Classification of qubit cellular automata on hypercubic lattices

Andrea Pizzamiglio

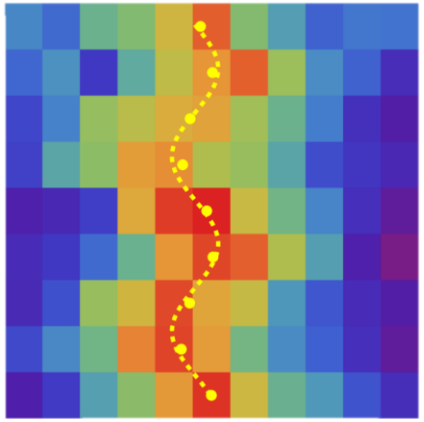
joint work with
Alessandro Bisio & Paolo Perinotti

Quantum Science Generation 2025, ETC Trento*

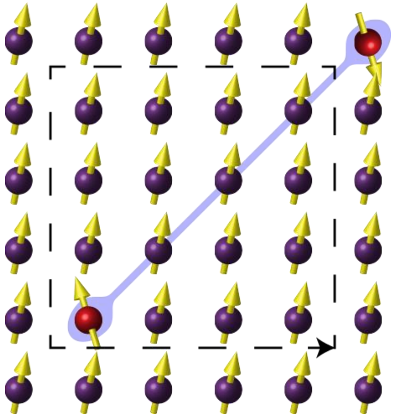
Finanziato dall'Unione europea
NextGenerationEU

Quantum Cellular Automata

Physics

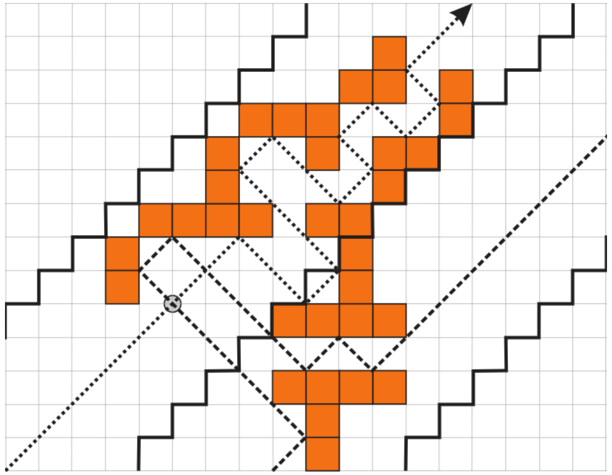


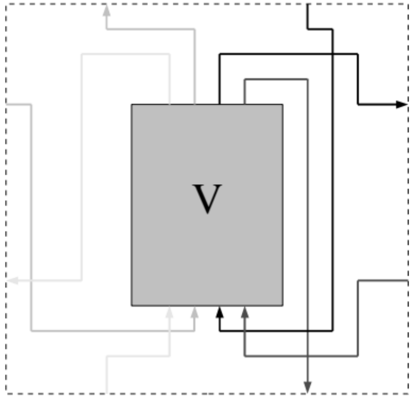
A. Bisio, P. Perinotti et al.,
PRR 6, 033136 (**2024**)

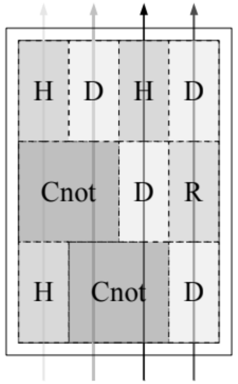


H. Po, L. Fidkowski et al.,
PRX 6, 041070 (**2016**)

Computer Science

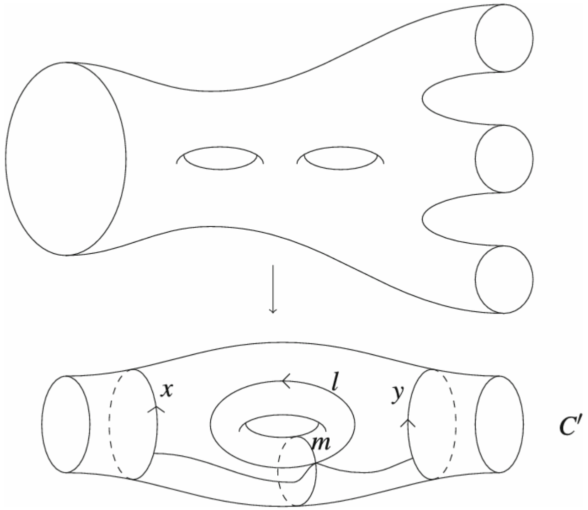


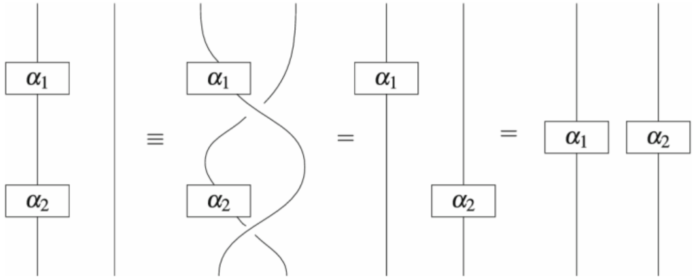




P. Arrighi,
Nat. Comput. 18, 885–899 (**2019**)

Maths



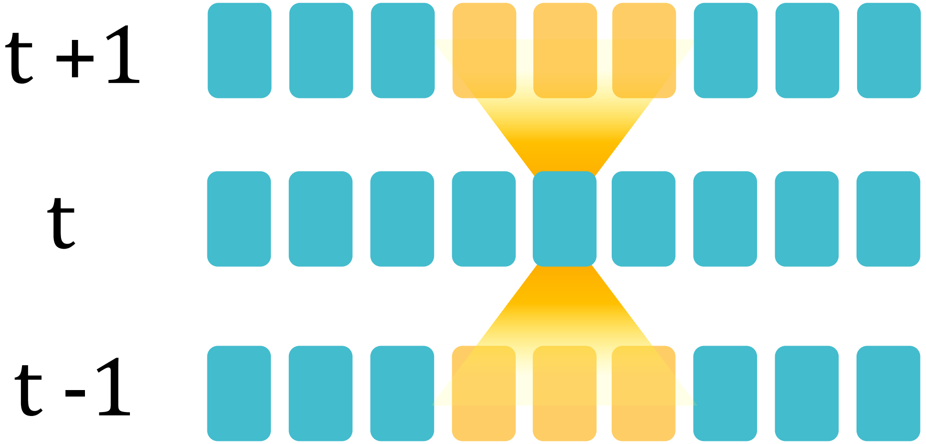


M. Freedman and M. B. Hastings,
CMP 376, 1171 (**2020**)

What is a Quantum Cellular Automaton?

