The chiral torsional anomaly and the Nieh-Yan invariant with and without boundaries

Based on JHEP 12 (2024) 149,
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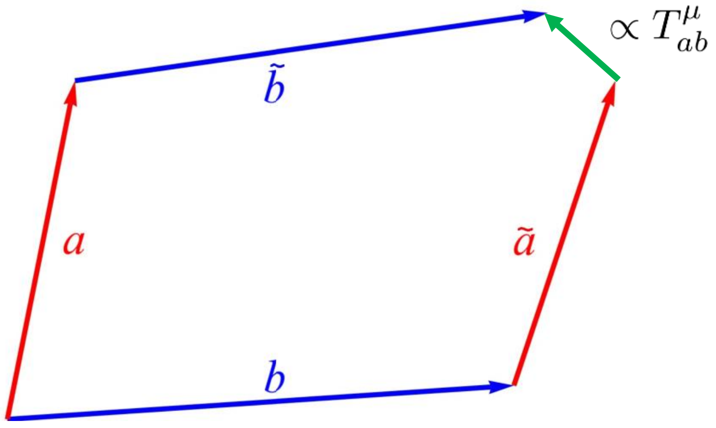
Motivation

- Anomalies strongly affect (3+1d) fermion transport
  [Landsteiner, Acta Phys. Polon. B47 (2016) 2617, Chernodub, Ferreiros, Grushin, Landsteiner, and Vozmediano, Phys. Rept. 977 (2022) 1-58]

$$\partial_\mu J^\mu = c\mathcal{A}$$

- Must have consensus on anomaly existence and form

- Torsional Nieh-Yan anomaly controversial across two fields and ~3 decades



- Problem 1: Total derivative → Absence of boundaries = No anomaly

$$\mathcal{A}_{\mathrm{NY}} = \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}\left(\eta_{\alpha\beta}T^{\alpha}_{[\mu\nu}T^{\beta}_{\rho\lambda]} - 2R_{\mu\nu\lambda\rho}\right) = \nabla_\mu\left(\frac{1}{3!}\epsilon^{\mu\nu\rho\lambda}T_{[\nu\rho\lambda]}\right)$$

[Nieh, Yan, J. Math. Phys. 23 (1982) 373]

- Resolutions?:
  - "Singular" geometries with torsion flux [Y. N. Obukhov, E. W. Mielke, J. Budczies, and F. W. Hehl, Found. Phys. 27 (1997)]
  - Boundaries

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- Problem 2: Anomaly coefficient

$$c \propto p^2 \ , \quad \text{and regulator} - \text{dependent}$$

- Resolutions?:
  - Weyl invariance [Chandia, Zanelli, PRD 55 (1997) 7589, Kreimer and Milke PRD 63 (2001) 048501]
  - Explicit cutoff
  - Temp.-dependence (counter counterargument)[Chernodub, Ferreiros, Grushin, Landsteiner, and Vozmediano, Phys. Rept. 977 (2022) 1-58]

- Goal = Consistent/coherent interpretation

Outline

- Setup
- Torsional Anomaly sans boundaries
- Torsional Anomaly with boundaries
- Conclusions & Outlook

Setup

- Backround = Riemann-Cartan = independent vielbein and connection

[Shapiro, Phys. Rept. 357 (2002) 113]

$$g_{\mu\nu} = \delta_{\alpha\beta} e^\alpha_\mu e^\beta_\nu \qquad \partial_\mu e^\alpha_\rho + \omega^\alpha_{\mu\beta} e^\beta_\rho - e^\alpha_\lambda \Gamma^\lambda_{\mu\rho} = 0 \qquad T^\mu_{\nu\rho} = E^\mu_a \overset{\omega}{\nabla}_{[\nu} e^a_{\rho]} = \Gamma^\mu_{[\nu\rho]}$$

- Matter = single Dirac fermion

$$S_E = \int_{\mathcal{M}} d^4x \, |e| \, \psi^\dagger \gamma^\mu i D_\mu \psi = \int_{\mathcal{M}} dx^4 \, |e| \, \psi^\dagger \not{D} \psi$$

