurements at the M2 beam line of the CERN SPS Phase-1: 2022-2024.

# DF of kaon

- ➢ Rainbow-Ladder (RL) is insufficient for reliably describing kaon structure.
- ➢ Kaon parton distribution amplitude:

Cui et al. *Eur. Phys. J. C* 80 (2020) 11, 1064; *Eur. Phys. J. A* 57 (2021) 1, 5.

| | $\langle u_\Delta^m \rangle$ | $m=1$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| CSMs | RL | 0.11 | 0.24 | 0.064 | 0.12 | 0.045 | 0.076 |
| | DB | 0.040 | 0.23 | 0.021 | 0.11 | 0.013 | 0.063 |
| Lattice-QCD | [40] | 0.027(2) | 0.26(2) | | | | |
| | [41] | 0.036(2) | 0.26(2) | | | | |
| | [10] | 0.036(2) | 0.26(2) | 0.020(2) | 0.13(2) | 0.014(2) | 0.085(15) |
| | [42–45] | 0.04(8) | | | | | |
| | [46] | 0.04(2) | 0.24(1) | | | | |
| | $\varphi = \varphi_{\mathrm{ms}}$ | 0.33 | 0.33 | 0.2 | 0.2 | 0.14 | 0.14 |
| | $\varphi = \varphi^{\mathrm{asy}}$ | 0 | 0.2 | 0 | 0.086 | 0 | 0.048 |

- ➢ CSMs with RL is both too asymmetric around x = 1/2 and too dilated.

# DF of kaon

➢ Kaon parton distribution amplitude

$$\varphi_K^u(x,\zeta_H) = \frac{1}{16\pi^3 f_K}\int d^2k_\perp \psi^{\uparrow\downarrow}(x,k_\perp^2;\zeta_H)$$

➢ Kaon parton distribution function

$$q_K^u(x,\zeta_H) = \int d^2k_\perp \,|\,\psi^{\uparrow\downarrow}(x,k_\perp^2;\zeta_H)\,|^2$$

➢ Factorised representation of light front wave function

$$\psi^{\uparrow\downarrow}(x,k_\perp^2;\zeta_H) = \varphi_K^u(x;\zeta_H)\,\psi^{\uparrow\downarrow}(k_\perp^2;\zeta_H)$$

➢ It is a good approximation to write

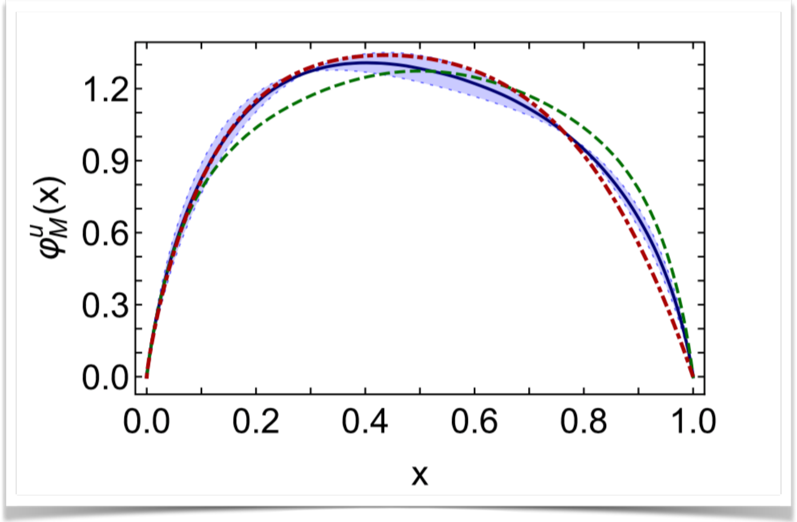
$$q_K^u(x,\zeta_H) \propto |\,\varphi_K^u(x;\zeta_H)\,|^2$$

# DF of kaon

➢ Starting point: kaon parton distribution amplitude Mellin moments

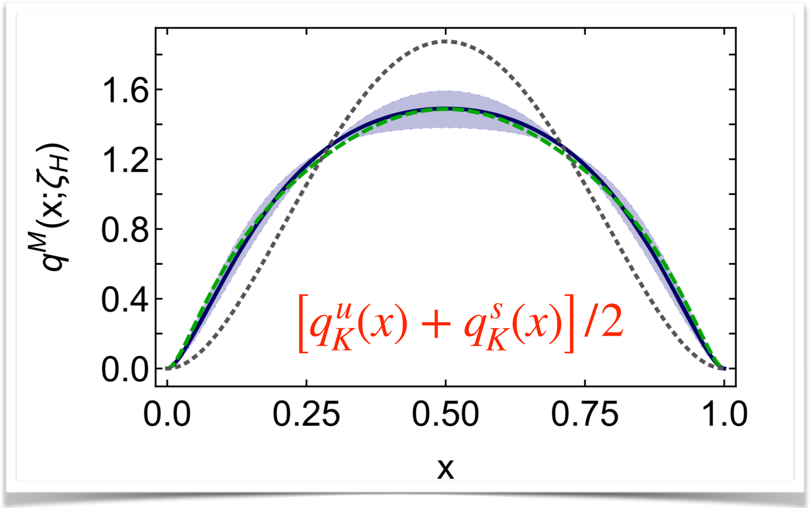
$$\langle \xi^n = (1-2x)^n \rangle_K^{u_\zeta} = \int_0^1 dx (1-2x)^n \varphi_K^u(x;\zeta) \qquad \langle [\xi, \xi^2] \rangle_K^{u_\zeta} = [0.035(5), 0.24(1)]$$