R. Feynman,
Int. J. Theor. Phys. 21, (1982)

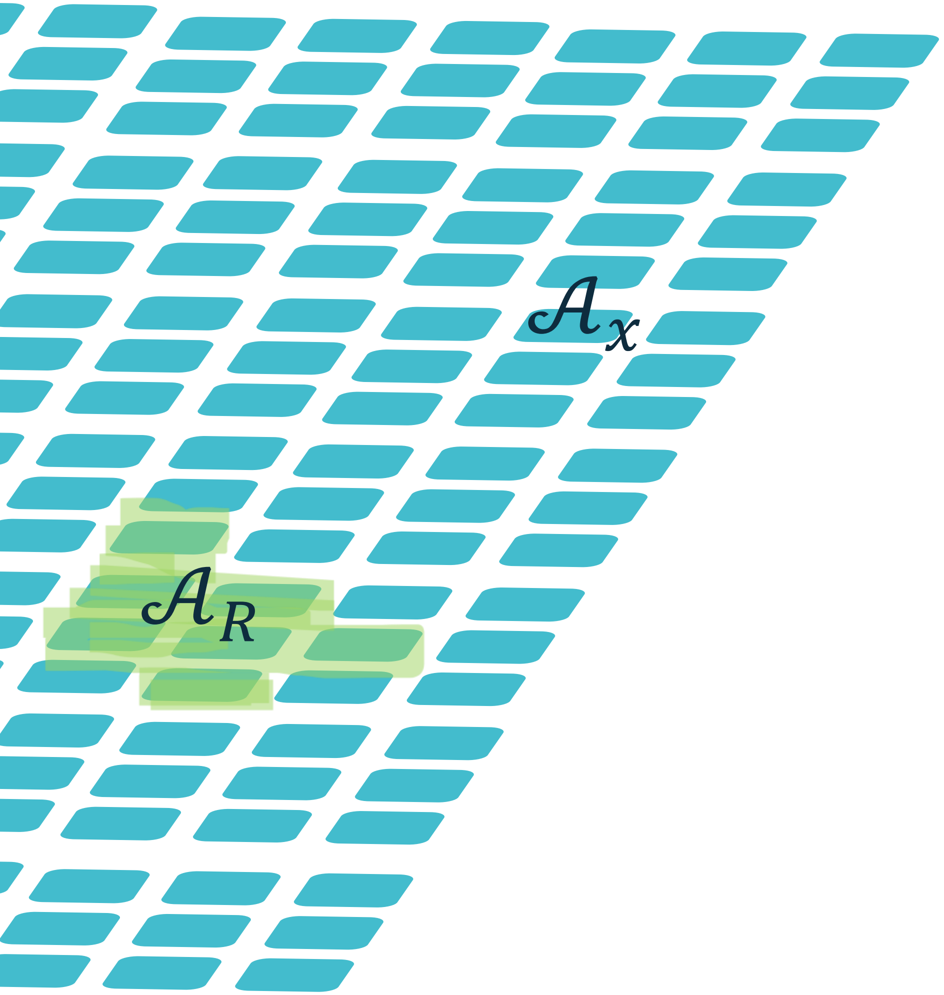
A Quantum Cellular Automaton (QCA) is the most general **discrete-time**, **local**, **reversible** dynamics of a lattice of quantum systems



Definition

B. Schumacher, R. F. Werner, arXiv 0405174 (2004)

∞ lattice $\mathbb{Z}^d$



 $= \mathcal{A}_x = \mathcal{L}(\mathbb{C}^D)$

D-level quantum systems

$\mathcal{A}_\mathrm{R} = \bigotimes_{x \in \mathrm{R}} \mathcal{A}_x$

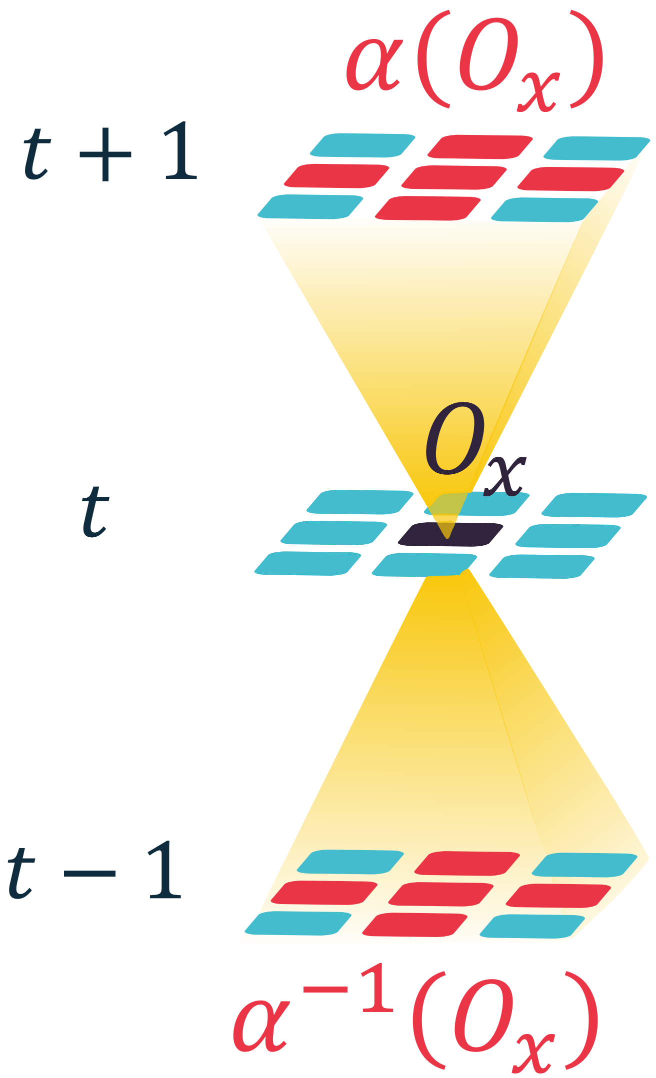
Local algebra of finite region R

$\mathcal{A}_{\mathbb{Z}^d} := \overline{\bigcup_{\mathrm{R} \subset \mathbb{Z}^d} \mathcal{A}_\mathrm{R}}^{\|\cdot\|_\infty}$

Algebra of the ∞ lattice $\mathbb{Z}^d$

Definition

B. Schumacher, R. F. Werner, arXiv 0405174 (2004)

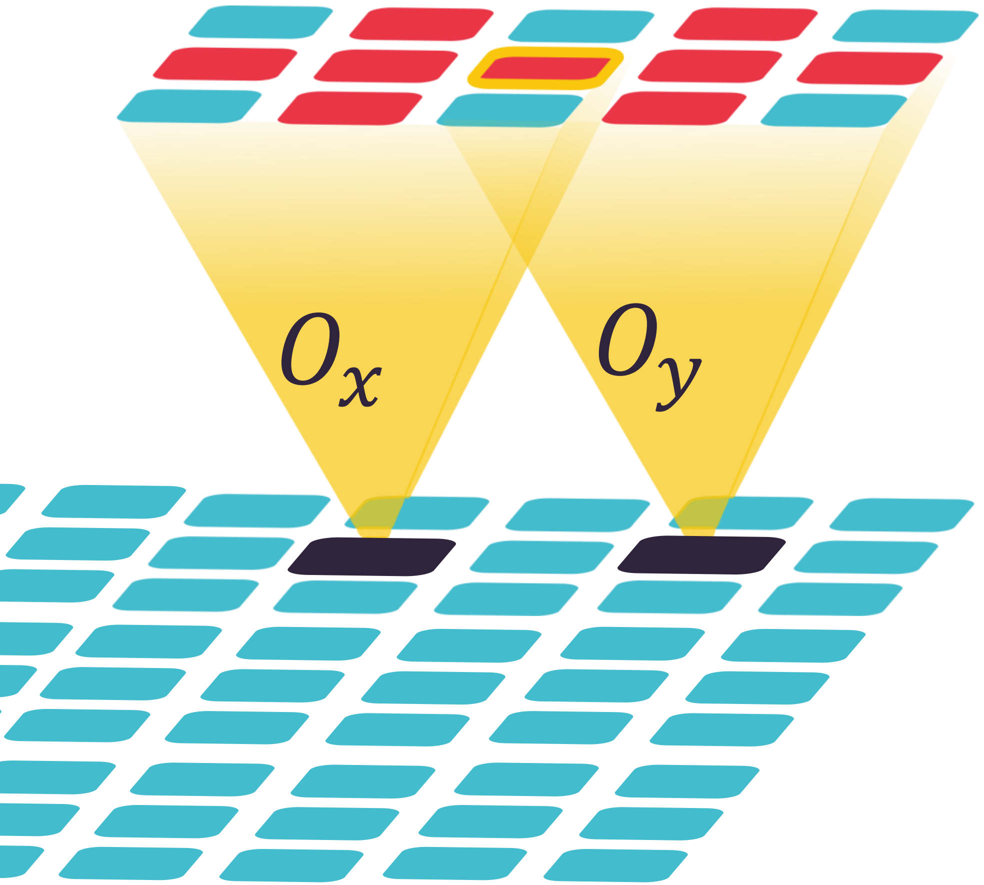


A Quantum Cellular Automaton α with neighborhood $\mathcal{N}$ is an automorphism of $\mathcal{A}_{\mathbb{Z}^d}$ such that:

- $O_x \in \mathcal{A}_x \Rightarrow \alpha(O_x) \in \mathcal{A}_{x+\mathcal{N}}$ (locality)
- $[\alpha, \tau^y] = 0 \;\; \forall$shift τ^y by $y \in \mathbb{Z}^d$ (translation invariance)

Definition

B. Schumacher, R. F. Werner, arXiv 0405174 (2004)



Update Rules of α:

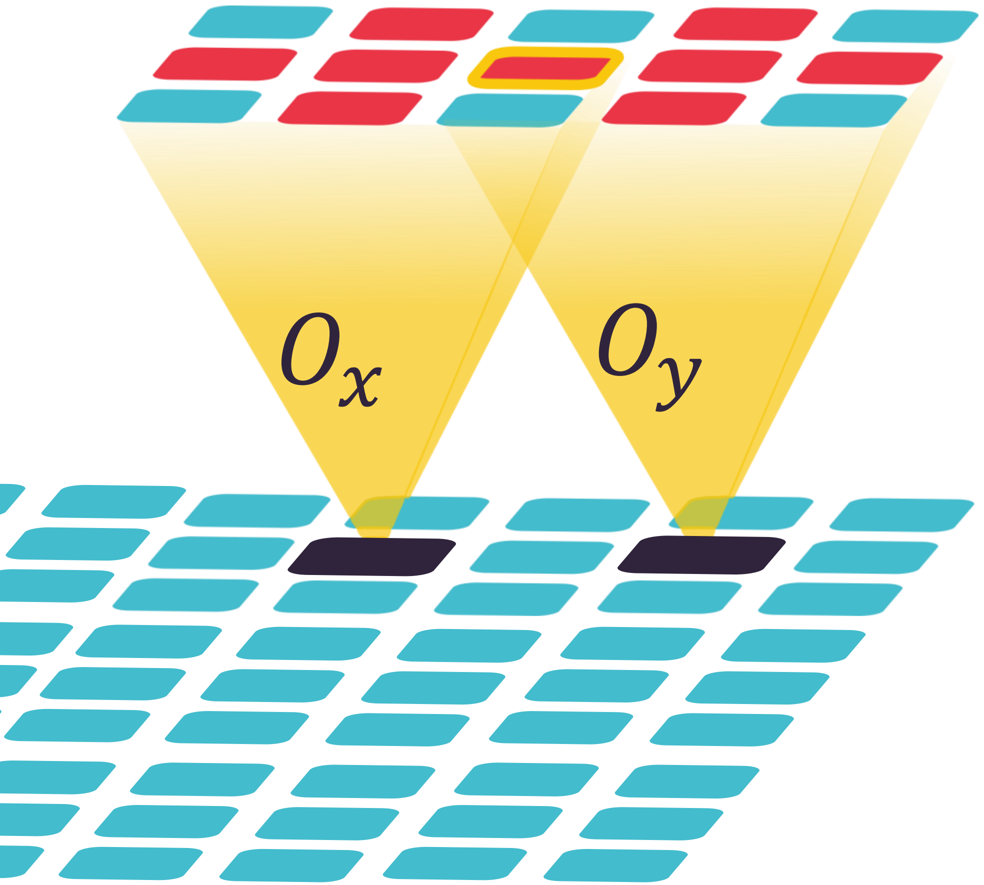
unitary operators

$$\alpha(O_x) = U^{\dagger}_{x+\mathcal{N}}(\,O_x \otimes I)\,U_{x+\mathcal{N}}$$

$$\alpha\left(O_x O_y\right) = \alpha(O_x)\alpha\left(O_y\right)$$

Definition

B. Schumacher, R. F. Werner, arXiv 0405174 (2004)



Update Rules of α:

unitary operators

$$\alpha(O_x) = U^\dagger_{x+\mathcal{N}}(\,O_x \otimes I)\,U_{x+\mathcal{N}}$$

Overlaps constrain the rules

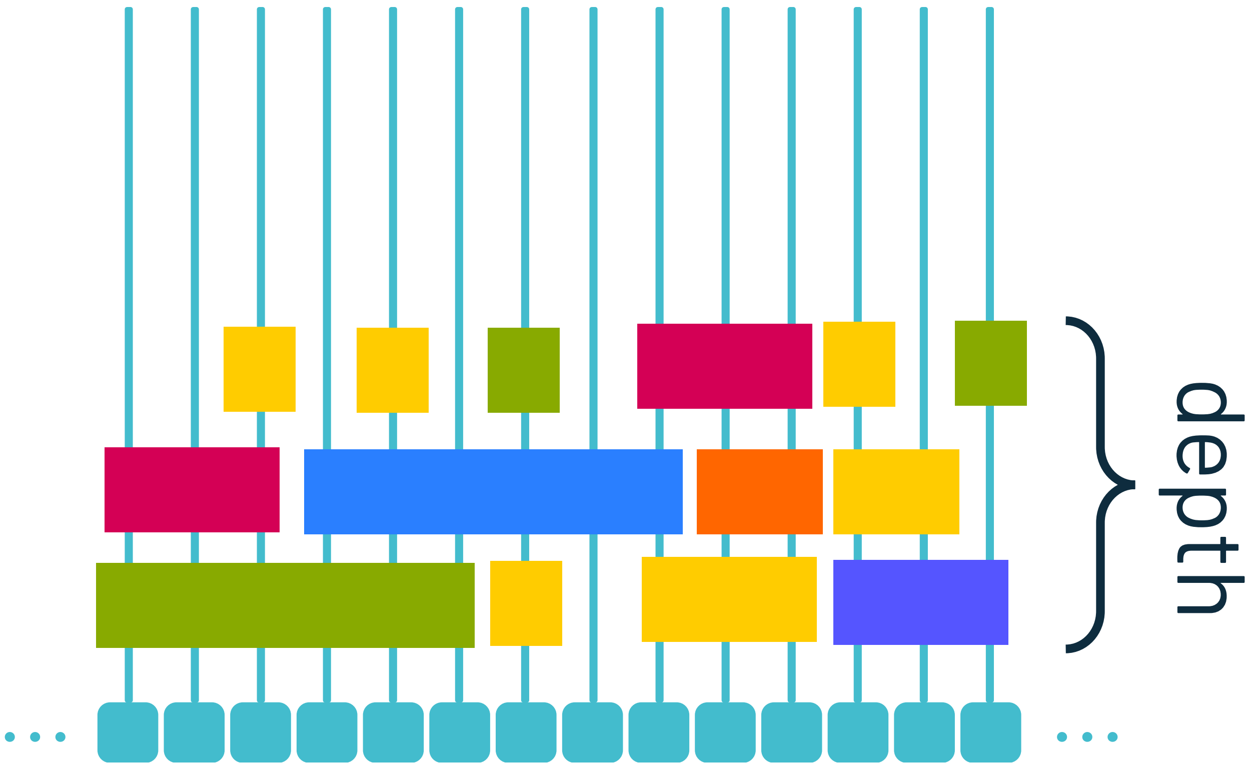
U gives an update rule of α iff

$$\left[\alpha(\mathcal{A}_x)\,,\alpha\left(\mathcal{A}_y\right)\right] = 0$$

for $y \neq x \in \mathbb{Z}^d$

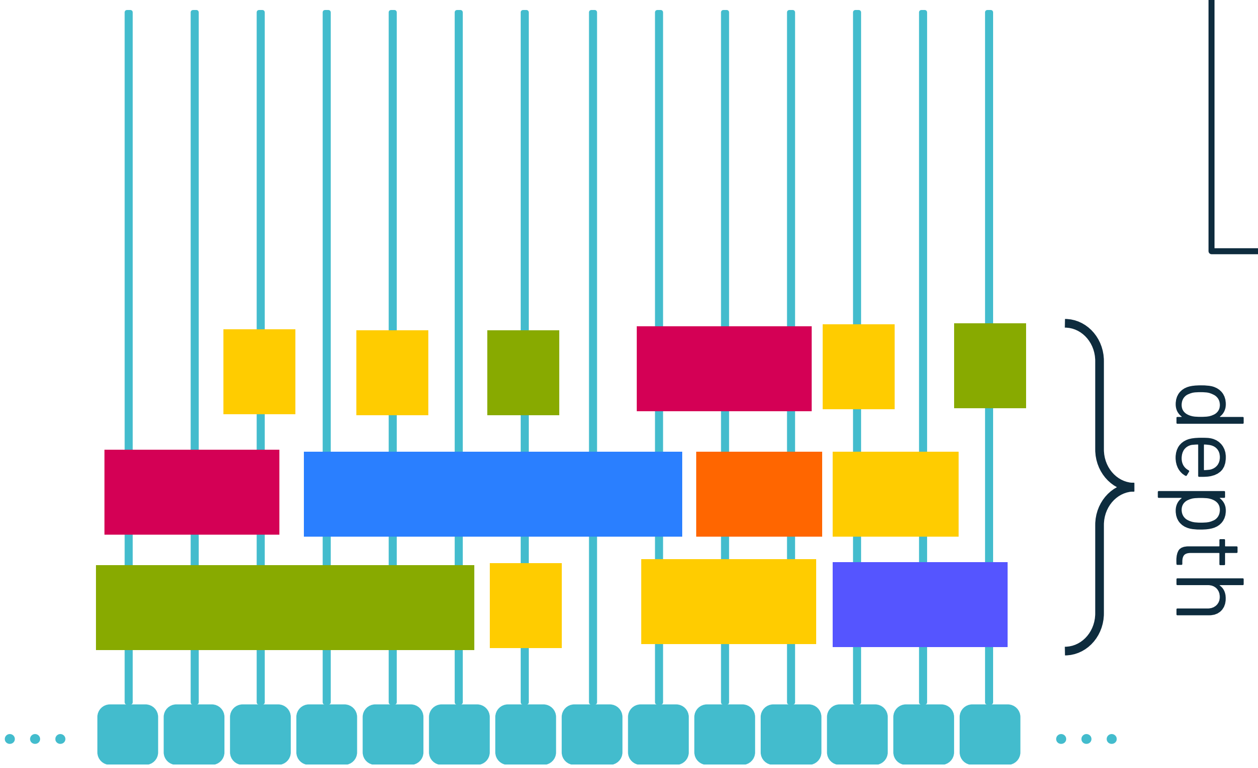
Examples

A quantum circuit is a QCA...



Examples

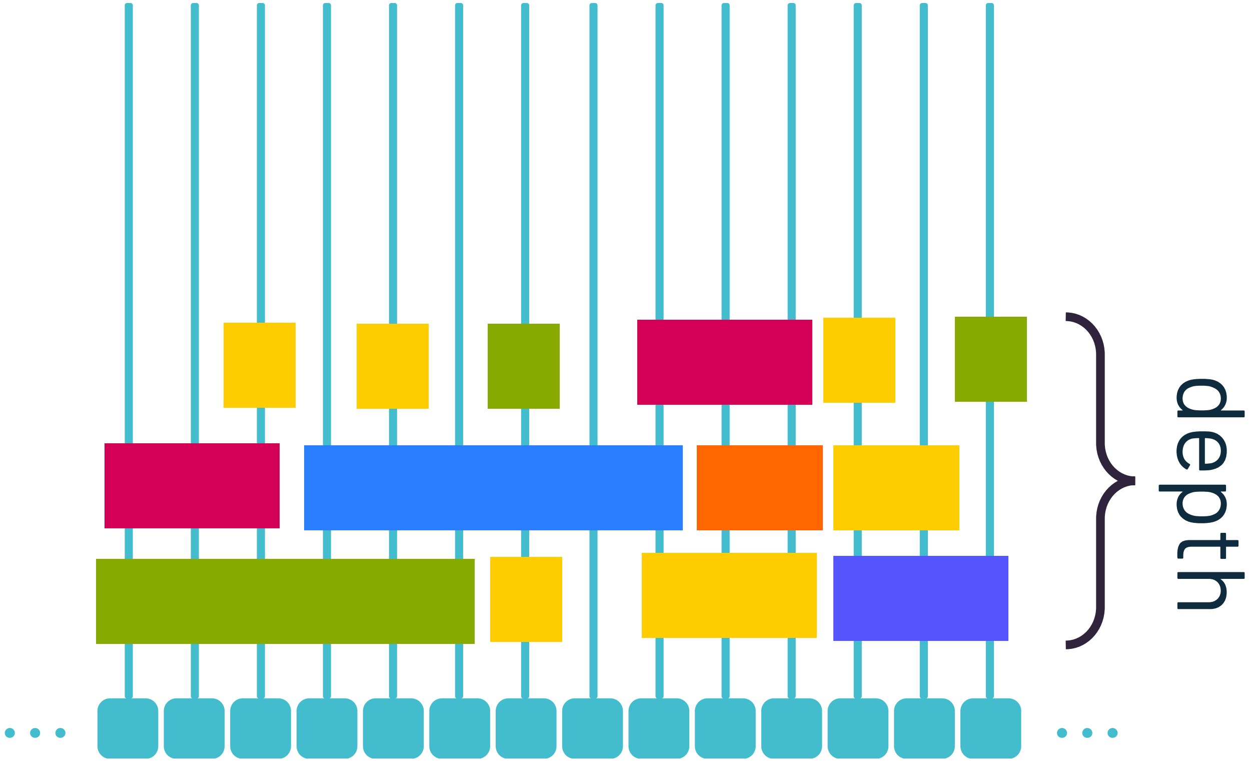
A quantum circuit is a QCA...



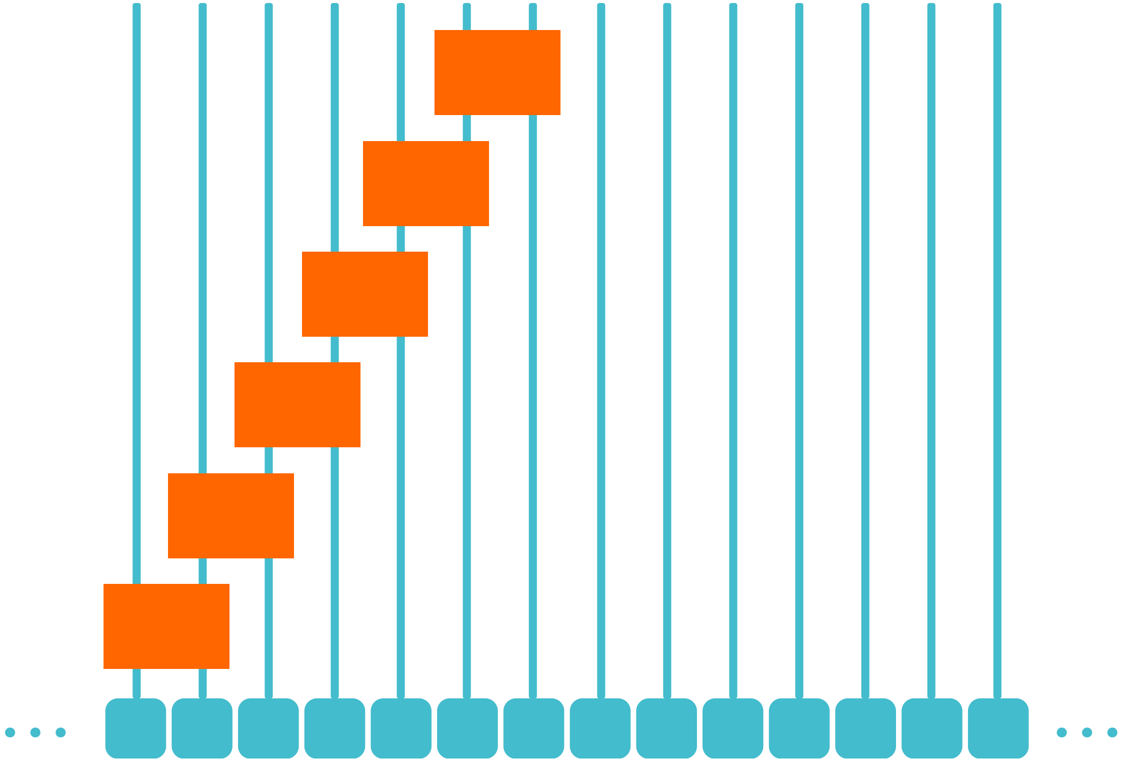
Circuit depth must be independent of the lattice size