- Couple to geometry and chiral gauge field

$$D_\mu = \partial_\mu + \omega_\mu + A_{5\mu}\gamma_* \longrightarrow \not{D} \to \mathring{\not{D}} + i\not{B}\gamma_*$$

$$\omega_\mu = \frac{1}{2}\omega^{\alpha\beta}_\mu \Sigma_{\alpha\beta} \; , \; \Sigma_{\alpha\beta} = \frac{1}{4}\left[\gamma^\alpha, \gamma^\beta\right] \qquad B_\mu = A_{5\mu} + \frac{1}{64}\varepsilon^{\lambda_1\lambda_2\lambda_3\mu} T_{[\lambda_1\lambda_2\lambda_3]} \qquad \gamma_* = -\gamma^1\gamma^2\gamma^3\gamma^4$$

Torsional Anomaly sans boundaries: Generalities

[Bertlmann, Anomalies in quantum field theory (1996)]

- (local) Chiral anomaly $\delta_{\delta\phi} \ln \mathcal{Z} \neq 0 \Leftrightarrow \int_{\mathcal{M}} \delta\phi(\partial_\mu J^\mu - c\mathcal{A}) = 0$
  $(\psi \to e^{i\gamma_* \delta\phi}\psi)$

- Can calculate directly $\langle \partial_\mu J^\mu \rangle = \ldots$

- Can calculate via (local) heat kernel

$$\int_{\mathcal{M}} \delta\phi \; c\mathcal{A} = \lim_{M\to\infty} \mathrm{Tr}\left[\delta\phi\gamma_* \mathrm{e}^{-\not{D}^2/M^2}\right] \sim \sum_{k=0}^{\infty} M^{n-k} a_k(\delta\phi\gamma_*, \not{D}^2)$$

- Nieh-Yan anomaly = Non-zero $a_2(\delta\phi\gamma_*, \not{D}^2)$

The caveat

- For constant $\delta\phi$ → Index theorem [Atiyah, Patodi, and Singer, Math. Proc. Cambridge Phil. Soc. 77 (1975) 43]

$$\int_{\mathcal{M}} c\mathcal{A} = \mathrm{Ind}\not{D} = n_+ - n_- = \mathrm{Tr}\left[\gamma_* \mathrm{e}^{-\not{D}^2/M^2}\right]$$

- Index = regulator-independent

- If heat-kernel index exists, $a_2(\gamma_*, \not{D}^2) = 0$ → $a_2(\delta\phi\gamma_*, \not{D}^2)$ renorm. artifact

  → No NY anomaly

- Nieh-Yan invariant exists =/= Nieh-Yan anomaly exists

The punchline

- Instanton-like spacetimes sans boudaries

[Y. N. Obukhov, E. W. Mielke, J. Budczies, and F. W. Hehl, Found. Phys. 27 (1997)]

$$g = h(r)^2 dr^2 + f(r)^2 d\Omega_3^2 \qquad (\text{E.g. } f = \frac{ar^2}{r^2 + c^2})$$

- Support Nieh-Yan flux but…

$$\int_{\mathcal{M}} \mathcal{A}_{\mathrm{NY}} = f(\infty) V_{S_3}$$

- … no global Index → **No Nieh-Yan anomaly in spaces without boundaries**

Adding the boundary

- Boundary $\partial\mathcal{M}$ with normal **n**

- Most general BCs (chiral bag)
  [Wipf and Durr, Nucl. Phys. B 443 (1995) 201-232]

$$\Pi_-(\theta)\psi|_{\partial\mathcal{M}} = 0, \qquad \Pi_-(\theta) = \tfrac{1}{2}\left(1 - i\gamma_* e^{\gamma_*\theta}\gamma^{\mathbf{n}}\right)$$

- Chiral angle = free $\partial\mathcal{M}$ parameter

$$\theta \to \theta - 2\phi$$

- Chirality not preserved globally → May/may not support global Index

- Focus on local Index

$$\int_{\mathcal{M}} \delta\phi\, c\mathcal{A} + \int_{\partial\mathcal{M}} \delta\phi\; c_B \mathcal{C}_B = \mathrm{Tr}\left[\delta\phi\gamma_* \mathrm{e}^{-\not{D}^2/M^2}\right]$$

The anomaly

- Sanity check: Use zeta and cutoff regulators (Reminder: $\not{D} \to \mathring{\not{D}} + i\not{B}\gamma_*$)

- Zeta function: $$\int_{\mathcal{M}} \delta\phi c\mathcal{A} + \int_{\partial\mathcal{M}} \delta\phi\, c_B \mathcal{C}_B = -2a_4(\delta\phi\gamma_*, \not{D}^2)$$

