Modelling vortex dynamics in neutron stars in 2D & 3D

ECT Workshop 2025

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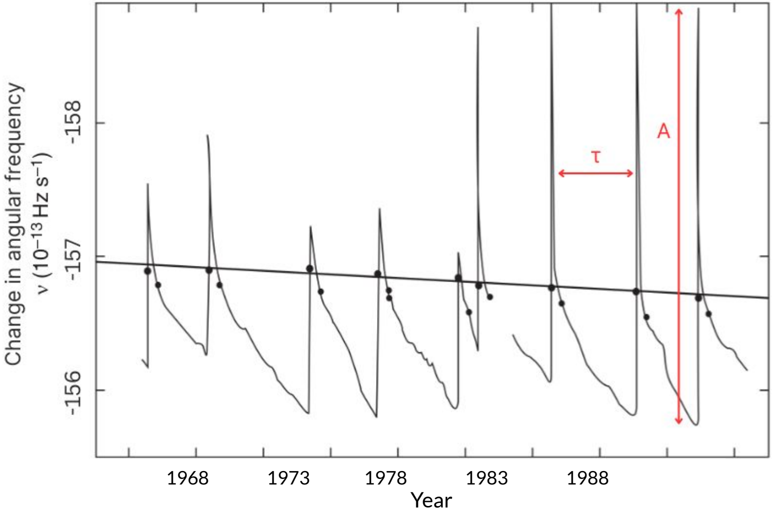
Newcastle University

UKRI Science and Technology Facilities Council

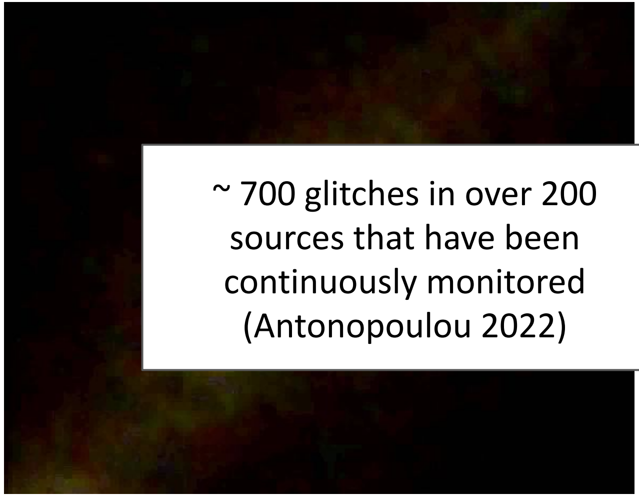
Pulsars

Cycle of pulsed gamma rays from the Vela pulsar. (Goddard Space Flight Center)



Increase in spin rate observed in the Vela Pulsar. (A. Haber 2018)

Pulsar Glitches



~ 700 glitches in over 200 sources that have been continuously monitored (Antonopoulou 2022)

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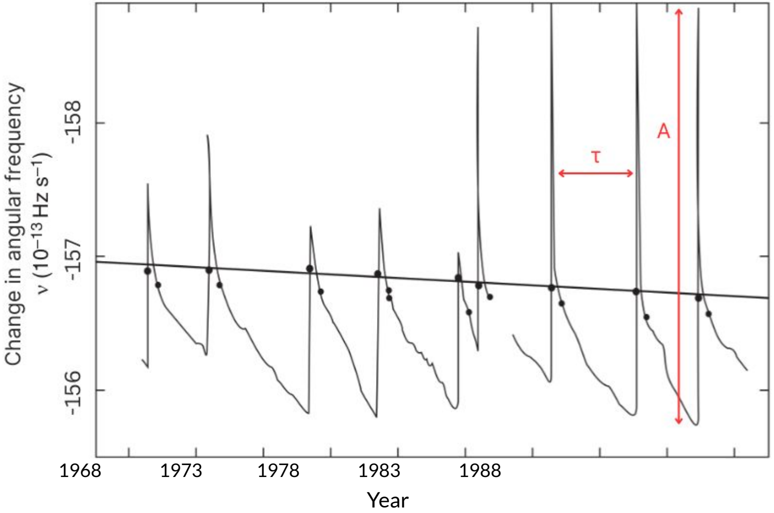

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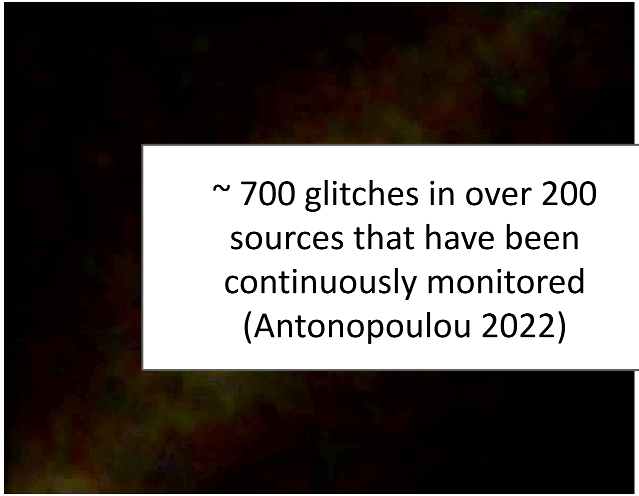

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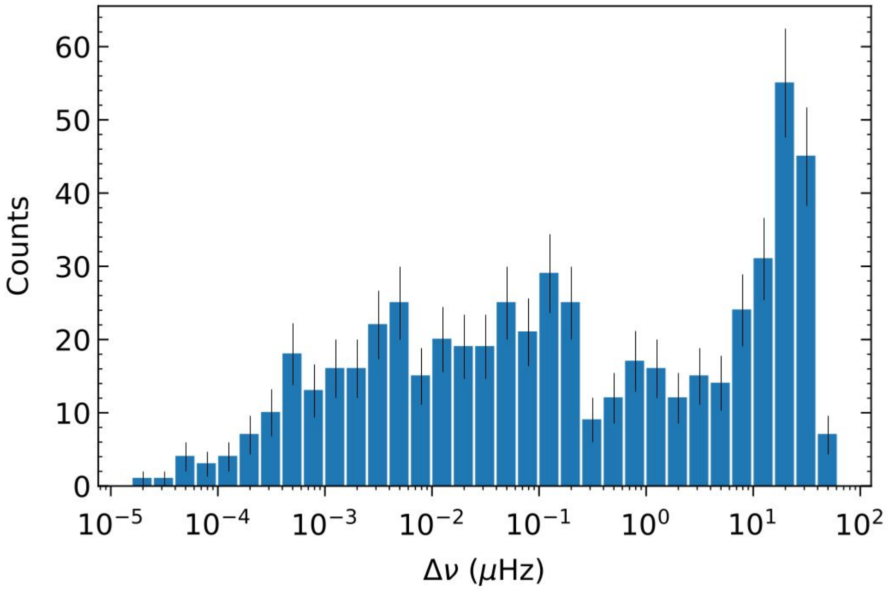
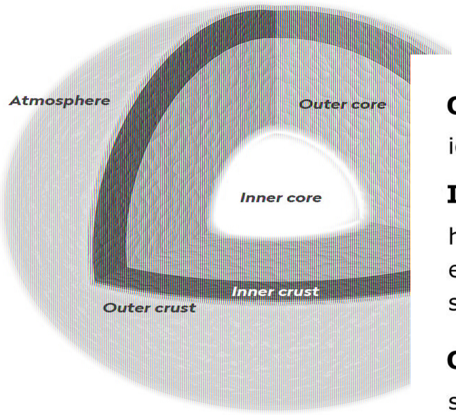
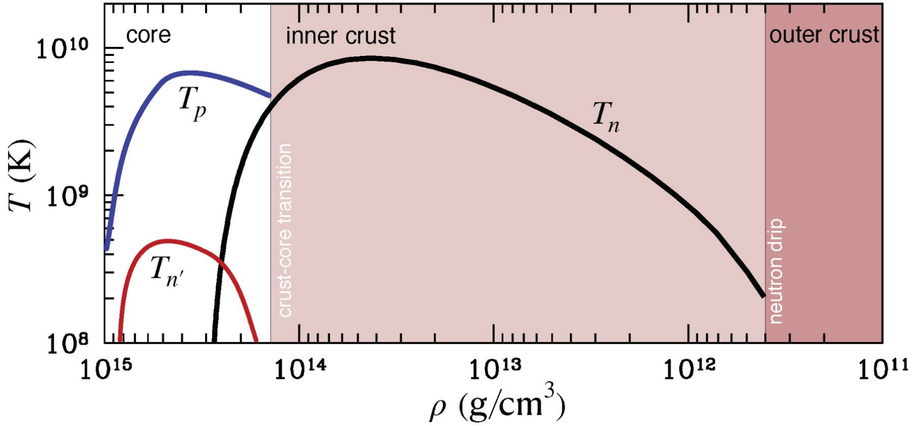
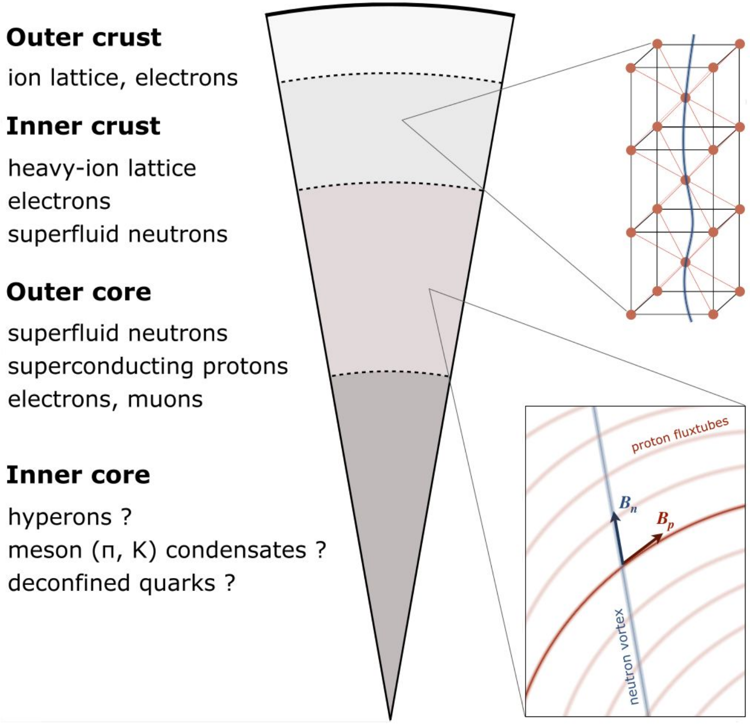


