# Unraveling the Emergence of Quantum State Designs in Systems with Symmetry

[Quantum 8, 1456 (2024)]

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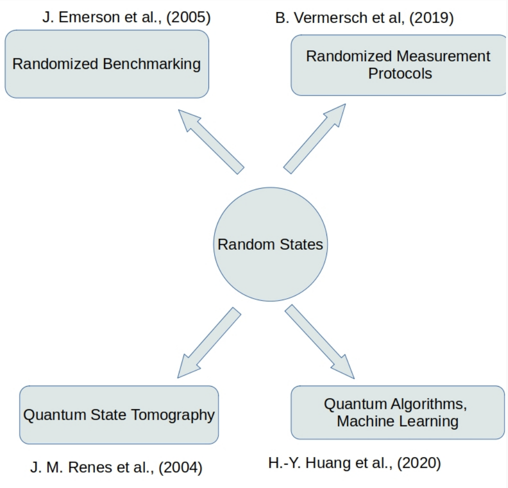
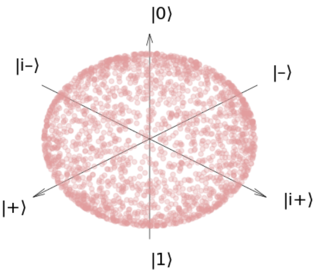
# Random Quantum States[1]




[1] J. Emerson et al., Journal of Optics B: Quantum and Semiclassical Optics **7**, S347 (2005), J. M. Renes et al., Journal of Mathematical Physics **45**, 2171–2180 (2004), B. Vermersch et al., Physical Review X **9**, 021061 (2019).

# (Haar) Random quantum states




- A pure quantum state $|\psi\rangle \in \mathcal{H}^d$ is a point on the surface of a hypersphere $\mathbb{R}^{2d-1}$.
- A Haar random state is a point picked uniformly at random from this hypersphere.
- Equivalently, applying a Haar random unitary ($U \in U(d)$) on a fixed fiducial state ($|\psi_0\rangle$) produces a Haar random state, i.e.,

$$|\psi\rangle = U|\psi_0\rangle$$

- For $d = 2$, the states lie on the Bloch sphere.

- Let $A_1, A_2, \cdots A_t$ be a set of operators and $|\psi\rangle \in \mathcal{H}$ be an arbitrary state,

$$f_t(|\psi\rangle) = \langle\psi|A_1|\psi\rangle\langle\psi|A_2|\psi\rangle\cdots\langle\psi|A_t|\psi\rangle$$

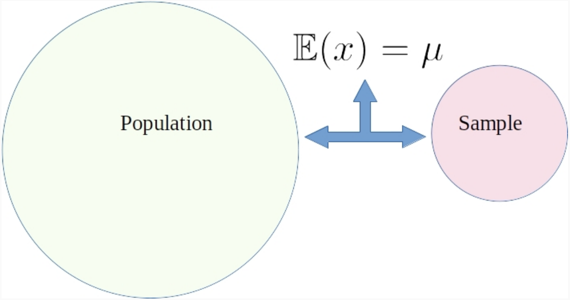
denotes a polynomial of order-$t$.

$$f_t(|\psi\rangle) = \left\langle \psi^{\otimes t} \middle| \bigotimes_{i=1}^{t} A_i \middle| \psi^{\otimes t} \right\rangle = \mathrm{Tr}\left[\underbrace{(|\psi\rangle\langle\psi|)^{\otimes t}}_{\in \mathcal{H}^{\otimes N}} \left(\bigotimes_{i=1}^{t} A_i\right)\right].$$

- One would often need to compute $\int_\psi d\mu(\psi) f_t(|\psi\rangle)$, which can be obtained by evaluating the following:

$$\underbrace{\boxed{\int_{|\psi\rangle} d\mu(\psi)\, (|\psi\rangle\langle\psi|)^{\otimes t}}}_{\text{t-th Moment}} = \mathbf{\Pi}_t \tag{1}$$

## Sampling and designs




- A sample should be representative of the entire population.
- Designs, in some sense, provide a "**smart way**" to sample from the population.

