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Dissipative electrically driven fluids

Based on:

- J. High Energ. Phys. 2023, 218 (2023)
- J. High Energ. Phys. 2024, 114 (2024)

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Motivation

- describe effect of externally applied electric field on a charged fluid → stationary configurations
- motivation: standard hydrodynamic description predicts that a stationary state is achieved when

[Kovtun '16]

$$\mathbb{E}_i - \partial_i \mu = 0$$

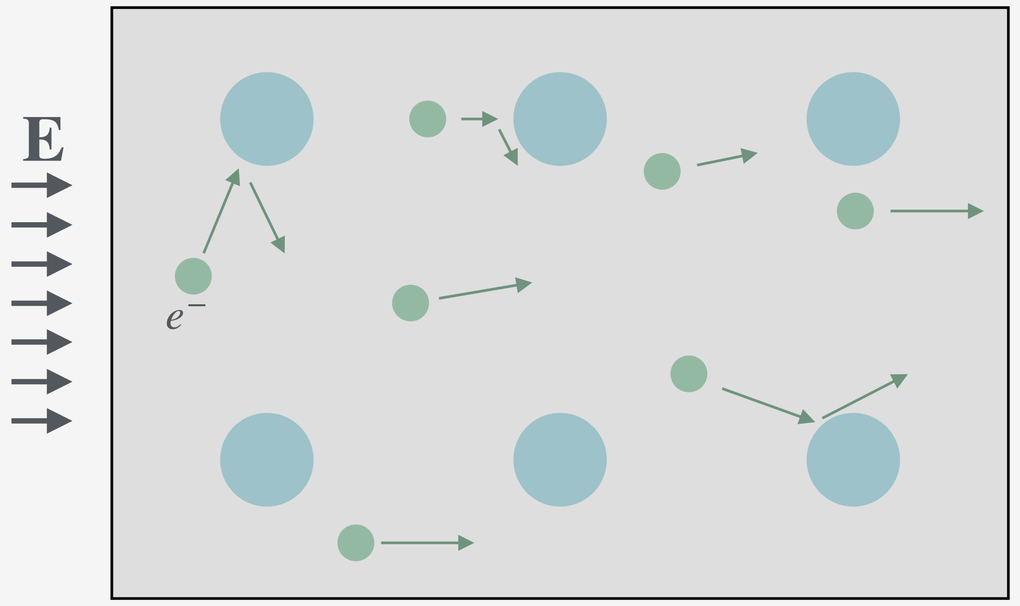
- removes the electric field from the dynamics
- velocity of fluid is unconstrained in magnitude

→ can take on arbitrary values independent of the driving electric field

- not what we observe in nature

Motivation

- Instead consider Drude's model



$$\langle \frac{dp}{dt} \rangle = q\mathbf{E} - \Gamma \langle \mathbf{p} \rangle$$

Charge acceleration through applied electric force

Relaxation term prevents indefinite acceleration of charge carriers

- Relaxation term accounts for the dissipation of momentum and energy
- electron fluid relaxes to a steady state which is described by a constant drift velocity

$$\langle \mathbf{v} \rangle = \frac{q\mathbf{E}}{\Gamma}$$

- conductivity by

DC conductivity σ_{DC}

Lack In hydrodynamics: manifests as an infinite DC conductivity

$$\mathbf{J} = nq\langle \mathbf{v} \rangle = \frac{nq^2}{\Gamma}\mathbf{E}$$

Motivation

- proprosal: incorporate relaxation terms for energy and momentum into the definition of stationarity
- presence of introduced sinks breaks boost invariance
  - no momentum relaxation no sources: stationary states: μ = const , T = const, $\mathbf{v}$ = const and arbitrary,
    → solutions with different velocities are equally good equilibria (related by boosts)
  - momentum relaxation: leads to vanishing of equilibrium velocity of the system
    → this solution cannot be related to others via boosts
- including external sources:
  - momentum relaxation and source constraint velocity to take a specific stationary value
    → boost symmetry broken

Boost agnostic hydrodynamics

- This approach necessitates that the fluid velocity becomes itself a thermodynamic variable

$$P(T, \mu, \vec{v}, \mathbb{E})$$

- velocity: thermodynamic variable - introduced as a chemical potential conjugate to momentum

[Boer, Hartong, Obers, Vandoren, Sybesma, Armas, Sonner,...]

