Relaxation dynamics of quasi-hydrodynamic mode

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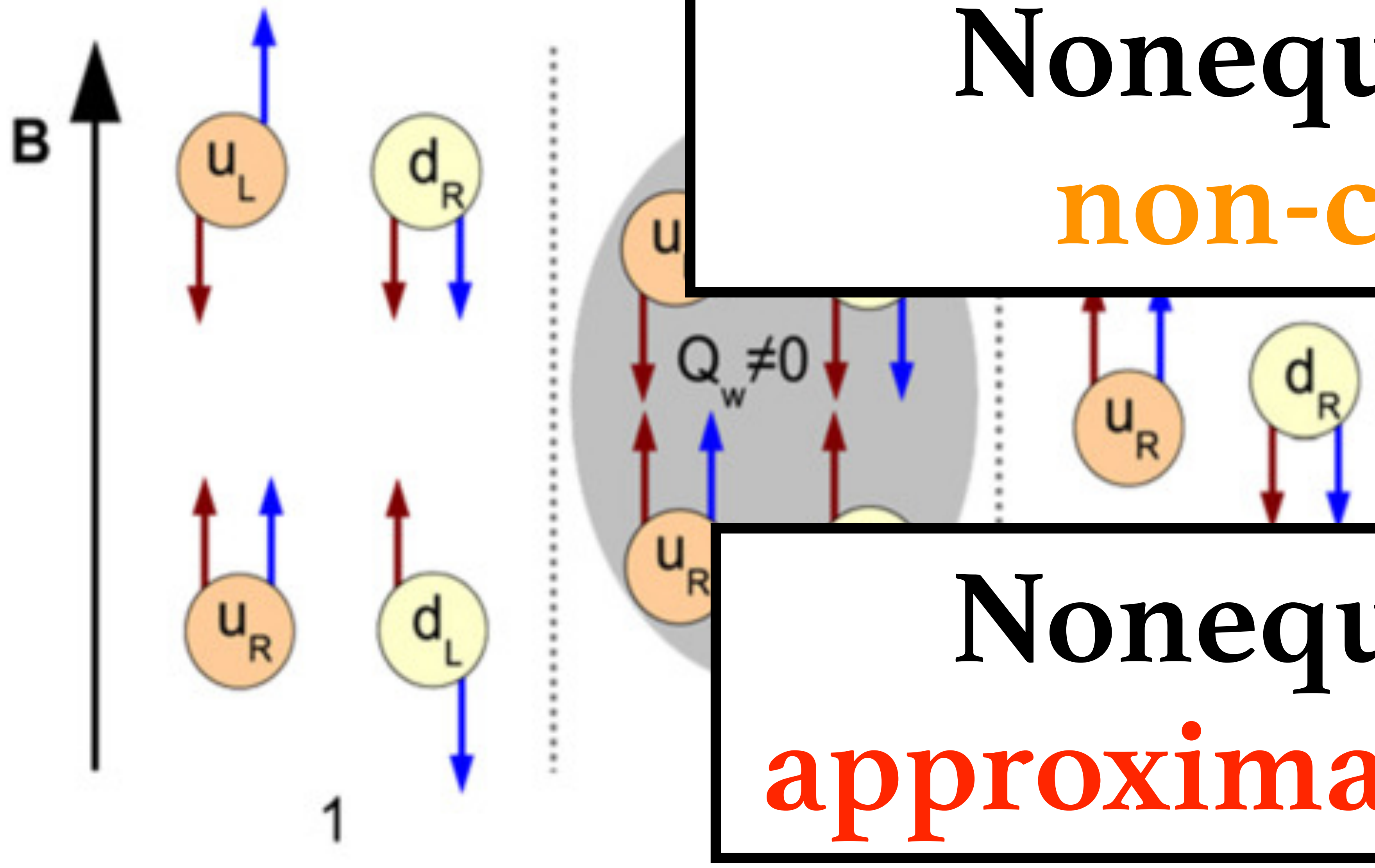
2025/3/27, Holographic perspectives on chiral transport and spin dynamics, ECT* at Trento

[Based on the ongoing work with Nishimura, Sogabe, Stephanov, and Yee]