➢ Kaon parton distribution amplitude



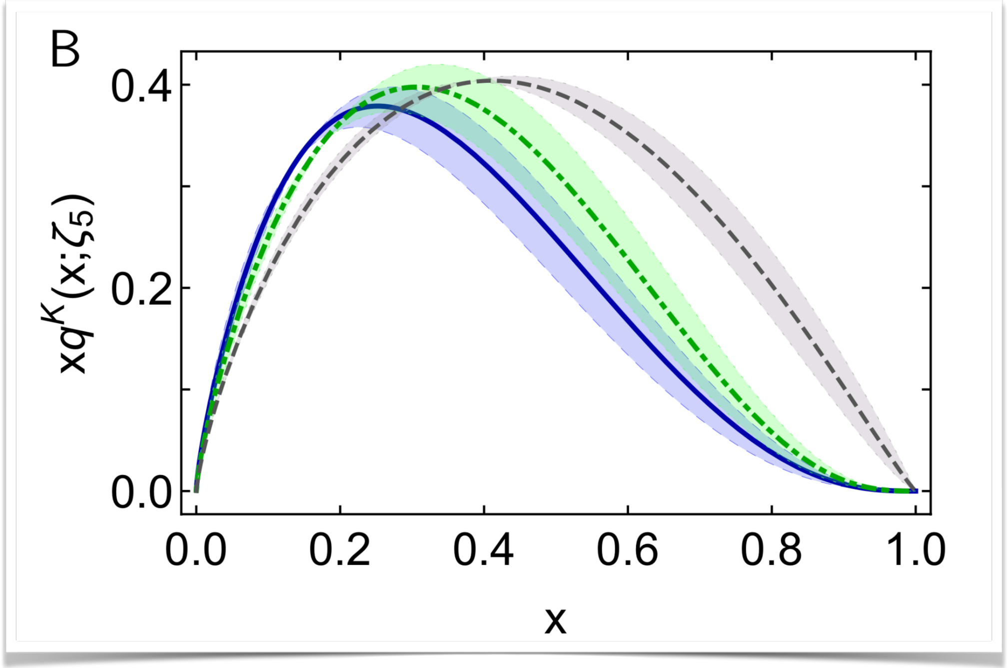

$$\varphi_K^u(x;\zeta_H) = 18.2x(1-x)$$
$$\left[1 + 5.0x^{\frac{0.0638}{2}}(1-x)^{\frac{0.0481}{2}} - 5.97x^{0.0638}(1-x)^{0.0481}\right]$$

➢ Kaon parton distribution function




$$q_K^u(x;\zeta_H) = 299.18x^2(1-x)^2$$
$$\left[1 + 5.0x^{\frac{0.0638}{2}}(1-x)^{\frac{0.0481}{2}} - 5.97x^{0.0638}(1-x)^{0.0481}\right]^2$$

# DF of kaon at $\zeta_5 = 5.2$ GeV




- Solid (blue) $u^K(x;\zeta_5)$,
- Dot- dashed (green), $s^K(x;\zeta_5)$,

$$q^K(x) = n_{q^\pi} x^\alpha (1-x)^\beta [1 + \rho x^{\alpha/4}(1-x)^{\beta/4} + \gamma x^{\alpha/2}(1-x)^{\beta/2}]$$

- Effective large-x exponent, $\beta_{eff}(\zeta_5) = 2.73(7)$
- Dashed (grey), Lattice QCD $\bar{s}$ DF

