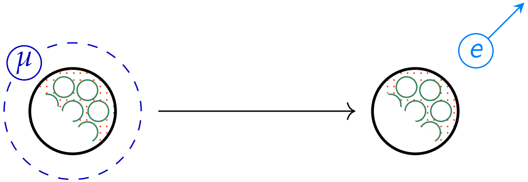
Uncertainty quantification of overlap integrals in $\mu \to e$ conversion

Frederic Noël

Universität Bern
Institute for Theoretical Physics



16.04.2025

ECT* Workshop: Lepton flavour change in nuclei

[Hoferichter, Menéndez, Noël; Phys. Rev. Lett. 130 (2023)]
[Noël, Hoferichter; JHEP 08 (2024)]
[Heinz, Hoferichter, Miyagi, Noël, Schwenk; 2412.04545 (nucl-th)]

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- Upcoming major experimental improvements [see: talks from Monday]
  $\rightarrow$ Increase NP reach by several orders of magnitude
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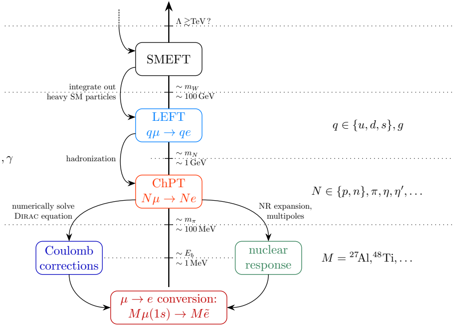
This Talk:

- Schematical Framework introduction including subleading responses
- Uncertainty assessment for leading responses [see also: talk: M. Heinz]

$\mu \to e$ conversion framework

$\mu \to e$ conversion framework components

Many different scales matter:



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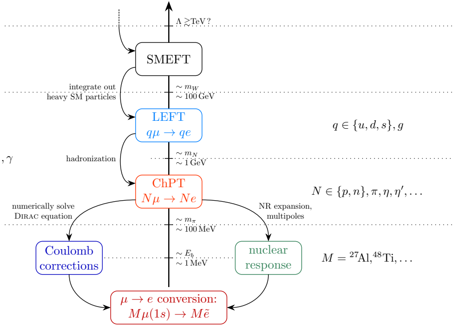

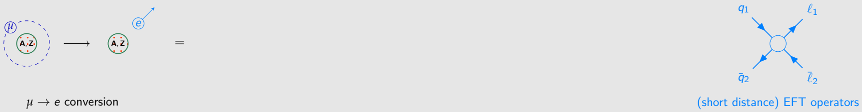

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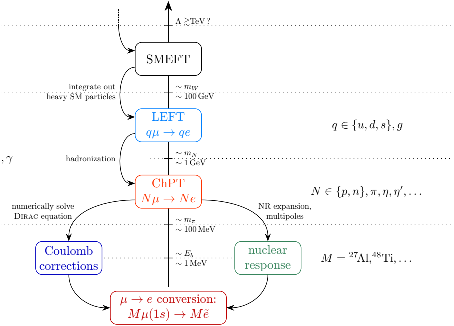

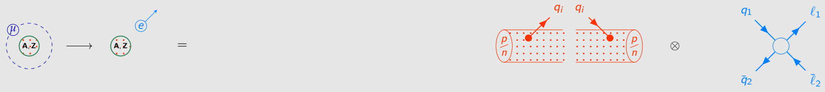

$\mu \to e$ conversion

hadronic matrix elements

(short distance) EFT operators

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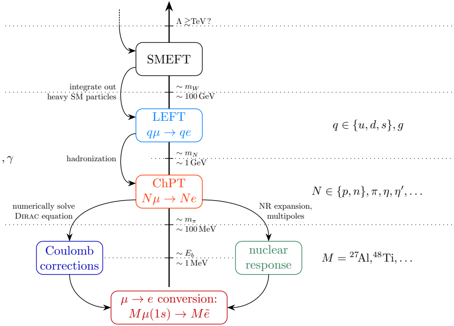

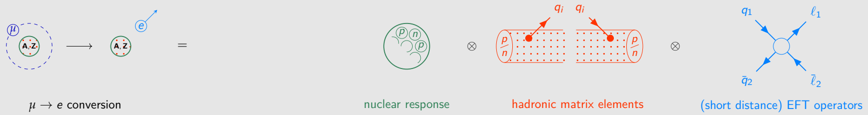

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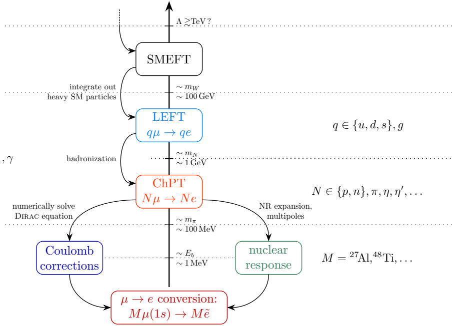

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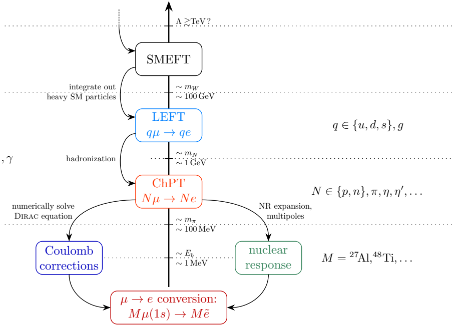

Objectives:

- Compare different probes:
  e.g.: $\mu \to e$ vs. $P \to \bar{\mu} e$
- Discriminate BSM operators

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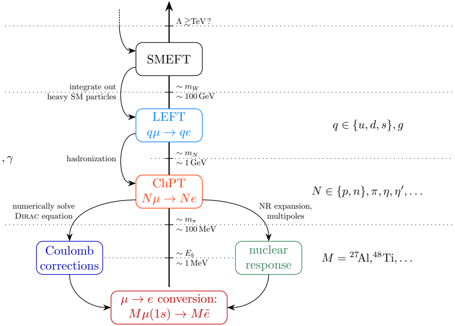

Objectives:

- Compare different probes: e.g.: $\mu \to e$ vs. $P \to \bar{\mu} e$
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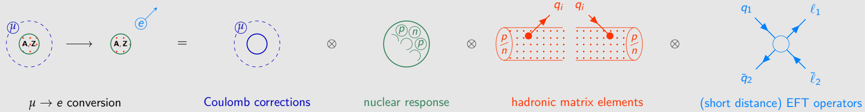
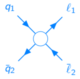
- Control theory uncertainties:
  - Hadronic matrix elements
  - Nuclear response
  - Coulomb corrections
- RG corrections



At all steps uncertainties need to be controlled!

Decomposition of the hadronic side



EFT operators

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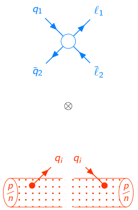

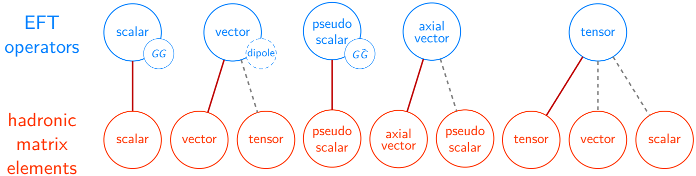

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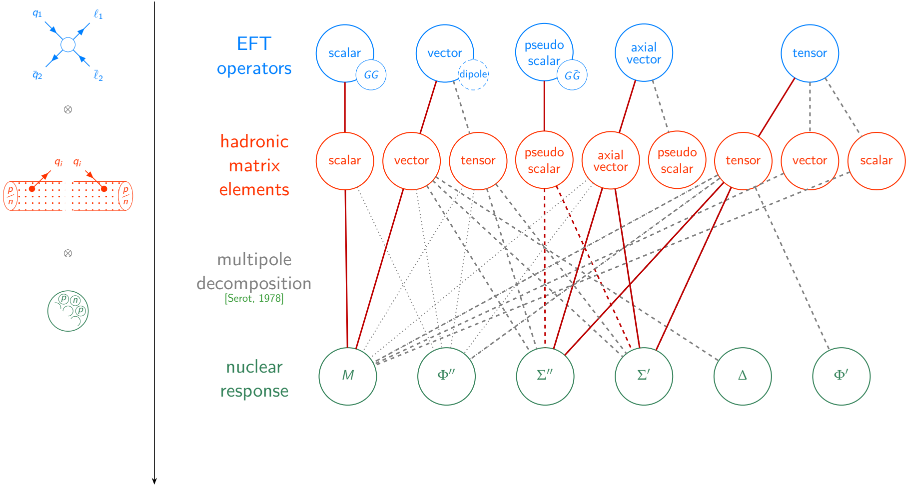