Distribution of glitch sizes (Antonopoulou 2022)

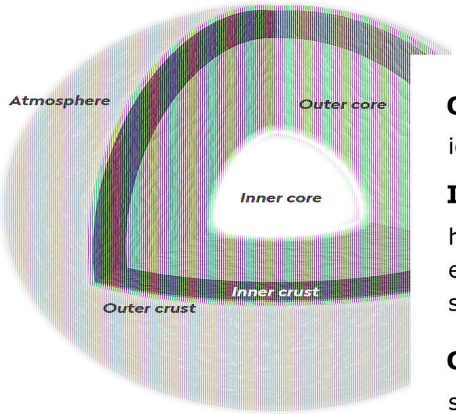
Why do pulsars glitch?

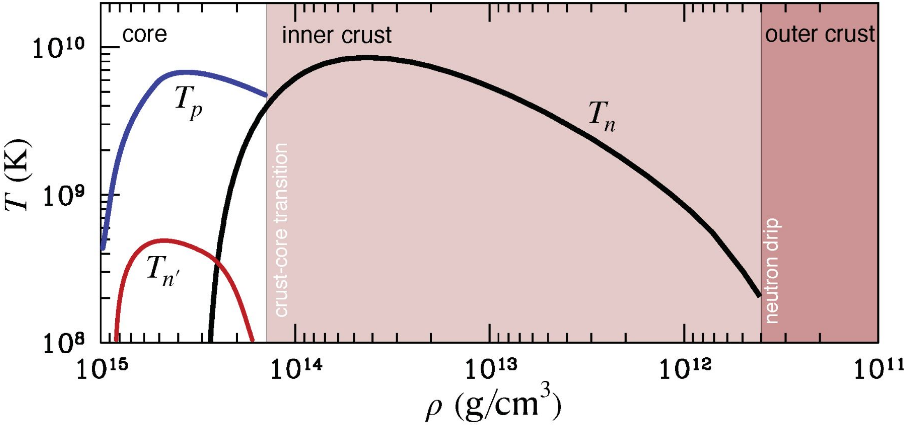


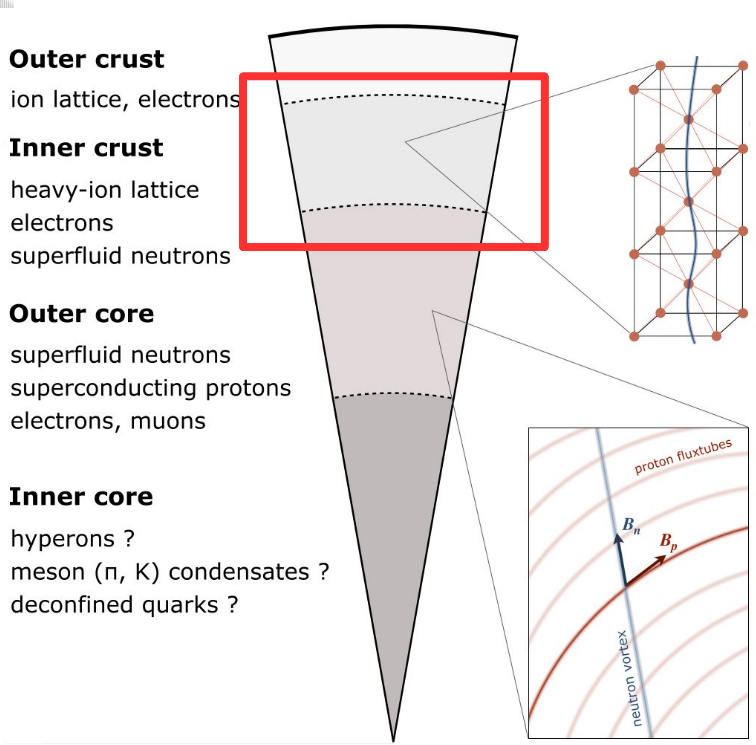




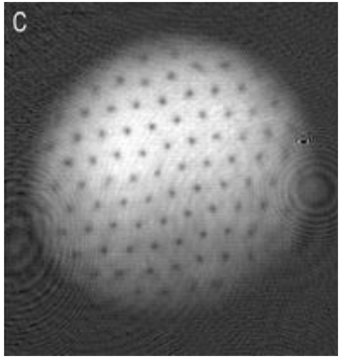
Neutron star composition







Simplified picture

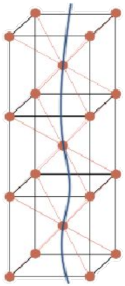


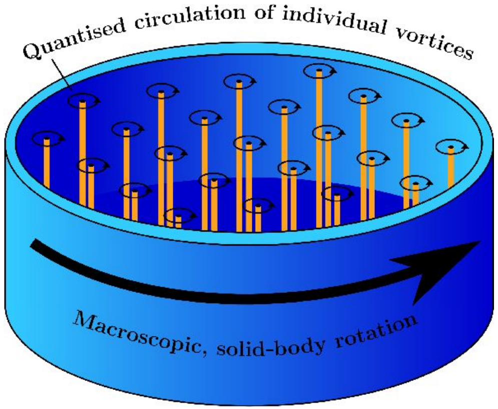
Top view of quantum vortices in a rotated quantum fluid in the lab.