Examples

A quantum circuit is a QCA...



but, not all QCA are circuits!



e.g. Shift $\tau^a(O_x) = O_{x+a}$

Classification of QCA

Classification Flavors

Update Rules

dynamics from information-processing principles

Invariants & Equivalence Classes modulo Circuits

topological phases of matter

Quantum Technologies

Classification: Update Rules

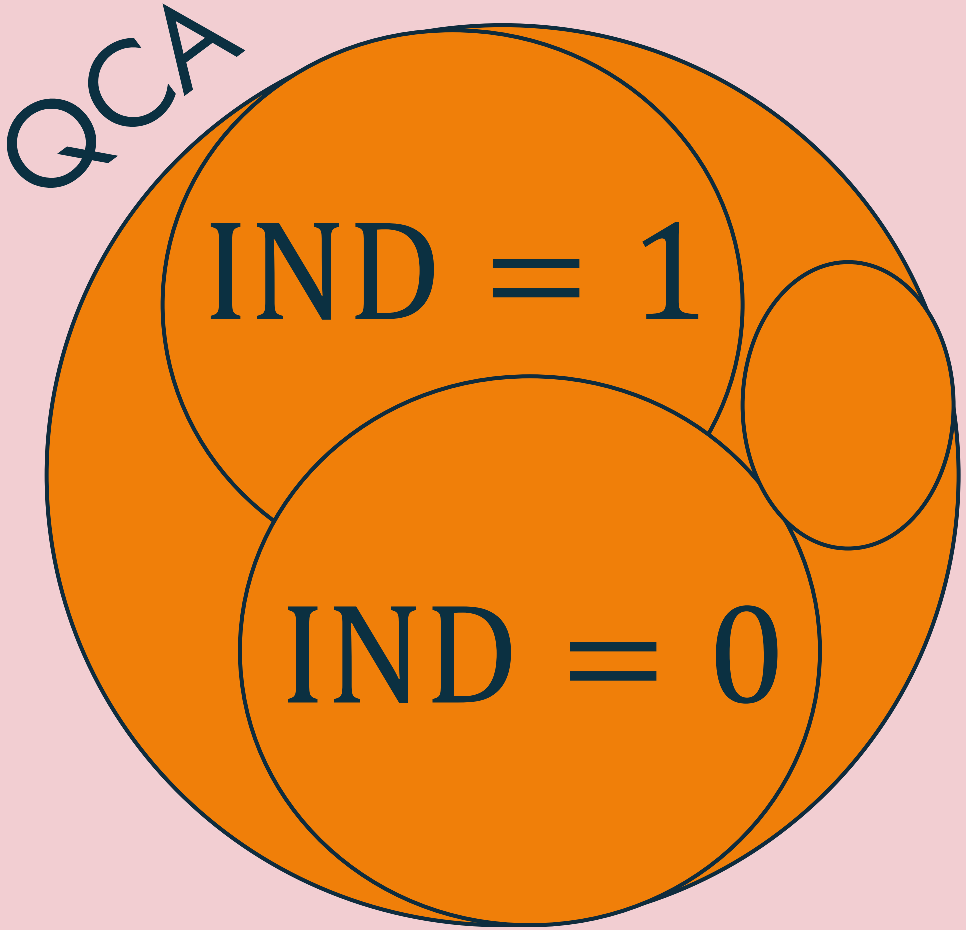
Fix:
- A lattice
- The cell algebra $\mathcal{L}(\mathbb{C}^D)$
- A neighborhood $\mathcal{N}$

Find:
All good unitaries in $\mathrm{U}\left(\mathrm{D}^{|\mathcal{N}|}\right)$ to update a cell.

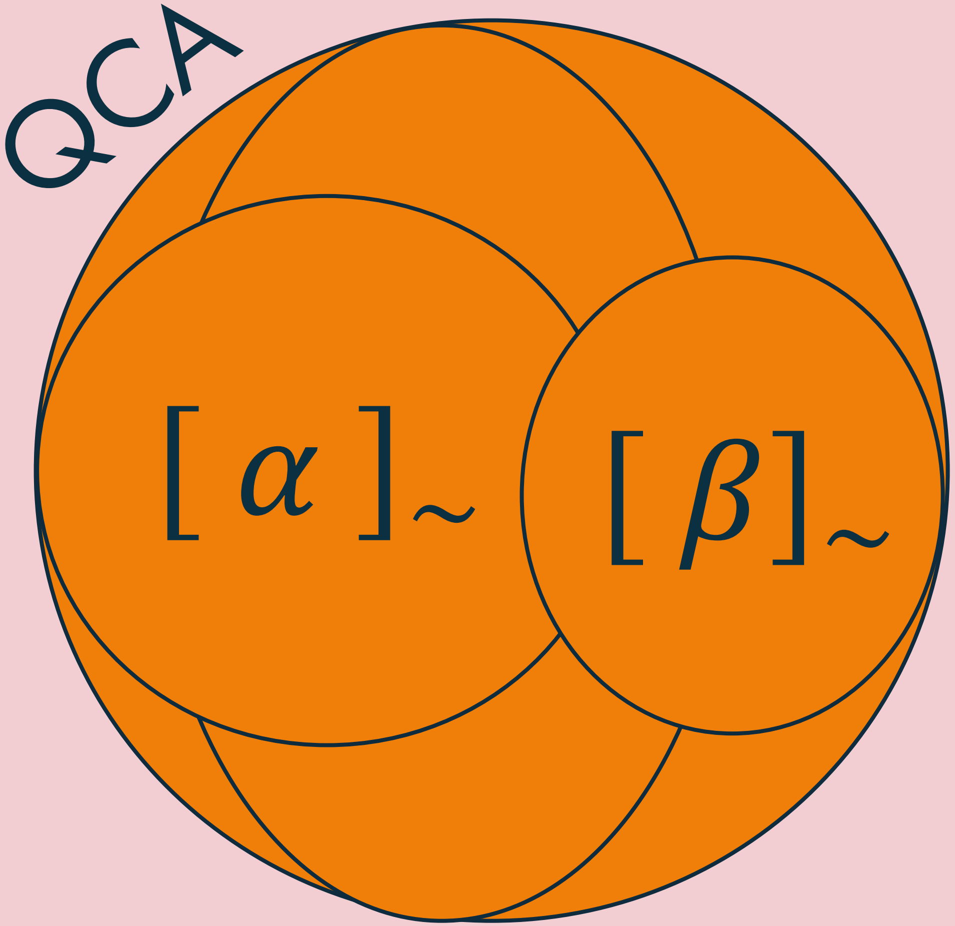
U gives a good update rule of a QCA α iff