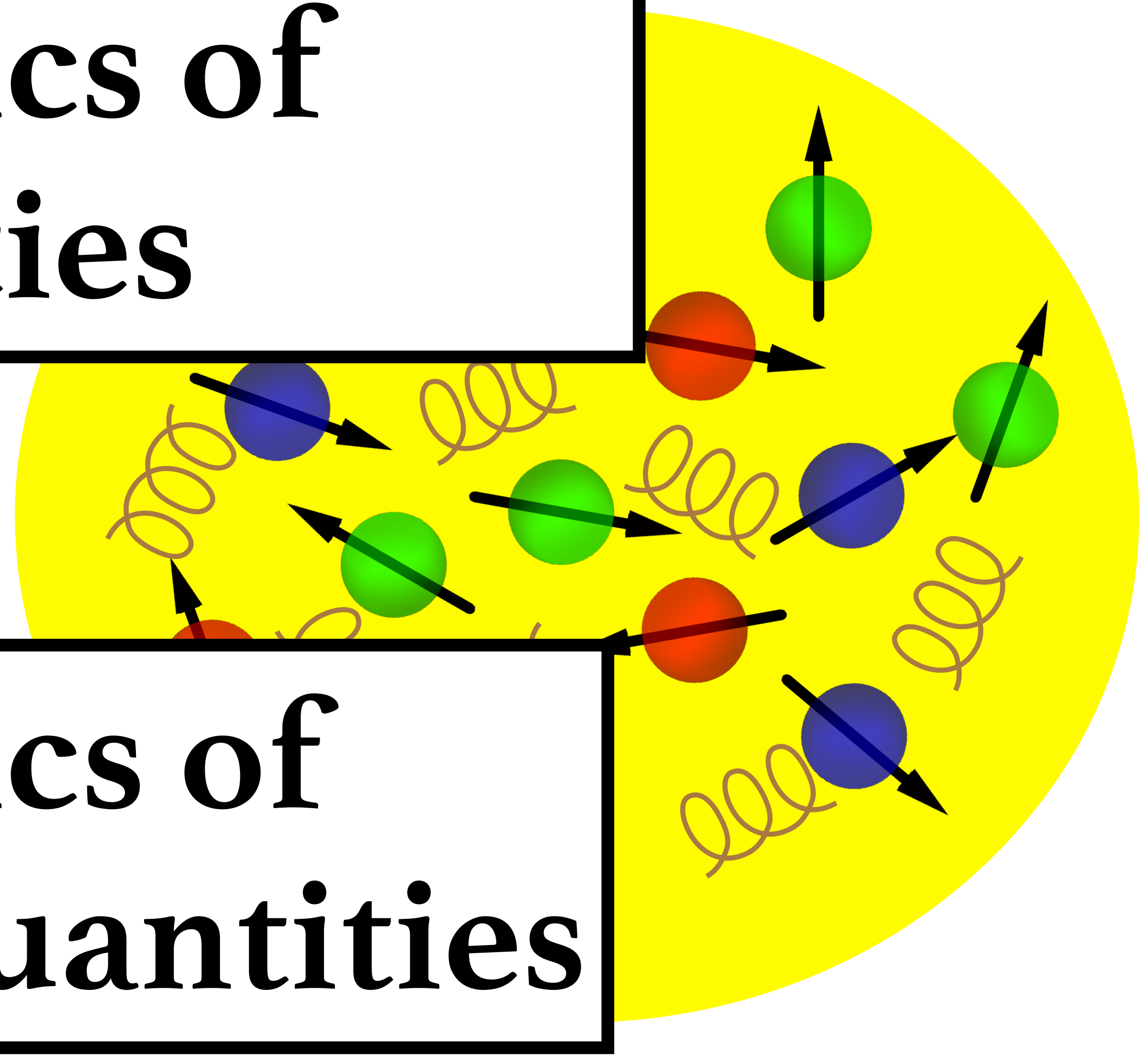
Motivation

"Holographic perspectives on chiral transport and spin dynamics"