Feynman's rule:

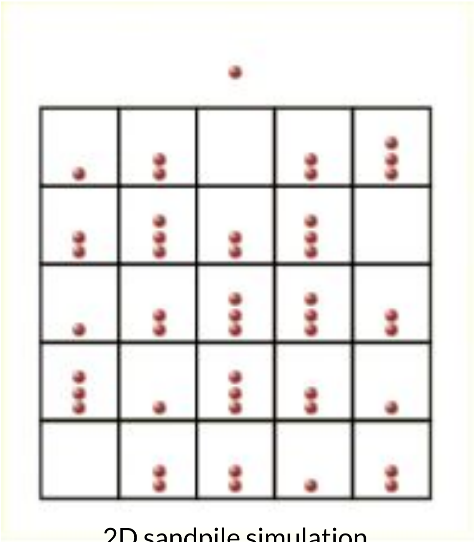
$$\Omega = \frac{N\kappa}{2\pi R^2}$$

$$\kappa = h/2m$$





Schematic of quantised rotation.
(Andersson 2021)



2D sandpile simulation.
Credit: A. Fronczak, P. Fronczak

3D Model: Vortex filament method

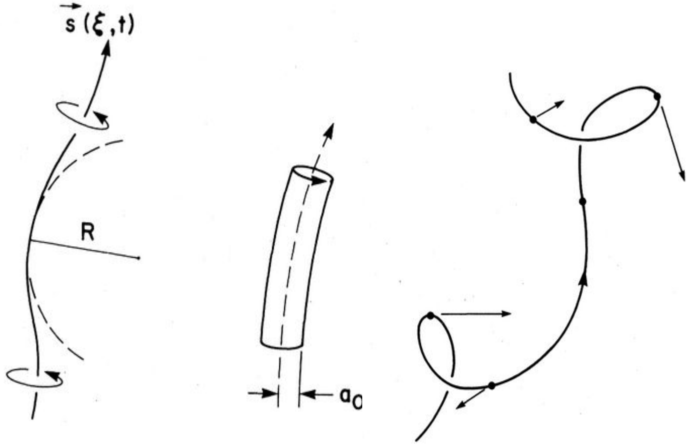
- Developed by K. W. Schwarz in 1985
- Vortices are modelled as space curves using the Schwarz Equation

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \mathbf{v} = \mathbf{v}_s + \alpha \mathbf{s}' \times (\mathbf{v}_n^{ext} - \mathbf{v}_s) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n^{ext} - \mathbf{v}_s)],$$

- Resolving the Magnus force and the drag force
- Superfluid velocity is calculated using the Biot-Savart equation (or Local Induction Approximation)

$$\mathbf{v}_s^{self}(\mathbf{s}_j) = \mathbf{v}_s^{loc}(\mathbf{s}_j) + \mathbf{v}_s^{non}(\mathbf{s}_j) = \frac{\kappa}{4\pi} \ln\left(\frac{2\sqrt{\ell_j \ell_{j+1}}}{e^{1/4} a_0}\right) \mathbf{s}_j' \times \mathbf{s}_j'' + \frac{\kappa}{4\pi} \oint_{\mathcal{L}'} \frac{(\mathbf{s}_j - \mathbf{r})}{|\mathbf{s}_j - \mathbf{r}|^3} \times \mathbf{dr}$$

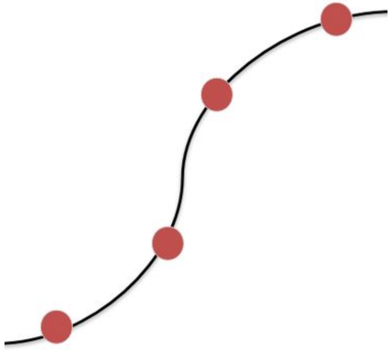
- Normal fluid contribution due to thermal excitation gas from mutual friction coupled to the superfluid

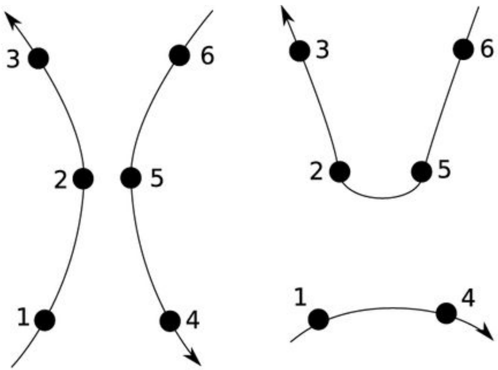


Vortex filaments as described by Vortex Filament Method (Schwarz, 1985)

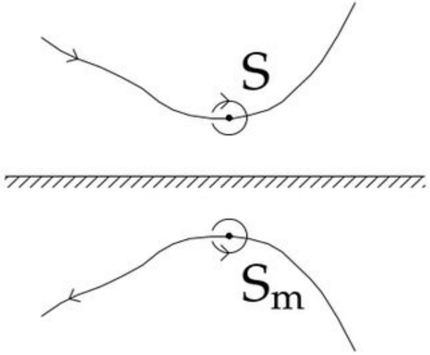
Continued

- Each vortex line is discretised into points
- Velocities calculated and combined in Schwarz equation
- Evolved in time using Adams-Bashforth time-stepping scheme
- Reconnections and solid boundaries



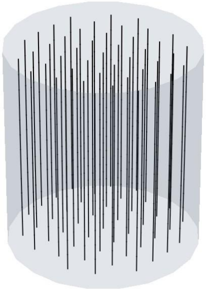


Vortex reconnections (A. Baggaley 2023)



Boundary fields are calculated as fields due to image vortices (Aarts, 1993).

Numerical set-up

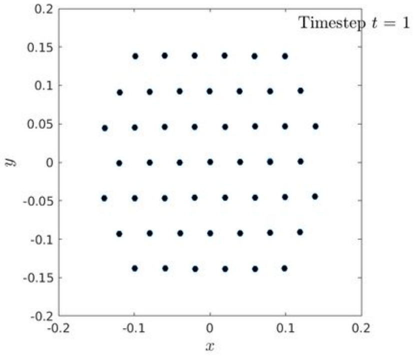


- Hexagonal lattice of vortices
- Equilibrium configuration obtained at high mutual friction; propagate in imaginary time
- Corotating frame

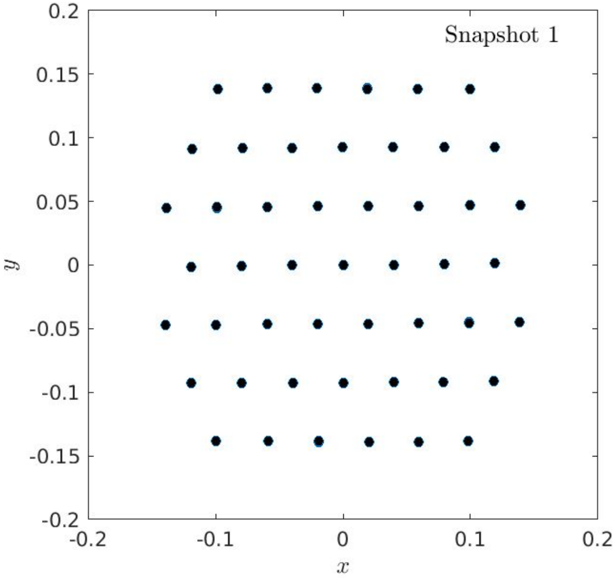
$$\dot{S}_x = V_{BS} + V_{NF} - \Omega y$$
$$\dot{S}_y = V_{BS} + V_{NF} + \Omega x$$
$$\dot{S}_z = V_{BS} + V_{NF}$$

- Solid boundary conditions
- Linear spin down

$$\Omega(t) = \Omega_0 - \dot{\Omega} t,$$



Initial lattice configuration



Initial lattice of vortices reorient into equilibrium position until perfect solid body rotation

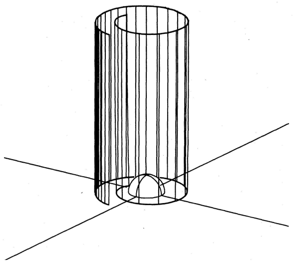
Vortex pinning

- Schwarz's model:

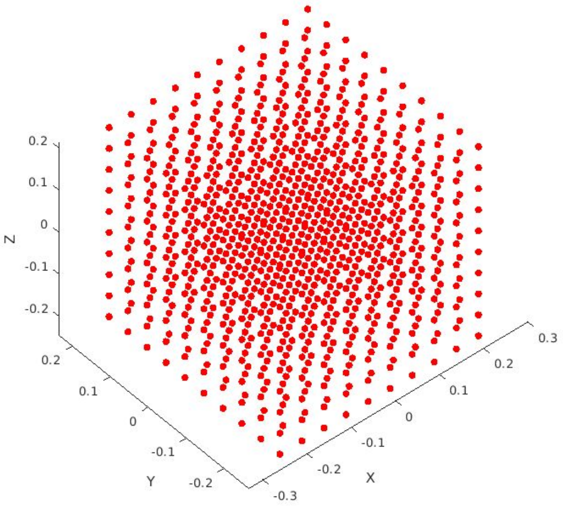
$$v_{\text{pin}} = \beta\kappa = \frac{\kappa}{2\pi D}\ln\left(\frac{R_p}{a_0}\right)$$

- Our model: penalisation scheme
- Pinning radius R_p
- Pinning strength
- Pinning density/spacing η_p