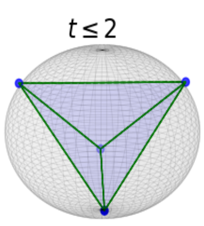
## Spherical Designs

* A set of $N$ points is called a ***spherical*** $t$***-design*** if the integral of any polynomial of degree at most $t$ over the sphere $\mathbb{S}^d$ is equal to the average value of the polynomial over the set of $N$ points[2], i.e.,

$$\int_{x\in\mathbb{S}^d} f(x)d\mu(x) = \frac{1}{|\mathcal{E}|}\sum_{x\in\mathcal{E}} f(x),$$

where,

- $f(x)$ is a $t$-th degree polynomial.
- $d\mu$ is the normalized surface measure over $\mathbb{S}^d$.
- $\mathcal{E}$ is the finite set of points on $\mathbb{S}^d$.




* In $\mathbb{S}^2$, vertices of a regular tetrahedron embedded inside form a spherical $2$-design.

[2] R. H. Hardin and N. J. Sloane, Discrete & Computational Geometry **15**, 429–441 (1996).

## Quantum state designs

**Def**:

[3] J. M. Renes et al., Journal of Mathematical Physics **45**, 2171–2180 (2004).

# Quantum state designs

**Def**: A finite ensemble of states $\mathcal{E} \equiv \{p_j, |\psi_j\rangle\}$ is called a state $t$-design, if[3]

$$\sum_{j=1}^{|\mathcal{E}|} p_j \left(|\phi_j\rangle\langle\phi_j|\right)^{\otimes t} = \int_{|\phi\rangle} d\mu_\phi \left(|\phi\rangle\langle\phi|\right)^{\otimes t} \qquad \propto \sum_{j=1}^{t!} \Pi_j$$

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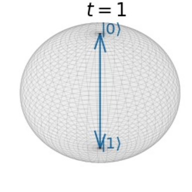


$$\int_{|\phi\rangle} d\mu_\phi |\phi\rangle\langle\phi| = \frac{\mathbb{I}}{d}$$

- A complete orthonormal basis is an exact $1$-design.
- Single qubit example: $\{|0\rangle, |1\rangle\}$.

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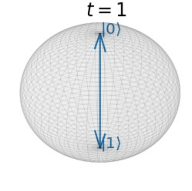


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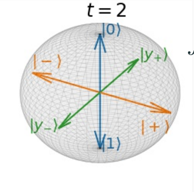
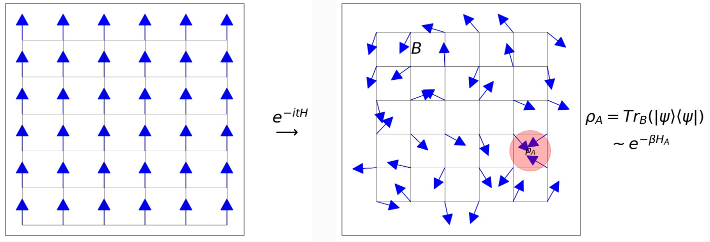



$$\int_{|\phi\rangle} d\mu_\phi \left(|\phi\rangle\langle\phi|\right)^{\otimes 2} = \frac{\mathbb{I} + \mathsf{F}}{d(d+1)} \qquad (F \sim \mathsf{SWAP})$$

- Ex. $\left\{|0\rangle, |1\rangle, \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}\right\}$.
- MUBs are 2-designs.

[3] J. M. Renes et al., Journal of Mathematical Physics **45**, 2171–2180 (2004).

- Isolated generic quantum systems locally equilibrate to the steady states, which are statistically indistinguishable from the Gibbs ensemble, i.e.,




$$\rho_{\text{local}} = \text{Tr}_{\overline{\text{local}}}(|\psi\rangle\langle\psi|) \sim \frac{e^{-\beta H_{\text{local}}}}{\mathcal{Z}}$$