- Cutoff ("flat" $\partial\mathcal{M}$): $$\int_{\mathcal{M}} \delta\phi c\mathcal{A} + \int_{\partial\mathcal{M}} \delta\phi\, c_B \mathcal{C}_B \simeq -2\sum_{k=0}^{4} \Lambda^{4-k} a_k(\delta\phi\gamma_*, \not{D}^2)$$

Nieh-Yan →
$$= -\frac{\Lambda^2}{2\pi^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{g}(\delta\phi)\mathring{\nabla}_\mu B^\mu + \frac{\Lambda}{8\pi^{3/2}}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}(\delta\phi)\mathring{\nabla}_\mu B^\mu$$

Pontryagin →
$$+ \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{g}(\delta\phi)\Big[-\frac{i}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(B_{\mu\nu}B_{\rho\sigma} - \frac{1}{2}\mathring{R}_{\mu\nu\lambda\theta}\mathring{R}_{\rho\sigma}{}^{\lambda\theta}\right)$$

$$+ \frac{1}{12\pi^2}\left(\mathring{G}^{\mu\nu} + g^{\mu\nu}\mathring{\nabla}^2\right)\mathring{\nabla}_\mu B_\nu - \frac{2}{3\pi^2}\left(B^\mu B^\nu + \frac{1}{2}g^{\mu\nu}B^2\right)\mathring{\nabla}_\mu B_\nu\Big]$$

$$- \int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}\left[\delta\phi\left(\frac{i}{30\pi^2}\epsilon^{abc}B_b\mathring{\nabla}_a B_c + \frac{2}{15\pi^2}B^2 B_{\mathbf{n}}\right) + \frac{1}{12\pi^2}(\nabla_{\mathbf{n}}\delta\phi)\mathring{\nabla}_\mu B^\mu\right]$$

Chern-Simon + ?? → (last two lines)

Nieh-Yan = scheme artifact

$$W_{\rm c.t.} = -\frac{\Lambda^2}{4\pi^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{g}B^2 - \int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}\theta B_{\mathbf{n}}\right] + \frac{\Lambda}{16\pi^{3/2}}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}\theta\mathring{\nabla}_\mu B^\mu$$

Pontryagin = Chiral anomaly + $\mathring{\nabla}$ Torsion effects $\left(A_{\mu\nu} = \mathring{\nabla}_{[\mu}A^5_{\nu]}\right)$

$$\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\left(A_{\mu\nu}A_{\rho\sigma} + 2\xi A_{\mu\nu}\mathring{\nabla}_\kappa T^\kappa_{\rho\sigma} + \xi^2\mathring{\nabla}_\lambda T^\lambda_{\mu\nu}\mathring{\nabla}_\kappa T^\kappa_{\rho\sigma}\right)$$

?? = Pure boundary term

$$\int_{\partial\mathcal{M}} c_B\mathcal{C}_B = \frac{1}{60\pi^2}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}\Big[\delta\phi\left(-\frac{5}{2}g_{\mathbf{n}\mu}\mathring{R}_3 + 5\mathring{\nabla}_{\mathbf{n}}\mathring{\nabla}_\mu - 80\overleftarrow{\mathring{\nabla}}_{\mathbf{n}}\mathring{\nabla}_\mu + (30\overleftarrow{\mathring{\nabla}}^2 - 336B^2)n_\mu\right)B^\mu + \frac{i}{3}\delta\phi\epsilon^{abc}B_{[a}\mathring{\nabla}_b B_{c]}\Big]$$

$$\int_{\mathcal{M}}\delta\phi c\mathcal{A} + \int_{\partial\mathcal{M}}\delta\phi\, c_B\mathcal{C}_B \simeq -2\sum_{k=0}^{4}\Lambda^{4-k}a_k(\delta\phi\gamma_*, \not{D}^2)$$

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Consequences/Conclusions & Outlook

- Torsional anomaly exists, Nieh-Yan term doesn't with or without boundaries
- Caveats: Lattice, finite temperature, derivative expansion
- Adding boundary = non-trivial Hall effect transport…

$$j_5^a|_{\partial\mathcal{M}} = \frac{\delta\theta}{180\pi^2} i\epsilon^{abc}\mathring{\nabla}_{[b}B_{c]} \equiv \sigma_T i\epsilon^{abc}\mathring{\nabla}_{[b}B_{c]}$$

- ... and bulk-boundary transport via ??
- Dynamics of b.c.s?

Thank you!