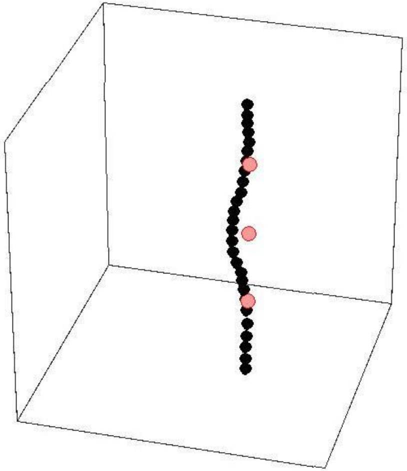
$$v_{\text{pin}} = \eta_p\kappa$$



Effect of pinning on a vortex in Schwarz's model (1985)

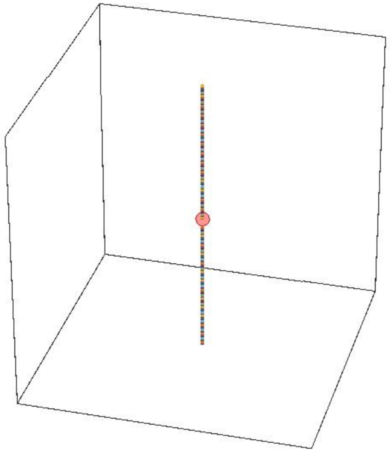


Pinning grid

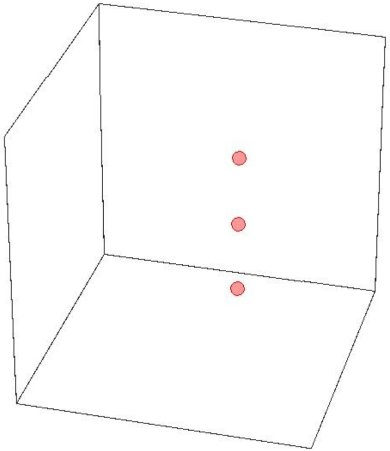


Vortex points (black) and pinning sites (red)

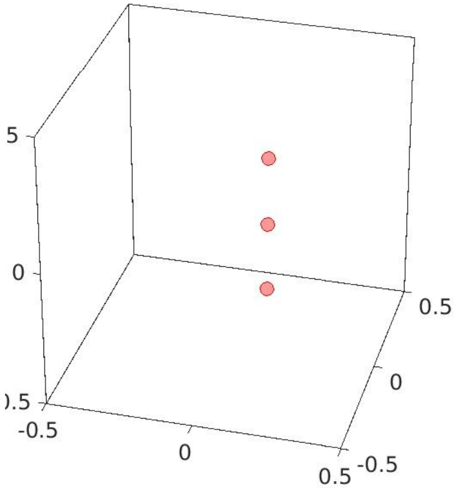
Effect of the pinning sites on vortices



Single vortex through a single pinning site with eta = 2.5



Single vortex through 3 pinning sites with eta = 2.5



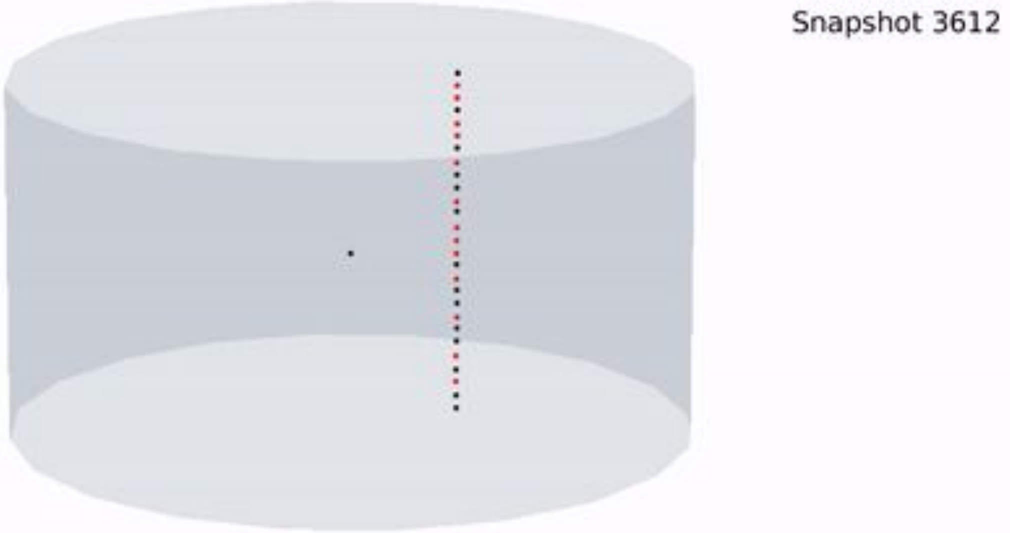
Single vortex through 3 pinning sites with eta = 5

Single vortex

Pinning radius Rpin = 0.02

Pinning velocity Vpin = 16V
(V – avg velocity of vortices
in the absence of pinning

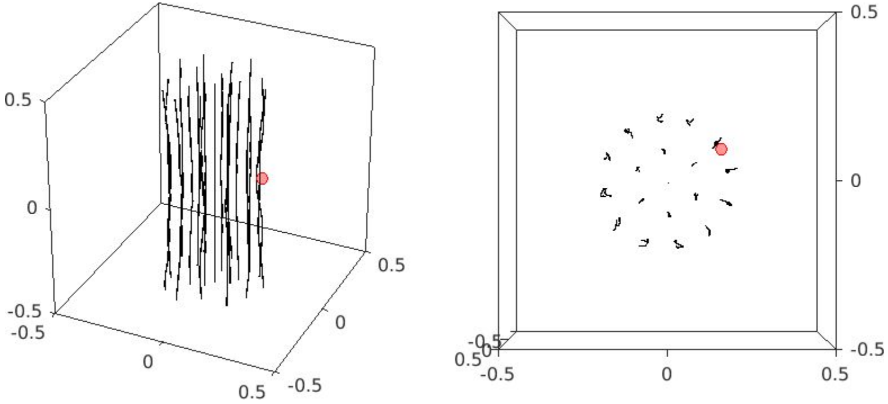
Inter-pinning site spacing
Lpin = 0.04



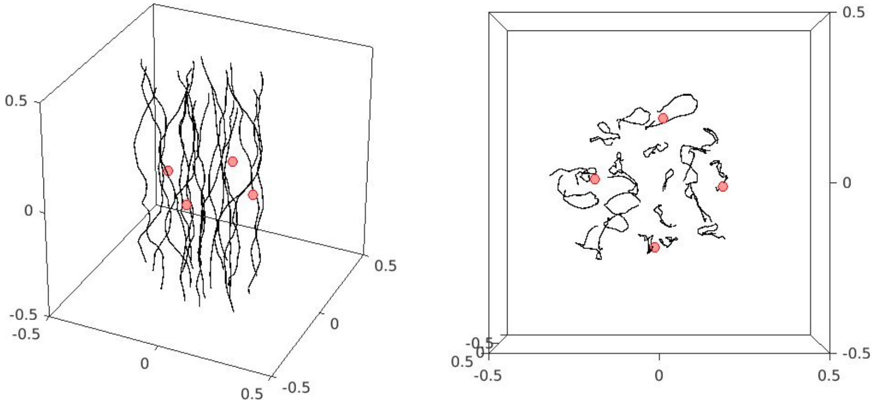
Single vortex encountering a
pinning grid as it spins out.

Many vortices

$\eta_p = 10$



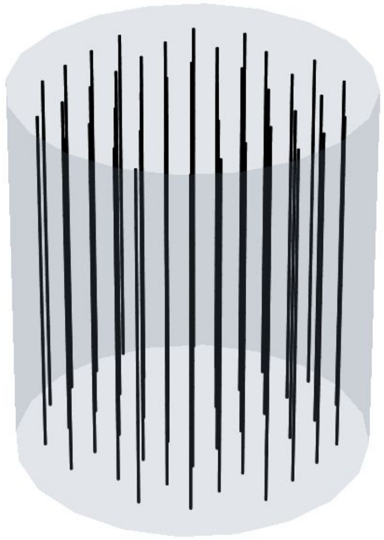
$\eta_p = 14$



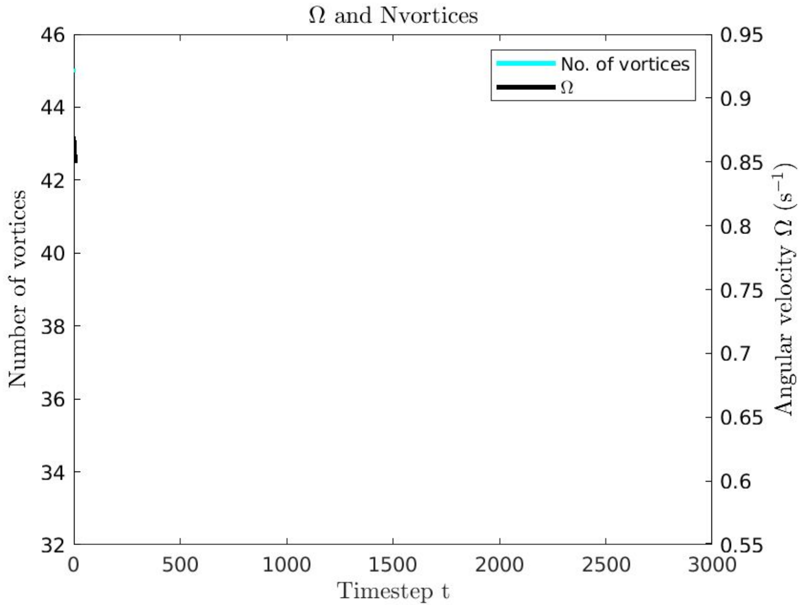
Small rotating lattice of vortices encountering pinning sites (non-rotating frame)