Chiral transport



Spin dynamics



Nonequilibrium dynamics of non-conserved quantities

↓

Nonequilibrium dynamics of approximately-conserved quantities

↓

$SU(2)_L \times SU(2)_R$-sym at $m \simeq 0$

$SU(2)_{HQ}$-sym at $M \to \infty$

Q. What is the hydro description for approximate symmetry!?

A. Quasi-hydrodynamic eqs. (& potential holographic outlook)!

Recap: Hydro regime and Hydro+ regime

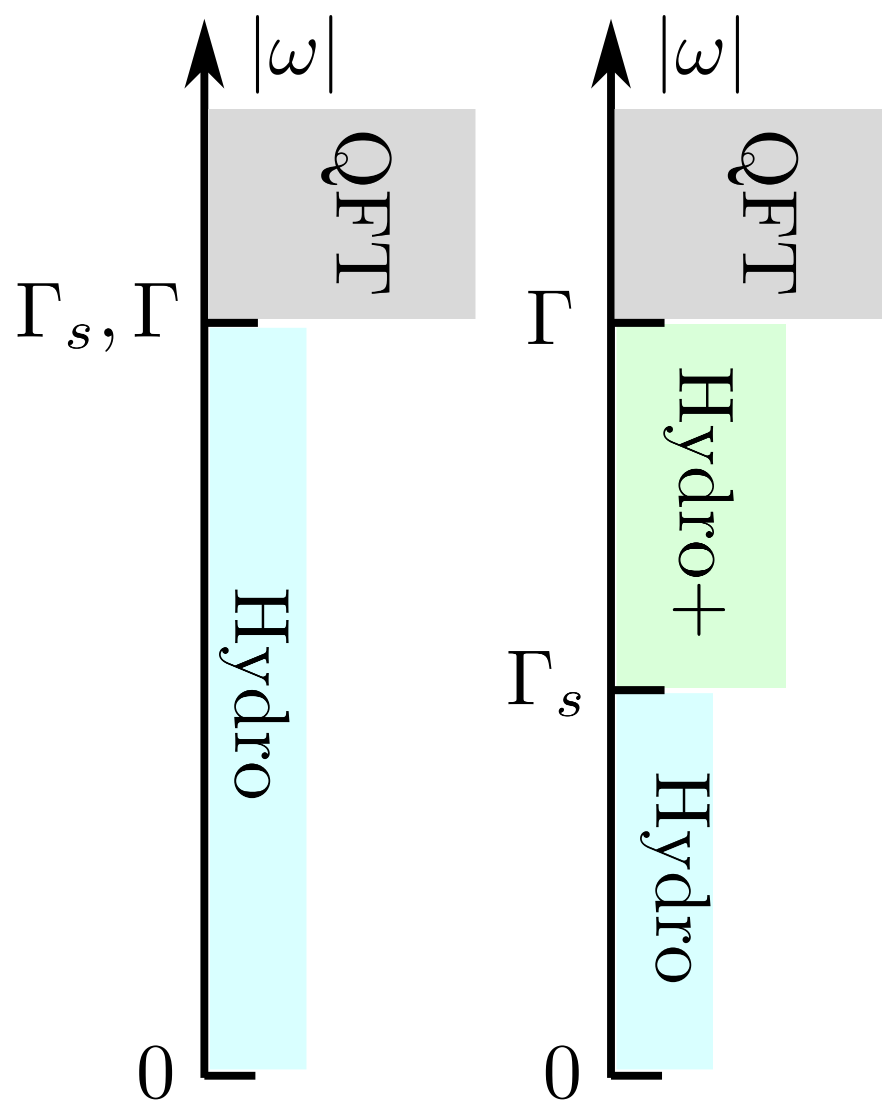
Two regimes

[Slide by M. Stephanov in the last workshop]

- Hydrodynamic regime: $\omega \ll \Gamma_s, \Gamma$.
  Breaks down when $\omega \sim \Gamma_s$.
- Hierarchy $\Gamma_s \ll \Gamma$ creates another regime

  $\Gamma_s \ll \omega \ll \Gamma$

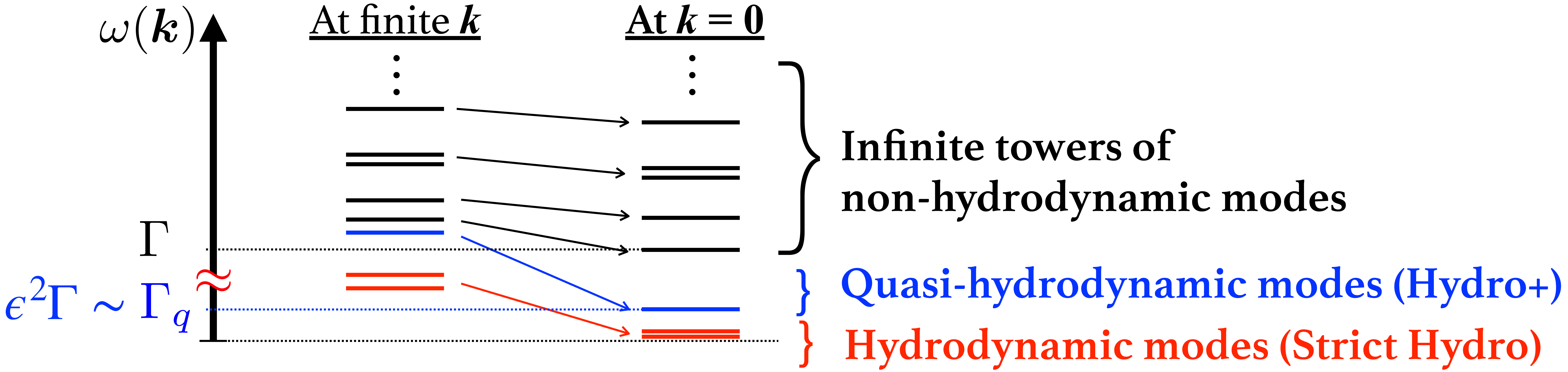
  for an effective theory with additional degree(s) of freedom, a.k.a. Hydro+
  (Hydro+ description is also valid in hydrodynamic regime, of course.)



- Unlike hydro variables these additional d.o.f. are not diffusive, i.e., they relax locally. But *very* slowly.
  - E.g., Israel-Stewart hydrodynamics is *not* a Hydro+ theory unless the rate $1/\tau_\Pi$ is *parametrically/controllably* small.

Hydro+ regime from approximate symmetry

◆ **Hydro, Quasi-hydro, and Non-hydro modes in generic interacting QFTs**



If $\Gamma_q \ll \Gamma$ **is satisfied, Hydro+ becomes well-defined!!**

This generally happens when symmetry emerges by tuning parameters (m, M, ...)!