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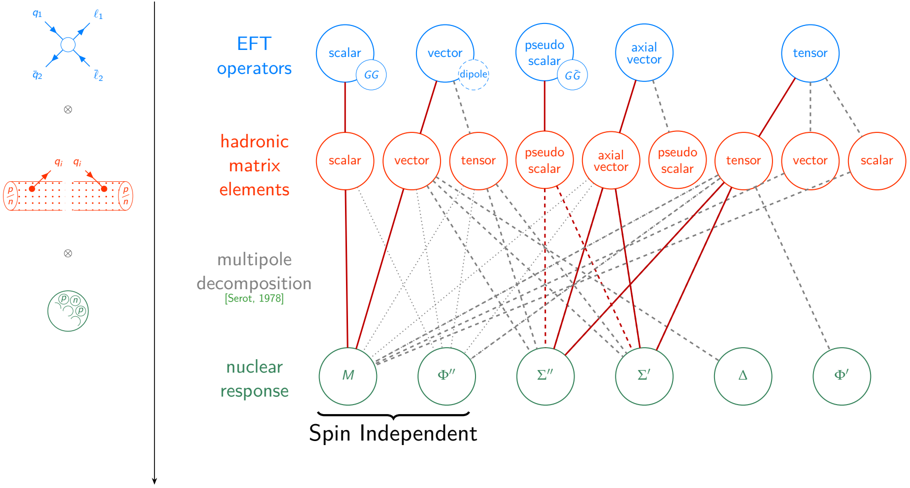

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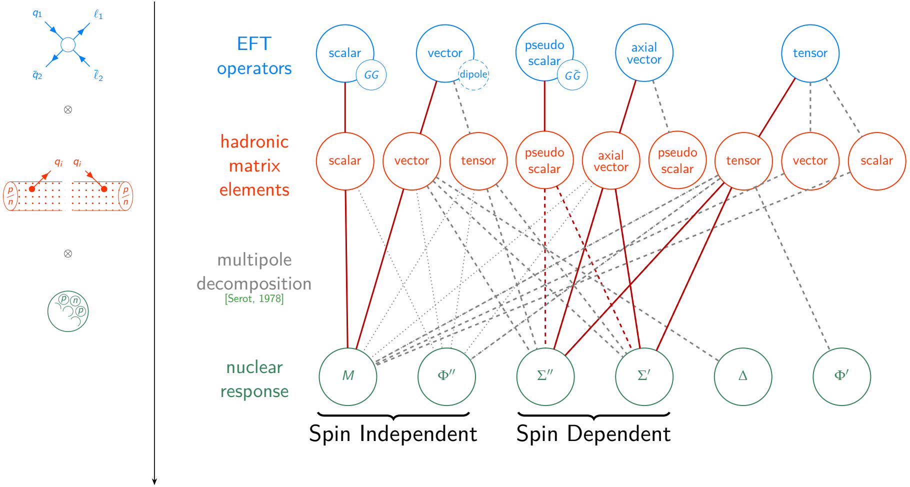

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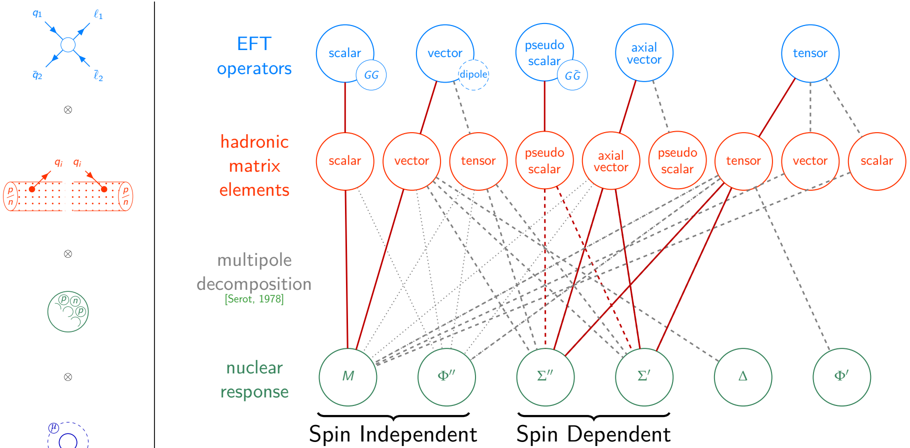

\+ Combine with Coulomb corrections on the leptonic side

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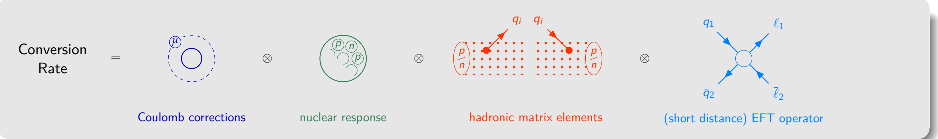
Overlap Integrals

- Overlap integrals combine nuclear responses and Coulomb corrections



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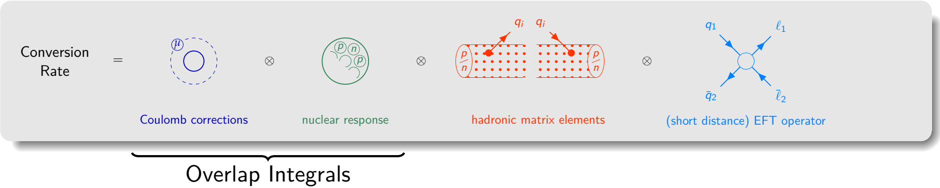

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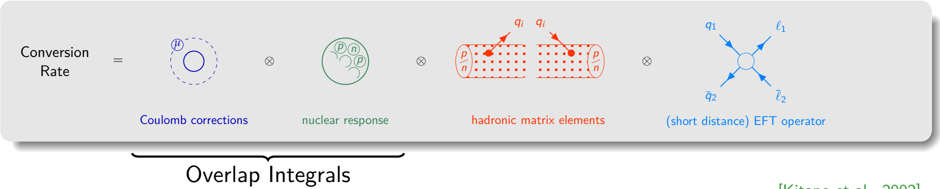

[Kitano et al., 2002]

Leading / SI overlap integrals:

$$\text{scalar:}\quad S^{(N)} = \frac{1}{2\sqrt{2}}\int_0^\infty \mathrm{d}r\,(\#N)\rho_N(r)\left[g_{-1}^{(e)}(r)\,g_{-1}^{(\mu)}(r) - f_{-1}^{(e)}(r)\,f_{-1}^{(\mu)}(r)\right]$$

$$\text{vector:}\quad V^{(N)} = \frac{1}{2\sqrt{2}}\int_0^\infty \mathrm{d}r\,(\#N)\rho_N(r)\left[g_{-1}^{(e)}(r)\,g_{-1}^{(\mu)}(r) + f_{-1}^{(e)}(r)\,f_{-1}^{(\mu)}(r)\right]$$

$$\text{dipole:}\quad D = -\frac{4m_\mu}{\sqrt{2}}\int_0^\infty \mathrm{d}r\quad E(r)\quad \underbrace{\left[g_{-1}^{(e)}(r)\,f_{-1}^{(\mu)}(r) + f_{-1}^{(e)}(r)\,g_{-1}^{(\mu)}(r)\right]}_{\text{electron and muon wave functions}}$$

- Development of subleading overlap integrals is currently in process

Example: Indirect limits for $P \to \bar{\mu} e$ from $\mu \to e$

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$\mathrm{BR}_{\mathrm{Ti}} < 6.1 \times 10^{-13}$	$\mathrm{BR}_{\pi^0} \lesssim 4 \times 10^{-17}$	$< 3.6 \times 10^{-10}$
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- For a rigorous limits we need to scan over all Wilson coefficients $\to \exists$ (fine-tuned) scenarios where $\mu \to e$ vanishes exactly
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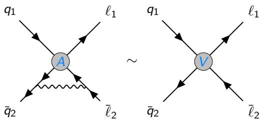
- For $\eta^{(\prime)} \to \bar{\mu} e$: in principle, no strict limits
- Cancellation easily lifted by RG corrections

  [Crivellin et al., 2017; Cirigliano et al., 2017]



Future projection for $\pi^0 \to \bar{\mu} e$

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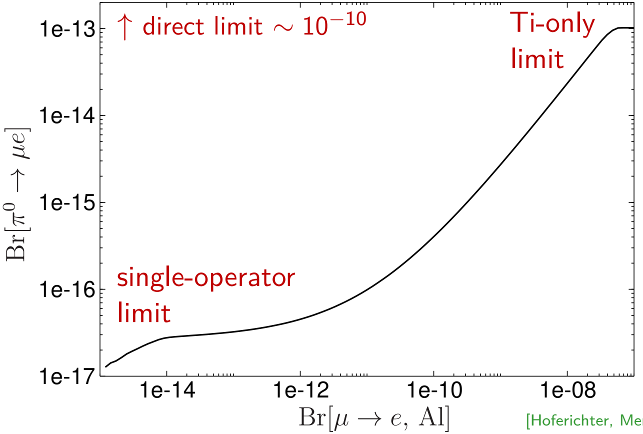
- Combining the limits from Ti and Al we find:



[Hoferichter, Menéndez, FN, 2023]

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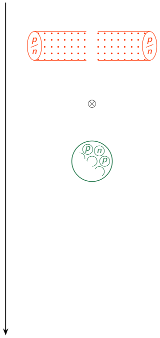