Toy model of rotating superfluid

Spin down rate $\dot{\Omega} = 10^{-6}$ s^{-2}, pinning strength $\eta_p = 0.00001$



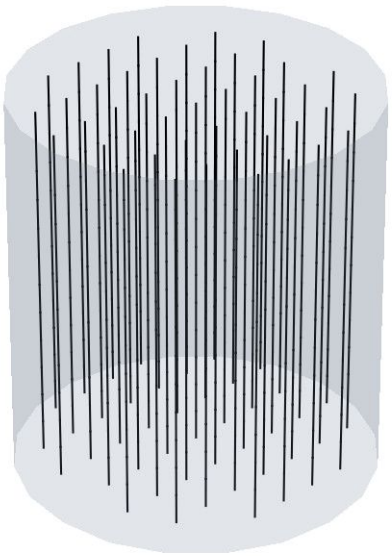
~ 50 vortices, solid cylindrical boundaries



Evolution of angular momentum in time.

Continued

Spin down rate $\dot{\Omega} = 10^{-7}\ \mathrm{s}^{-2}$, pinning strength $\eta_p = 10$



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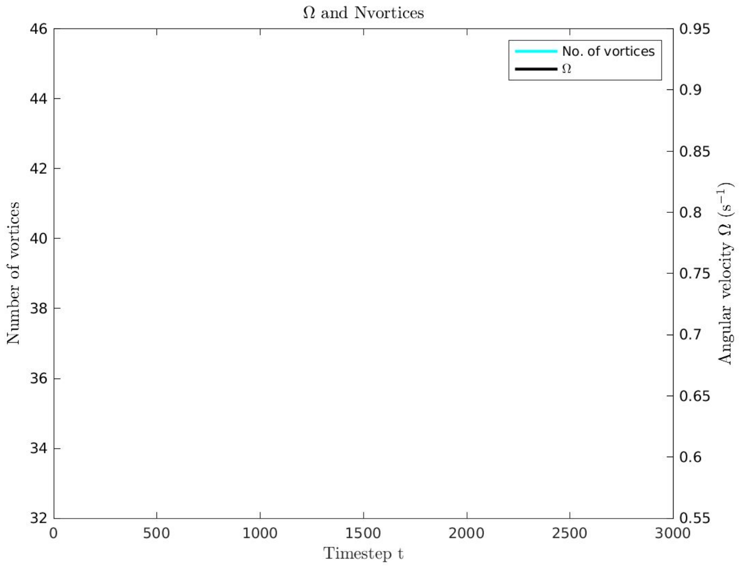

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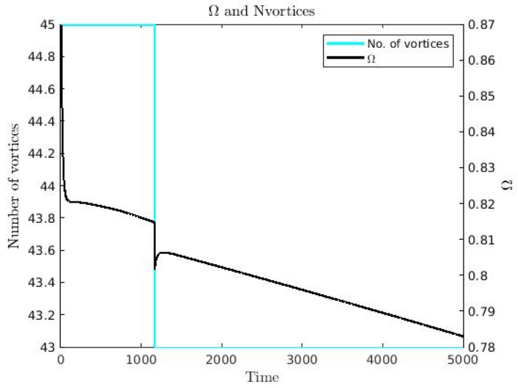
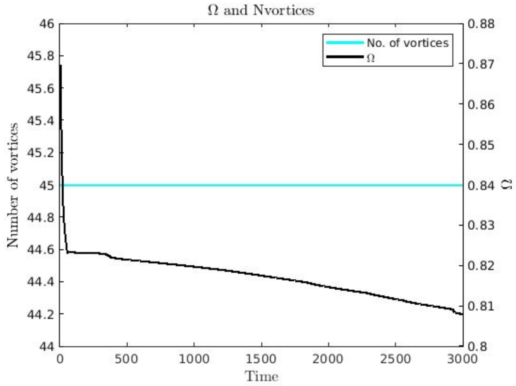
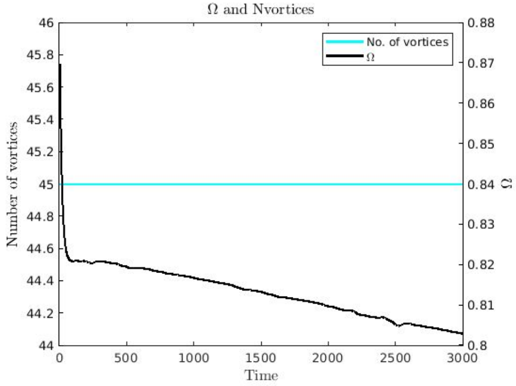


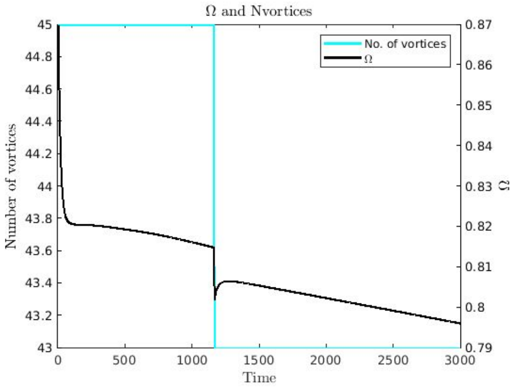
dxpin=0.08, ω=7*1E-8, 1gamma



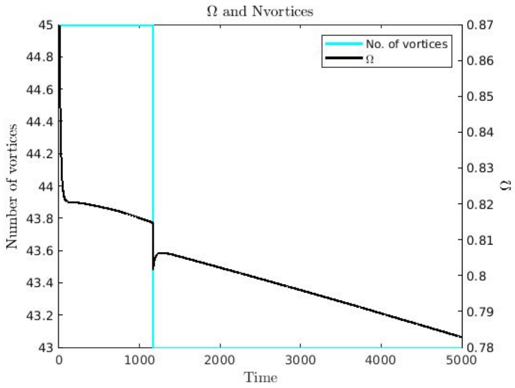
dxpin=0.08, ω=7*1E-8, 0.5gamma



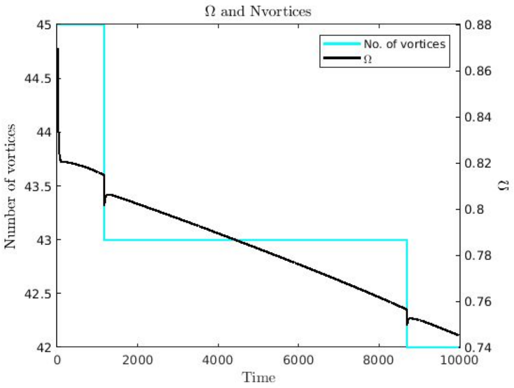
dxpin=0.08, ω=7*1E-8, 0.1gamma



dxpin=0.08, ω=7*1E-8, 0.05gamma



dxpin=0.08, ω=7*1E-8, 0.01gamma



dxpin=0.08, ω=7*1E-8, 0.1gamma

2D Model: Point Vortex Model

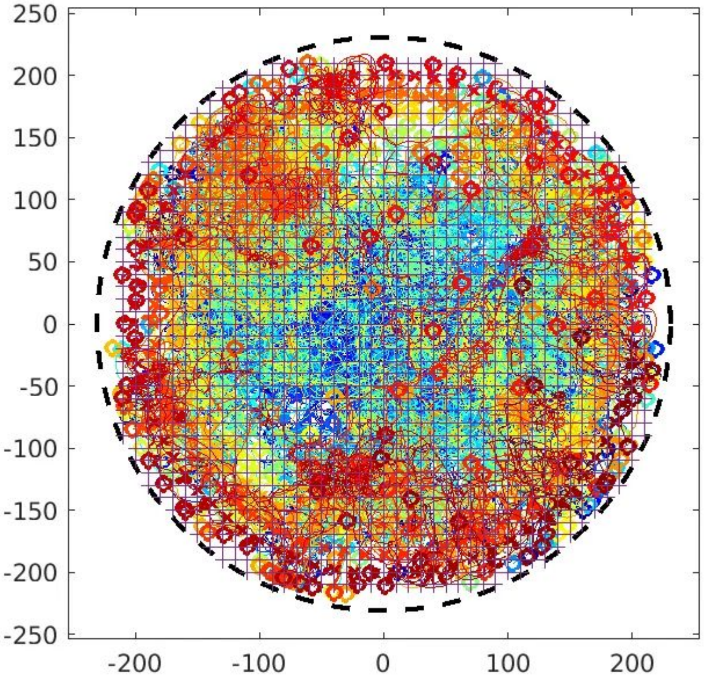
- Two dimensional equations of motion, Helmholtz (1858)