- Non-abelian chiral symmetry: $SU(2)_R \times SU(2)_L$ at $m = 0$
- Heavy-quark spin symmetry: $SU(2)_{HQ}$ at $M \to \infty$

[Remarks: Relativistic spin hydro, (ressumed-)MIS etc. does not generally fit into Hydro+ regime]

Outline

Approximate conservation law as Ward-Takahashi identity

(Quasi-)hydrodynamic equation from local thermodynamics

Dispersion relation and Green-Kubo formula

Microscopic Setup: $G = \mathrm{SU}(N)$

◆ **Action with approximate symmetry**

Dynamical fields: ψ, Symmetry-breaking parameters: $M = \epsilon \hat{M}$

$$\mathcal{S}[\psi; M] = \mathcal{S}_{\mathrm{sym}}[\psi] + \mathcal{S}_{\mathrm{asym}}[\psi; M]$$

Under SU(N)-symmetry: **Invariant** **Non-invariant**

$\psi \to U\psi,\ U \in \mathrm{SU}(N)$

- When $\epsilon = 0 \Rightarrow \mathcal{S}_{\mathrm{asym}} = 0 \Rightarrow \partial_\mu J^\mu_A = 0 \quad (A = 1, \cdots, N^2 - 1)$
- When $\epsilon \neq 0 \Rightarrow \mathcal{S}_{\mathrm{asym}} \neq 0 \Rightarrow \partial_\mu J^\mu_A \neq \epsilon \Theta_A \quad (A = 1, \cdots, N^2 - 1)$

Q. How to find Θ_A? ➡ A. Background field (spurion) method!

Background field (spurion) method

◆ **Gauged action and spurious local symmetry**

Dynamical fields: ψ, Symmetry-breaking parameters: $M = \epsilon \hat{M}$

$$\mathcal{S}_{\text{gauged}}[\psi; A_\mu, M] = \boxed{\mathcal{S}_{\text{sym}}[\psi; A_\mu]} + \boxed{\mathcal{S}_{\text{asym}}[\psi; A_\mu, M]}$$

Under SU(N) gauge-symmetry: **Invariant** **Invariant**

$$\begin{cases} \psi \to U\psi \\ A_\mu \to g A_\mu g^\dagger - i g \partial_\mu g^\dagger \quad \text{with } U, g \in \mathrm{SU}(N) \\ M \to gM \quad \text{(or appropriate symmetry transformation for } M\text{)} \end{cases}$$

We assumed the promotion of M to a SU(N)-fund. rep. field makes $\mathcal{S}_{\text{asym}}$ inv!

➡ $\mathcal{S}_{\text{gauged}}$ now enjoys the *spurious* SU(N) gauge symmetry!

The transformation of M is *spurious*, but we treat it as if it is a field (spurion)!

Ward-Takahashi identity

$\mathcal{S}_{\text{gauged}}$: invariant under the *spurious* SU(N) gauge symmetry!

⬇

◆ **Approximate conservation law as WT identity**

$$\partial_\mu J^\mu_A - f_{ABC} A^B_\mu J^\mu_C = \epsilon \Theta_A \quad \text{with} \quad \Theta_A := \mathrm{i}(\hat{M}^\dagger t_A \Phi - \Phi^\dagger t_A \hat{M})$$

Current and source defined by $J^\mu_A := \dfrac{\delta \mathcal{S}_{\text{gauged}}}{\delta A^A_\mu}$ $\Phi^n := \dfrac{\delta \mathcal{S}_{\text{asym}}}{\delta M^\dagger_n}$

Derivation:

$$0 = \delta_\theta \mathcal{S}_{\text{gauged}} := \int \mathrm{d}^{d+1}x \left[\frac{\delta \mathcal{S}_{\text{gauged}}}{\delta \psi} \delta_\theta \psi + \frac{\delta \mathcal{S}_{\text{gauged}}}{\delta A^A_\mu} \delta_\theta A^A_\mu + \frac{\delta \mathcal{S}_{\text{gauged}}}{\delta M^n} \delta_\theta M^n + \delta_\theta M^\dagger_n \frac{\delta \mathcal{S}_{\text{gauged}}}{\delta M^\dagger_n} \right]$$

$= 0$ (by eom) $\quad =: J^\mu_A \quad =: \Phi^\dagger_n \quad =: \Phi^n$

$$= \cdots = \int \mathrm{d}^{d+1}x \theta^A \left[-\partial_\mu J^\mu_A + f_{ABC} A^B_\mu J^\mu_C + \mathrm{i}(M^\dagger t_A \Phi - \Phi^\dagger t_A M) \right]$$

Outline

Approximate conservation law as Ward-Takahashi identity

$$\partial_\mu J^\mu_A - f_{ABC} A^B_\mu J^\mu_C = \epsilon \Theta_A \quad \text{with} \quad \Theta_A := \mathrm{i}(\hat{M}^\dagger t_A \Phi - \Phi^\dagger t_A \hat{M})$$

(Quasi-)hydrodynamic equation from local thermodynamics

Dispersion relation and Green-Kubo formula

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(Quasi-)hydrodynamic equation from local thermodynamics

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We need constitutive relation!