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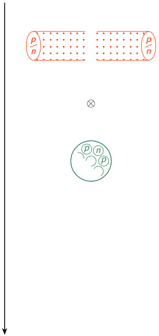

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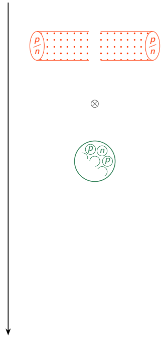

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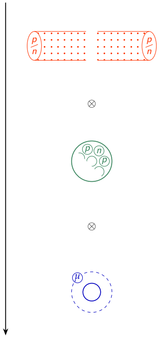

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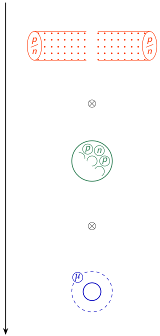

Charge densities with quantified uncertainties required

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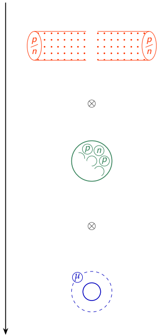
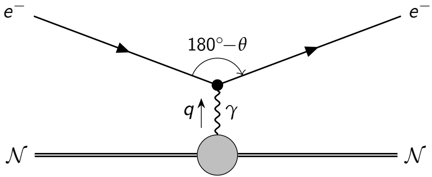


Charge densities with quantified uncertainties required

So far: As Fourier-Bessel series without uncertainties [Vries et al., 1987]
→ Redo extraction from elastic electron nucleus scattering

How to describe elastic electron scattering?

Typical description via Plane Wave Born Approximation



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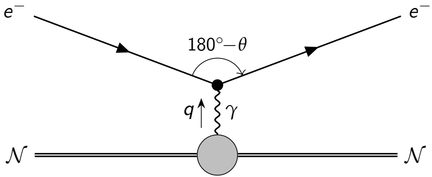

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$\underline{J = 0:}$

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- strongly dominating

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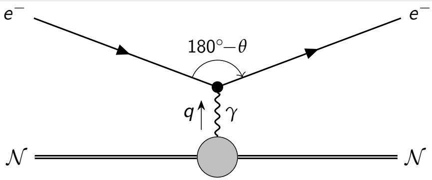

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$$F(q,\theta) \supset F_{L>0}^{\text{ch}}, F_L^{\text{mag}}$$

- become relevant where F_0^{ch} small (zeroes, high q, high θ)
- subtract before extraction

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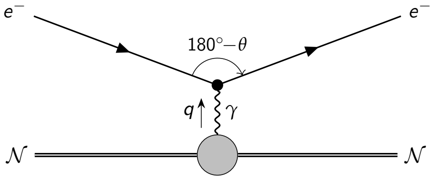

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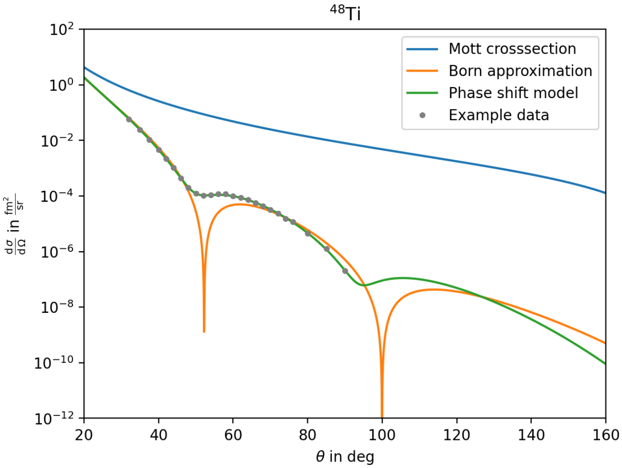
Even for $J = 0$ insufficient $\rightarrow$ Coulomb corrections

Coulomb corrections

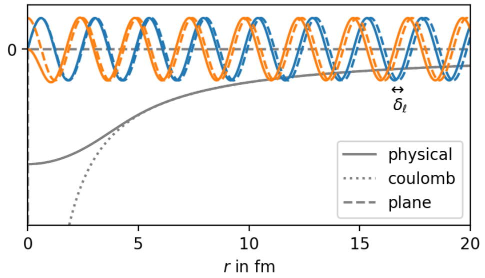


- Coulomb corrections fill out minima and shift the crosssection
- Not properly accounted for by approximative methods

Phase-shift model

- Born approximation assumes plane waves
- Finite extend of the nucleus distorts wave functions

$\rightarrow$ results in phase shift δ_ℓ, contains info about $d\sigma/d\Omega$



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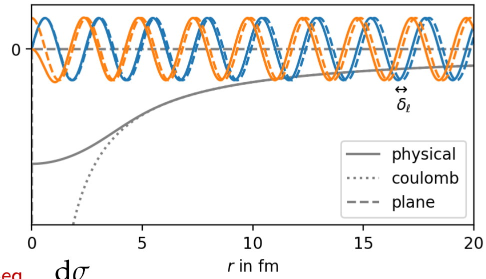

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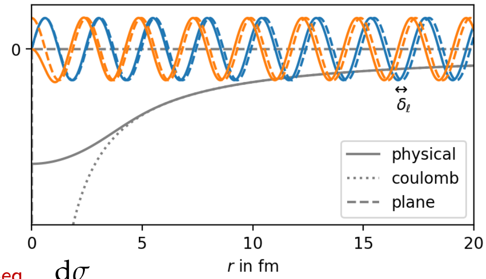

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Solve Dirac equation numerically:

$$\forall_\ell : \left[\psi_\ell \sim \begin{pmatrix} g_\ell(r) \\ i f_\ell(r) \end{pmatrix} \xrightarrow{r\to\infty} \begin{pmatrix} \cos(kr + \delta'_\ell(r)) \\ i\sin(kr + \delta'_\ell(r)) \end{pmatrix} \rightarrow \begin{array}{r} \delta'_\ell(r) = \delta'_{\ell,c}(r) + \bar{\delta}_\ell \\ \delta_\ell = \delta_{\ell,c} + \bar{\delta}_\ell \end{array} \right]$$

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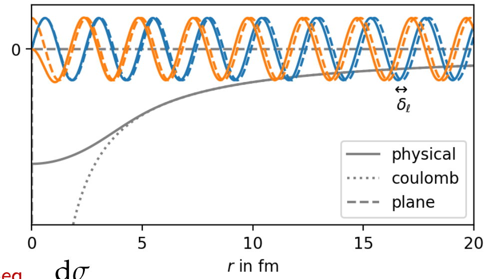

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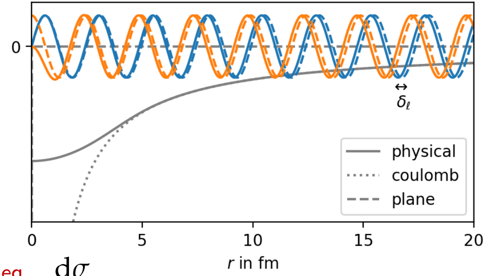

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$$\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \left(1 + \tan^2\left(\tfrac{\theta}{2}\right)\right)|f(\theta)|^2 \quad \text{with} \quad f(\theta) \sim \sum_\ell P_\ell(\cos(\theta))\, e^{2i\delta_\ell}$$

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Python package `phasr` [https://pypi.org/project/phasr]

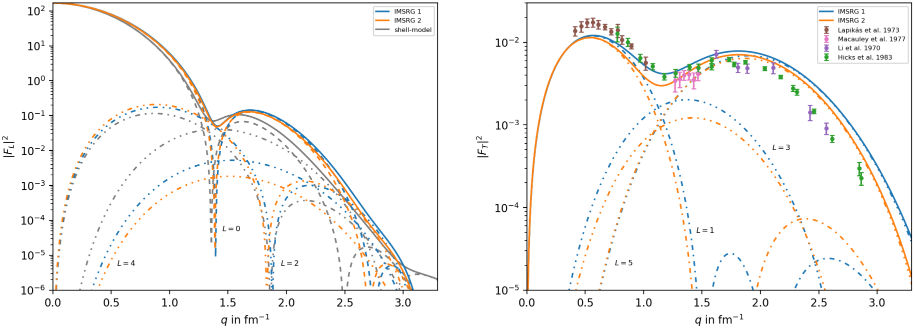
- different charge distribution parameterizations implemented
- calculates bound state and continuums solutions → overlap integrals
- calculates elastic scattering cross sections using the phase shift model
- Recent addition: Parity violating electron scattering (PVES)

Nuclei with $J > 0$

^{27}Al ($J = \frac{5}{2}$) requires $L > 0$ contributions

- Employ ab-initio calculations (using IMSRG) [talk: M. Heinz]

ZF_L^{ch} | F_L^{mag}



- Subtract and remove data points dominated by $L > 0$
- So far: No Coulomb corrections for $L > 0$

Extracting charge densities from electron scattering

- Fourier-Bessel parameterization:
  ($q_n = \frac{n\pi}{R}$ s.t. $j_0(q_n R) = 0$) [Dreher et al., 1974]

$$\rho_0(r) = \begin{cases} \sum_{n=1}^{N} a_n \, j_0(q_n r) & , \ r \leq R \\ 0 & , \ r > R \end{cases}$$

