Electrodynamics of vortices in quasi-two-dimensional scalar Bose-Einstein condensates

Seong-Ho Shinn (Seongho Shin)
University of Luxembourg, Department of Physics and Materials Science
Quantum Theory Group

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- For $i\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2M}\nabla^2+V(\boldsymbol{r},t)\right]\psi(\boldsymbol{r},t)$, with $[V(\boldsymbol{r},t)\psi(\boldsymbol{r},t)]^*=\psi^*(\boldsymbol{r},t)V(\boldsymbol{r},t)$.

$$\frac{\partial n(\boldsymbol{r},t)}{\partial t}+\nabla\cdot[n(\boldsymbol{r},t)\boldsymbol{v}_s(\boldsymbol{r},t)]=0,$$

$$M\frac{\partial\boldsymbol{v}_s(\boldsymbol{r},t)}{\partial t}=-\nabla\left[\Phi_Q(\boldsymbol{r},t)+V(\boldsymbol{r},t)+\frac{M}{2}v_s^2(\boldsymbol{r},t)\right].$$

E. Madelung, Z. Phys. **40**, 322 (1927).

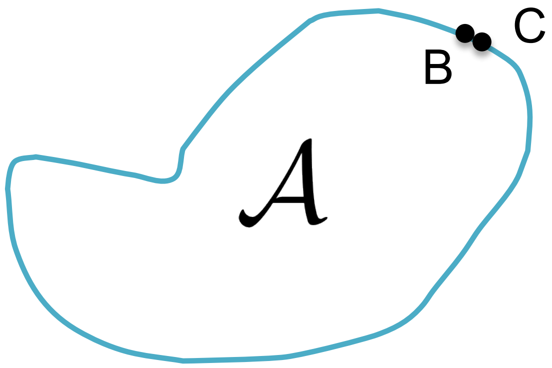
D. Bohm, Phys. Rev. **85**, 166 (1952).

T. Takabayasi, Prog. Theor. Phys. **8**, 143 (1952).

$$n(\boldsymbol{r},t):=|\psi(\boldsymbol{r},t)|^2,\qquad n(\boldsymbol{r},t)\boldsymbol{v}_s(\boldsymbol{r},t):=\boldsymbol{J}(\boldsymbol{r},t)=\frac{\hbar}{M}\mathrm{Im}\left[\psi^*(\boldsymbol{r},t)\nabla\psi(\boldsymbol{r},t)\right],$$

$$\Phi_Q(\boldsymbol{r},t):=-\frac{\hbar^2}{2M}\frac{\nabla^2\sqrt{n(\boldsymbol{r},t)}}{\sqrt{n(\boldsymbol{r},t)}}.$$

- Let us write $\psi(\boldsymbol{r},t) = \sqrt{n(\boldsymbol{r},t)}e^{i\phi(\boldsymbol{r},t)}.$ $\Rightarrow$ $\boldsymbol{v}_s(\boldsymbol{r},t) = \frac{\hbar}{M}\nabla\phi(\boldsymbol{r},t).$



$$\psi(\boldsymbol{r}_B,t) = \psi(\boldsymbol{r}_C,t).$$

$$\Rightarrow \quad \phi(\boldsymbol{r}_C,t) - \phi(\boldsymbol{r}_B,t) = 2\pi q, \quad q \in \mathbb{Z}.$$

$$\Rightarrow \quad \oint_{\partial\mathcal{A}} d\boldsymbol{l} \cdot \boldsymbol{v}_s(\boldsymbol{r},t) = \frac{2\pi\hbar}{M}q.$$

$$\int_{\mathcal{A}} d^2r\, [\nabla \times \boldsymbol{v}_s(\boldsymbol{r},t)] \cdot \boldsymbol{e}_\perp = \frac{2\pi\hbar}{M}\sum_{j=1}^{N_v(\mathcal{A};t)} q_j, \quad q_j \in \mathbb{Z}.$$

- For a nonrotating 3D liquid ^{4}He without dissipation, vortices can be mapped to electrodynamics (under the limit where $n(\boldsymbol{r},t)$ is constant outside vortices + perturbation)

B. I. Halperin, *Superfluidity, melting, and liquid-crystal phases in two dimensions* (1979)

V. Ambegokar et al., Phys. Rev. B **21**, 1806 (1980)

H. Kleinert, *Gauge Fields in Condensed Matter* (1989)

H. Kleinert, *Multivalued Fields* (2008)

- For a nonrotating scalar Bose-Einstein condensate (BEC) without dissipation, a similar mapping exists under the following conditions:
  - (1) When $n(\boldsymbol{r},t)$ is approximately constant

3D system: U. R. Fischer, Ann. Phys. **278**, 62 (1999)

Quasi-2D system: T. Simula, Phys. Rev. A **101**, 063616 (2020)

  - (2) When $n(\boldsymbol{r},t)$ is time-independent (it may change in space)

Quasi-2D system: E. G. Johansen, arXiv:2405.18090

- From $n(\boldsymbol{r},t)\nabla\times\boldsymbol{J}(\boldsymbol{r},t)$ and $[\nabla n(\boldsymbol{r},t)]\times\boldsymbol{J}(\boldsymbol{r},t)\,,\quad n^2(\boldsymbol{r},t)\nabla\times\boldsymbol{v}_s(\boldsymbol{r},t)=0.$

  ➡ At the vortex core, $n(\boldsymbol{r},t)=0.$

  ➡ Those mappings hold for static vortices (vortex motions are negligible)