Approximate conservation law as Ward-Takahashi identity

$$\partial_\mu J^\mu_A - f_{ABC} A^B_\mu J^\mu_C = \epsilon \Theta_A \quad \text{with} \quad \Theta_A := \mathrm{i}(\hat{M}^\dagger t_A \Phi - \Phi^\dagger t_A \hat{M})$$

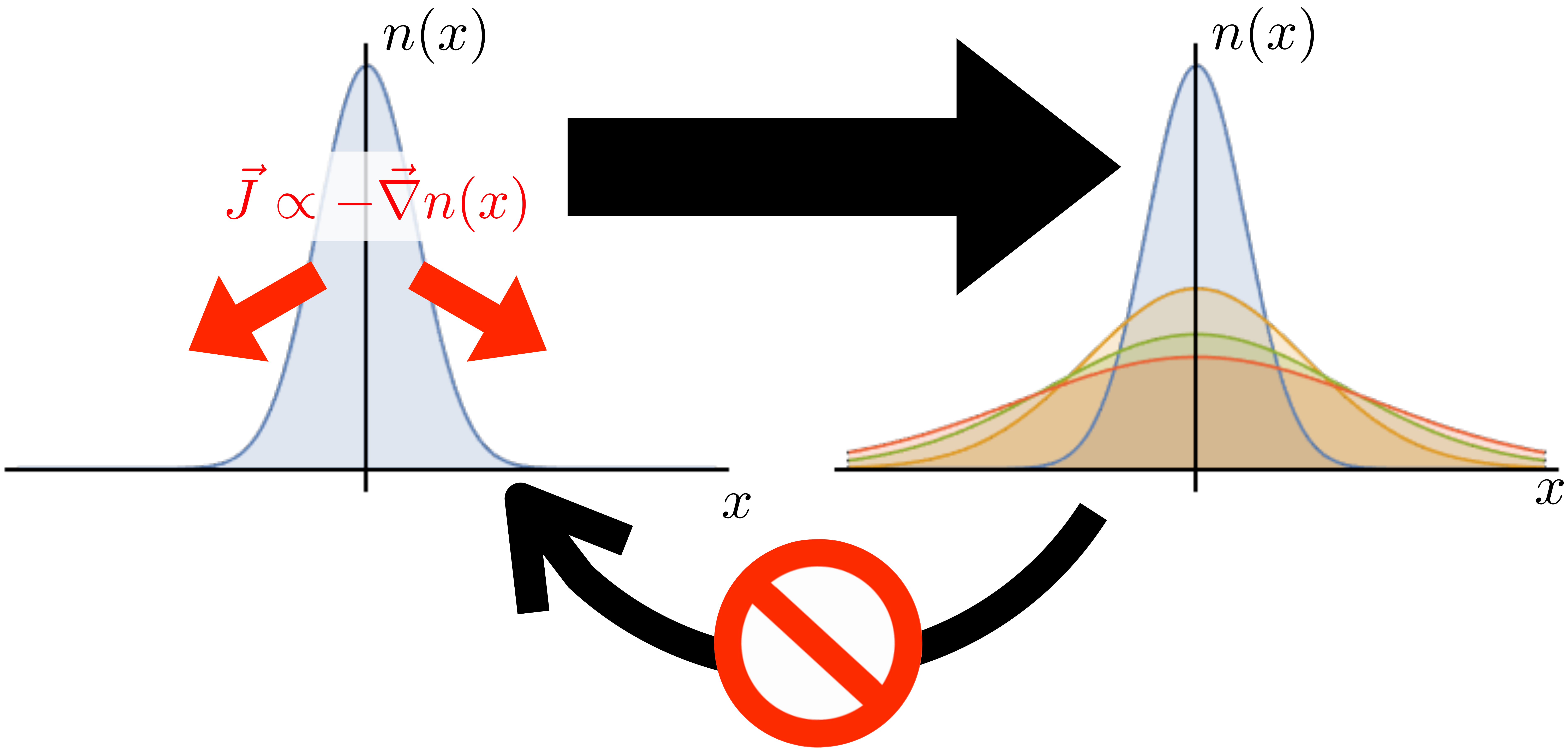
To make this WT identity as a solvable set of equation,
we need to express J^i_A and Φ (or Θ_A) in terms of $n (:= J^0_A), A^A_\mu$, and M

➡ Derivation of hydro ≈ Derivation of constitutive relations!

You can choose any of **Various approaches:**

- Holography (Fluid/gravity corresponsence)
- Kinetic theory (Boltzmann equation)
- Nonequilibrium statisitical operator
- Schwinger-Keldysh Effective Field Theory
- **Local thermodynamics (1st law and 2nd law)** [← I will use]

Irreversibility of diffusion



No-go for time-reversal process!

Thermodynamic concepts, especially, The 2_{nd} law, should be there!

Deriving quasi-hydrodynamic equation

Step 1: Identify dynamical d.o.m & its equation of motion

$$\partial_\mu \langle J_A^\mu \rangle = \epsilon \langle \Theta_A \rangle \ \text{ with } \ n_A := \langle J_A^0 \rangle \ \text{ and } \ \epsilon \langle \Theta_A \rangle = i \left[M^\dagger t_A \langle \Phi \rangle - \langle \Phi^\dagger \rangle t_A M \right]$$

QFT interpretation

$\simeq$ Ward-Takahashi identity

Step 2: Introduce entropy & conjugate variable with 1st law

$$S_{\text{ent}}[n_A] = S_{\text{ent}}^{\text{sym}}[n_A] + S_{\text{ent}}^{\text{asym}}[n_A; M] \ \Rightarrow \ \beta\mu^A := \frac{\delta S_{\text{ent}}}{\delta n_A}$$

$\simeq$ Effective Lagrangian (Hamiltonian)