- Total charge fulfilled by construction
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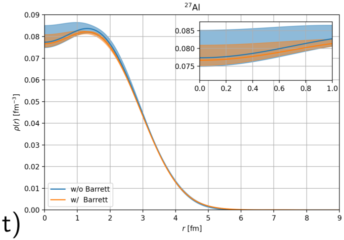

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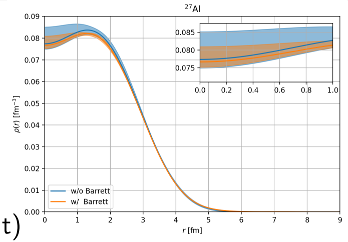

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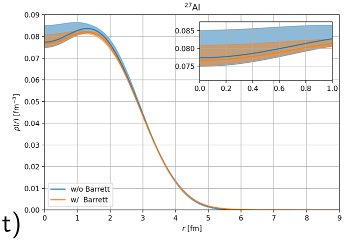

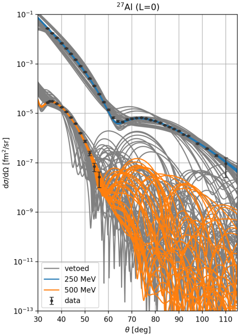

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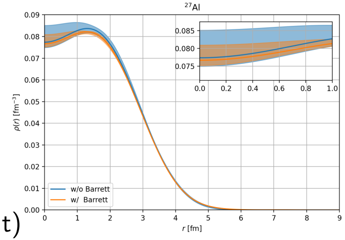
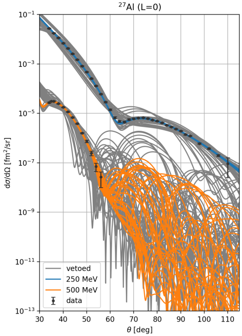


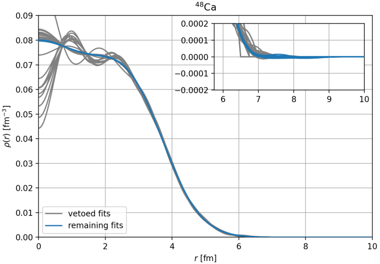


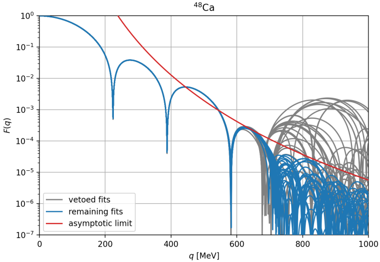
Carried out for ^{27}Al, ^{40,48}Ca, ^{48,50}Ti

Results available in python notebook [2406.06677]

Estimate uncertainties

Suppress overparametrization:





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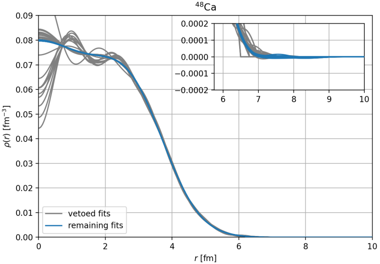
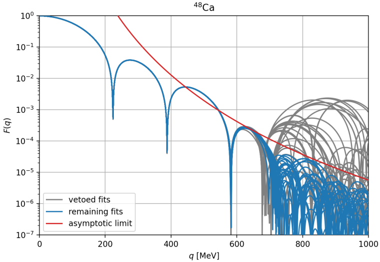
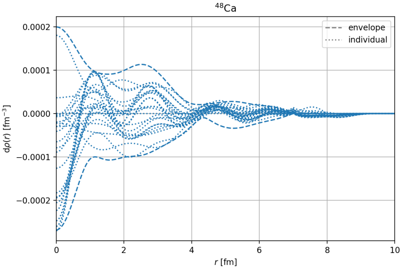
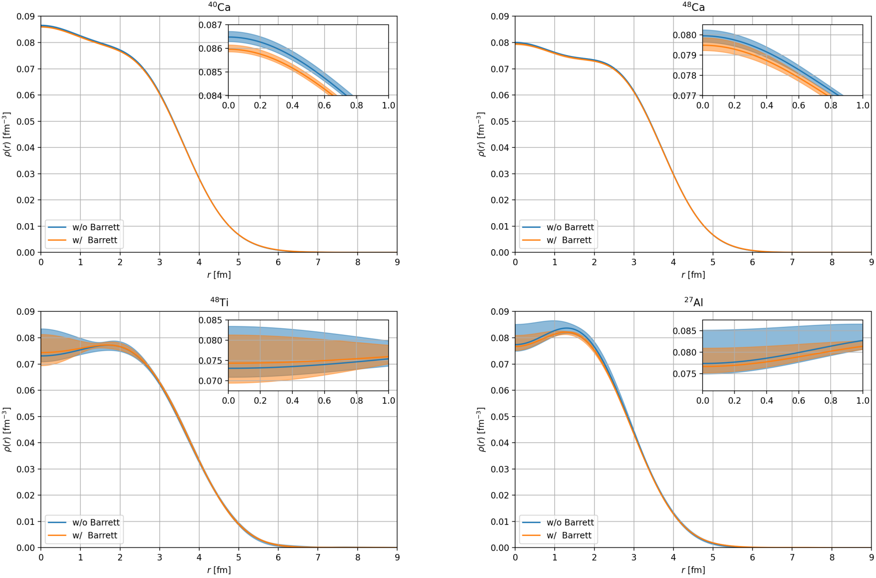




- "statistical" uncertainties: From data uncertainties (stat., syst. & corr.)
- systematical uncertainties: From scan over R, N (envelope or individual)



Charge density results



Application to leading overlap integrals

Dipole overlap integral

Dipole: $$D = -\frac{4}{\sqrt{2}} m_\mu \int_0^\infty \mathrm{d}r \; E(r) \left[g_{-1}^{(e)}(r) \, f_{-1}^{(\mu)}(r) + f_{-1}^{(e)}(r) \, g_{-1}^{(\mu)}(r) \right]$$

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- Only depends on charge density ρ_0:
  - electric field $E(r)$ from $\rho_0(r)$
  - wavefunctions $g_\kappa^{(\ell)}$, $f_\kappa^{(\ell)}$ from solving Dirac equation with $V(r)$

$$E(r) = \frac{\sqrt{4\pi\alpha}}{r^2} \int_0^r \mathrm{d}r' \, r'^2 \rho_0(r')$$

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Results with propagated uncertainties

$$D(^{40}\mathrm{Ca}) = 0.07531(5) \qquad D(^{48}\mathrm{Ca}) = 0.07479(10)$$

$$D(^{48}\mathrm{Ti}) = 0.0864(1) \qquad D(^{27}\mathrm{Al}) = 0.0359(2)$$

- For the first time: Fully quantified uncertainties
- Consistent with results from [Kitano et al., 2002]
- Contain individual uncertainty components and correlations from ρ_0

Scalar and Vector overlap integrals

$$\underline{\text{Scalar:}} \quad S^{(N)} = \frac{\#N}{2\sqrt{2}} \int_0^\infty \mathrm{d}r \; \rho_N(r) \left[g_{-1}^{(e)}(r) g_{-1}^{(\mu)}(r) - f_{-1}^{(e)}(r) f_{-1}^{(\mu)}(r) \right]$$

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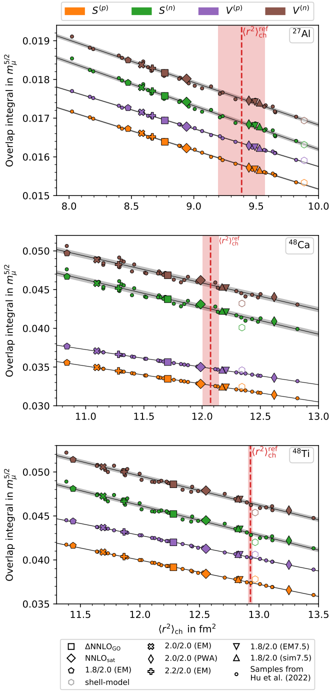
Establish correlation using IMSRG [talk: M. Heinz]

Correlations

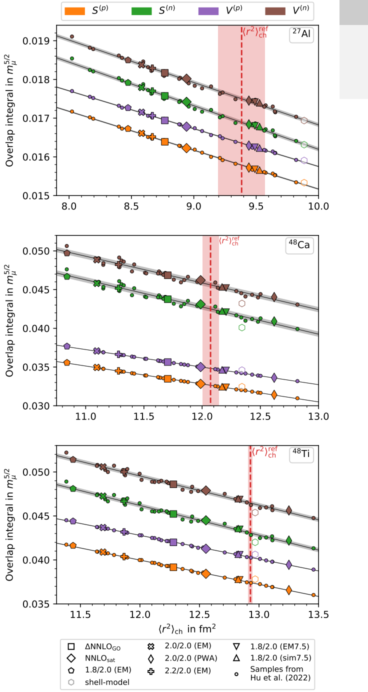
- observed strong correlation between overlap integrals and $\langle r^2 \rangle_{\mathrm{ch}}$