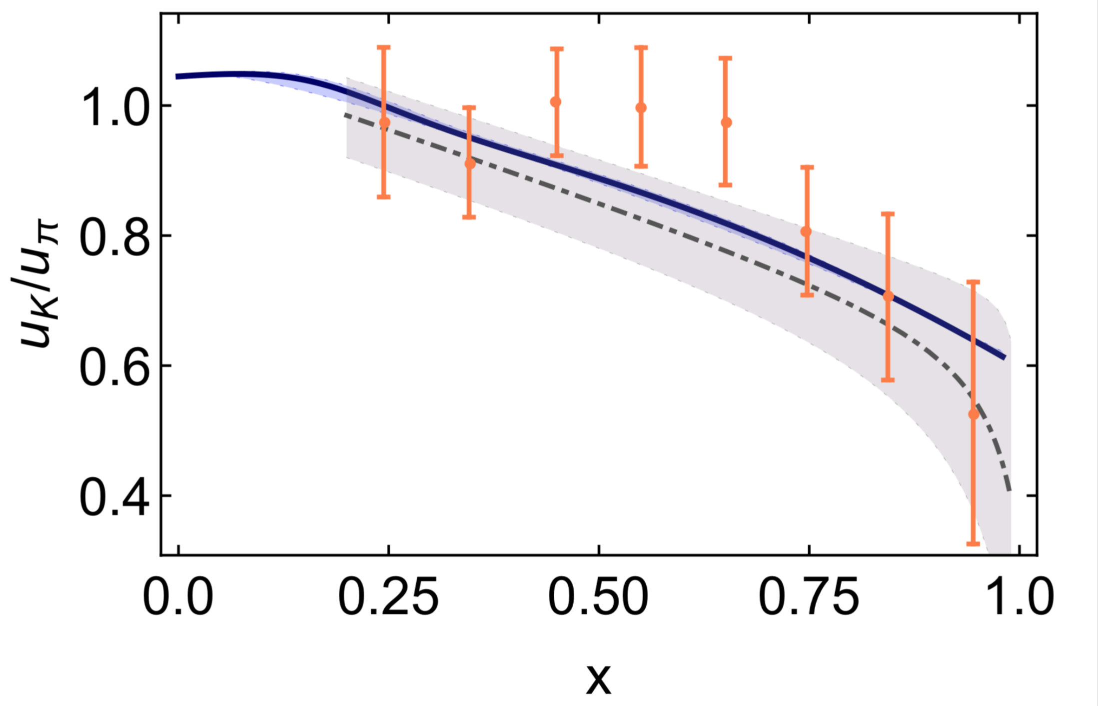
Lin, Chen, Fan, Zhang and Zhang, *Phys. Rev. D* 103 (2021) 1, 014516

- Low-order moments in comparison with Lattice QCD

| | $q\backslash\zeta_5$ | $\langle x q^K \rangle$ | $\langle x^2 q^K \rangle$ | $\langle x^3 q^K \rangle$ |
|---|---|---|---|---|
| | $u$ | 0.19(2) | 0.067(09) | 0.030(5) |
| | $\bar{s}$ | 0.22(2) | 0.081(11) | 0.038(7) |
| Lattice QCD | u | 0.193(8) | 0.080(7) | 0.042(6) |
| | $\bar{S}$ | 0.267(8) | 0.123(7) | 0.070(6) |

- Lattice QCD vs. CSMs, lattice QCD DF = much harder.
- Lattice QCD DF …$(1-x)^\beta$, $\beta = 1.13(16)$

# DF of kaon at $\zeta_5 = 5.2$ GeV

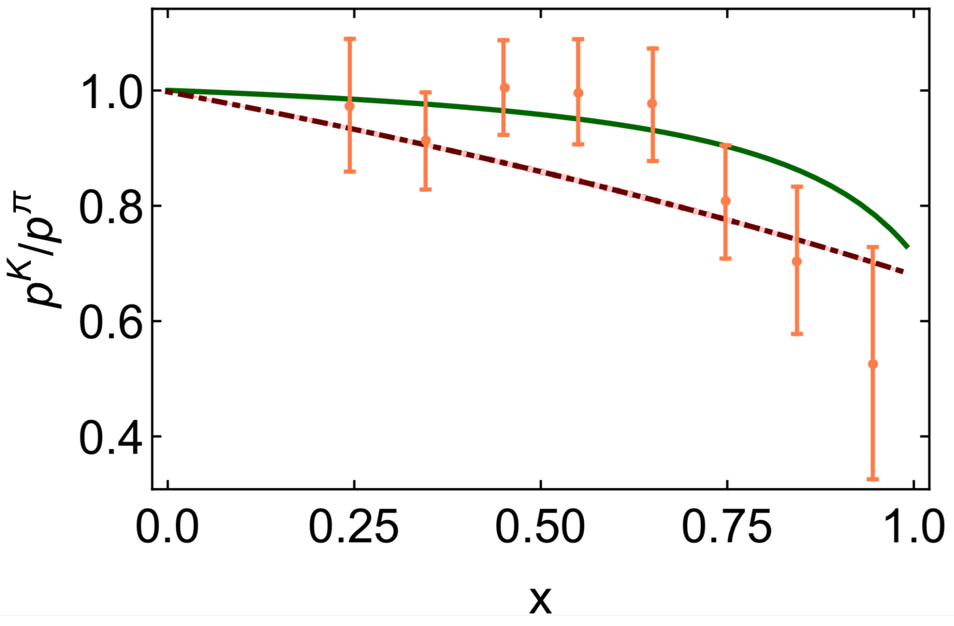



- Solid (blue), $u^K(x;\zeta_5)/u^\pi(x;\zeta_5)$.
- Dot-dashed (grey), Lattice QCD

  Lin, Chen, Fan, Zhang and Zhang, *Phys. Rev. D* 103 (2021) 1, 014516
- Experimental extraction (orange)

  J. Badier et al., Phys. Lett. B 93, 354 (1980).
- Relative difference between the central Lattice QCD result and CSMs result is ≈ 5% … despite fact that individual lQCD DFs are very different from continuum results.
- $u^K(x;\zeta_5)/u^\pi(x;\zeta_5)$ is very forgiving of even large differences between the individual DFs used to produce the ratio, results for $u^K(x;\zeta_5)$and $u^\pi(x;\zeta_5)$ separately have greater discriminating power.