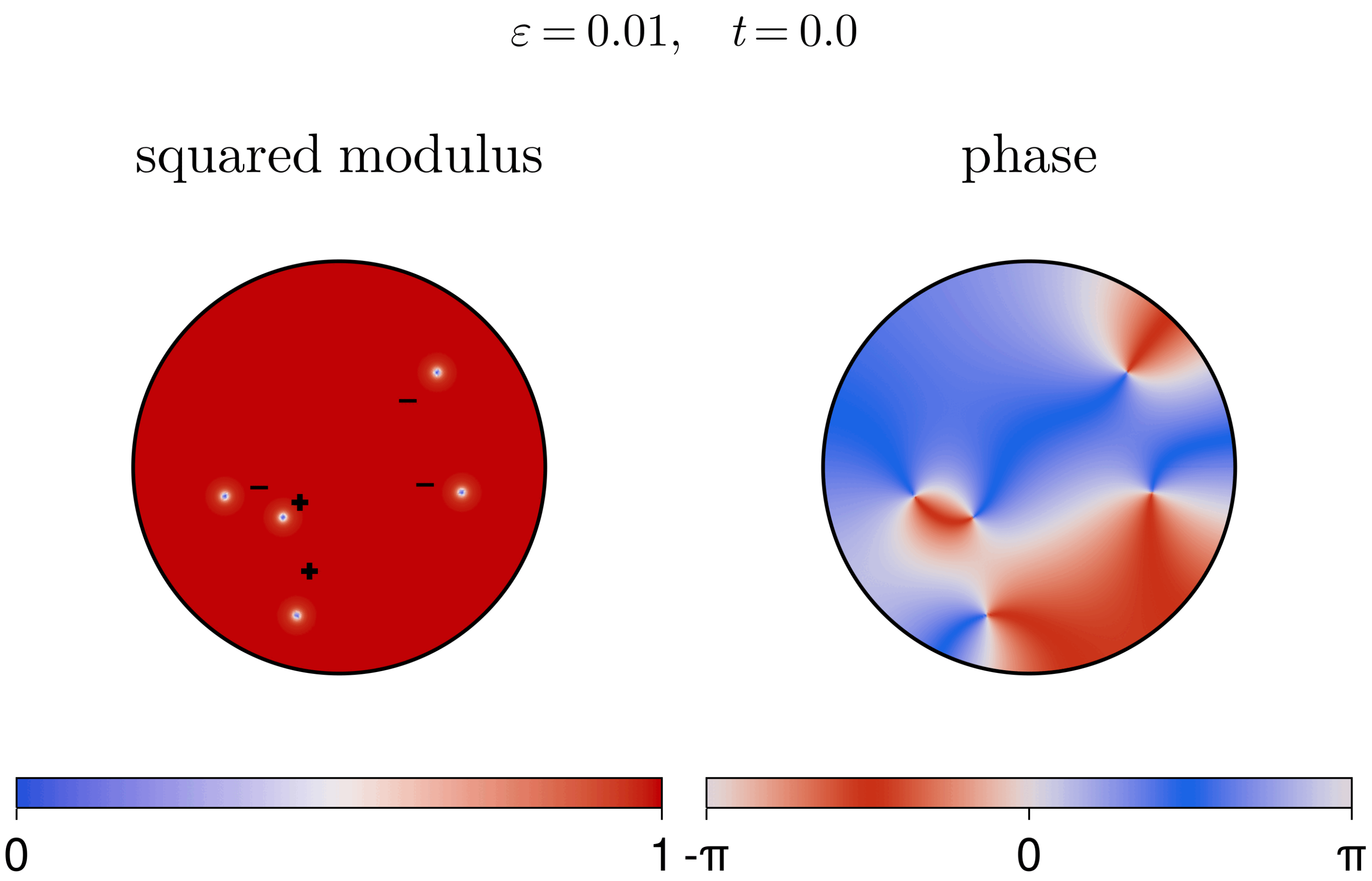
  ➡ Vortex interaction $V_{v,int}(r_1,r_2)\propto q_1q_2\ln(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\,.$

- In the point-vortex model (PVM), for nonrotating dissipationless quasi-2D scalar BEC, vortices do not collide

P. J. Torres, et al., Commun. Pure Appl. Math. **10**, 1589 (2011).

A. Richaud, V. Penna, and A. L. Fetter, Phys. Rev. A **103**, 023311 (2021).

T. C. Corso et al., arXiv:2404.02133.

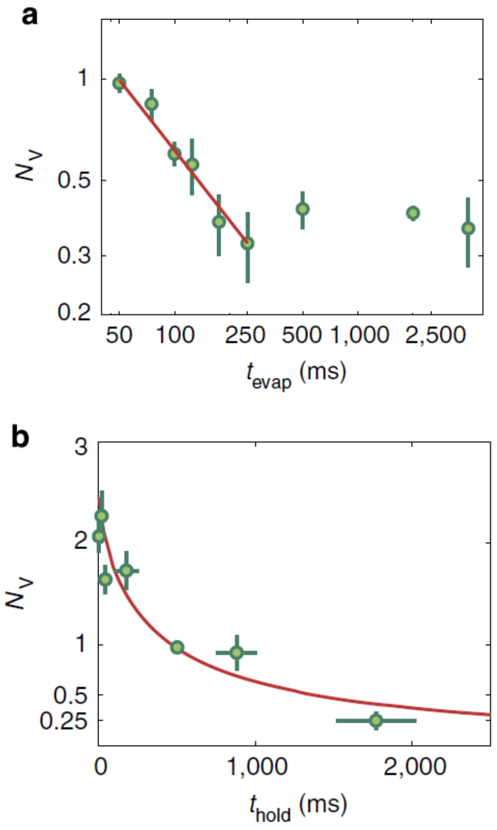


$$i\frac{\partial \psi(\boldsymbol{r},t)}{\partial t} = -\nabla^2 \psi(\boldsymbol{r},t) + \frac{1}{\epsilon^2}\left(1 - |\psi(\boldsymbol{r},t)|\right)^2 .$$

Gif from Gaspard Kemlin's homepage

https://gkemlin.pages.math.cnrs.fr/research/

- In general, the number of vortices is not conserved.
  - Phase transition



Left figures are from L. Chomaz et al., Nat. Commun. **6**, 6162 (2015).