- The temperature $\beta$ is fixed by the mean energy of the system

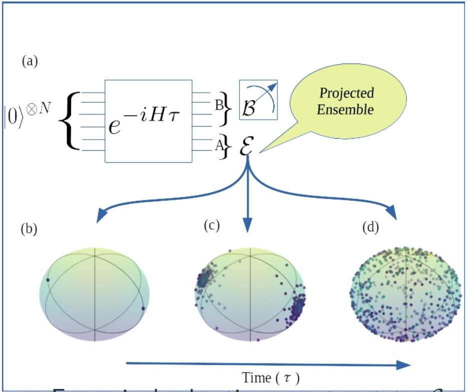
$$E = \langle\psi|H|\psi\rangle.$$

- When $E = 0$, we have $\rho_{\text{local}} \sim \frac{\mathbb{I}}{\mathcal{D}} \implies 1$-designs.

$$\rho_{\text{local}} = \sum_{|b\rangle} \langle b| (|\psi\rangle\langle\psi|) |b\rangle \tag{2}$$

# Projected ensemble Framework

* Consider a measurement basis $\mathcal{B} \equiv \{|b\rangle\}$, where $|b\rangle \in \mathcal{H}^{\otimes N_B}$ and $|\psi_{AB}\rangle \in \mathcal{H}^{\otimes N}$ such that $|\psi_{AB}\rangle = e^{-i\tau H}|0\rangle^{\otimes N}$.




- $\mathcal{E} \equiv \{p_b, |\phi_A(b)\rangle\}$, where

$$|\phi_A(b)\rangle = \frac{\langle b|\psi\rangle}{\sqrt{\langle\psi|b\rangle\langle b|\psi\rangle}}$$

and

$$p_b = \langle\psi|b\rangle\langle b|\psi\rangle.$$

- For a single chaotic generator state, $\mathcal{E}$ approximates a state $t$-designs[4]:

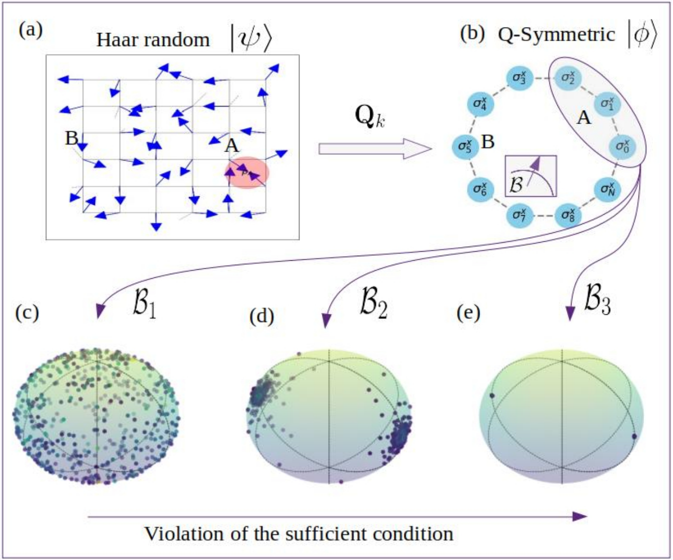
$$\Delta^{(t)} = \left\| \sum_{|b\rangle\in\mathcal{B}} p_b \left(|\phi(b)\rangle\langle\phi(b)|\right)^{\otimes t} - \int_{|\phi\rangle} d\mu_\phi \left(|\phi\rangle\langle\phi|\right)^{\otimes t} \right\|_1 \leq \varepsilon \sim 2^{-N_B/2}$$

[4] J. S. Cotler et al., PRX Quantum 4, 010311 (2023).

# Emergence of quantum state designs in systems with symmetry[5]

- *What's the general choice of measurement basis for the emergence of $t$-designs when the generator state abides by a symmetry?*
- To address this question, we first adhere our analysis to random generator states with translation symmetry.

[5]N. D. Varikuti and S. Bandyopadhyay, **"Unraveling the emergence of quantum state designs in systems with symmetry",** Quantum **8**, 1456 (2024).