$$\left[\alpha(\mathcal{A}_x)\,,\alpha\left(\mathcal{A}_y\right)\right] = 0$$

Classification: Invariants & Equivalence classes

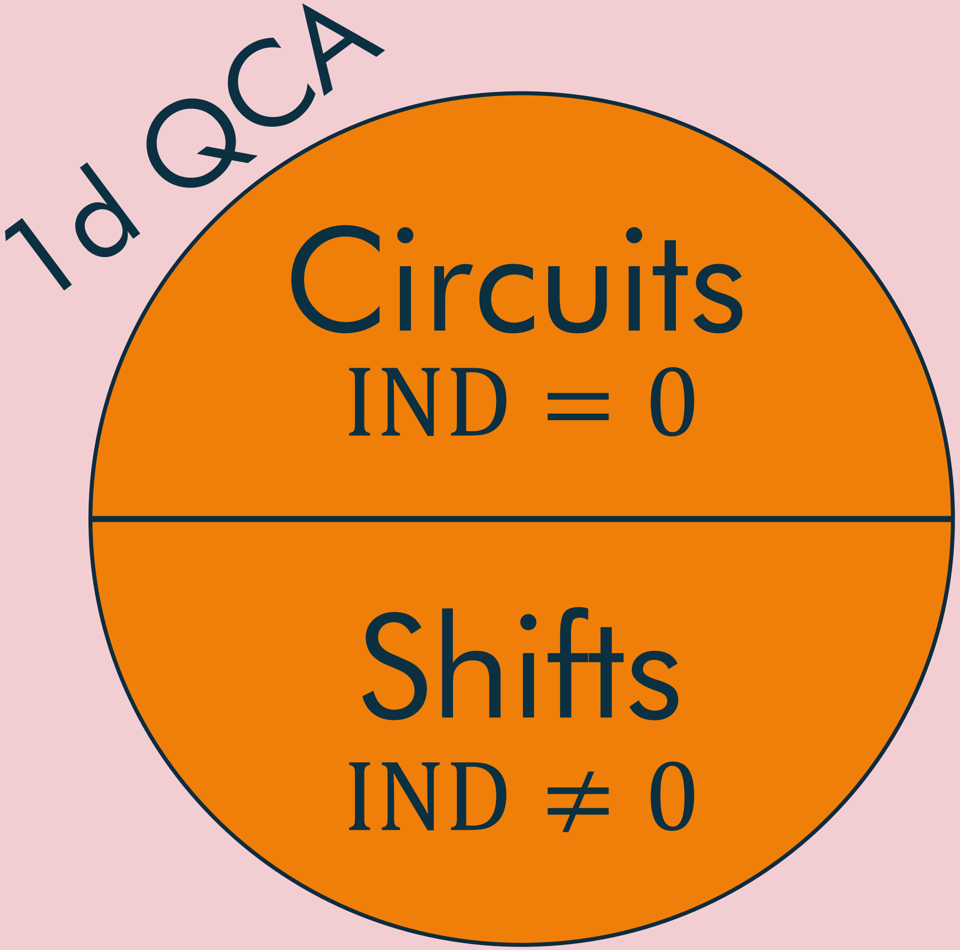


invariant IND



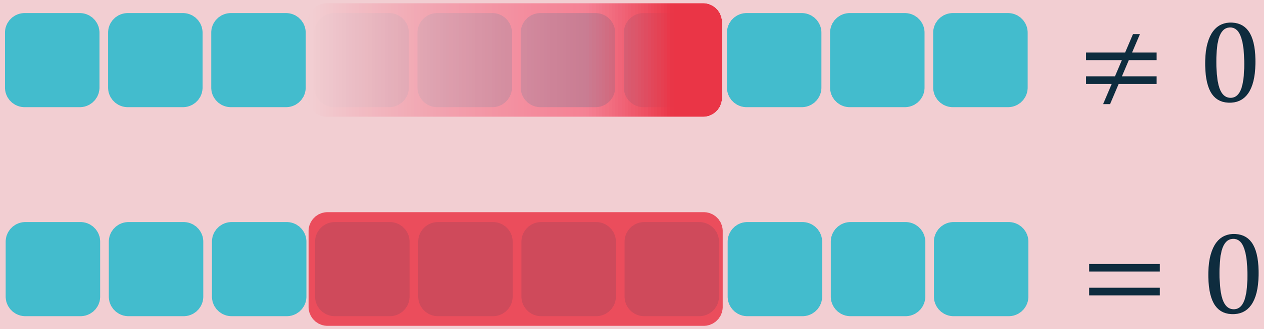
equivalence relation $\sim$

Classification: Invariants & Equivalence classes



1d QCA fully classified

$$\alpha \sim \beta \iff$$
$$\alpha = \mathrm{Circuit} \cdot \beta \iff$$
$$\mathrm{IND}(\alpha) = \mathrm{IND}(\beta)$$



GNVW Index = information flux

D. Gross, V. Nesme, H. Vogts, R.F. Werner, CMP 310, 419 (2012)

Our Classification

RESULTS:

- Update Rules
- Multi-Index
- Equivalence classes modulo Circuits

ASSUMING:

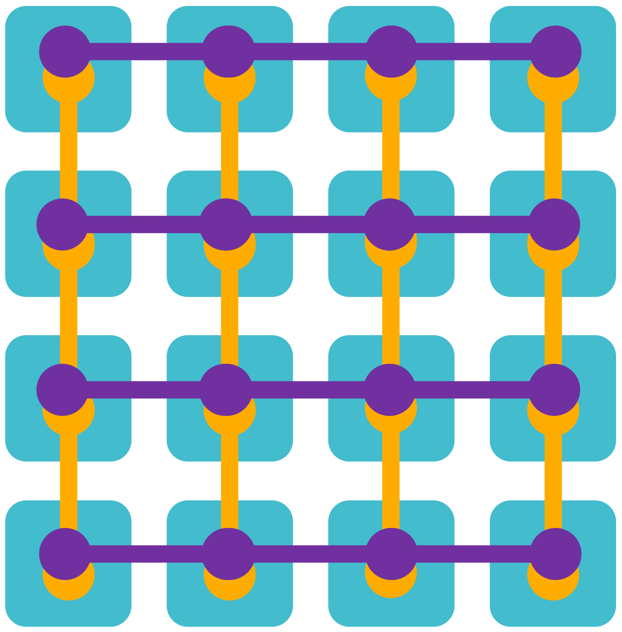
- Lattice $\mathbb{Z}^d$
- Qubits $\mathcal{L}(\mathbb{C}^2)$
- Nearest neighbor