Vortex number growth can be explained via the Kibble-Zurek mechanism

A. del Campo and W. H. Zurek, Int. J. Mod. Phys. A **29**, 1430018 (2014).

Vortex number decay might be due to coarsening

G. Biroli, L. F. Cugliandolo, and A. Sicilia, Phys. Rev. E **81**, 05101 (2010).

P. M. Chesler, A. M. García-García, and H. Liu, Phys. Rex. X **5**, 021015 (2015).

- SPGPE simulation for a nonrotating quasi-2D scalar BEC

- Vortices may collide in the damped-PVM, where vortices move with velocity

$$\boldsymbol{v}_P\left(\boldsymbol{r},t\right)=\boldsymbol{v}_s\left(\boldsymbol{r},t\right)-q\Gamma\boldsymbol{e}_\perp\times\boldsymbol{v}_s\left(\boldsymbol{r},t\right).$$

S. Rica and E. Tirapegui, Phys. Rev. Lett. **64**, 878 (1990).

Z. Mehdi et al., Phys. Rev. Res. **5**, 013184 (2023).

- Rica and Tirapegui started from the Ginzburg-Landau equation in the 2D system with

$$\frac{\partial A\left(\boldsymbol{r},t\right)}{\partial t}=\mu A\left(\boldsymbol{r},t\right)+\left(1+i\alpha\right)\nabla^2 A\left(\boldsymbol{r},t\right)-\left(1+i\beta\right)\left|A\left(\boldsymbol{r},t\right)\right|^2 A\left(\boldsymbol{r},t\right).$$

- Mehdi et al. started from the SPGPE in the quasi-2D scalar BEC

1. Introduction

- Would the mapping between vortices in quasi-2D scalar BEC and electrodynamics exist in the system with dissipation or under rotation?

⇒ Yes!

S.-H. Shinn and A. del Campo, Phys. Rev. Res. **7**, 013217 (2025).

- (Starting point) The mean-field wavefunction $\psi(\boldsymbol{r},t)$ satisfies

$$i\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}=H(\boldsymbol{r},t)\,\psi(\boldsymbol{r},t)\,,\quad H^*(\boldsymbol{r},t)\neq H(\boldsymbol{r},t)\ \text{in general.}$$

$$H_{GPE}(\boldsymbol{r},t)=-\frac{\hbar^2}{2M}\nabla^2+V_{tr}(\boldsymbol{r},t)+gn(\boldsymbol{r},t)-\mu(t)\,,\quad H^*_{GPE}(\boldsymbol{r},t)=H_{GPE}(\boldsymbol{r},t)\,.$$

L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (2003).

$$H_{d,GPE}(\boldsymbol{r},t)=(1-i\gamma)\,H_{GPE},\quad H^*_{d,GPE}(\boldsymbol{r},t)\neq H_{d,GPE}(\boldsymbol{r},t)\,.$$

L. P. Pitaevskii, ZhETF **35**, 408 (1959) [Sov. Phys. JETP **8**, 282 (1959)].

$$\frac{\partial n(\boldsymbol{r},t)}{\partial t} + \nabla \cdot [n(\boldsymbol{r},t)\,\boldsymbol{v}_s(\boldsymbol{r},t)] = G(\boldsymbol{r},t)$$

$$U_{\mathrm{sf}}(\boldsymbol{r},t) := V_{tr}(\boldsymbol{r},t) + gn(\boldsymbol{r},t) - \mu(t)\,.$$

$$M\frac{\partial \boldsymbol{v}_s(\boldsymbol{r},t)}{\partial t} = \boldsymbol{f}_K(\boldsymbol{r},t) - \nabla U_{\mathrm{sf}}(\boldsymbol{r},t) + \boldsymbol{F}_{\mathrm{sf}}(\boldsymbol{r},t)$$

$$\boldsymbol{f}_K(\boldsymbol{r},t) := -\nabla\left[\Phi_Q(\boldsymbol{r},t) + \frac{M}{2}v_s^2(\boldsymbol{r},t)\right].$$

- From the vortex quantization condition $\oint_{\partial\mathcal{A}} d\boldsymbol{l}\cdot\boldsymbol{v}_s(\boldsymbol{r},t) = \frac{2\pi\hbar}{M}\sum_{j=1}^{N_v(\mathcal{A};t)} q_j(t)\,,$

$$\nabla\times[\boldsymbol{f}_K(\boldsymbol{r},t) + \boldsymbol{F}_{\mathrm{sf}}(\boldsymbol{r},t)] = 2\pi\hbar\boldsymbol{e}_\perp\frac{\partial\rho_v(\boldsymbol{r},t)}{\partial t}, \qquad \int_{\mathcal{A}} d^2r\,\rho_v(\boldsymbol{r},t) = \sum_{j=1}^{N_v(\mathcal{A};t)} q_j(t)\,.$$

- As vortices may collide, we need (at least) two different velocities

Introduce the effective electric field, effective electric displacement field, and effective polarization density

$$\boldsymbol{E}_{\mathrm{sf}}(\boldsymbol{r},t) := \frac{M}{2\pi\hbar\epsilon_{\mathrm{sf}}}\boldsymbol{v}_P(\boldsymbol{r},t)\times\boldsymbol{e}_{\perp}, \quad \boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t) := \frac{M}{2\pi\hbar}\boldsymbol{v}_s(\boldsymbol{r},t)\times\boldsymbol{e}_{\perp}, \quad \boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t) := \boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t)-\epsilon_{\mathrm{sf}}\boldsymbol{E}_{\mathrm{sf}}(\boldsymbol{r},t)\,.$$

$$\nabla\cdot\boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t) = \rho_v(\boldsymbol{r},t) - \sum_{j=1}^{\infty} q_j(t)\,\delta(\boldsymbol{r}-\boldsymbol{r}_j(t)), \quad \nabla\cdot\boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t) = \rho_v(\boldsymbol{r},t)\,.$$

- In the PVM, the necessary condition is $\nabla\cdot\boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t) = 0$.