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = \frac{1}{2\pi}\sum_{j\neq i}\frac{\kappa_j(x_i - x_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\kappa_j(y_i - y_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}$$

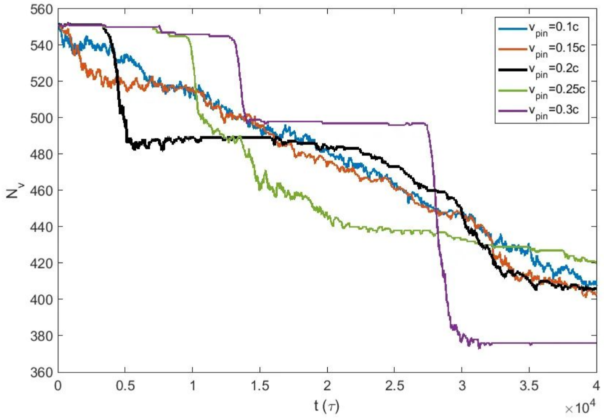
- Additional dissipation and rotation terms

$$i\dot{z}_k = e^{-i\gamma}\left(-\frac{\hbar}{m}\frac{z_k}{R^2 - |z_k|^2} - \frac{\hbar}{m}\sum_{j\neq k}^{N}\frac{z_k - z_j}{|z_k - z_j|^2} + \Omega z_k\right)$$

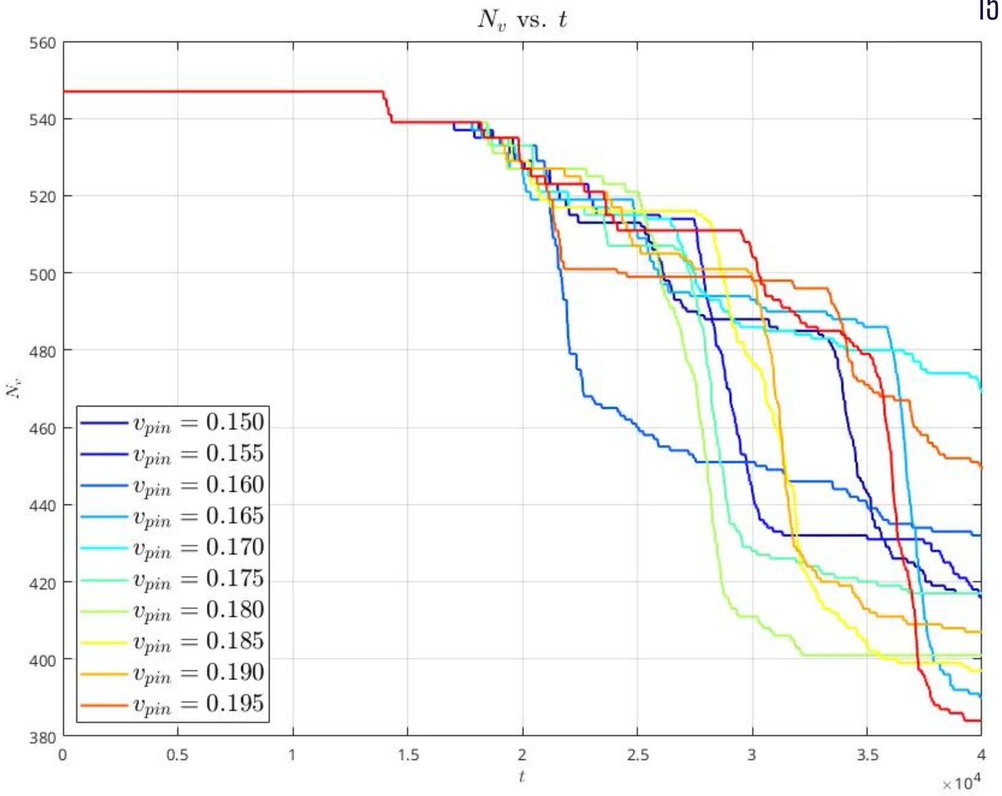


Vortex trajectories in a PVM simulation.