# DF of kaon at $\zeta_5 = 5.2$ GeV

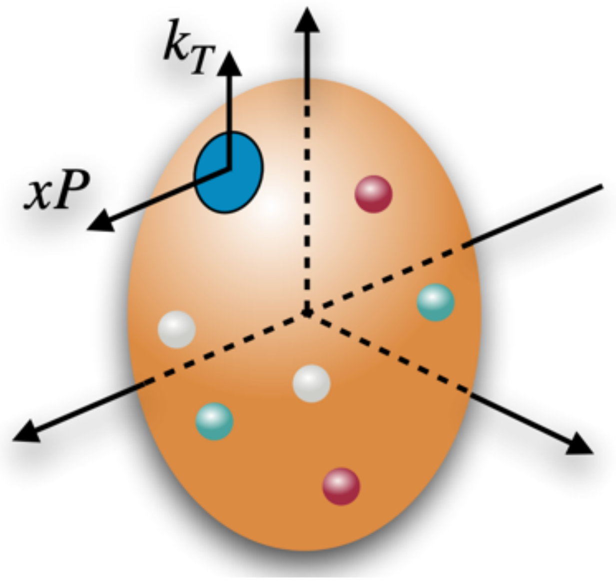


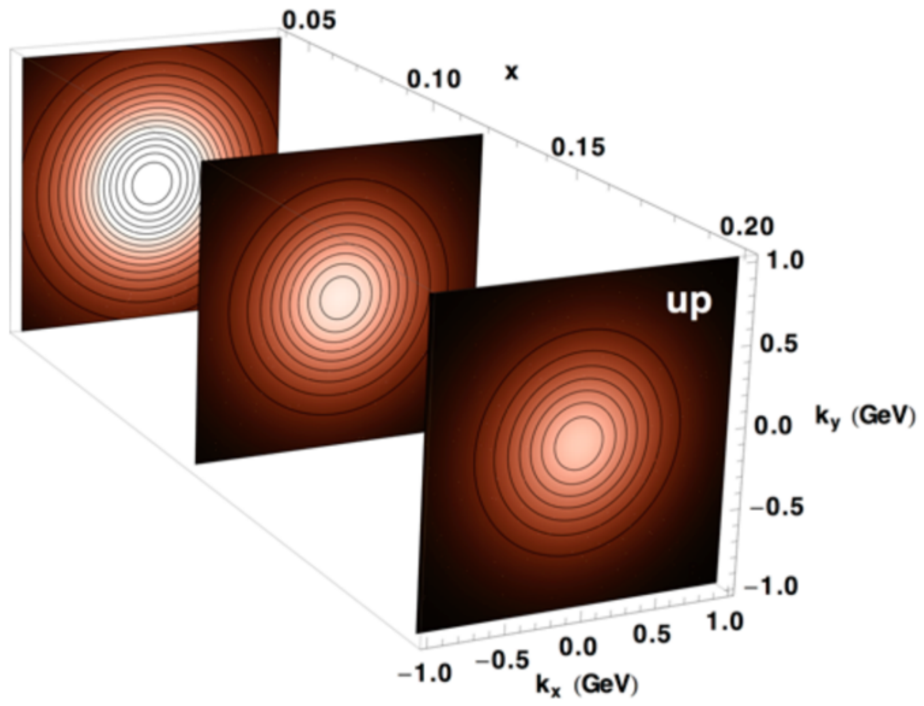

- Curious: each of the gluon and sea quark PDF ratios is point-wise similar to the experimentally extracted value of $u^K(x;\zeta_5)/u^\pi(x;\zeta_5)$ .

- Solid (green), $g^K(x;\zeta_5)/g^\pi(x,\zeta_5)$.
- Dot-dashed (red), $S^K(x;\zeta_5)/S^\pi(x,\zeta_5)$
- Experimental extraction (orange) $u^K(x;\zeta_5)/u^\pi(x;\zeta_5)$
- Kaon's gluon and sea distributions differ from those of the pion only on the valence region $x > 0.2$.
- Valence DFs are negligible at low-$x$, where gluon and sea distributions are large, and vice versa.
- The biggest impact of a change in the valence DFs must lie at large-$x$.
- First Moment $\langle x \rangle_g^K = 0.44(2)$ , $\langle x \rangle_{\text{sea}}^K = 0.14(2)$ .