- From the vortex quantization condition, $\nabla \times [\boldsymbol{f}_K(\boldsymbol{r},t) + \boldsymbol{F}_{\mathrm{sf}}(\boldsymbol{r},t)] = 2\pi\hbar \boldsymbol{e}_\perp \frac{\partial \rho_v(\boldsymbol{r},t)}{\partial t}$.

$$\frac{\partial \rho_v(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{J}_{\mathrm{sf}}(\boldsymbol{r},t) = 0. \quad \Rightarrow \quad \boldsymbol{J}_{\mathrm{sf}}(\boldsymbol{r},t) := \frac{\boldsymbol{e}_\perp}{2\pi\hbar} \times [\boldsymbol{f}_K(\boldsymbol{r},t) + \boldsymbol{F}_{\mathrm{sf}}(\boldsymbol{r},t)] - \frac{\partial \boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t)}{\partial t}.$$

$$\nabla \times \boldsymbol{H}_{\mathrm{sf}}(\boldsymbol{r},t) = \boldsymbol{J}_{\mathrm{sf}}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t)}{\partial t}.$$

$$\Rightarrow \quad \boldsymbol{H}_{\mathrm{sf}}(\boldsymbol{r},t) := -\frac{U_{\mathrm{sf}}(\boldsymbol{r},t) - \bar{U}_{\mathrm{sf}}(t)}{2\pi\hbar}\boldsymbol{e}_\perp, \qquad \bar{U}_{\mathrm{sf}}(t) := \frac{1}{|\mathcal{A}|}\int_{\mathcal{A}} d^2r\, U_{\mathrm{sf}}(\boldsymbol{r},t).$$

$$\Rightarrow \quad \nabla \cdot \boldsymbol{H}_{\mathrm{sf}}(\boldsymbol{r},t) = 0.$$

$$\nabla \times \boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t) = -\frac{1}{c_{\mathrm{sf}}^2}\left[\boldsymbol{J}_{m,\mathrm{sf}}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{H}_{\mathrm{sf}}(\boldsymbol{r},t)}{\partial t}\right].$$

$$\Rightarrow \quad \boldsymbol{J}_{m,\mathrm{sf}}(\boldsymbol{r},t) := c_{\mathrm{sf}}^2\left[\frac{M}{2\pi\hbar}\boldsymbol{e}_\perp \nabla \cdot \boldsymbol{v}_P(\boldsymbol{r},t) - \nabla \times \boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t)\right] + \frac{\boldsymbol{e}_\perp}{2\pi\hbar}\frac{\partial}{\partial t}\left[U_{\mathrm{sf}}(\boldsymbol{r},t) - \bar{U}_{\mathrm{sf}}(t)\right].$$

$$\Rightarrow \quad \nabla \cdot \boldsymbol{J}_{m,\mathrm{sf}}(\boldsymbol{r},t) = 0.$$

For more details, please refer to
S.-H. Shinn and A. del Campo,
Phys. Rev. Res. **7**, 013217 (2025).

- The effective Poynting vector is

$$\boldsymbol{S}_{\mathrm{sf}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{sf}}(\boldsymbol{r},t) \times \boldsymbol{H}_{\mathrm{sf}}(\boldsymbol{r},t) = \frac{M}{(2\pi\hbar)^2 \epsilon_{\mathrm{sf}}}\left[U_{\mathrm{sf}}(\boldsymbol{r},t) - \bar{U}_{\mathrm{sf}}(t)\right] \boldsymbol{v}_P(\boldsymbol{r},t).$$

velocity of the vortex!

- From the GPE (no dissipation), $\nabla \cdot \boldsymbol{v}_s(\boldsymbol{r},t) = 0.$

H. Kleinert, *Gauge Fields in Condensed Matter*, 1989.

- If the dissipation is small, $\nabla \cdot \boldsymbol{v}_s(\boldsymbol{r},t) \sim 0.$

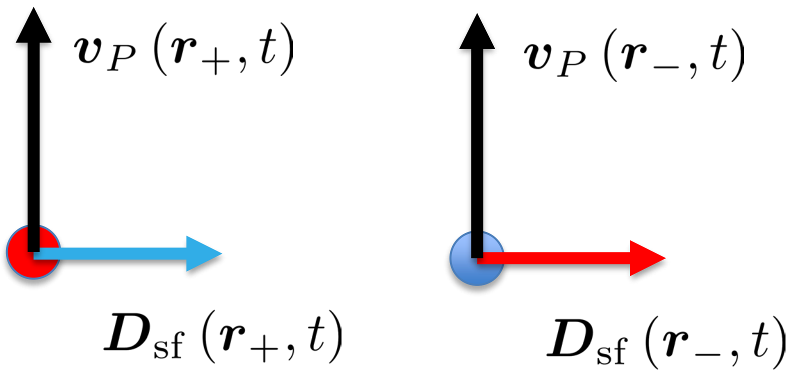
- In the PVM, $\nabla \cdot \boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t) = 0.$ ⇒ $\boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t) \sim c_1(t)\,\boldsymbol{v}_s(\boldsymbol{r},t)\,.$