Correlations

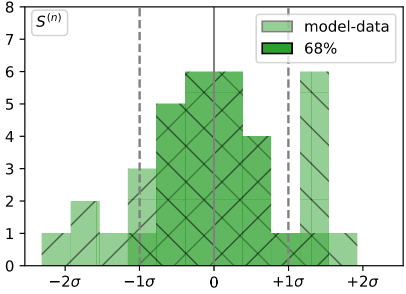
- observed strong correlation between overlap integrals and $\langle r^2 \rangle_{\mathrm{ch}}$
- uncertainties propagated including covariances: [talk: M. Heinz] from correlation / fit and propagated from charge distribution

	I_i	our result	Kitano et al.
^{27}Al	$S^{(p)}$	0.01579(2)(19)	0.0155
	$S^{(n)}$	0.01689(5)(21)	0.0167
	$V^{(p)}$	0.01635(2)(18)	0.0161
	$V^{(n)}$	0.01750(5)(21)	0.0173
^{48}Ti	$S^{(p)}$	0.03742(05)(5)	0.0368
	$S^{(n)}$	0.04305(25)(6)	0.0435
	$V^{(p)}$	0.04029(04)(5)	0.0396
	$V^{(n)}$	0.04646(24)(5)	0.0468



Uncertainties and Covariances

Uncertainties & Covariances
from correlations & nuclear structure:



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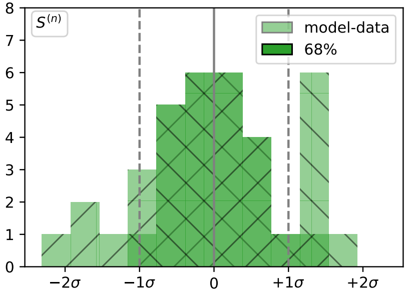

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Combined Covariances:

^{27}Al					
	D	$S^{(p)}$	$S^{(n)}$	$V^{(p)}$	$V^{(n)}$
D	1.0000	0.7205	0.7030	0.7210	0.7028
$S^{(p)}$		1.0000	0.9656	1.0000	0.9645
$S^{(n)}$			1.0000	0.9664	1.0000
$V^{(p)}$				1.0000	0.9654
$V^{(n)}$					1.0000

^{48}Ti					
	D	$S^{(p)}$	$S^{(n)}$	$V^{(p)}$	$V^{(n)}$
D	1.0000	0.4657	0.1169	0.5003	0.1163
$S^{(p)}$		1.0000	0.1118	0.9991	0.0916
$S^{(n)}$			1.0000	0.1176	0.9997
$V^{(p)}$				1.0000	0.0978
$V^{(n)}$					1.0000

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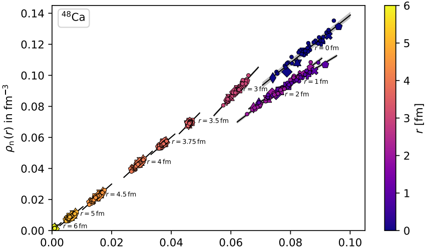

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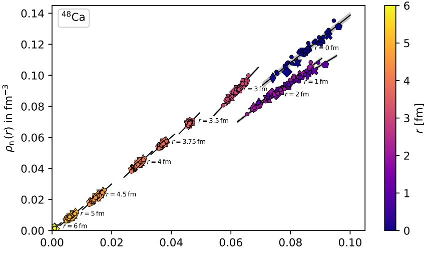
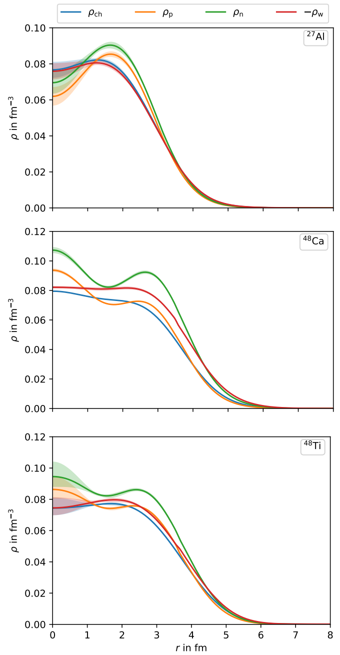


$\rightarrow$ direct comparison to experiment



Outlook: Subleading responses

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- integral weights are Bessel transforms of structure functions:

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Outlook: Subleading responses

- Overlap integrals for subleading responses well on the way:

  depend on: $\mathcal{S}_L = M_L, \Phi''_L, \Sigma'_L, \Sigma''_L, \Delta_L, \ldots$

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Unclear: Correlation to which experimentally accessible quantities?

Conclusion

Summary:

- Comprehensive EFT framework with nuclear responses and Coulomb corrections at the same time
  - Goal: Discriminate BSM operators
  - Controlled uncertainty estimates

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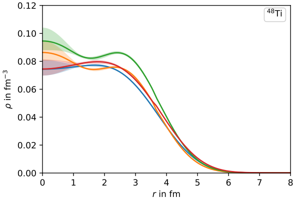

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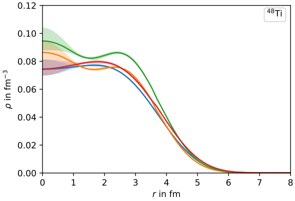

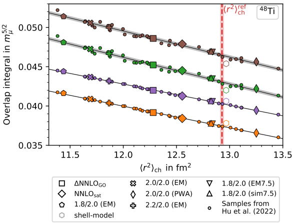

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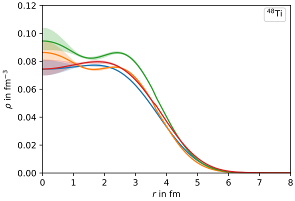
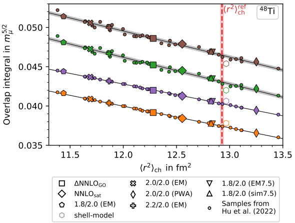
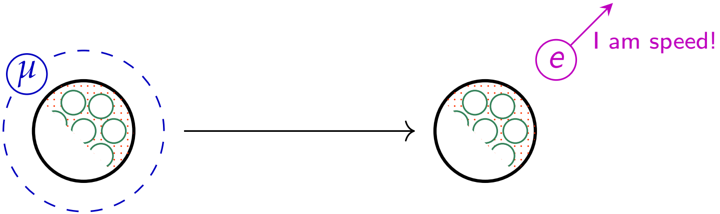




Outlook:

- Subleading nuclear responses
- Relevance of 2-body currents
- Coulomb corrections in PVES

Thank you for your attention!



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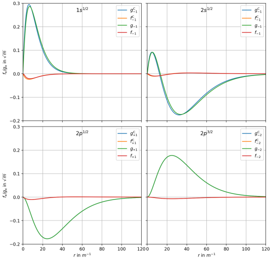
Radii

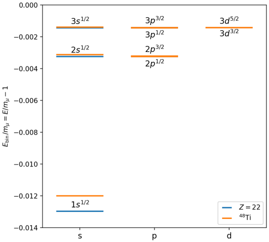
- Qualitative radii for the considered nuclei
- Statistical uncertainties
  - based on fit statistics and data uncertainties
- Systematical uncertainties
  - based on different R, N with two strategies

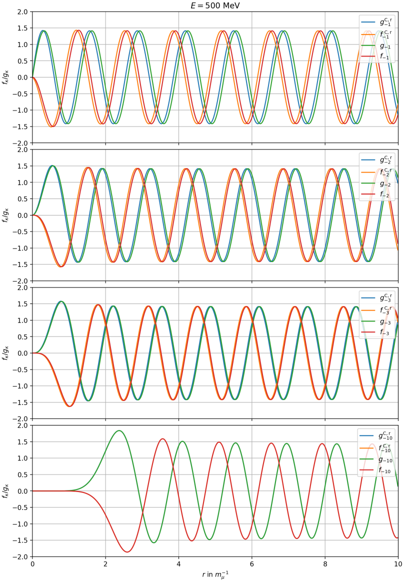
All parameterizations with uncertainties and correlations are made available in a `python` notebook

Nucleus	$\sqrt{\langle r^2 \rangle}$ [fm]	Refs.
^{27}Al	$2.996(11)\,^{(43)[44]}_{(^{+26}_{-33})[35]}$	3.035(2)
	$3.063(3)\,^{(30)[31]}_{(^{+0}_{-1})[3]}$	3.0610(31)
^{40}Ca	$3.452(3)\,^{(8)[9]}_{(^{+1}_{-9})[10]}$	3.450(10)
	$3.4771(17)\,^{(17)[24]}_{(^{+0}_{-5})[17]}$	3.4776(19)
^{48}Ca	$3.4499(29)\,^{(31)[42]}_{(^{+42}_{-52})[60]}$	3.451(9)
	$3.475(2)\,^{(10)[10]}_{(^{+0}_{-3})[4]}$	3.4771(20)
^{48}Ti	$3.62(3)\,^{(8)[8]}_{(^{+2}_{-3})[4]}$	3.597(1)
	$3.596(3)\,^{(57)[57]}_{(^{+1}_{-1})[3]}$	3.5921(17)