Step 3: Find J_A^i, Θ_A compatible with 2nd law (+derivative exp.)

$$\frac{\mathrm{d}S_{\text{ent}}}{\mathrm{d}t} \geq 0 \ \forall n_A(x) \ \Rightarrow \ \begin{cases} \langle J_A^i \rangle = -T\lambda \partial^i (\beta\mu^A), \\ \epsilon \langle \Theta_A \rangle = -\gamma M^\dagger (t_A t_B + t_B t_A) M \mu^B \end{cases} \ \text{with } \lambda, \gamma > 0$$

$\simeq$ A kind of symmetry constraints

$$\frac{\mathrm{d}S_{\text{ent}}}{\mathrm{d}t} = \int \mathrm{d}^d x \frac{\delta S_{\text{ent}}}{\delta n_A} \partial_t n_A = \cdots = \int \mathrm{d}^d x \left[-\boxed{\langle J_A^i \rangle} \partial_i (\beta\mu^A) - i\beta\mu^A M^\dagger t_A \boxed{\langle \Phi \rangle} + i \boxed{\langle \Phi^\dagger \rangle} t_A M \beta\mu^A \right] \geq 0$$

$= -T\lambda \partial^i (\beta\mu^A)$; $= iT\gamma t_B M \beta\mu^B$; $= iT\gamma\beta\mu^B M^\dagger t_B$

Deriving quasi-hydrodynamic equation

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Step 4: Identify how we can compute Onsager coefficients λ, γ

QFT interpretation

$\simeq$ Ward-Takahashi identity

$\simeq$ Effective Lagrangian (Hamiltonian)

$\simeq$ A kind of symmetry constraints

$\simeq$ Matching condition for low-energy coeff.

Outline

Approximate conservation law as Ward-Takahashi identity

$$\partial_\mu J^\mu_A - f_{ABC} A^B_\mu J^\mu_C = \epsilon \Theta_A \quad \text{with} \quad \Theta_A := \mathrm{i}(\hat{M}^\dagger t_A \Phi - \Phi^\dagger t_A \hat{M})$$

(Quasi-)hydrodynamic equation from local thermodynamics

$$\langle J^i_A \rangle = -T\lambda \delta^{ij} \partial_j (\beta \mu^A) \quad \text{and} \quad \epsilon \langle \Theta_A \rangle = -\gamma M^\dagger (t_A t_B + t_B t_A) M \mu^B$$

$[M = \epsilon \hat{M}]$

Dispersion relation and Green-Kubo formula

Matching condition for Onsager coeffs.

(Quasi-)hydrodynamic equation from local thermodynamics

$$\langle J^i_A \rangle = -T\lambda\delta^{ij}\partial_j(\beta\mu^A) \quad \text{and} \quad \epsilon\langle\Theta_A\rangle = -\gamma M^\dagger(t_A t_B + t_B t_A)M\mu^B$$

$[M = \epsilon\hat{M}]$

Two Onsager coefficients:
- Charge conductivity : λ
- Kinetic coefficient : γ

⇒ We want to find the correlation function expression of γ

(= Green-Kubo formula for γ)

◆ **Several familiar methods**

(1) Perturb the system with the background field
(2) Petrurb the density operator around local equilibrium (noneq. statistical operator)
(3) Add noise and compute the hydro correlation function [←I will use]

Effective field theory at Hydro+ regime

◆ **Stochastic quasi-hydro equations**

(1) WT identity: $\partial_\mu \check{n}_A + \partial_i \check{J}^i_A = \epsilon \check{\Theta}_A$

(2) Constitutive relation:
with noise

$$\begin{cases} \check{J}^i_A = -T\lambda \partial^i(\beta \check{\mu}^A) + \check{\xi}^i_A \\ \epsilon \check{\Theta}_A = -\gamma M^\dagger (t_A t_B + t_B t_A) M \check{\mu}^B + \check{\xi}_A \end{cases}$$

(3) Fluctuation-Dissipation Relation:

$$\begin{cases} \langle \check{\xi}^i_A(x) \check{\xi}^i_A(y) \rangle = 2T\lambda \delta^{ij} \delta_{AB} \delta^{(d+1)}(x-y) \\ \langle \check{\xi}_A(x) \check{\xi}_A(y) \rangle = 2TM^\dagger (t_A t_B + t_B t_A) M \gamma \delta_{AB} \delta^{(d+1)}(x-y) \end{cases}$$

⇒ Correlation function at Hydro+ regime (including nonlinear term)

⇒ To find the matching condition, let us solve the linearized equation!