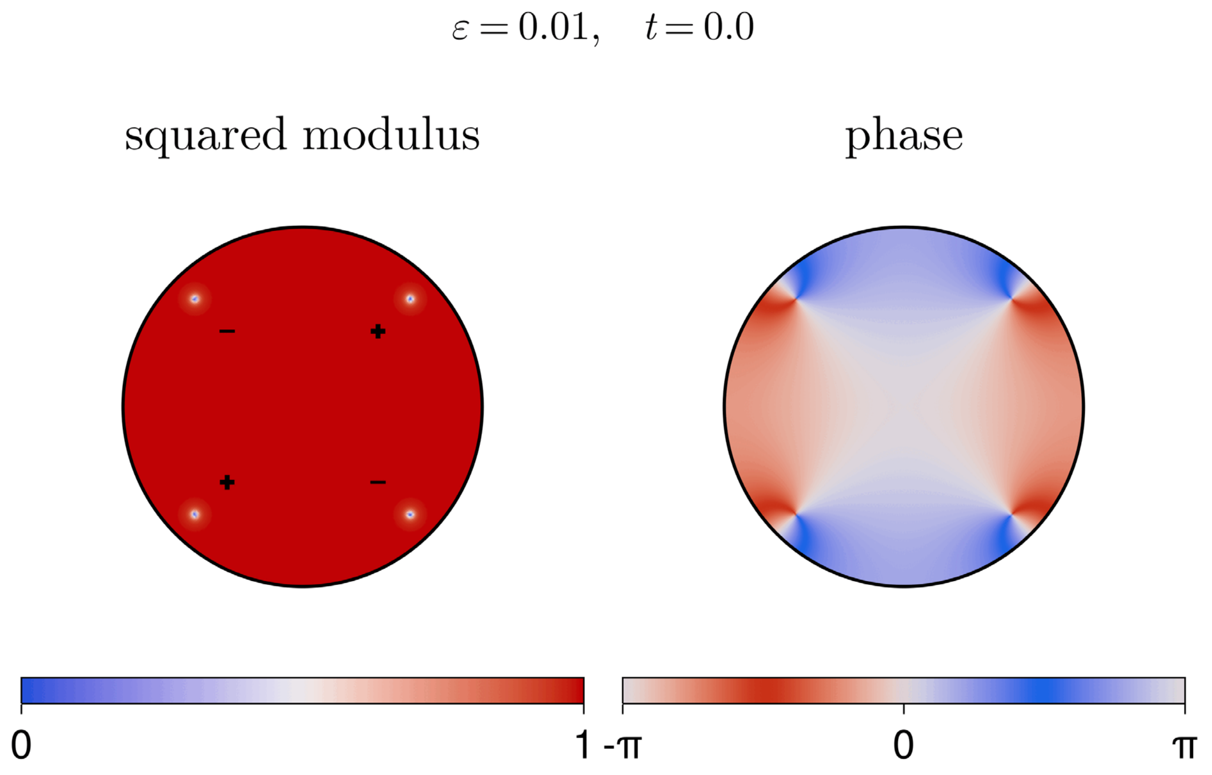
$$\boldsymbol{v}_P(\boldsymbol{r},t) \sim \boldsymbol{v}_s(\boldsymbol{r},t) - \frac{2\pi\hbar}{M} c_1(t)\,\boldsymbol{e}_\perp \times \boldsymbol{v}_s(\boldsymbol{r},t)\,.$$

- The general motion can be obtained from the effective Lorentz force

$$\boldsymbol{f}_v(\boldsymbol{r},t) := \rho_v(\boldsymbol{r},t)\,\boldsymbol{E}_{\mathrm{sf}}(\boldsymbol{r},t) + \boldsymbol{J}_{\mathrm{sf}}(\boldsymbol{r},t) \times \boldsymbol{B}_{\mathrm{sf}}(\boldsymbol{r},t)\,, \quad \boldsymbol{B}_{\mathrm{sf}}(\boldsymbol{r},t) := \mu_{\mathrm{sf}}\left[\boldsymbol{H}_{\mathrm{sf}}(\boldsymbol{r},t) + M_{\mathrm{sf}}(\boldsymbol{r},t)\,\boldsymbol{e}_\perp\right].$$

- In the PVM, $\boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t)=\epsilon_{\mathrm{sf}}\boldsymbol{E}_{\mathrm{sf}}(\boldsymbol{r},t)\,,\quad \boldsymbol{v}_P(\boldsymbol{r},t)=\frac{2\pi\hbar}{M}\boldsymbol{e}_\perp\times\boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t)\,.$