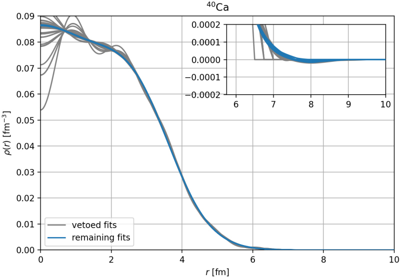
Solve Coulomb numerically

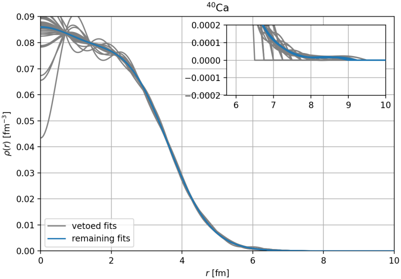


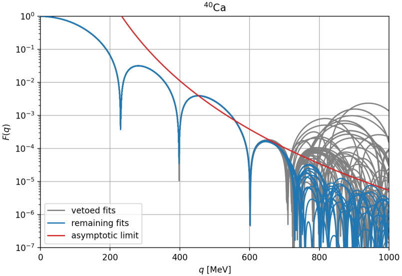




Comparison ^{40}Ca

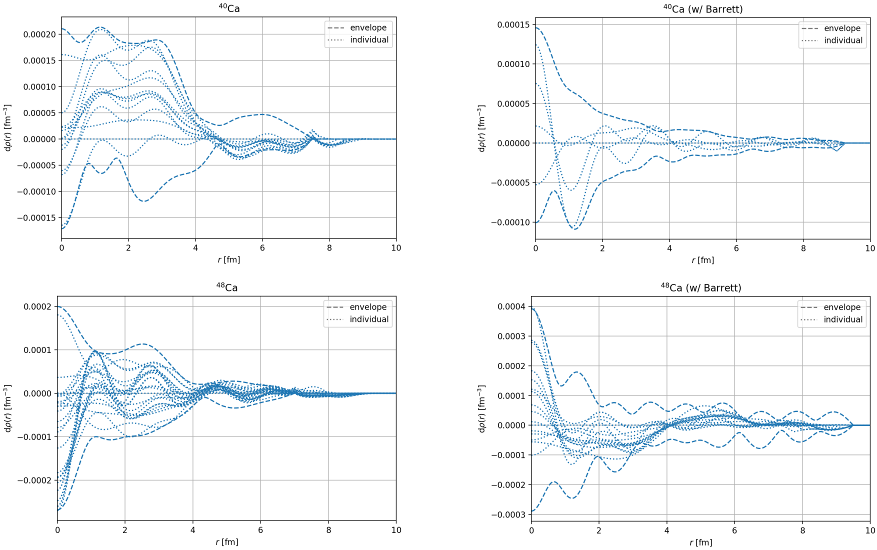




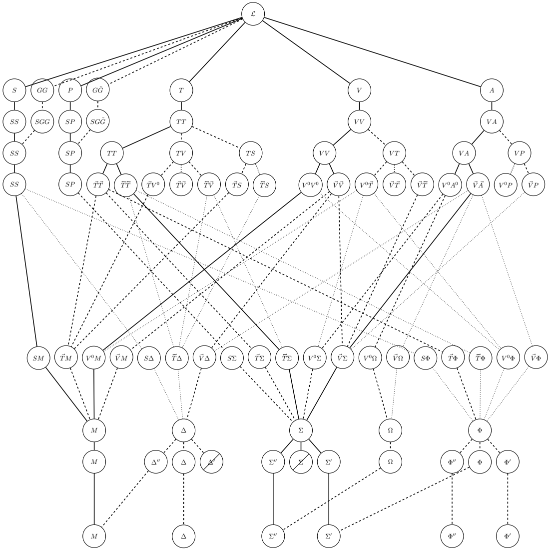


^{40}Ca							
$R \backslash N_x$	7	8	9	10	11	12	13
6.50				1.416			
6.75		1.153	1.109	1.129			
7.00		1.101	1.077	1.096			
7.25		1.071	1.082	1.099	1.119		
7.50	1.107	1.085	1.087	1.101	1.121		
7.75		1.123	1.085	1.103	1.117		
8.00			1.087	1.104	1.120	1.141	
8.25		1.140	1.096	1.105	1.121	1.139	
8.50		1.107	1.111	1.128	1.119	1.137	
8.75			1.135	1.113	1.113	1.131	
9.00			1.139	1.122	1.107	1.127	1.148
9.25			1.208	1.150	1.135	1.149	1.171
9.50						1.196	1.186

Systematic uncertainty bands



Full tree of $\mu \to e$ conversion



$\mu \to e$ conversion Lagrangian

$$\mathcal{L}_{\text{eff}}^{\mu\to e} = \sum_{Y=L,R}\left(\sum_{q=u,d,s}\ \sum_{\substack{X=S,P,\\V,A,T}} C_Y^{X,q}\ L_Y^X\ Q^{X,q} + \sum_{\substack{X=D,\\GG,G\tilde{G}}} C_Y^X\ L_Y^X\ Q^X\ +\ \text{h.c.}\right),$$

$$\begin{aligned}
L_Y^S &= \Lambda^{-2}\ \overline{e_Y}\mu, & Q^{S,q} &= \bar{q}q,\\
L_Y^P &= L_Y^S, & Q^{P,q} &= \bar{q}\gamma^5 q,\\
L_Y^{V,\mu} &= \Lambda^{-2}\ \overline{e_Y}\gamma^\mu\mu, & Q_\mu^{V,q} &= \bar{q}\gamma_\mu q,\\
L_Y^{A,\mu} &= L_Y^{V,\mu}, & Q_\mu^{A,q} &= \bar{q}\gamma_\mu\gamma^5 q,\\
L_Y^{T,\mu\nu} &= \Lambda^{-2}\ \overline{e_Y}\sigma^{\mu\nu}\mu, & Q_{\mu\nu}^{T,q} &= \bar{q}\sigma_{\mu\nu}q,\\
L_Y^{GG} &= \Lambda^{-1}\ L_Y^S, & Q^{GG} &= \alpha_s G_{\alpha\beta}^a G_a^{\alpha\beta},\\
L_Y^{G\tilde{G}} &= \Lambda^{-1}\ L_Y^S, & Q^{G\tilde{G}} &= i\alpha_s G_{\alpha\beta}^a \tilde{G}_a^{\alpha\beta},\\
L_Y^{D,\mu\nu} &= \Lambda\ L_Y^{T,\mu\nu}, & Q_{\mu\nu}^D &= F_{\mu\nu},
\end{aligned}$$

hadronic matrix elements

$$\langle N|\, \bar{q}q \,|N\rangle = \bar{u}_N(p',s')\left(\frac{m_N}{m_q} f_q^N(q)\right) u_N(p,s),$$

$$\langle N|\, \bar{q}i\gamma^5 q \,|N\rangle = \bar{u}_N(p',s')\left(\frac{m_N}{m_q} G_5^{q,N}(q) i\gamma^5\right) u_N(p,s),$$

$$\langle N|\, \bar{q}\gamma^\mu q \,|N\rangle = \bar{u}_N(p',s')\left(\gamma^\mu F_1^{q,N}(q) - \frac{i\sigma^{\mu\nu}q_\nu}{2m_N} F_2^{q,N}(q)\right) u_N(p,s),$$

$$\langle N|\, \bar{q}\gamma^\mu\gamma^5 q \,|N\rangle = \bar{u}_N(p',s')\left(\gamma^\mu\gamma^5 G_A^{q,N}(q) - \gamma^5 \frac{q^\mu}{2m_N} G_P^{q,N}(q)\right) u_N(p,s),$$

$$\langle N|\, \bar{q}\sigma^{\mu\nu} q \,|N\rangle = \bar{u}_N(p',s')\left(\sigma^{\mu\nu} F_{1,T}^{q,N}(q) - 2\gamma^{[\mu} \frac{iq^{\nu]}}{m_N} F_{2,T}^{q,N}(q) - 4p^{[\mu} \frac{iq^{\nu]}}{m_N^2} F_{3,T}^{q,N}(q)\right) u_N(p,s$$

$$\langle N|\, G^a_{\mu\nu} G_a^{\mu\nu} \,|N\rangle = \bar{u}_N(p',s')\left(\frac{4\pi}{\alpha_s} a_N(q)\right) u_N(p,s),$$

$$\langle N|\, G^a_{\mu\nu} \tilde{G}_a^{\mu\nu} \,|N\rangle = \bar{u}_N(p',s')\left(i\frac{4\pi}{\alpha_s} \tilde{a}_N(q)\gamma^5\right) u_N(p,s).$$