Dispersion relation & Kubo formula

◆ Linearized eom and its solution

$$\left[D := \frac{\lambda}{\chi}, \quad \Gamma_{AB} := \frac{\gamma}{\chi}\hat{M}^{\dagger}(t_A t_B + t_B t_A)\hat{M}\right]$$

$$\left[(\partial_t - D\nabla^2)\delta_{AB} + \epsilon^2\Gamma_{AB}\right]\check{n}_B + \partial_i\check{\xi}^i_A - \check{\xi}_A = 0$$

◆ Disspersion relation

$$\widetilde{n}_A(\omega,\vec{k}) = G_R^{n_A n_B}(\omega,\vec{k})\left[ik_i\widetilde{\xi}^i_A(\omega,\vec{k}) + \widetilde{\xi}_A(\omega,\vec{k})\right]$$

with $G_R^{n_A n_B}(\omega,\vec{k}) := \left[(-i\omega + D\vec{k}^2)\delta_{AB} + \epsilon^2\Gamma_{AB}\right]$

Pole is located at

$$0 = \det\left[(-i\omega + Dk^2)\delta_{AB} + \epsilon^2_{AB}\right]$$

[For SU(2) case, $\omega(k) = -\frac{i}{2}\epsilon^2|\hat{M}|^2\frac{\gamma}{\chi} - iDk^2$]

◆ Green-Kubo formula

$$\epsilon\widetilde{\Theta}_A(\omega,0) = -i\omega(-i\omega\delta_{AB} + \epsilon^2\Gamma_{AB})^{-1}\widetilde{\xi}_B(\omega,0)$$

$$\begin{cases} \xrightarrow{\omega\to 0,\ \epsilon\neq 0} 0 \\ \xrightarrow{\epsilon\to 0,\omega\neq 0} \widetilde{\xi}_B(\omega,0) \end{cases}$$

$$\hat{M}^{\dagger}(t_A t_B + t_B t_A)\hat{M}\gamma = \frac{1}{2T}\lim_{\omega\to 0}\lim_{\epsilon\to 0}\langle\widetilde{\Theta}(-\omega)\widetilde{\Theta}_B(\omega)\rangle$$

$$= \lim_{\omega\to 0}\lim_{\epsilon\to 0}\frac{1}{\omega}\,\mathrm{Im}\,G_R^{\Theta_A\Theta_B}(\omega)$$

Summary

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$[M = \epsilon \hat{M}]$

Dispersion relation and Green-Kubo formula

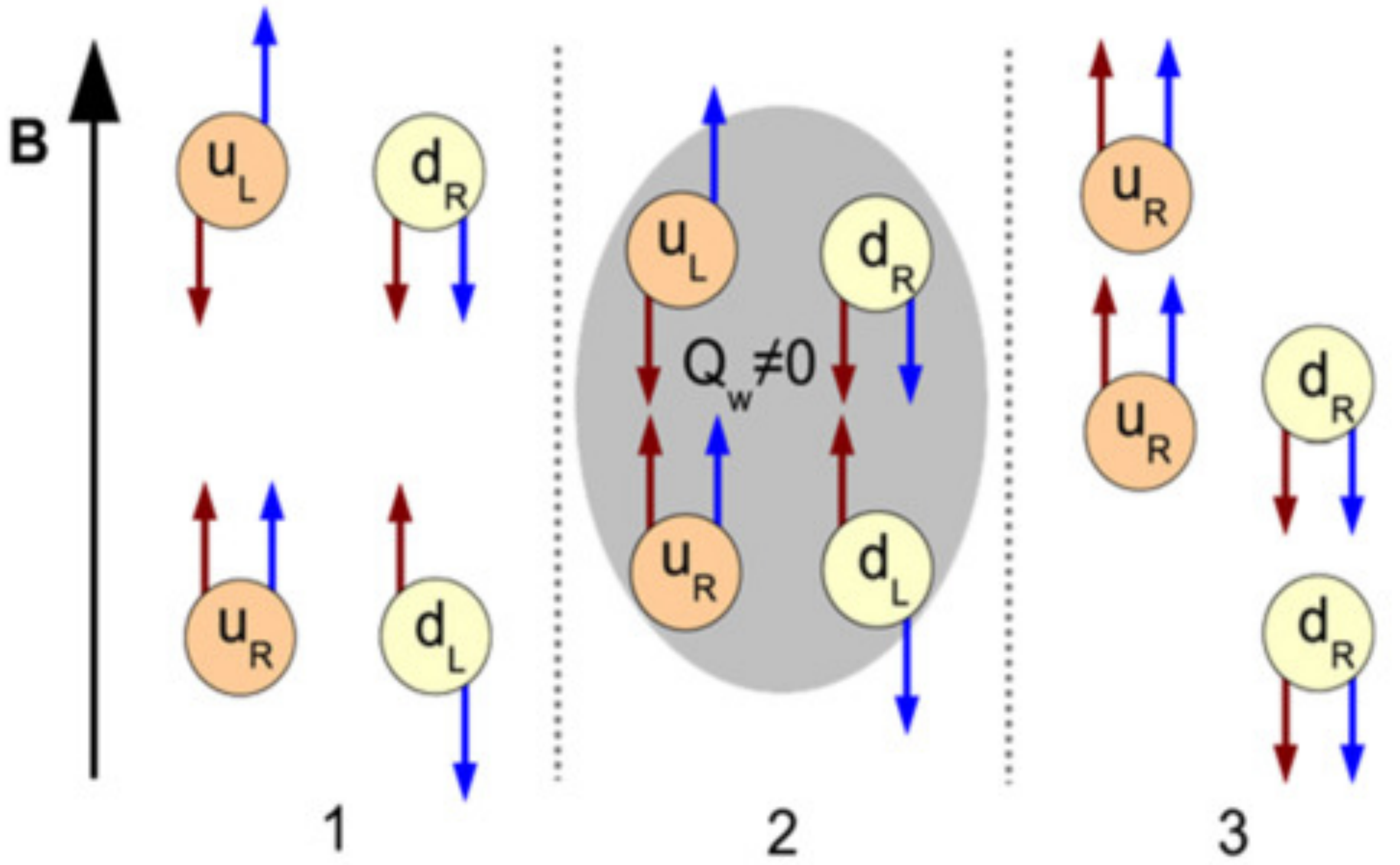
$$0 = \det\left[(-i\omega + Dk^2)\delta_{AB} + \epsilon^2_{AB}\right], \quad \hat{M}^\dagger (t_A t_B + t_B t_A) \hat{M} \gamma = = \lim_{\omega\to 0} \lim_{\epsilon \to 0} \frac{1}{\omega} \operatorname{Im} G_R^{\Theta_A \Theta_B}(\omega)$$