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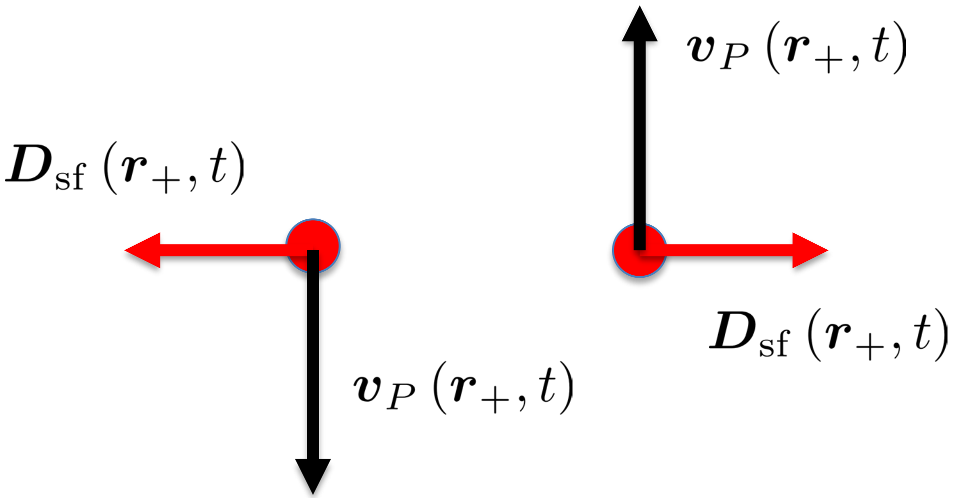
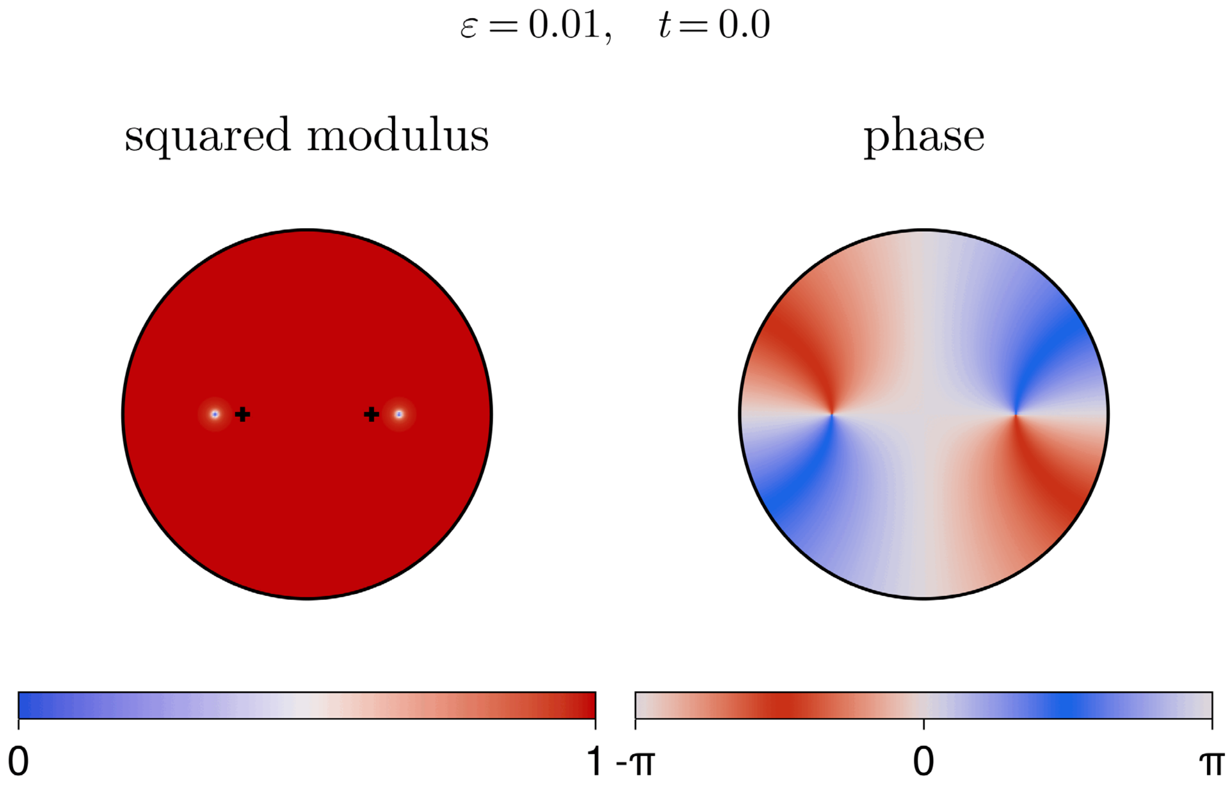
3. Applications – Vortex stability in a box trap

- In the PVM, $\boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t)=\epsilon_{\mathrm{sf}}\boldsymbol{E}_{\mathrm{sf}}(\boldsymbol{r},t)\,, \quad \boldsymbol{v}_P(\boldsymbol{r},t)=\frac{2\pi\hbar}{M}\boldsymbol{e}_{\perp}\times\boldsymbol{D}_{\mathrm{sf}}(\boldsymbol{r},t)\,.$





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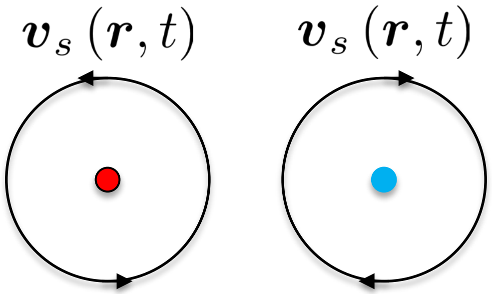
https://gkemlin.pages.math.cnrs.fr/research/

- The effective Poynting vector satisfies $\boldsymbol{S}_{\mathrm{sf}}(\boldsymbol{r},t) \propto \left[U_{\mathrm{sf}}(\boldsymbol{r},t) - \bar{U}_{\mathrm{sf}}(t)\right] \boldsymbol{v}_P(\boldsymbol{r},t)\,.$
  - $U_{\mathrm{sf}}(\boldsymbol{r},t) = gn(\boldsymbol{r},t) - \mu(t)\,.$

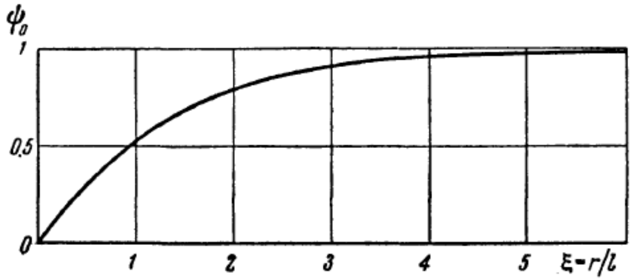
    ➡ No effective radiation outside vortices as $U_{\mathrm{sf}}(\boldsymbol{r},t) \sim \bar{U}_{\mathrm{sf}}(t)$.
  - In the PVM, $\boldsymbol{v}_P(\boldsymbol{r},t) = \boldsymbol{v}_s(\boldsymbol{r},t)\,.$

$$\oint d\boldsymbol{l} \cdot \boldsymbol{v}_s(\boldsymbol{r},t) = \frac{2\pi\hbar}{M} q.$$

➡



The result for n = 1 is given in the figure.