Non-rel. expansion

$$\bar{u}_{N'}\mathbb{1}u_N = \chi^\dagger_{s'}\left(\mathbb{1} + \frac{-\vec{q}^2 - 4\vec{p}^2 + 4(\vec{q}\cdot\vec{p}) - 2i(\vec{q}\cdot(\vec{\sigma}\times\vec{p}))}{8m_N^2}\right)\chi_s,$$

$$\bar{u}_{N'}\gamma^5 u_N = \chi^\dagger_{s'}\left(\frac{(\vec{q}\cdot\vec{\sigma})}{2m_N}\right)\chi_s,$$

$$\bar{u}_{N'}\gamma^0 u_N = \chi^\dagger_{s'}\left(\mathbb{1} + \frac{-\vec{q}^2 + 2i(\vec{q}\cdot(\vec{\sigma}\times\vec{p}))}{8m_N^2}\right)\chi_s,$$

$$\bar{u}_{N'}\gamma^i u_N = \chi^\dagger_{s'}\left(\frac{-q_i + 2p_i + i(\vec{q}\times\vec{\sigma})_i}{2m_N}\right)\chi_s,$$

$$\bar{u}_{N'}\gamma^0\gamma^5 u_N = \chi^\dagger_{s'}\left(\frac{-(\vec{q}\cdot\vec{\sigma}) + 2(\vec{\sigma}\cdot\vec{p})}{2m_N}\right)\chi_s,$$

$$\bar{u}_{N'}\gamma^i\gamma^5 u_N = \chi^\dagger_{s'}\left(\sigma_i + \frac{-\vec{q}^2\sigma_i + 2i(\vec{q}\times\vec{p})_i + 2(\vec{q}\times(\vec{\sigma}\times\vec{p}))_i}{8m_N^2}\right.$$
$$\left. + \frac{-4\vec{p}^2\sigma_i - 2q_i(\vec{\sigma}\vec{p}) + 2(\vec{q}\cdot\vec{p})\sigma_i + 4(\vec{\sigma}\cdot\vec{p})p_i}{8m_N^2}\right)\chi_s,$$

$$\bar{u}_{N'}\sigma^{0i} u_N = \chi^\dagger_{s'}\left(\frac{iq_i + (\vec{q}\times\vec{\sigma})_i + 2(\vec{\sigma}\times\vec{p})_i}{2m_N}\right)\chi_s,$$

$$\epsilon_{ijk}\ \bar{u}_{N'}\sigma^{ij} u_N = 2\ \chi^\dagger_{s'}\left(\sigma_k + \frac{-\vec{q}^2\sigma_k - 2i(\vec{q}\times\vec{p})_k - 2(\vec{q}\times(\vec{\sigma}\times\vec{p}))_k}{8m_N^2}\right.$$
$$\left. + \frac{+2q_k(\vec{\sigma}\cdot\vec{p}) + 2(\vec{q}\cdot\vec{p})\sigma_k - 4(\vec{\sigma}\cdot\vec{p})p_k}{8m_N^2}\right)\chi_s.$$

Non-relativistic Operators

$$h^M = \mathbb{1}, \qquad h^\Sigma = \sigma_i,$$

$$h_i^\Delta = \frac{ip_i}{q}, \qquad h^\Omega = \frac{i(\vec{\sigma}\cdot\vec{p})}{q}, \qquad h_i^\Phi = \frac{-(\vec{\sigma}\times\vec{p})_i}{q},$$

$$h^{\Omega'} = \frac{i(\vec{q}\cdot\vec{\sigma})(\vec{\sigma}\cdot\vec{p})}{q^2}, \qquad h^{\Omega''} = \frac{-\vec{p}^2}{q^2}, \qquad h_i^\Theta = \frac{-\sigma_i\vec{p}^2}{q^2},$$

$$h_i^\Pi = \frac{-(\vec{\sigma}\cdot\vec{p})p_i}{q^2}, \qquad h_i^\Xi = \frac{-(\vec{q}\cdot\vec{\sigma})p_i}{q^2}, \qquad h_i^\Gamma = \frac{i(\vec{q}\cdot\vec{p})\sigma_i}{q^2},$$

$q_i, p_i \leftrightarrow -i\vec{\nabla}_i$ acting on either one or both nucleons

Amplitude

$$i\mathcal{M}_Y = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3} \sum_{i=1}^{A} \sum_{S} \tilde{l}^{(m)}_{S,Y}(\vec{q})\; e^{-i\vec{q}\cdot\vec{x}} \left\langle JM_f \right| \hat{h}^{S,N_i}_{(m)}(\vec{x}) \left| JM_i \right\rangle$$

$$= \int \frac{\mathrm{d}^3q}{(2\pi)^3} \sum_{i=1}^{A} \sum_{S=M,\Omega} \tilde{l}_{S,Y}(\vec{q}) \int \mathrm{d}^3x\, e^{-i\vec{q}\cdot\vec{x}} \left\langle JM_f \right| \hat{h}^{S,N_i}(\vec{x}) \left| JM_i \right\rangle$$

$$+ \int \frac{\mathrm{d}^3q}{(2\pi)^3} \sum_{i=1}^{A} \sum_{S=\Sigma,\Delta,\Phi} \tilde{l}^{j}_{S,Y}(\vec{q}) \int \mathrm{d}^3x\, e^{-i\vec{q}\cdot\vec{x}} \left\langle JM_f \right| \hat{h}^{S,N_i}_{j}(\vec{x}) \left| JM_i \right\rangle$$

$$\tilde{l}^{(m)}_{S,Y}(\vec{q}) = \sum_{X} C^{(n,m)}_{Y,X,S}(\vec{q}) \int \mathrm{d}^3x'\, \overline{\Psi^{\kappa',t'}_{e_Y}(x')}\, \Gamma^{(n)}_X \Psi^{(1s),t}_{\mu}(x')\; e^{i\vec{q}\cdot\vec{x'}}$$

Multipole Names

		$\mathbb{1}$	$\vec{\sigma}\vec{\nabla}$	$\vec{\nabla}$	$\vec{\sigma}$	$\vec{\sigma} \times \vec{\nabla}$
$h_{\mu=0}$	$\mathcal{M}$	M	Ω			
$h_{\mu=\pm}$	$\mathcal{T}^{\text{mag}}$			Δ	Σ	Φ
	$\mathcal{T}^{\text{el}}$			Δ'	Σ'	Φ'
$h_{\mu=3}$	$\mathcal{L}$			Δ''	Σ''	Φ''

For an elastic process:

$$\langle M||\Sigma||M\rangle = 0, \qquad \langle M||\Delta'||M\rangle = 0,$$

$$\langle M||\Omega||M\rangle = -\frac{1}{2}\langle M||\Sigma''||M\rangle\,, \qquad \langle M||\Delta''||M\rangle = +\frac{1}{2}\langle M||M||M\rangle\,,$$

$$\langle M||\Phi||M\rangle = +\frac{1}{2}\langle M||\Sigma'||M\rangle\,.$$

Multipoles

$$M^i_{JM} = j_J(qr_i)Y_{JM}(\hat{r}_i),$$

$$\Omega^i_{JM} = j_J(qr_i)Y_{JM}(\hat{r}_i)\left(\vec{\sigma}\cdot\frac{\vec{\nabla}_i}{q}\right),$$

$$\Delta^i_{JM} = j_J(qr_i)\vec{Y}_{JJM}(\hat{r}_i)\cdot\frac{\vec{\nabla}_i}{q},$$

$$\Delta'^{\,i}_{JM} = -i\left(\frac{\vec{\nabla}_i}{q}\times j_J(qr_i)\vec{Y}_{JJM}(\hat{r}_i)\right)\cdot\frac{\vec{\nabla}_i}{q},$$

$$\Delta''^{\,i}_{JM} = \left(\frac{\vec{\nabla}_i}{q}j_J(qr_i)Y_{JM}(\hat{r}_i)\right)\cdot\frac{\vec{\nabla}_i}{q},$$

$$\Sigma^i_{JM} = j_J(qr_i)\vec{Y}_{JJM}(\hat{r}_i)\cdot\vec{\sigma},$$

$$\Sigma'^{\,i}_{JM} = -i\left(\frac{\vec{\nabla}_i}{q}\times j_J(qr_i)\vec{Y}_{JJM}(\hat{r}_i)\right)\cdot\vec{\sigma},$$

$$\Sigma''^{\,i}_{JM} = \left(\frac{\vec{\nabla}_i}{q}j_J(qr_i)Y_{JM}(\hat{r}_i)\right)\cdot\vec{\sigma},$$

$$\Phi^i_{JM} = i\, j_J(qr_i)\vec{Y}_{JJM}(\hat{r}_i)\cdot\left(\vec{\sigma}\times\frac{\vec{\nabla}_i}{q}\right),$$

$$\Phi'^{\,i}_{JM} = \left(\frac{\vec{\nabla}_i}{q}\times j_J(qr_i)\vec{Y}_{JJM}(\hat{r}_i)\right)\cdot\left(\vec{\sigma}\times\frac{\vec{\nabla}_i}{q}\right),$$

$$\Phi''^{\,i}_{JM} = i\left(\frac{\vec{\nabla}_i}{q}j_J(qr_i)Y_{JM}(\hat{r}_i)\right)\cdot\left(\vec{\sigma}\times\frac{\vec{\nabla}_i}{q}\right),$$