(a)
Haar random $|\psi\rangle$
B
A
$\mathbf{Q}_k$
(b) Q-Symmetric $|\phi\rangle$
A
B
$\mathcal{B}$
(c)
$\mathcal{B}_1$
(d)
$\mathcal{B}_2$
(e)
$\mathcal{B}_3$
Violation of the sufficient condition

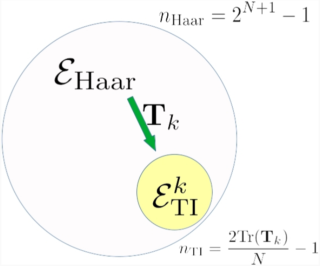
- Translation operator: $T|i_1 i_2 \cdots i_N\rangle = |i_N i_1 \cdots i_{N-1}\rangle$.
- Eigenvalues: $N$-th roots of unity — $\{e^{-2\pi ik/N}\}$ for $k = 0, 1, \cdots, N-1$.
- $T$ generates a cyclic group of order $N$ — $\{\mathbb{I}, T, T^2, \cdots T^{N-1}\}$.

We construct a subspace projector

$$\mathbf{T}_k = \sum_{j=0}^{N-1} e^{2\pi ijk/N} T^j.$$

- Given a Haar random $|\psi\rangle \in \mathcal{H}^{\otimes N}$, $|\phi\rangle = \mathbf{T}_k|\psi\rangle / \sqrt{\langle\psi|\mathbf{T}_k^\dagger \mathbf{T}_k|\psi\rangle}$ is a translation symmetric state with eigenvalue $e^{-2\pi ik/N}$.

$$T|\phi\rangle = e^{-2\pi ik/N}|\phi\rangle$$



# Main Result

* For a given measurement basis $\mathcal{B}$ supported over $B$, we intend to verify the equality of moments when the generator states have symmetry:

$$\mathbb{E}_{|\phi\rangle\in\mathcal{E}_{\mathsf{TI}}^{k}}\left(\underbrace{\sum_{|b\rangle\in\mathcal{B}} p_b\left(|\phi(b)\rangle\langle\phi(b)|\right)^{\otimes t}}_{t\text{-th moment of proj. ensemble}}\right) = \int_{|\psi\rangle\in\mathcal{E}_{\mathsf{Haar}}} d\psi\left(|\psi\rangle\langle\psi|\right)^{\otimes t},$$

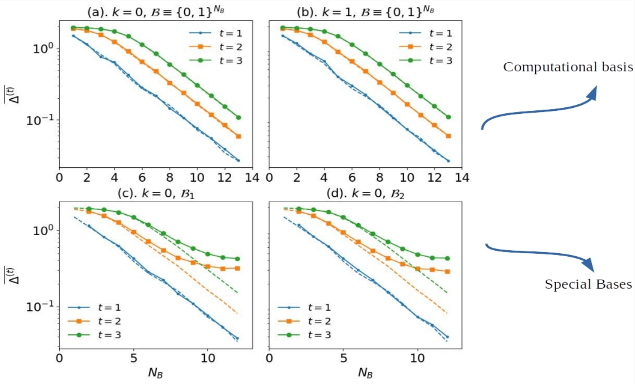
<u>*Sufficient condition:*</u>

* The above identity holds if for all $|b\rangle\in\mathcal{B}$, $\langle b|\mathbf{T}_k|b\rangle = \mathbb{I}_{2^{N_A}}$.
    - If there exists $|a\rangle\in\mathcal{H}^{\otimes N_A}$ such that $\langle ab|\mathbf{T}_k|ab\rangle = N$ for a given $|b\rangle$, then $|b\rangle$ maximally violates the condition.
    - If $|ab\rangle$ is an eigenstate of $T$ with different $k$, then $\langle ab|\mathbf{T}_k|ab\rangle = 0$.
    - If the above equality holds, Levy's lemma ensures that the projected ensemble approximates a higher-order state design.

# Designs from random T-invariant states [arXiv:2402.08949]

* Figure of merit [Cotler et al. (2021)]:

$$\Delta^{(t)} = \left\| \sum_{|b\rangle \in \mathcal{B}} \frac{\left(\langle b|\phi\rangle\langle\phi|b\rangle\right)^{\otimes t}}{\left(\langle\phi|b\rangle\langle b|\phi\rangle\right)^{t-1}} - \int_{|\phi\rangle} d\mu_\phi \left(|\phi\rangle\langle\phi|\right)^{\otimes t} \right\|_1$$