# Transverse momentum dependent (TMD) distribution function

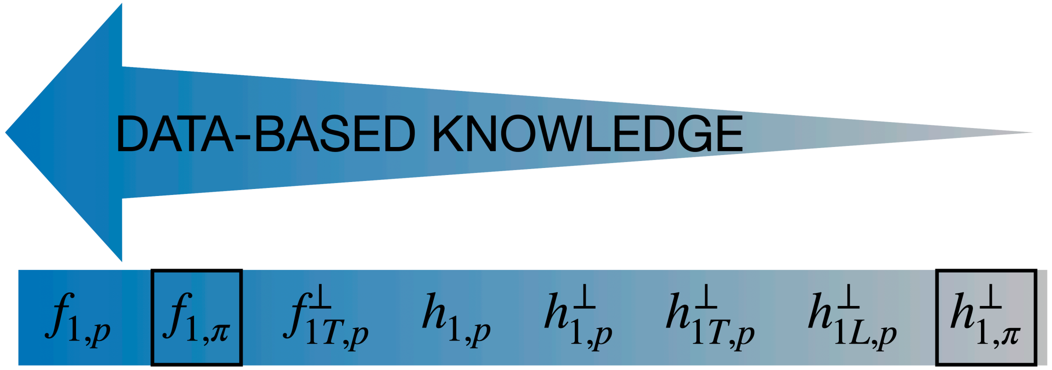
➢ Transverse momentum dependent (TMD) distribution function, longitudinal momentum x, transverse momentum $k_T$.



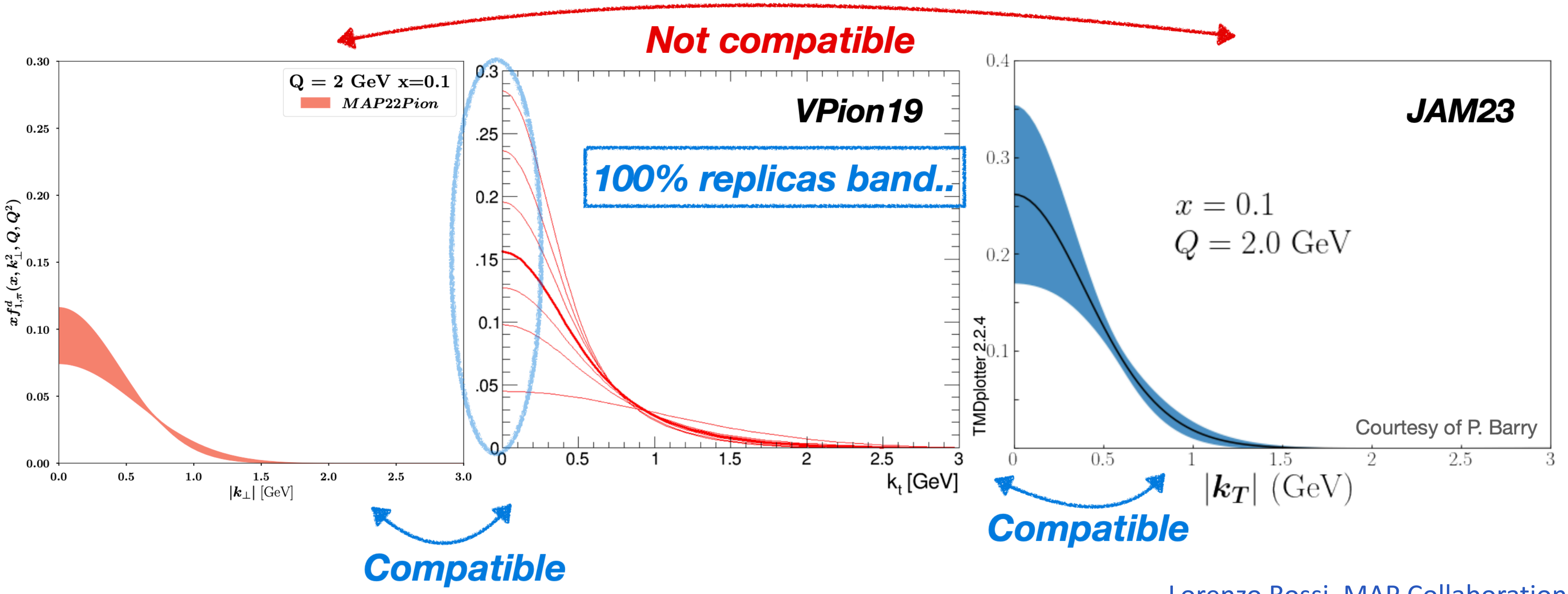



# TMDs of pion

➢ Data-based knowledge about the TMD of pion is limited.