arXiv: 2408.04493

Update Rules

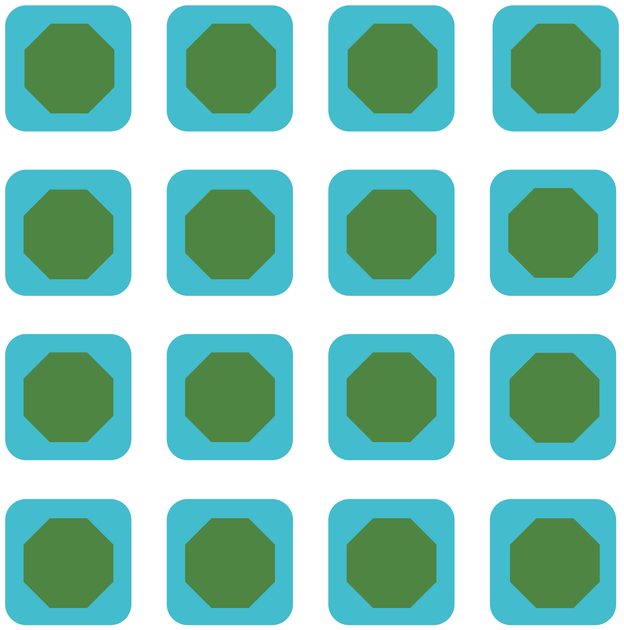
Multiply Controlled-Phase

$CP(\vec{\varphi})$



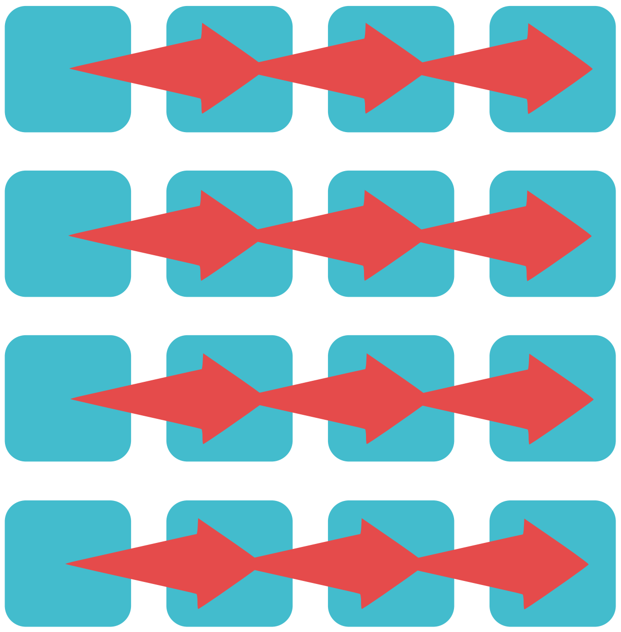
Site-wise unitary

$V(\vec{\theta})$

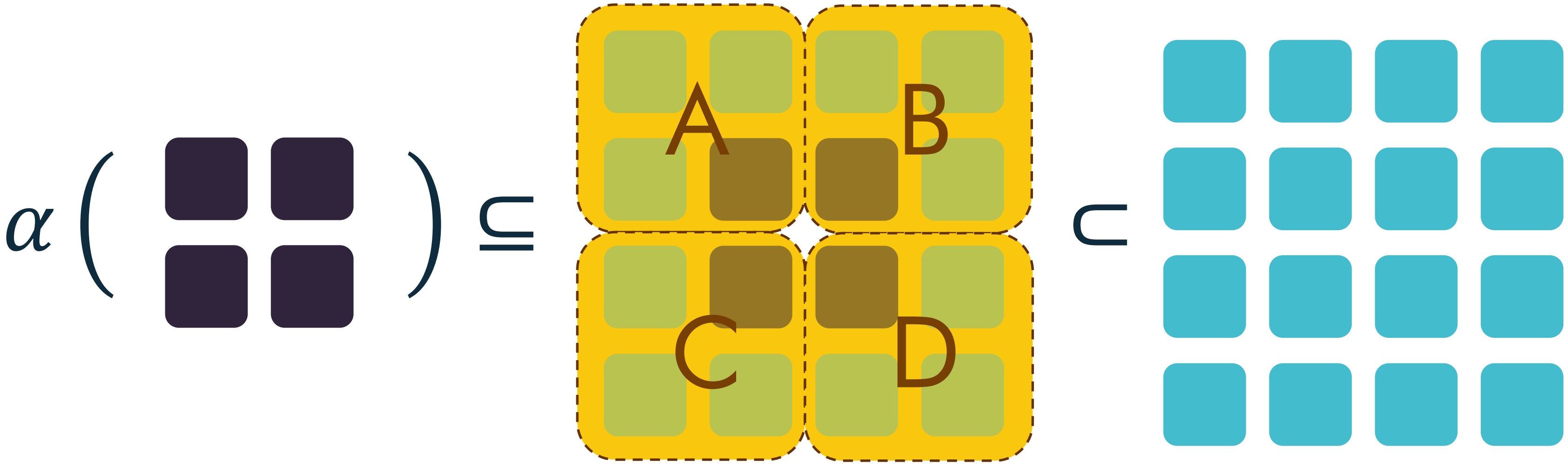


Shift

$Swap$

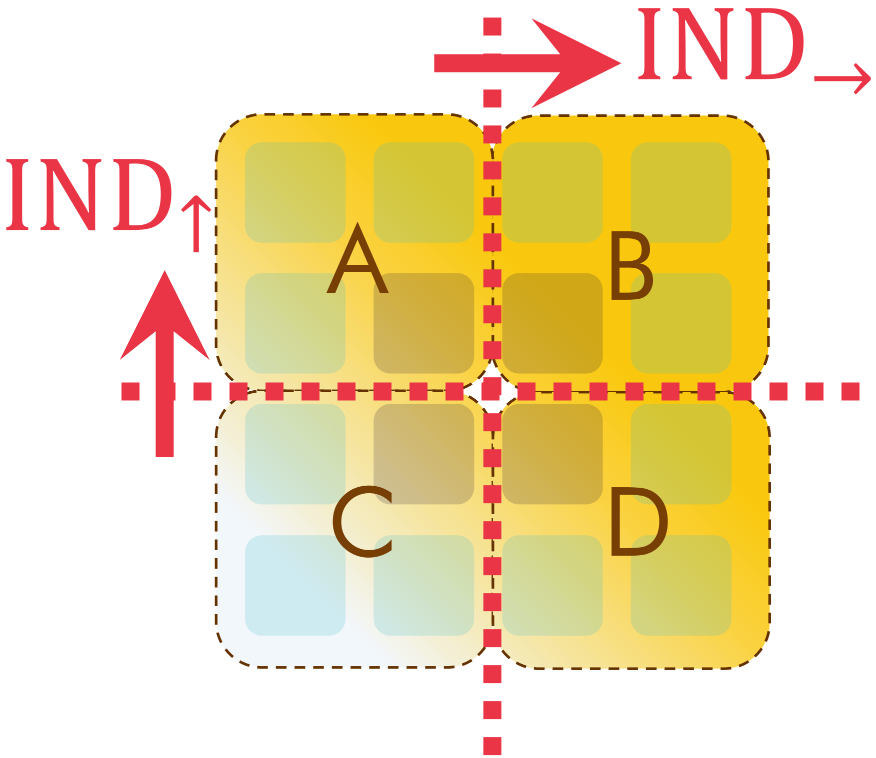


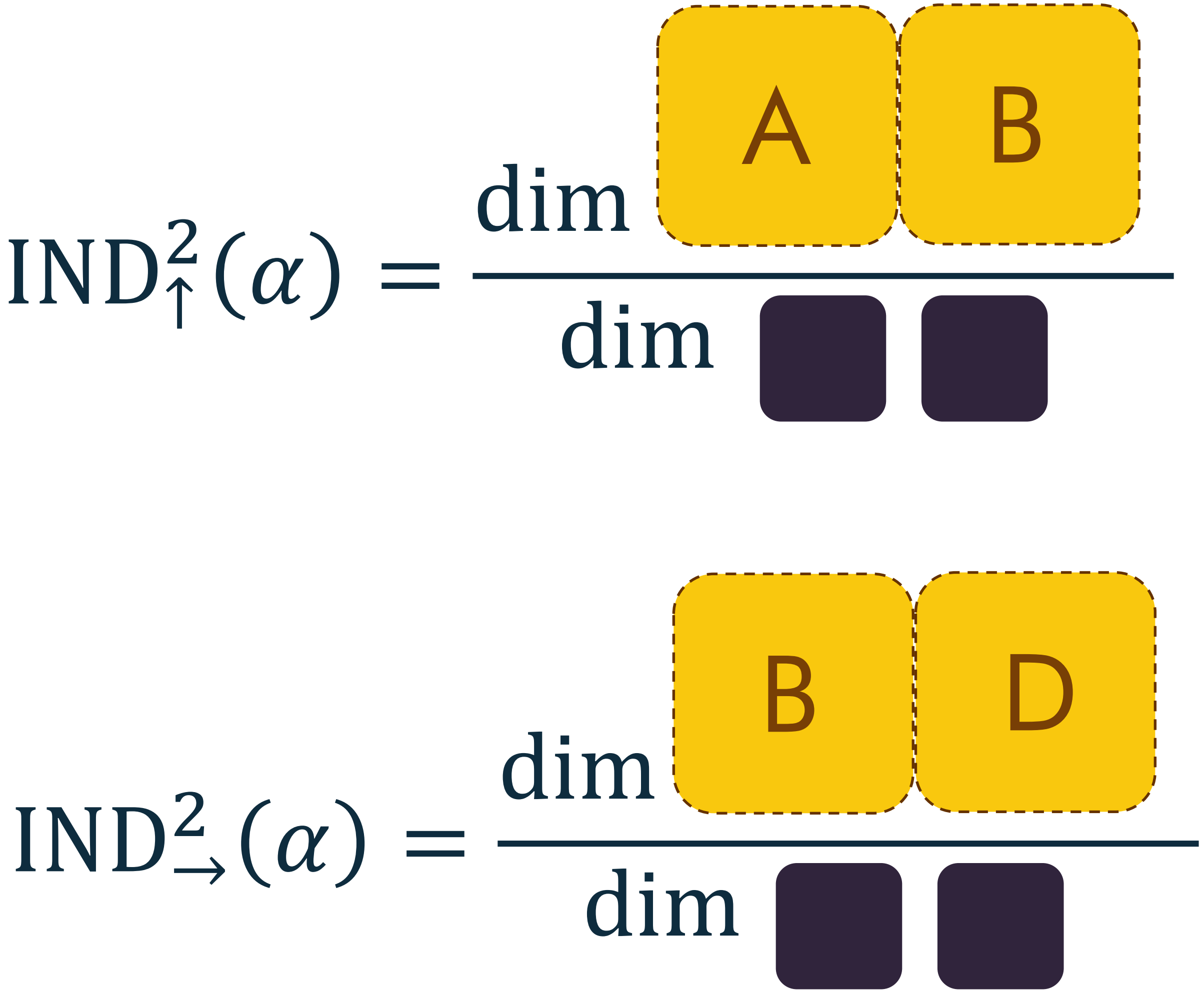
Multi-Index



Multi-Index

Information Flux as GNVW



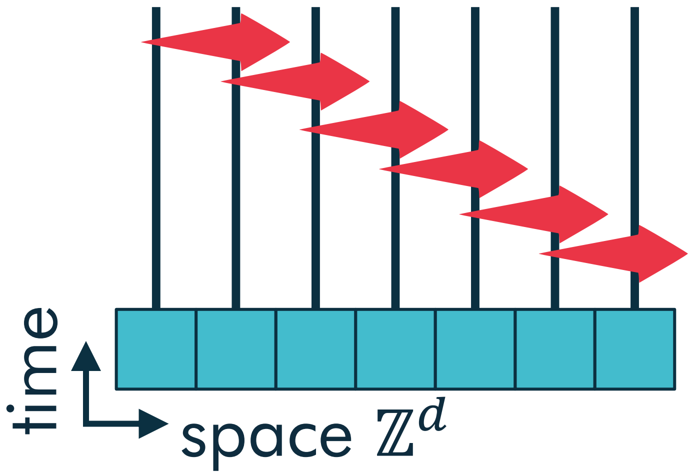


Multi-Index

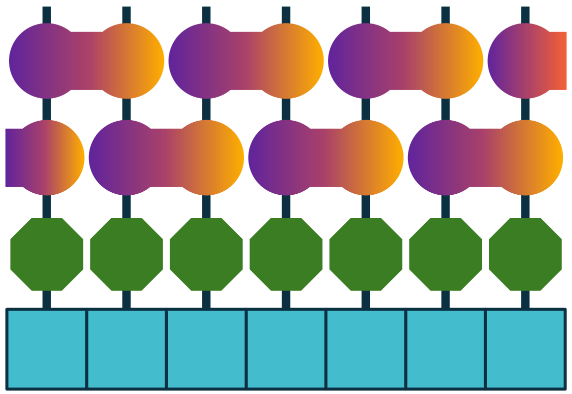
Unique Equivalence Classes

$$\alpha \sim \text{Circuit} \Leftrightarrow \log \overrightarrow{\text{IND}}\,(\alpha) = \vec{0}$$
$$\alpha \sim \text{Shift} \Leftrightarrow \log \overrightarrow{\text{IND}}\,(\alpha) \neq \vec{0}$$

Shifts