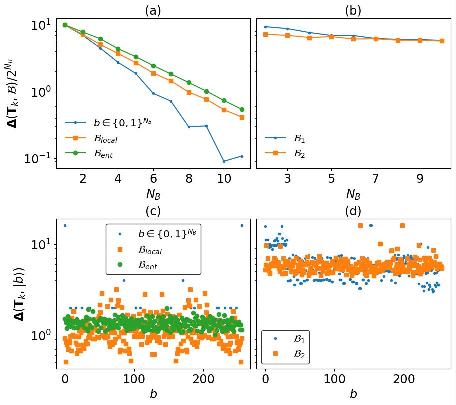

Average trace distance $\overline{\Delta^{(t)}}$ versus $N_B$

# Numerical results

- We quantify the violation using $\mathbf{\Delta}(\mathbf{T}_k, \mathcal{B})/2^{N_B}$, where

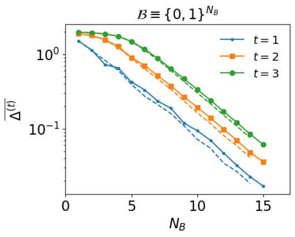
$$\mathbf{\Delta}(\mathbf{T}_k, \mathcal{B}) = \sum_{|b\rangle \in \mathcal{B}} \| \langle b | \mathbf{T}_k | b \rangle - \mathbb{I}_{2^{N_A}} \|_1$$

**Violation**:



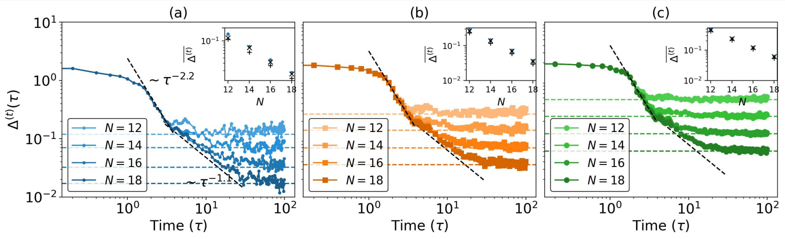

- $\mathcal{B}_{\text{local}} \sim (u_1 \otimes \cdots \otimes u_{N_B})$.
- $\mathcal{B}_{\text{Ent}} \sim U_B$ (global unitary)
- $\mathcal{B}_1 \sim$ Eigenbasis of $T_B$.
- $\mathcal{B}_2 \sim$ Eigenbasis of $(u \otimes \mathbb{I}_{B-1})T_B$.

$$H = \sum_{i=0}^{N} \sigma_i^x \sigma_{i+1}^x + h_x \sum_{i=0}^{N} \sigma_i^x + h_y \sum_{i=0}^{N} \sigma_i^y$$




- Initial state: $|\psi(0)\rangle = |0\rangle^{\otimes N}$.

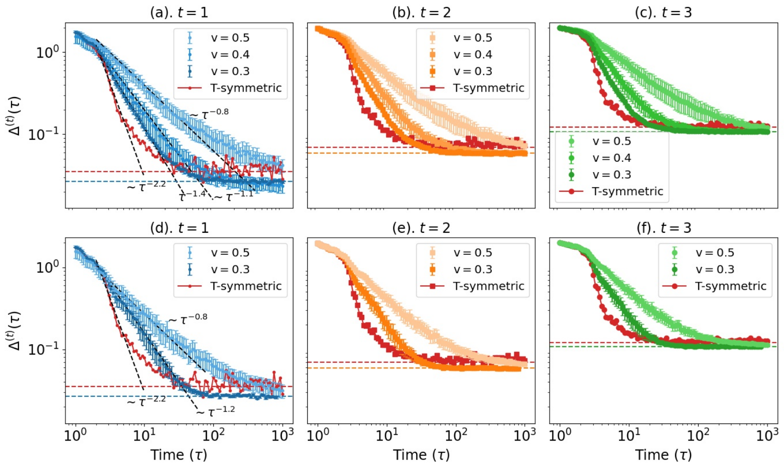
Translation + Ref - symmetric vs Haar states




Deep thermalization of a quantum state $|\psi(t)\rangle$ evolved under the dynamics of a chaotic Ising Hamiltonian with periodic boundary conditions.

$$H = \sum_{i=1}^{N} (J + \eta_i)\sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^{N} h_x \sigma_i^x + \sum_{i=1}^{N} (h_y + \xi_i)\sigma_i^y, \tag{3}$$

(a)-(c) — $\eta_i \in \mathcal{N}(0, v)$ , $\xi_i = 0$. (d)-(f) — $\eta_i = 0$ , $\xi_i \in \mathcal{N}(0, v)$.