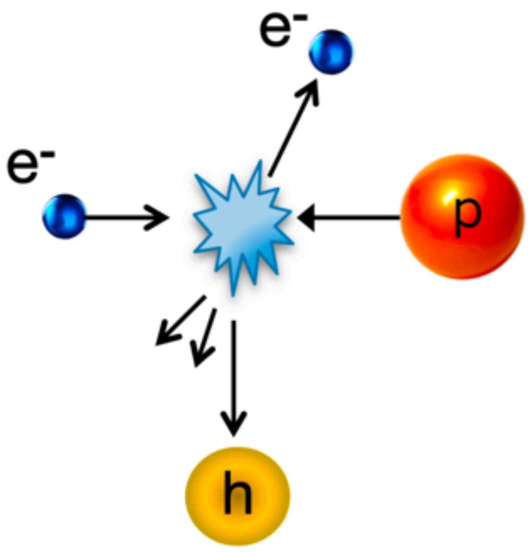



## Comparison of pion TMD PDFs




Lorenzo Rossi, MAP Collaboration (slide)

# TMDs of pion




- ➢ Semi-inclusive deep inelastic scattering (SIDIS)
- ➢ Transverse momentum dependent
- ➢ Wilson line, process-dependent, not universal

- ➢ Gauge-invariant TMD correlation function

Bacchetta, Trento lectures, 2012; Mulders, lectures at the Galileo Galilei Institute (2015).

$$\Phi(x,k_\perp) = \int \frac{d^3r}{8\pi^3} e^{-ixP^+r^- + ik_\perp\cdot r_\perp} \langle P|\bar{\psi}(0^+, r^- r_\perp)\widetilde{U^\dagger_{(+\infty;r)}}\widetilde{U_{(+\infty;0)}}\psi(0)|P\rangle\,,$$

$$\widetilde{U^\dagger_{(+\infty;r)}} = U^\perp_{(+\infty^-,+\infty_\perp;+\infty^-,0_\perp)} U^-_{(+\infty^-,0_\perp;0^-,0_\perp)}\,,$$

$$\widetilde{U_{(+\infty;0)}} = U^{-\dagger}_{(+\infty^-,r_\perp;r^-,r_\perp)} U^{\perp\dagger}_{(+\infty^-,+\infty_\perp;+\infty^-,r_\perp)}\,.$$

- ➢ $U^-$ and $U^{-\dagger}$: resummation of collinear gluons
- ➢ $U^\perp$and $U^{\perp\dagger}$: resummation of soft transversal gluons
- ➢ Choosing light-cone gauge, only transversal Wilson lines remain
- ➢ Choosing Feynman gauge, only longitudinal Wilson lines remain.

# TMDs of pion: Boer-Mulders function

- Working at leading twist one may in general rewrite quark + quark correlation function as:

$$\Phi_{q/\pi}(x, \mathbf{k}_\perp) = \frac{1}{2}\left[f^q_{1\pi}(x, k_\perp^2) i\gamma\cdot n + h^{q\perp}_{1\pi}(x, k_\perp^2)\frac{1}{M}\sigma_{\mu\nu}k_{\perp\mu}n_\nu\right],$$

- $f^q_{1\pi}(x, k_\perp^2)$ is helicity-independent TMD, $h^{q\perp}_{1\pi}(x, k_\perp^2)$ is Boer-Mulders function.
- M is a dynamically generated mass scale that is characteristic of EHM. A sensible choice for pion is $M \approx 4f_\pi$ where $f_\pi$ is the pion leptonic decay constant.
- It is plain that

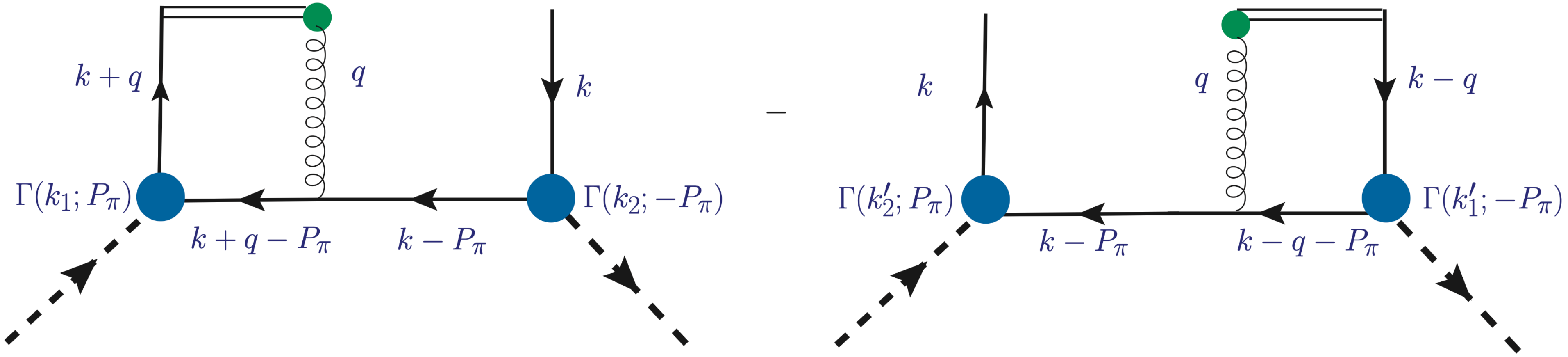
$$f^q_{1\pi}(x, k_\perp^2) = \mathrm{tr}\frac{1}{2} i\gamma\cdot\bar{n}\Phi_{q/\pi}(x, \mathbf{k}_T),$$

$$\frac{k_\perp^2}{M^2}h^{q\perp}_{1\pi}(x, k_\perp^2) = \mathrm{tr}\frac{1}{2M}\sigma_{\mu\nu}k_{\perp\mu}\bar{n}_\nu\Phi_{q/\pi}(x, \mathbf{k}_T).$$

- It is plain that many studies use $M \to m_\pi$, i.e., the pion mass; however, we judge this to be ill-advised because $m_\pi \to 0$ in the chiral limit whereas $f_\pi$ does not.