V. L. Ginzburg and L. P. Pitaevskii, ZhETP **34**, 1240 (1958) [Sov. Phys. JETP **7**, 858 (1958)].

  - Vortices cannot collide in the PVM

    ➡ Effective radiation around any infinitesimal closed curve around the vortex is zero

    No effective radiation. Vortices do not lose energy!

3. Applications – Vortex stability in a box trap

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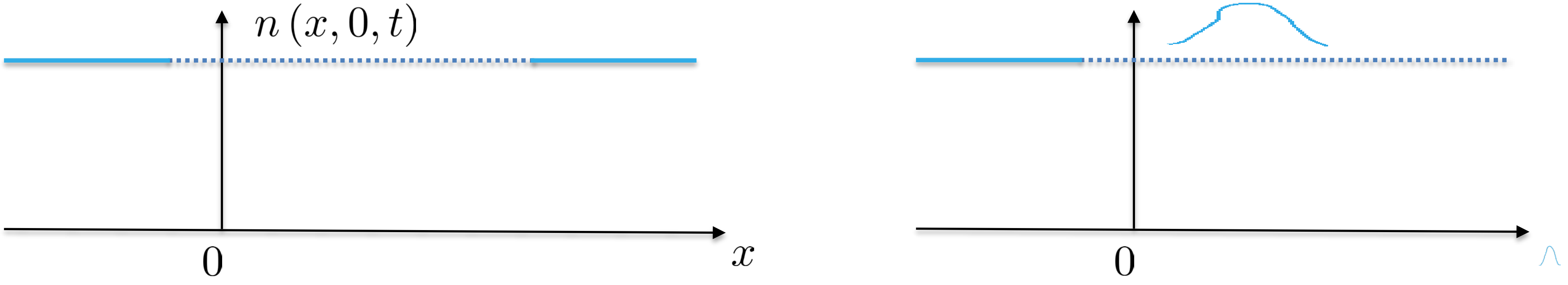
- The effective Poynting vector satisfies $\boldsymbol{S}_{\mathrm{sf}}(\boldsymbol{r},t) \propto \left[U_{\mathrm{sf}}(\boldsymbol{r},t) - \bar{U}_{\mathrm{sf}}(t)\right] \boldsymbol{v}_P(\boldsymbol{r},t)$.
  - $U_{\mathrm{sf}}(\boldsymbol{r},t) = gn(\boldsymbol{r},t) - \mu(t)$.

    ➡ No effective radiation outside vortices as $U_{\mathrm{sf}}(\boldsymbol{r},t) \sim \bar{U}_{\mathrm{sf}}(t)$.

  - In the damped PVM or other models where vortices can collide, $\boldsymbol{v}_P(\boldsymbol{r},t) = \boldsymbol{v}_s(\boldsymbol{r},t) - \frac{2\pi\hbar}{M}\boldsymbol{e}_\perp \times \boldsymbol{P}_{\mathrm{sf}}(\boldsymbol{r},t)$.



➡ Effective radiation around any infinitesimal closed curve around the vortex may not not zero

Effective radiation occurs when vortices are about to collide. Vortices lose energy!

3. Applications – Vortex stability in a box trap

- It is known that the phonon emission plays a role in the annihilation/creation of vortices

E. Kozik and B. Svistunov, Phys. Rev. B **72**, 172505 (2005).

W. J. Kwon et al., Nature (London) **600**, 64 (2021).

➡ Phonons are emitted when the effective radiation occurs

- For vortices created by the phase transition, their initial spacing distribution follows the Poisson point process (PPP).

A. del Campo, F. J. Gómez-Ruiz, and H.-Q. Zhang, Phys. Rev. B **106**, L140101 (2022).

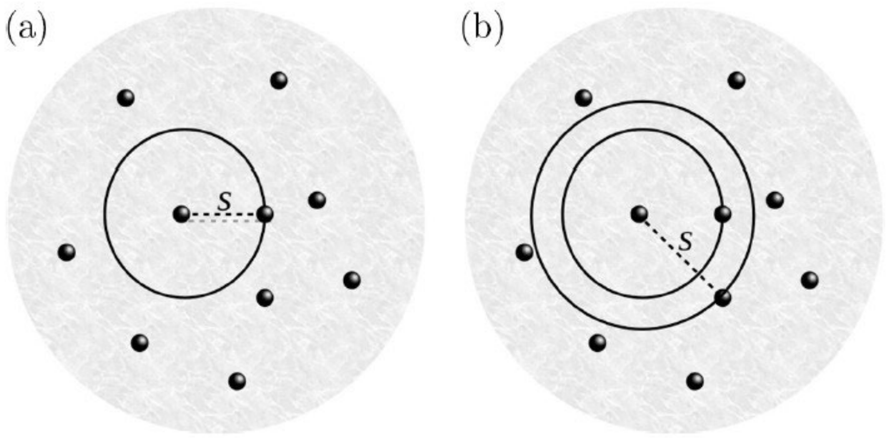
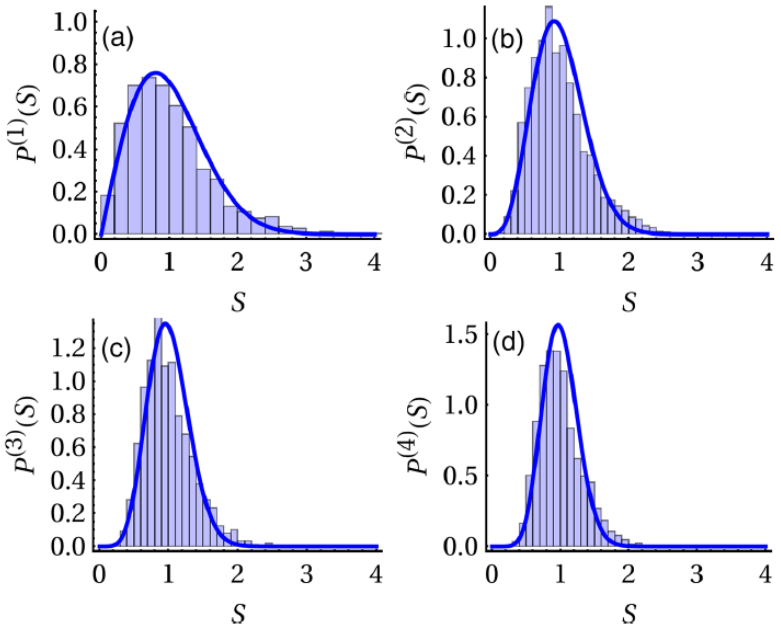