Prefactors

$$C_{Y,S,M} = \left(1 - \frac{\vec{q}^2}{8m_N^2}\right) C_Y^{S,S}, \qquad C^i_{Y,S,\Sigma} = \frac{q^i}{2m_N} C_Y^{S,P},$$

$$C^i_{Y,S,\Delta} = \frac{-iqq^i}{2m_N^2} C_Y^{S,S}, \qquad C^i_{Y,S,\Phi} = \frac{iqq^i}{4m_N^2} C_Y^{S,S},$$

$$C_{Y,V^0,M} = C_Y^{V,V} + \frac{\vec{q}^2}{8m_N^2}\left(4C_Y^{V,T} - C_Y^{V,V}\right), \qquad C^i_{Y,V^0,\Sigma} = \frac{-q^i}{2m_N} C_Y^{V,A},$$

$$C^i_{Y,V^0,\Phi} = \frac{iqq^i}{4m_N^2}\left(4C_Y^{V,T} - C_Y^{V,V}\right), \qquad C_{Y,V^0,\Omega} = \frac{-iq}{m_N} C_Y^{V,A},$$

$$C^{i,j}_{Y,\vec{V},\Delta} = \delta^{ij}\frac{iq}{m_N} C_Y^{V,V} + \frac{-q\epsilon^{ilj}q_l}{4m_N^2} C_Y^{V,A}, \qquad C^i_{Y,\vec{V},M} = \frac{q^i}{2m_N} C_Y^{V,V},$$

$$C^{i,j}_{Y,\vec{V},\Phi} = \frac{q\epsilon^{ilj}q_l}{4m_N^2} C_Y^{V,A}, \qquad C^i_{Y,\vec{V},\Omega} = \frac{-iqq^i}{4m_N^2} C_Y^{V,A},$$

$$C^{i,j}_{Y,\vec{V},\Sigma} = (-\delta^{ij})\left(1 - \frac{\vec{q}^2}{8m_N^2}\right) C_Y^{V,A} + \frac{i\epsilon^{ilj}q_l}{2m_N}\left(2C_Y^{V,T} - C_Y^{V,V}\right) + \frac{-iq^iq^j}{2m_N^2} C_Y^{V,P},$$

$$C^i_{Y,\vec{T},M} = \frac{iq^i}{m_N}\left(C_Y^{T,S} - C_Y^{T,T} + C_Y^{T,V}\right), \qquad C^{i,j}_{Y,\vec{T},\Sigma} = \frac{-\epsilon_{ilj}q_l}{m_N} C_Y^{T,T},$$

$$C^{i,j}_{Y,\vec{T},\Phi} = \frac{2q}{m_N}\delta^{ij} C_Y^{T,T},$$

$$C^{i,j}_{Y,\overleftrightarrow{T},\Sigma} = \delta^{ij}\left(1 - \frac{\vec{q}^2}{8m_N^2}\right) C_Y^{T,T} + \frac{\delta^{ij}\vec{q}^2 - q^iq^j}{4m_N^2} C_Y^{T,V}, \qquad C^{i,j}_{Y,\overleftrightarrow{T},\Phi} = \frac{qq_l\epsilon^{ilj}}{4m_N^2} C_Y^{T,T}$$

$$C^{i,j}_{Y,\overleftrightarrow{T},\Delta} = \frac{q\epsilon_{ilj}q^l}{2m_N^2}\left(C_Y^{T,S} - C_Y^{T,T} + C_Y^{T,V}\right).$$

Leading Terms

$$\overline{|\mathcal{M}|^2} = \sum_{Y=L,R} \frac{2}{\Lambda^4} \left| \Lambda \eta_e C_Y^D D + \sum_{N=n,p} \left(C_Y^{S,S} S^{(N)} + C_Y^{V,V} V^{(N)} \right) \right|^2$$

$$= \sum_{Y=L,R} \frac{2}{\Lambda^4} \left| \Lambda \eta_e C_Y^D D + \sum_{\substack{N=n,p \\ q=u,d,s}} \left(\frac{m_N}{m_q} C_Y^{S,q} f_q^N(0) S^{(N)} + C_Y^{V,q} F_1^{q,N}(0) V^{(N)} \right) \right|^2$$

$$S^{(N)} = \frac{1}{2\sqrt{2}} \int_0^\infty \mathrm{d}r \, (\#N) \rho_N(r) \left[g_{-1}^{(e)}(r)\, g_{-1}^{(\mu)}(r) - f_{-1}^{(e)}(r)\, f_{-1}^{(\mu)}(r) \right]$$

$$V^{(N)} = \frac{1}{2\sqrt{2}} \int_0^\infty \mathrm{d}r \, (\#N) \rho_N(r) \left[g_{-1}^{(e)}(r)\, g_{-1}^{(\mu)}(r) + f_{-1}^{(e)}(r)\, f_{-1}^{(\mu)}(r) \right]$$

$$D = -\frac{4 m_\mu}{\sqrt{2}} \int_0^\infty \mathrm{d}r \quad E(r) \quad \left[g_{-1}^{(e)}(r)\, f_{-1}^{(\mu)}(r) + f_{-1}^{(e)}(r)\, g_{-1}^{(\mu)}(r) \right]$$

PVES

$$A_{\text{PVES}} = \frac{(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega})_R - (\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega})_L}{(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega})_R + (\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega})_L} \approx -\frac{G_F q^2}{4\pi\alpha_{\text{el}}\sqrt{2}} \frac{Q^{\text{w}} F_0^{\text{w}}(q^2)}{Z F_0^{\text{ch}}(q^2)},$$

$$Z F_L^{\text{ch}}(q^2) = \left(1 - \frac{\langle r_p^2\rangle}{6} q^2 - \frac{1}{8m_N^2} q^2\right) \mathcal{F}_L^{M_p}(q^2) - \frac{\langle r_n^2\rangle}{6} q^2 \mathcal{F}_L^{M_n}(q^2)$$
$$+ \frac{1+2\kappa_p}{4m_N^2} q^2 \mathcal{F}_L^{\Phi''_p}(q^2) + \frac{2\kappa_n}{4m_N^2} q^2 \mathcal{F}_L^{\Phi''_n}(q^2) + \mathcal{O}(q^4).$$

$$Q^{\text{w}} F_L^{\text{w}}(q^2) = \left(Q_p^{\text{w}}\left(1 - \frac{\langle r_p^2\rangle}{6} q^2 - \frac{1}{8m_N^2} q^2\right) - Q_n^{\text{w}} \frac{\langle r_n^2\rangle + \langle r_{s,N}^2\rangle}{6} q^2\right) \mathcal{F}_L^{M_p}(q^2)$$
$$+ \left(Q_n^{\text{w}}\left(1 - \frac{\langle r_n^2\rangle + \langle r_{s,N}^2\rangle}{6} q^2 - \frac{1}{8m_N^2} q^2\right) - Q_p^{\text{w}} \frac{\langle r_n^2\rangle}{6} q^2\right) \mathcal{F}_L^{M_n}(q^2)$$
$$+ \frac{Q_p^{\text{w}}(1+2\kappa_p) + 2Q_n^{\text{w}}(\kappa_n + \kappa_{s,N})}{4m_N^2} q^2 \mathcal{F}_L^{\Phi''_p}(q^2)$$
$$+ \frac{Q_n^{\text{w}}(1+2\kappa_p + 2\kappa_{s,N}) + 2Q_p^{\text{w}}\kappa_n}{4m_N^2} q^2 \mathcal{F}_L^{\Phi''_n}(q^2)$$

Detailed Limits

	π^0	η	η'
$C_Y^{A,3}$	1.3×10^{-17}	–	–
$C_Y^{A,8}$	–	1.5×10^{-17}	4.0×10^{-20}
$C_Y^{A,0}$	–	2.9×10^{-19}	2.1×10^{-19}
$C_Y^{P,3}$	4.1×10^{-17}	–	–
$C_Y^{P,8}$	–	1.6×10^{-12}	2.1×10^{-14}
$C_Y^{P,0}$	–	4.1×10^{-12}	5.4×10^{-13}
$C_Y^{G\tilde{G}}$	–	5.8×10^{-15}	4.7×10^{-16}

Cancelation & RG corrections

$$C_Y^{A,u} = C_Y^{A,d}, \qquad C_Y^{A,s} = -\frac{2C_Y^{A,u} g_A^{u,0}}{g_A^{s,N}},$$

$$\frac{C_Y^{P,u}}{m_u} = \frac{C_Y^{P,d}}{m_d}, \qquad \frac{C_Y^{P,s}}{m_s} = \frac{4\pi}{\Lambda} C_Y^{G\tilde{G}} \frac{2g_A^{u,0}}{g_A^{u,0} - g_A^{s,N}}.$$

$$C_Y^{V,q} \simeq -3Q_q \frac{\alpha}{\pi} \log \frac{M_W}{m_N} C_Y^{A,q},$$