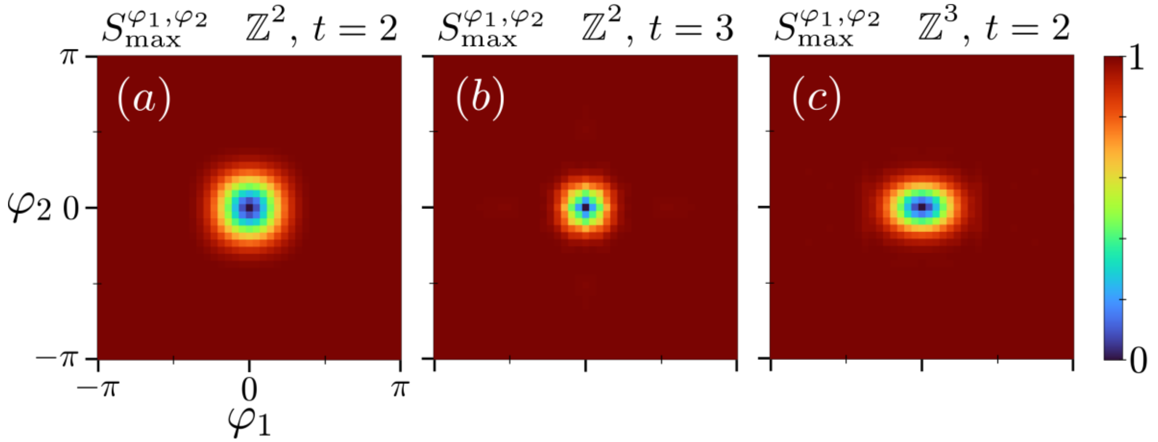
(Finite-depth) Circuits

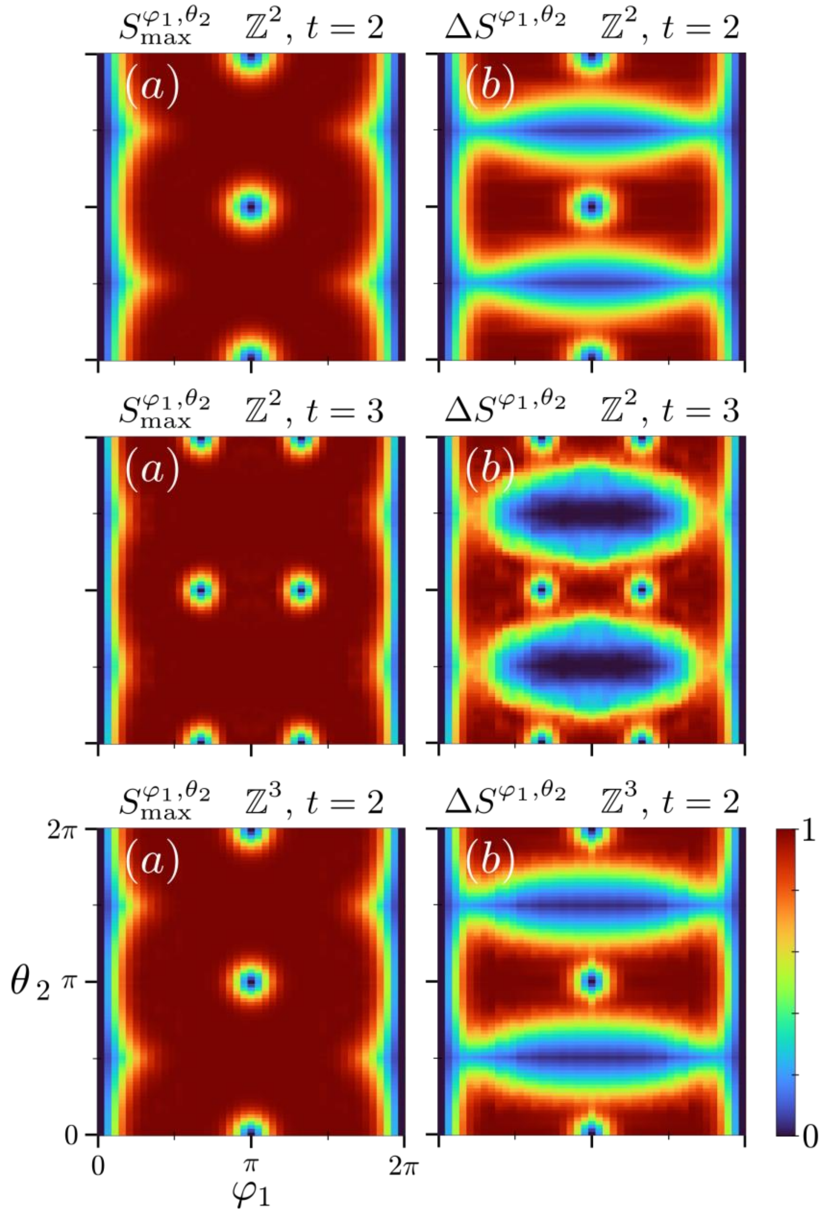


Simulation & Entanglement production

<u>Entanglement entropy of a qubit</u>
vs QCA parameters,
maximized over input states
(pure, separable, homogeneous)

Entanglement grows with time t and lattice size $\mathbb{Z}^d$





Conclusions & Outlooks

- Quantum Cellular Automata are the most general local, discrete dynamics
- Their classification in $d \geq 2$ is largely unexplored
- Our qubit QCA classification generalizes the results on $\mathbb{Z}$ to $\mathbb{Z}^d$

What's next ? (in preparation)

- Classification of Fermionic QCA (Majorana modes)
- Renormalization of QCA (coarse-grained dynamics)
- Index Theory in $d \geq 2$

Thank you for the attention!