- Can define:

$$\rho_{\mathrm{m}} = 2\left(\frac{\partial P}{\partial \vec{v}^2}\right), \quad \kappa_{\mathbb{E}} = 2\left(\frac{\partial P}{\partial \vec{\mathbb{E}}^2}\right)$$

- Important: different inertial frames represent distinct hydrodynamic states

Boost agnostic hydrodynamics

- For a boost agnostic fluid the natural curved background it couples to is an Aristotelean geometry
- Aristotelian spacetime: manifold equipped with two metrics (incorporating space and time on different grounds) [Penrose '68]
  - one-form τ_μ, spatial metric $h_{\mu\nu}$ (signature $(0,1,...,1)$)
  - no longer insist on local Lorentzian symmetry (no tangent space transformations rule)
  - We can rewrite the spatial metic in terms of vielbeins

$$h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu \,, \qquad e = det(\tau, e^a_\mu)$$

  - In these geometries energy-momentum tensor can be decomposed as $T^\mu_\nu = -T^\mu \tau_\nu + T^{\mu\rho} h_{\rho\nu}$

Stationarity

- As I anticipated we are interested in the stationarity configurations (reason for introducing relaxations)

- For this introduce a notion of dynamical evolution:

  time-direction: time-like Killing vector β^μ

- stationary once it satisfies the stationary condition given by

$$\mathcal{L}_\beta \tau_\mu = 0$$
$$\mathcal{L}_\beta h_{\mu\nu} = 0$$
$$\mathcal{L}_\beta A_\mu + \partial_\mu \Lambda = 0$$

- in FSCC using thermodynamic variables we can reformulate the hydrostatic constraints

$$\partial_\mu T = 0, \quad \partial_t v^i = 0, \quad \partial_i v_j + \partial_j v_i = 0, \quad \partial_t \mathbb{E}_i + v^j \partial_j \mathbb{E}_i + \mathbb{E}_j \partial_i v^j = 0$$
$$\mathbb{E}_i - \partial_i \mu = 0$$

Relaxations

- Diffeomorphism and gauge invariance of generating functional lead to conservation equations
- To move away from conservation add non-conservative forces while remaining $U(1)$ charge conservation

$$e^{-1}\partial_\mu\left(eT^\mu_\rho\right) + T^\mu\partial_\rho\tau_\mu - \frac{1}{2}T^{\mu\nu}\partial_\rho h_{\mu\nu} - F_{\rho\mu}J^\mu = \Gamma_\rho$$

$$e^{-1}\partial_\mu\left(eJ^\mu\right) = 0$$

[Boer, Hartong, Obers, Vandoren, Sybesma, Armas]

- In FSCC

$$\partial_t\varepsilon + \partial_i J^i_\varepsilon - \mathbb{E}_i J^i = -\hat{\Gamma}_\varepsilon$$

$$\partial_t P_i + \partial_j T^j_i - n\mathbb{E}_i = -\hat{\Gamma}^i_{\mathbf{P}}$$

$$\partial_t n + \partial_i J^i = 0$$

(recall Drude $\langle\frac{dp}{dt}\rangle = q\mathbf{E} - \Gamma\langle\mathbf{p}\rangle$)

- Can parametrize relaxation as $\hat{\Gamma}^i_{\mathbf{P}} = \Gamma_{\mathbf{P}} P_i$

Relaxation at order zero

- At order zero in derivatives ((non-)conservation equations)

$$nv^i\left(\mathbb{E}_i-\partial_i\mu\right)=\hat{\Gamma}_\varepsilon+\mathcal{O}(\partial)$$
$$n\left(\mathbb{E}_i-\partial_i\mu\right)=\Gamma_{\mathbf{P}}P_i+\mathcal{O}(\partial)$$

- Assuming that neither of the sites is zero on their own we treat these expressions as conditions for hydrostaticity → modify our hydrostaticity condition by

$$\mathbb{E}_i-\partial_i\mu=0\rightarrow\mathbb{E}_i-\partial_i\mu-\Gamma_{\mathbf{P}}P_i=0$$

- energy and momentum relaxations related through

$$\boxed{\hat{\Gamma}_\varepsilon=\Gamma_{\mathbf{P}}v_i}$$

At higher order?