Initial conditions

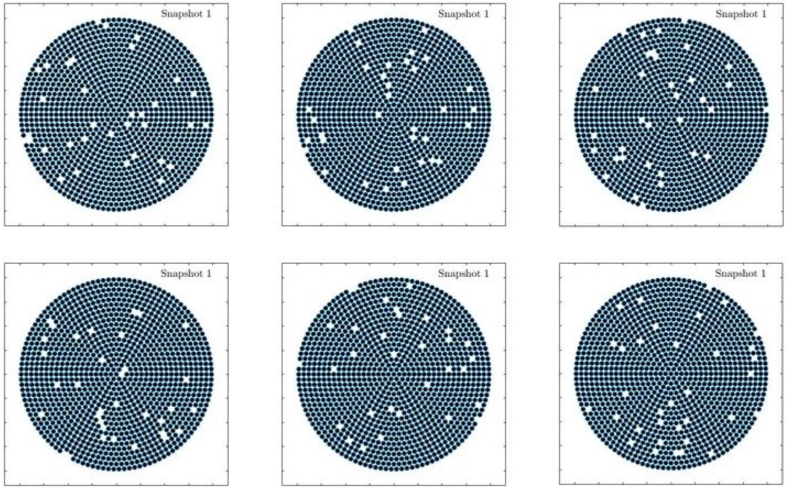


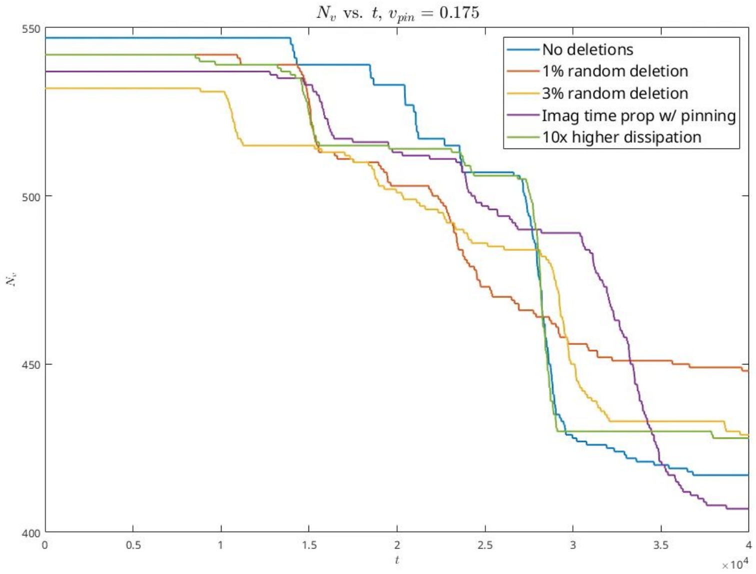
v_pin = 0.1 to 0.3



v_pin = 0.15 to 0.20

Generating initial conditions

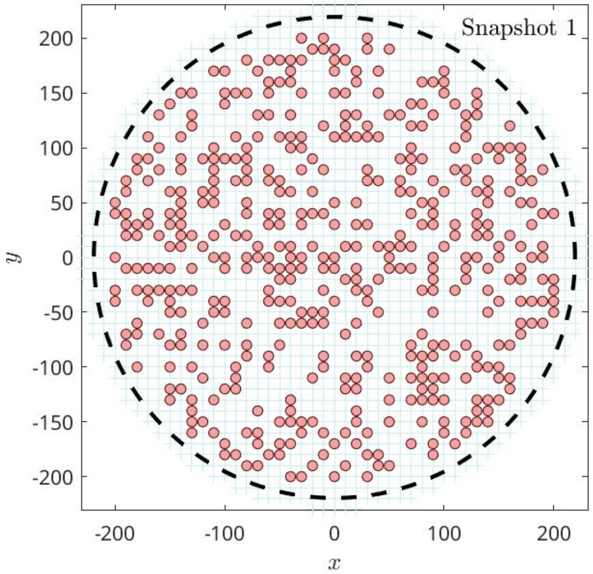


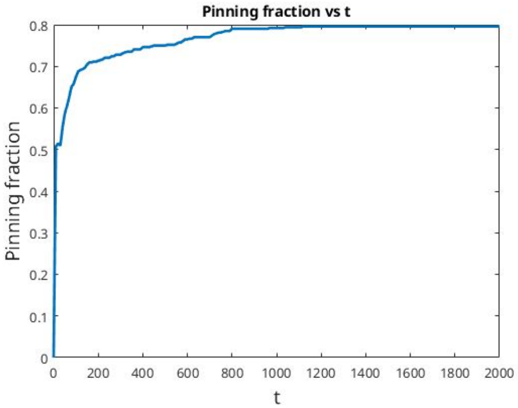


Production run

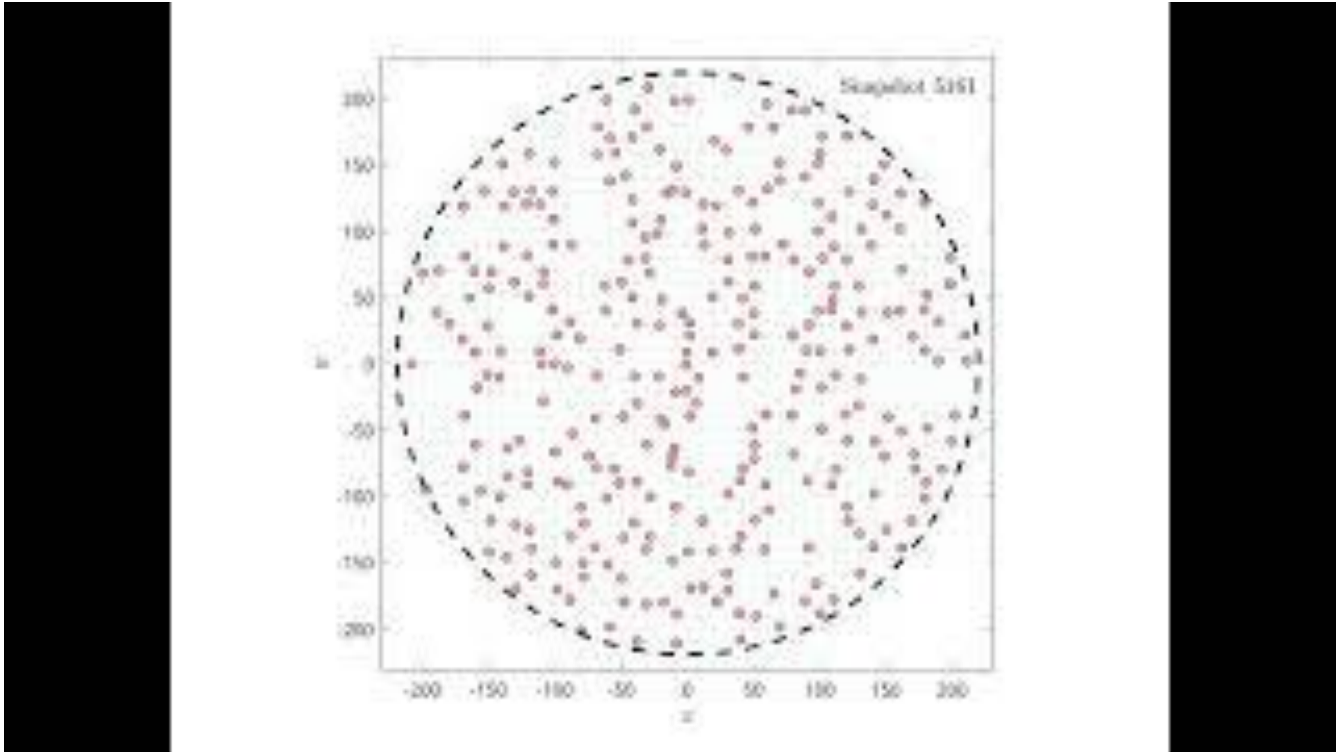
- Assign vortices to random pinning sites (square grid)
- Propagate in imaginary time until ~20% vortices are freely moving
- Pinning by penalisation scheme (critical depinning velocity v_pin)

$$\Omega_0 = 2 \times 10^{-3},$$
$$N_v \approx 500,$$
$$v_{pin} = 0.175,\ w_{pin} = 1.875,$$
$$\gamma = -a \sin 0.01,$$
$$t = 8 \times 10^4 \text{ steps}$$

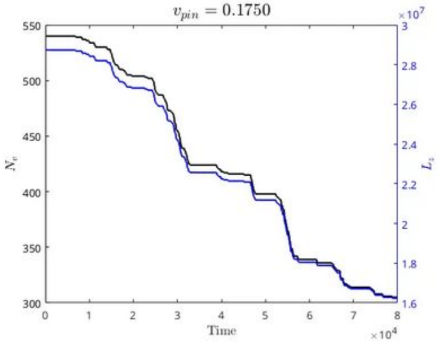


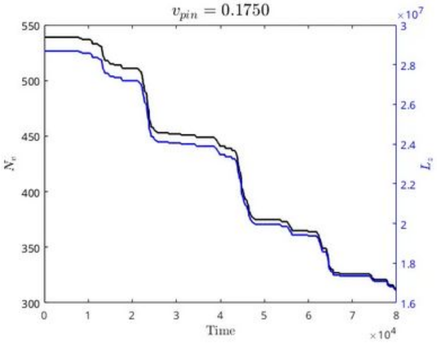


Results: Vortex Avalanches



Results


$v_{pin} = 0.1750$
N_v
L_z
Time
$\times 10^4$
$\times 10^7$


$v_{pin} = 0.1750$
N_v
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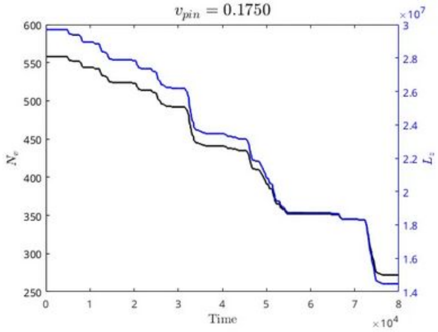
Time
$\times 10^4$
$\times 10^7$


$v_{pin} = 0.1750$
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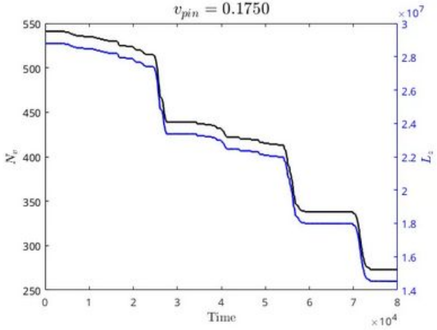
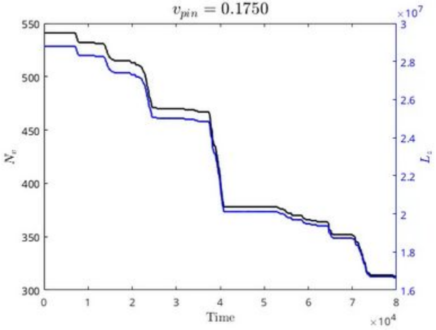
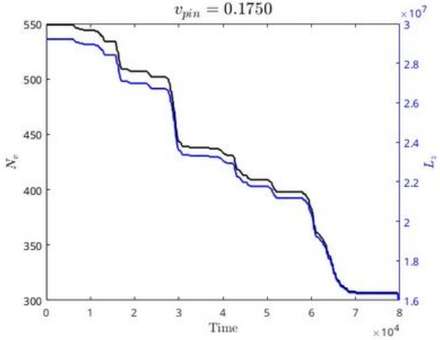