# Extension to other symmetries

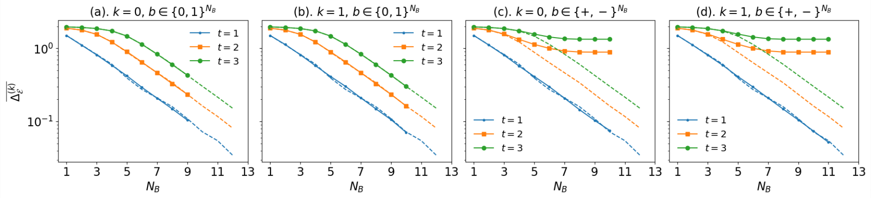
$Z_2$ (spin-flip)-symmetry: $\Sigma = \underbrace{\sigma^x \otimes \sigma^x \otimes \cdots \otimes \sigma^x}_{N\text{-times}}.$

- Projector onto an invariant subspace: $\boldsymbol{Z}_\pm = \mathbb{I} \pm \Sigma.$
- Violation in computational ($\sigma^z$) basis:

$$\checkmark\ \boldsymbol{\Delta}(\mathbf{Z}_\pm, \mathcal{B}) = \sum_{b \in \{0,1\}^N} \| \pm \langle b | \Sigma | b \rangle \|_1 = \sum_{b \in \{0,1\}^N} \left\| (\sigma^x)^{\otimes N_A} \prod_{j=N_A+1}^{N_B} \langle b_j | \sigma^x | b_j \rangle \right\|_1 = 0.$$

- Violation in $\sigma^x$ basis:

$$\times\ \boldsymbol{\Delta}(\mathbf{Z}_\pm, \mathcal{B}) = \sum_{b \in \{+,-\}^N} \| \pm \langle b | \Sigma | b \rangle \|_1 = \sum_{b \in \{+,-\}^N} \left\| (\sigma^x)^{\otimes N_A} \prod_{j=N_A+1}^{N_B} \langle b_j | \sigma^x | b_j \rangle \right\|_1 = 2^{N_A}.$$



## Summary

- We established a sufficient condition for the emergent state designs from random T-invariant quantum states.
- Identified measurement bases that do not produce the state designs.
- Examined the deep thermalization in the chaotic Ising chain with periodic boundary conditions and contrasted the results with no-symmetry cases.

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# Future prospects

- Verifying robustness of projected ensemble framework against 'False positives' of quantum chaos.
- Faster unitary designs from translation invariant Hamiltonians?
- Interplay between measurements and dynamics when both the former and the latter exhibit symmetries.
- Deep thermalization in systems with non-abelian symmetries.

# Moments of the projected ensembles of $Z_2$-symmetric states

Symmetry group: $\{\mathbb{I}_{2^N}, \sigma_x^{\otimes N}\}$.

Projectors onto symmetric subspaces:

$$\mathbf{Z}_0 = \frac{\mathbb{I}_{2^N} + \sigma_x^{\otimes N}}{2} \text{ and } \mathbf{Z}_1 = \frac{\mathbb{I}_{2^N} - \sigma_x^{\otimes N}}{2}$$

Measurement basis: $\mathcal{B} \equiv \{|b\rangle\}$

$$\mathcal{M}_{\mathbb{Z}_2}^t \propto \sum_{|b\rangle \in \mathcal{B}} \langle b|\mathbf{Z}_k|b\rangle^{\otimes t} \mathbf{\Pi}_A^t,$$

If the measurements are performed in $\sigma_x$-basis, then the moments of the projected ensemble are

$$\mathcal{M}_{\mathbb{Z}_2}^t = \frac{1}{\mathcal{N}} \left( \mathbf{Z}_{0,N_A}^{\otimes t} + \mathbf{Z}_{1,N_A}^{\otimes t} \right) \mathbf{\Pi}_A^t,$$

where $\mathcal{N}$ is the normalizing constant.