Relaxation at order one

- simplification: assume hydrostaticity condition and constitutive relation for momentum relaxation term to be exact
- i.e. true at all orders in derivatives
- FSCC:

$$\mathbb{E}_i - \partial_i \mu - \frac{\Gamma P_i}{\mathrm{n}} = 0, \quad \hat{\Gamma}^i_{\vec{P}} = \Gamma P_i$$

- Still: $\hat{\Gamma}_\epsilon$ receives derivative correction as it was derived as a consequence of the equations of motion on hydrostatic solutions → constitutive relation cannot be freely specified

Relaxation at order one

- To obtain first order corrections: require fluid to locally obey second law of thermodynamics

[Boer, Hartong, Have, Obers, Sybesma, Armas, Jain,...]

$$e^{-1}\partial_\mu (eS^\mu) \geq 0$$

- The entropy current can be split into $S^\mu = S^\mu_{can} + S^\mu_{non}$

- S^μ_{can} from covariantising Euler relation

$$S^\mu_{can} = -T^\mu{}_\nu \beta^\nu + P\beta^\mu - \frac{\mu}{T} J^\mu - \kappa_{\mathbb{E}} \mathbb{E}^\nu \mathbb{E}_\nu \beta^\mu$$

- S^μ_{non} together with relaxation scalar and non-canonical entropy current cancel hydrostatic contributions to entropy production

Relaxation at order one

- Using (non-)conservation equation of energy-momentum tensor and charge current, divergence of canonical entropy current in terms of altered stationarity condition is

$$e^{-1}\partial_\mu\left(eS^\mu_{\text{can}}\right)+\left(\beta^\rho+\frac{1}{nT}\left(J^\nu-J^\nu_{(0)}\right)h_{\nu\sigma}h^{\sigma\rho}\right)\Gamma_\rho$$
$$=\left(T^\mu-T^\mu_{(0)}\right)\mathscr{L}_\beta\tau_\mu-\frac{1}{2}\left(T^{\mu\nu}-T^{\mu\nu}_{(0)}\right)\mathscr{L}_\beta h_{\mu\nu}-\left(J^\mu-J^\mu_{(0)}\right)\delta'_{\mathscr{B}}A_\mu$$

where

$$\delta_{\mathscr{B}}A_\mu := \mathscr{L}_\beta A_\mu - \partial_\mu\Lambda = \mathscr{L}_\beta A_\mu - \partial_\mu\left(\frac{u^\nu A_\nu-\mu}{T}\right)$$

$$\delta'_{\mathscr{B}}A_\mu = \delta_{\mathscr{B}}A_\mu - \frac{1}{nT}h_{\mu\nu}h^{\nu\rho}\Gamma_\rho$$

- Rewriting divergence in this way allows us to isolate non-hydrostatic contributions to the constitutive relations of $T^\mu, T^{\mu\nu}, J^\mu$

Relaxation at order one

- decompose each constitutive relations into: hydrostatic, non-hydrostatic non-dissipative and dissipative corrections

$$T^{\mu} - T^{\mu}_{(0)} = T^{\mu}_{\text{HS}} + T^{\mu}_{\text{NHS}} + T^{\mu}_{\text{D}}$$
$$T^{\mu\nu} - T^{\mu\nu}_{(0)} = T^{\mu\nu}_{\text{HS}} + T^{\mu\nu}_{\text{NHS}} + T^{\mu\nu}_{\text{D}}$$
$$J^{\mu} - J^{\mu}_{(0)} = J^{\mu}_{\text{HS}} + J^{\mu}_{\text{NHS}} + J^{\mu}_{\text{D}}$$

- Similarly: assume that we can separate relaxation contributions into two types: those that can be expressed in terms of stationary tensor structures and those that vanish at stationarity

- What we find:

$$\hat{\Gamma}_{\varepsilon} = \rho_m \Gamma v_j \left(n v^j + J^j_{(1),\text{NHS}} + J^j_{(1),\text{D}} \right) + \mathcal{O}(\partial^3)$$

Conductivities

- To compute the AC conductivity's (needed to compare to Drude) we employ linear response theory
- study how each of the charge currents δJ^i, $\delta Q^i = \delta J^i_\epsilon - \mu \delta J^i \equiv \delta T^i_0 - \mu \delta J^i$, P^i responds to perturbations of the $\mathbb{E}, T, v_{0j}$
- captured in the response matrix

$$\begin{pmatrix} \delta J_i \\ \delta Q_i \\ \delta P_i \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & T\alpha_{ij} & \zeta^1_{ij} \\ T\bar{\alpha}_{ij} & T\kappa_{ij} & \zeta^2_{ij} \\ \zeta^3_{ij} & \zeta^4_{ij} & \zeta^5_{ij} \end{pmatrix} \begin{pmatrix} \delta E_j \\ \delta(-\partial_j T/T) \\ \delta v_{0j} \end{pmatrix}$$