# TMDs of pion: Boer-Mulders function

- The helicity-independent TMD, $f_{1\pi}^{q}(x, k_\perp^2)$, is always nonzero; but if the gauge links are omitted, then the Boer-Mulders function vanishes.
- This is the signal that interactions between the spectator of the initial scattering event and the degree-of-freedom struck by the probe are crucial to obtaining $h_{1\pi}^{q\perp}(x, k_\perp^2) \neq 0$.



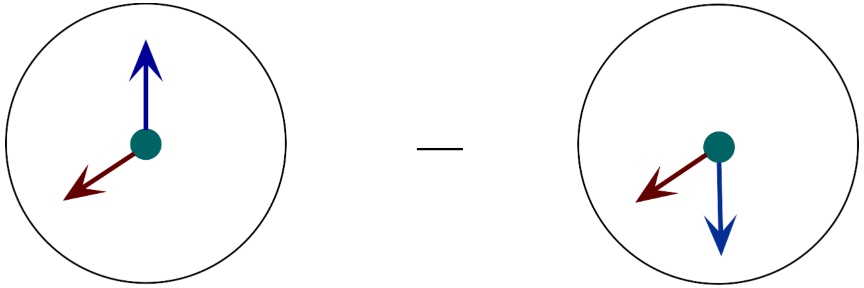

$$h_{1\pi}^{\perp}(x, k_\perp^2)\frac{\vec{k}_{\perp\alpha}}{M} = N_c \mathrm{tr}_\mathrm{D} \int \frac{d^4q}{(2\pi)^4}\frac{dk_3 dk_4}{(2\pi)^4} \delta_n^x(k) S(k-P_\pi)\Gamma(-P_\pi)S(k)\sigma_{\alpha+}\frac{[-gn_\mu]}{n\cdot q}$$
$$\times S(k+q)\Gamma(P_\pi)S(k+q-P_\pi)[-g\gamma_\nu]D_{\mu\nu}(q)\,.$$

# TMDs of pion: Boer-Mulders function

- The number density asymmetry corresponds to the following difference:

$$f_{q^{\uparrow}/\pi}(x,\vec{k}_{\perp}) - f_{q^{\downarrow}/\pi}(x,\vec{k}_{\perp}) = -\,h_{1\pi}^{q\perp}(x,k_{\perp}^2)\frac{|\vec{k}_{\perp}|}{M}\sin(\phi_{k_{\perp}} - \phi_S)\,,$$

  - ✓ where the azimuthal angles are measured between the indicated transverse vector and $\hat{P}$.



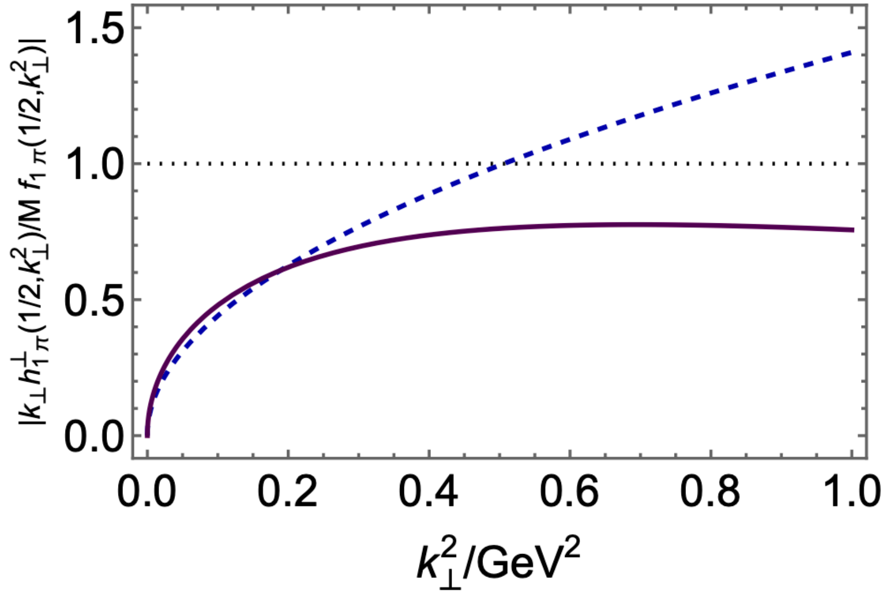
  - ✓ Number density interpretation of the Boer Mulders function. Vertical blue arrows - transverse polarisation of the quark; oblique red vectors - quark transverse momentum vector.