FIG. 3. Schematic representation of the defect spacing s with respect to the reference vortex at the center of the circle in the case of (a) $P(s) = P^{(1)}(s)$ and (b) $P^{(2)}(s)$. The black points represent vortices without accounting for their topological charges.



Figures are from M. Thudiyangal and A. del Campo, Phys. Rev. Res. **6**, 033152 (2024).

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- Spacing distribution of the 2D Coulomb gas follows PPP at infinite temperature.

This is our main result for the 2DCG at $N = 2$, shown in Fig. 1 left for several values of β. It correctly reproduces the 2D Poisson distribution (red bottom curve) when setting $\beta = 0$,

$$p_{\mathrm{Poisson}}^{\mathrm{2D}}(s) = \frac{\pi}{2} s \exp\left[-\frac{1}{4}\pi s^2\right], \qquad (24)$$

From G. Akemann, A. Mielke, and P. Päßler, Phys. Rev. E **106**, 014146 (2022).

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PHYSICAL REVIEW RESEARCH **7**, 023107 (2025)

Spatial form factor for point patterns: Poisson point process, Coulomb gas, and vortex statistics

Matteo Massaro[1,*] and Adolfo del Campo[1,2]

[1]*Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, Luxembourg*

[2]*Donostia International Physics Center, E-20018 San Sebastián, Spain*

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Point processes have broad applications in science and engineering. In physics, their use ranges from quantum chaos to statistical mechanics of many-particle systems. We introduce a spatial form factor (SFF) for the characterization of spatial patterns associated with point processes. Specifically, the SFF is defined in terms

- Vortices may be created or annihilated in nonequilibrium conditions (phase transition, …)

- Vortices in a quasi-2D scalar BEC can be mapped to electrodynamics without using the perturbative approach, even in the nonequilibrium conditions (dissipation, phase transition, …)

- Is there a similar duality between topological defects and electrodynamics?
  - Defects in crystalline solids in the Hermitian case of elastic media have similar duality

    L. Tsaloukidis and P. Surówka, Phys. Rev. B **109**, 104118 (2024).

  - Multi-component BEC?

$$\oint_{\partial\mathcal{M}} d\boldsymbol{l} \cdot \boldsymbol{v}_s(\boldsymbol{r},t) = \frac{2\pi\hbar}{M} \sum_{j=1}^{N_v(\mathcal{M};t)} q_j(t) \iff \oint_{\partial\mathcal{M}} (d\boldsymbol{l} \times \boldsymbol{e}_\perp) \cdot \boldsymbol{D}(\boldsymbol{r},t) = \sum_{j=1}^{N_f(\mathcal{M};t)} Q_j$$

$$M \frac{\partial \boldsymbol{v}_s(\boldsymbol{r},t)}{\partial t} = \boldsymbol{f}_K(\boldsymbol{r},t) - \nabla U_{\mathrm{sf}}(\boldsymbol{r},t) + \boldsymbol{F}_{\mathrm{sf}}(\boldsymbol{r},t) \iff \nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_f(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

$$\oint_{\partial\mathcal{M}} d\boldsymbol{l} \cdot \left[-\frac{U_{\mathrm{sf}}(\boldsymbol{r},t) - \bar{U}_{\mathrm{sf}}(t)}{2\pi\hbar} \boldsymbol{e}_\perp \right] = 0 \iff \oint_{\partial\mathcal{M}} (d\boldsymbol{l} \times \boldsymbol{e}_\perp) \cdot \boldsymbol{H}(\boldsymbol{r},t) = 0$$

$$\int_{\mathcal{M}} d^2r\, \rho_v(\boldsymbol{r},t) = \sum_{j=1}^{N_v(\mathcal{M};t)} q_j(t) \iff \int_{\mathcal{M}} d^2r\, \rho_f(\boldsymbol{r},t) = \sum_{j=1}^{N_f(\mathcal{M};t)} Q_j$$

$$\frac{\partial \rho_v(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{J}_{\mathrm{sf}}(\boldsymbol{r},t) = 0 \iff \frac{\partial \rho_f(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{J}_f(\boldsymbol{r},t) = 0$$

Prof. Dr. Adolfo del Campo

Dr. András Grabarits

Dr. Seongho Shin

Dr. Kazutaka Takahashi

Preethi Gopalakrishnan

Matteo Massaro

Kasturi Ranjan Swain

Thank you!

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