- To obtain the matrix we linearise and solve the hydrodynamic equations in the presence of the sources
- Consider small fluctuations of our fluid away from a stationary configuration with $T = \text{const},\ \mu = \text{const},\ v_{0j} = 0$

Conductivities

- The AC conductivities given by the $\mathbf{k} \to \mathbf{0}$ limit are

$$\sigma(\omega, \mathbf{0}) = \sigma_0 + \frac{n(n - \Gamma \rho_m \sigma_0)}{\rho_m(\Gamma - i\omega)}$$

(no Onsager reciprocity yet)

- Noticing that $\sigma(\omega \to 0) = \sigma_{DC} = n^2/\rho_m\Gamma$

- can write

$$\sigma(\omega) = \sigma_0 + \frac{\sigma_{\mathrm{DC}} - \sigma_0}{1 - i\omega\tau}$$

(sum of incoherent term and Drude term)

Imposing time-reversal invariance

- want system to respect microscopic time reversal symmetry in effective correlates at $\omega \neq 0$ for a state at zero velocity

- In this case the conductivity becomes

$$\sigma = \frac{\sigma_{DC}}{1 - i\omega\Gamma^{-1}}$$

(Drude with DC conductivity)

$$\sigma_{DC} = \frac{n^2}{\rho_m \Gamma}$$

- Incoherent conductivity disappeared → can only appear if the system does not form a steady state or if we violate Onsager reciprocity

- Main result: thermo-electric conductivities of our model assume Drude form when imposing positivity of entropy production and Onsager reciprocity

Conclusion

- Considered hydrodynamic model of a charged fluid in an external electric field in the presence of impurities that relax momentum and energy.
- Looked for steady states

  → find that stationarity constraints need to be modified to incorporate relaxations

- included dissipative corrections
- allows us to consider conductivity of fluids that reach a stationary state in a driving electric field
- positivity of entropy production and Onsager reciprocity constrained transport in the fluid

  → no incoherent conductivity to make a contribution to the DC

- <u>Further:</u> stability of the model? hydrodynamical realisation of steady states in prope brane models?

Thermodynamics

- temperature, chemical potential and the fluid velocity

$$T = \frac{1}{\tau_\mu \beta^\mu}, \quad \mu = T\left(A_\mu \beta^\mu + \Lambda\right), \quad u^\mu = T\beta^\mu$$

In FSCC: $u^\mu = (1, v^i)$

- Electric field $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} = \mathbb{E}_\mu \tau_\nu - \mathbb{E}_\nu \tau_\mu$

Relaxation

$$\Gamma_\rho = -T\hat{\Gamma}_\sigma\left(\left(\beta^\sigma + \frac{1}{nT}\left(J^\mu_{\text{NHS}} + J^\mu_{\text{D}}\right)h_{\mu\nu}h^{\nu\sigma}\right)\tau_\rho - \frac{1}{T}h^{\sigma\mu}h_{\mu\rho}\right)$$
$$-\Gamma_{\rho\sigma}\left(\beta^\sigma + \frac{1}{nT}\left(J^\mu_{\text{NHS}} + J^\mu_{\text{D}}\right)h_{\mu\nu}h^{\nu\sigma}\right),$$

$$\Gamma_{\mu\nu} = \Gamma\left(c_1\tau_\mu\tau_\nu + c_2h_{\mu\nu}\right) + \mathcal{O}\left(\partial^3\right)$$

Generating functional

- generating functional $W[\tau, h, A]$: correlation functions

(leading term) $$W_{(0)}[\tau, h, A] = \int d^{d+1}x\, e\, P\left(T, \mu, \vec{\mathbb{E}}^2, \vec{v}^2, \vec{v} \cdot \vec{\mathbb{E}}\right)$$

- define one-point functions

$$T^{\mu\nu} = \frac{2}{e}\frac{\delta W}{\delta h_{\mu\nu}}, \quad T^{\mu} = -\frac{1}{e}\frac{\delta W}{\delta \tau_{\mu}}, \quad J^{\mu} = \frac{1}{e}\frac{\delta W}{\delta A_{\mu}}$$

Hydrostatic part

- Hydrostatic part has to satisfy following non-conservation equation

$$\partial_\mu {T_{\rm HS}}^\mu{}_\nu - F_{\nu\mu} J^\mu_{\rm HS} - \Gamma^{\rm HS}_\nu = 0,$$
$$\partial_\mu J^\mu_{\rm HS} = 0$$