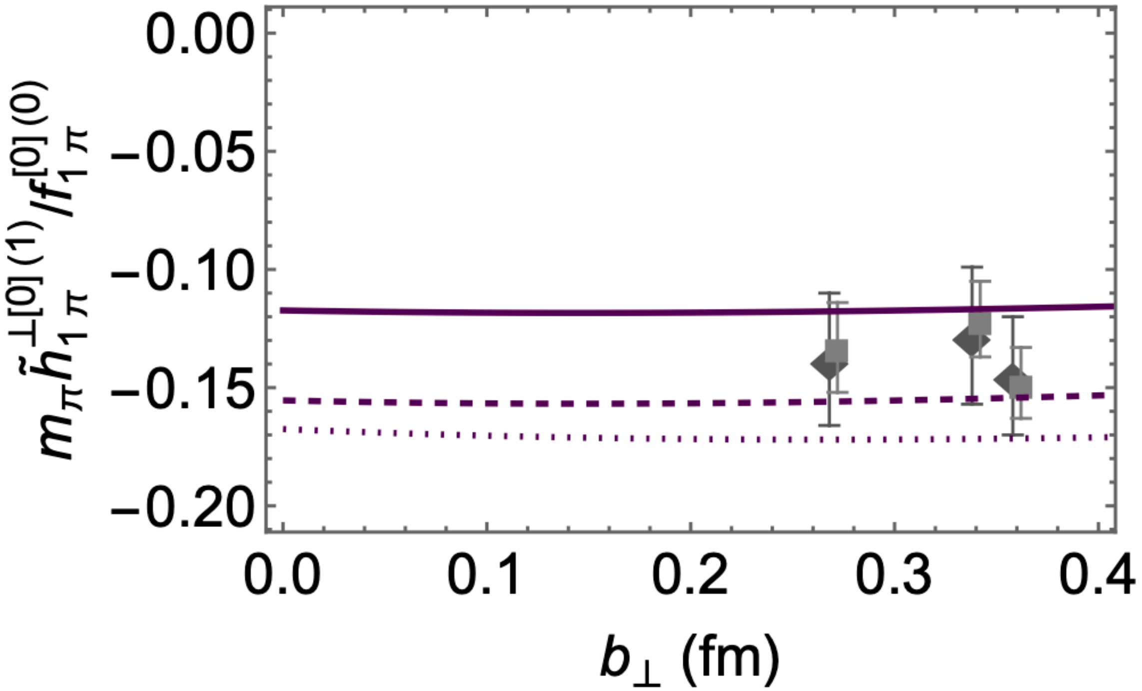
- Exploiting the requirements imposed by positivity of the defining matrix element, one arrives at the pointwise positivity bound:

$$|k_{\perp}h_{1\pi}^{q\perp}(x,k_{\perp}^2)/M| \le f_{1\pi}^{q}(x,k_{\perp}^2)$$

# TMDs of pion: Boer-Mulders function

Cheng et al, *Eur. Phys. J. C* 85 (2025) 1, 115




- Positivity constraint: the bound is violated if the curve crosses the horizontal dotted line. $|k_\perp h_{1\pi}^{\perp \mathrm{MI}}(x=1/2,k_\perp^2)/Mf_{1\pi}(x=1/2,k_\perp^2)|$ -- dashed blue cf. $|k_\perp h_{1\pi}^{\perp}(x=1/2,k_\perp^2)/Mf_{1\pi}(x=1/2,k_\perp^2)|$ solid purple.

- Generalised Boer Mulders shift. Dotted purple curve -- hadron scale physical pion mass result; dashed purple curve -- hadron scale heavy pion ( $m_\pi = 0.518$ GeV); and solid purple curve -- heavy pion result evolved $\zeta_{\mathscr{H}} \to \zeta_2$. LQCD results, obtained with $m_\pi = 0.518$ GeV at a resolving scale $\zeta = \zeta_2 = 2$ GeV.

# Summary and Outlook

➢Summary

✓Using a Continuum Schwinger Function Methods, presented a symmetry-preserving calculation of the pion and kaon DFs, as well as the pion Boer-Mulders function.

✓Pion: valence quark DF shows pointwise agreement with both Lattice QCD as well as experimental extraction; gluon and sea distributions, also align with Lattice QCD.

✓Kaon: ratio $u_K(x,\zeta_5)/u_\pi(x,\zeta_5)$ ratio agrees with recent Lattice QCD and experimental results; gluon and sea distributions, no Lattice QCD data available.

✓New-era experiments (JLAB at 22 GeV, COMPASS++/AMBER, EIC, and EicC) are capable of discriminating between the results from theoretical studies.

➢Outlook

Hadron structure, such as transverse momentum dependent distribution (TMD), generalized parton distribution (GPD), fragmentation function (FF), etc..

# Thank you!