$v_{pin} = 0.1750$
N_v
L_z
Time
$\times 10^4$
$\times 10^7$


$v_{pin} = 0.1750$
N_v
L_z
Time
$\times 10^4$
$\times 10^7$


$v_{pin} = 0.1750$
N_v
L_z
Time
$\times 10^4$
$\times 10^7$

Avalanche finder algorithm

- Convolution of data with a top-hat window function

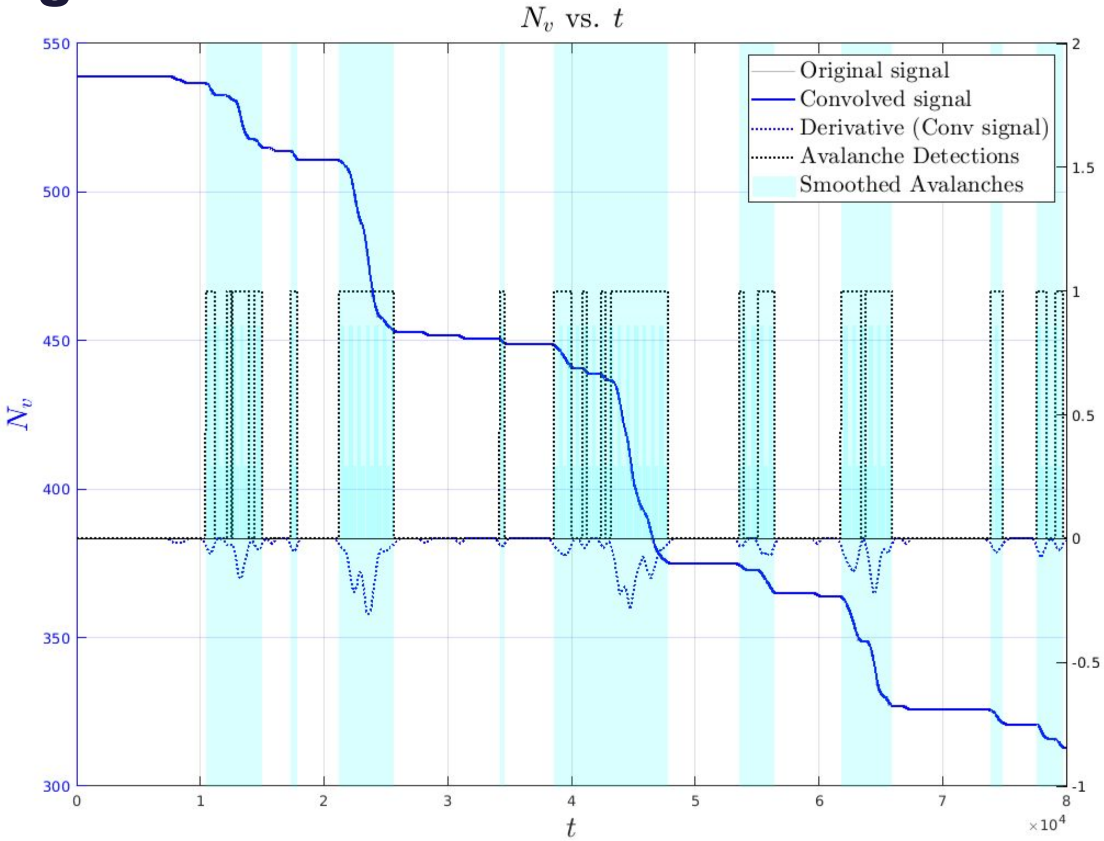
$$N_{v,\text{smooth}} = \int ds N_v(s) w(t-s)$$

- Threshold drop-rate dN_threshold
- Threshold waiting time dTau_threshold

Kernel width = 500

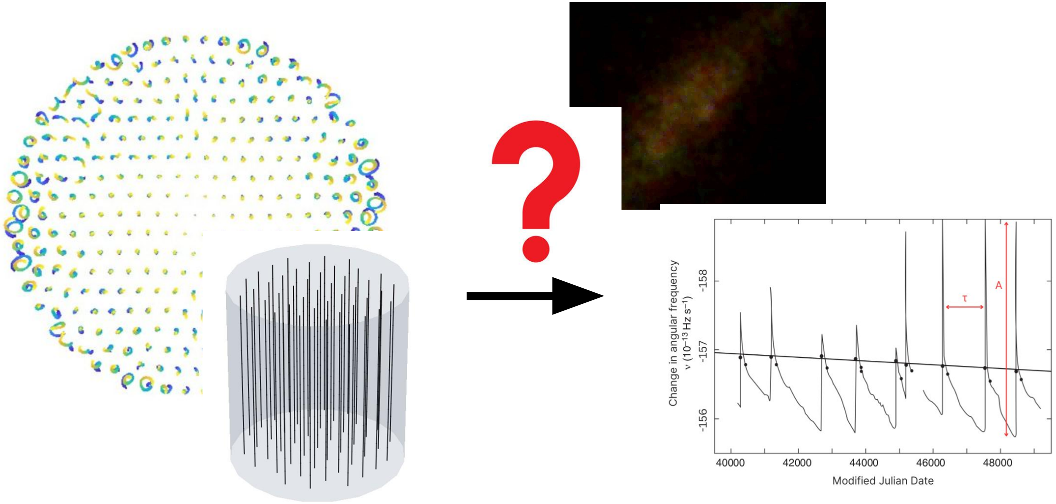
dN_threshold = 0.025

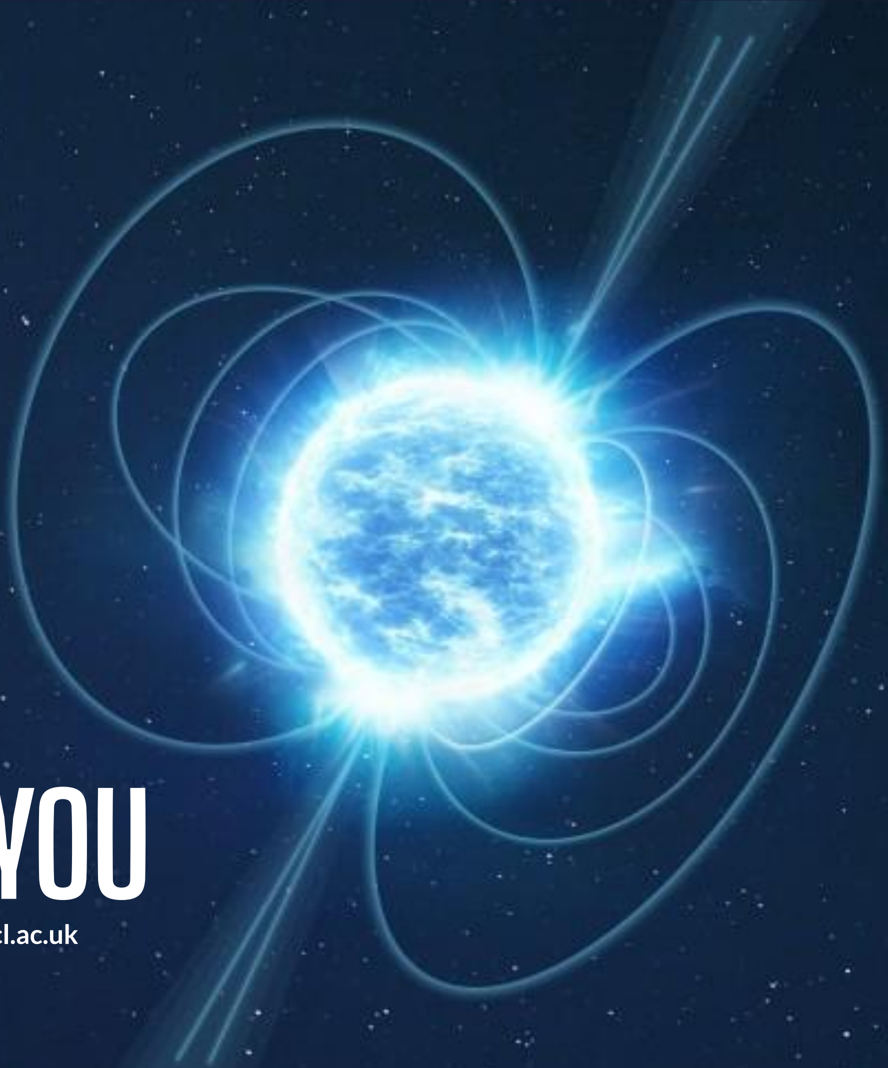
dTau_threshold = 150



Next steps and future outlook

- Statistical analysis of PVM simulation data to extract power-law avalanche sizes and exponential waiting times
- Constrain parameter space for 3D vortex filament model
- Extrapolate to real systems?





THANK YOU

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