- At order $\mathcal{O}(\partial^0)$ in constitutive relations: $\Gamma^{HS}_{(1),\nu} = \rho_{\rm m} \Gamma\left(\mathbf{v}^2, v_i\right)$

- At order $\mathcal{O}(\partial^1)$ in constitutive relations: find that using only hydrostatic conditions that do not involve relaxation term $\Gamma^{HS}_{(2),\nu} \equiv 0$

- Now considering entropy production in presence of relaxation terms

- Have freedom to define S^μ_{non}, Γ^{non} satisfying

$$e^{-1}\partial_\mu\left(e S^\mu_{\rm non}\right) + \Gamma^{\rm non} = -T^\mu_{\rm HS}\mathscr{L}_\beta \tau_\mu + \frac{1}{2} T^{\mu\nu}_{\rm HS}\mathscr{L}_\beta h_{\mu\nu} + J^\mu_{\rm HS}\delta'_{\mathscr{B}} A_\mu$$
$$\Gamma^{\rm non} = -\frac{1}{nT} J^\mu_{\rm HS} h_{\mu\sigma} h^{\sigma\rho}\Gamma_\rho$$

- In this way we eliminate all stationary configurations consistent with positivity of entropy production (by defining a relaxation scalar and non-canonical entropy current that cancels hydrostatic contributions to entropy production)

$$S^\mu = S^\mu_{can} + S^\mu_{non}$$

Non-hydrostatic, non-dissipative part

- Part that makes no contribution to entropy production but is not hydrostatic

$$T^{\mu}_{\rm NHS}\mathscr{L}_\beta\tau_\mu - T^{\mu\nu}_{\rm NHS}\frac{1}{2}\mathscr{L}_\beta h_{\mu\nu} - J^{\mu}_{\rm NHS}\delta'_{\mathscr{B}}A_\mu \equiv 0$$

- At order one: must be linear combinations of $\mathscr{L}_\beta\tau_\mu, \quad \mathscr{L}_\beta h_{\mu\nu}, \quad \delta'_{\mathscr{B}}A_\mu$
- Correspondingly equation above is quadratic form in hydrostatic constraints
- quadratic form: to fail to contribute to entropy production must be antisymmetric (in this way no entropy production)

$$\begin{pmatrix} T^{\mu}_{(1),\rm NHS} \\ T^{\mu\nu}_{(1),\rm NHS} \\ J^{\mu}_{(1),\rm NHS} \end{pmatrix} = \begin{pmatrix} 0 & N_2^{\mu(\rho\sigma)} & N_1^{\mu\rho} \\ -N_2^{\rho(\mu\nu)} & 0 & N_3^{\rho(\mu\nu)} \\ -N_1^{\rho\mu} & -N_3^{\mu(\rho\sigma)} & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_\beta\tau_\rho \\ -\frac{1}{2}\mathscr{L}_\beta h_{\rho\sigma} \\ -\delta'_{\mathscr{B}}A_\rho \end{pmatrix}$$

[Armas]

- We obtained most general tensor structures consistent with our symmetries and defined 24 non-hydrostatic, non-dissipative transport coefficients

Dissipative part

- Dissipative terms lead production of entropy
- Analogously dissipative contributions can be written in quadratic form in terms of symmetric coefficient matrix, allowing for entropy production

$$\begin{pmatrix} T^{\mu}_{(1),\mathrm{D}} \\ T^{\mu\nu}_{(1),\mathrm{D}} \\ J^{\mu}_{(1),\mathrm{D}} \end{pmatrix} = \begin{pmatrix} D_1^{\mu\rho} & D_2^{\mu(\rho\sigma)} & D_3^{\mu\rho} \\ D_2^{\rho(\mu\nu)} & D_4^{(\mu\nu)(\rho\sigma)} & D_5^{\rho(\mu\nu)} \\ D_3^{\rho\mu} & D_5^{\mu(\rho\sigma)} & D_6^{\mu\rho} \end{pmatrix} \begin{pmatrix} \mathscr{L}_\beta \tau_\rho \\ -\frac{1}{2}\mathscr{L}_\beta h_{\rho\sigma} \\ -\delta'_{\mathscr{B}} A_\rho \end{pmatrix}$$

- Obtained most general structures consistent with our symmetries and defined 42 dissipative transport coefficient terms