Outlook for Holography

◆ (Almost) Everything I know about holography

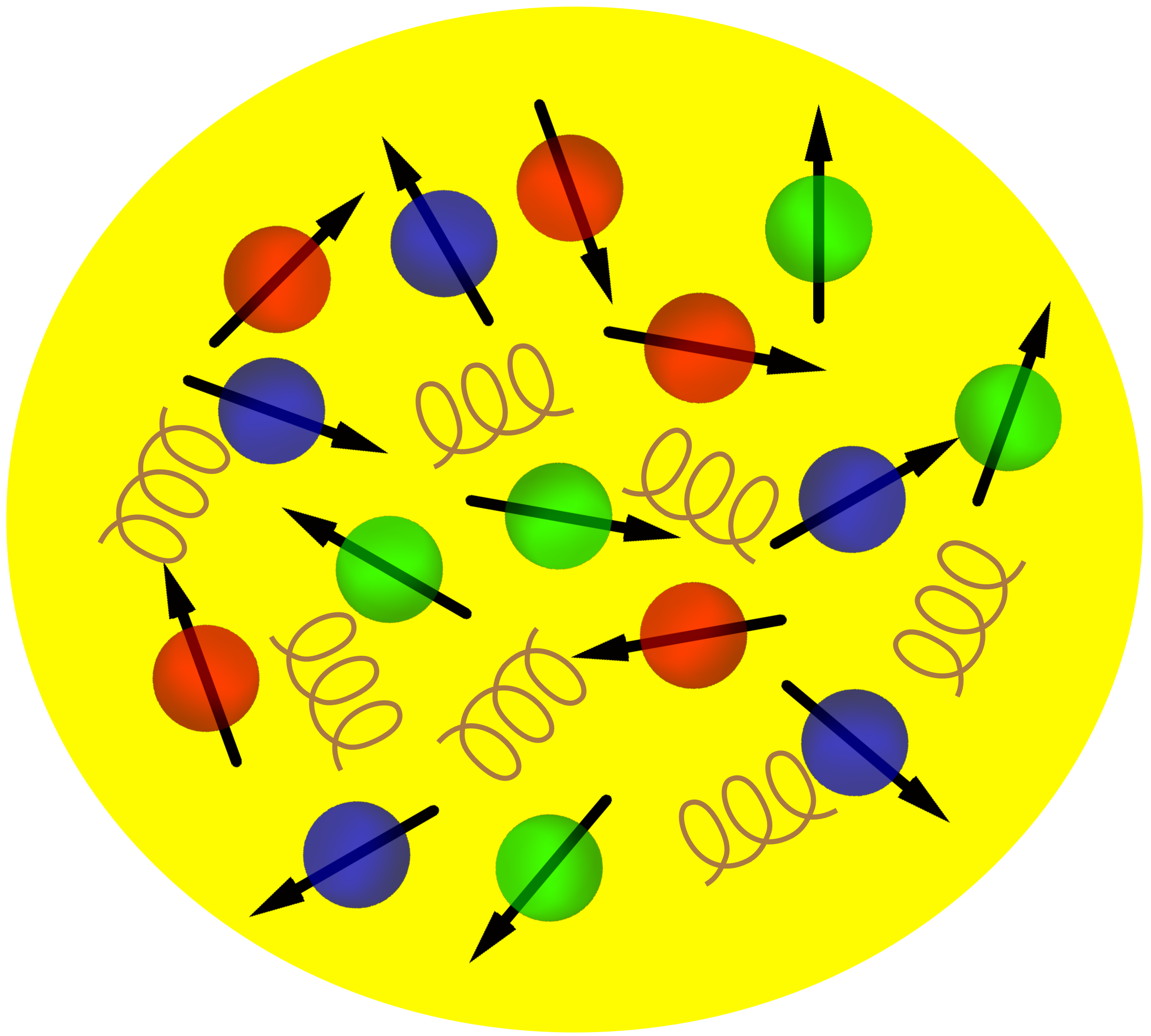
Let our background field be the dynamical one in AdS_5 spacetime!
(including spurion fields!)

Chiral transport



$SU(2)_L \times SU(2)_R$-sym at $m \simeq 0$

Spin dynamics



$SU(2)_{HQ}$-sym at $M \to \infty$

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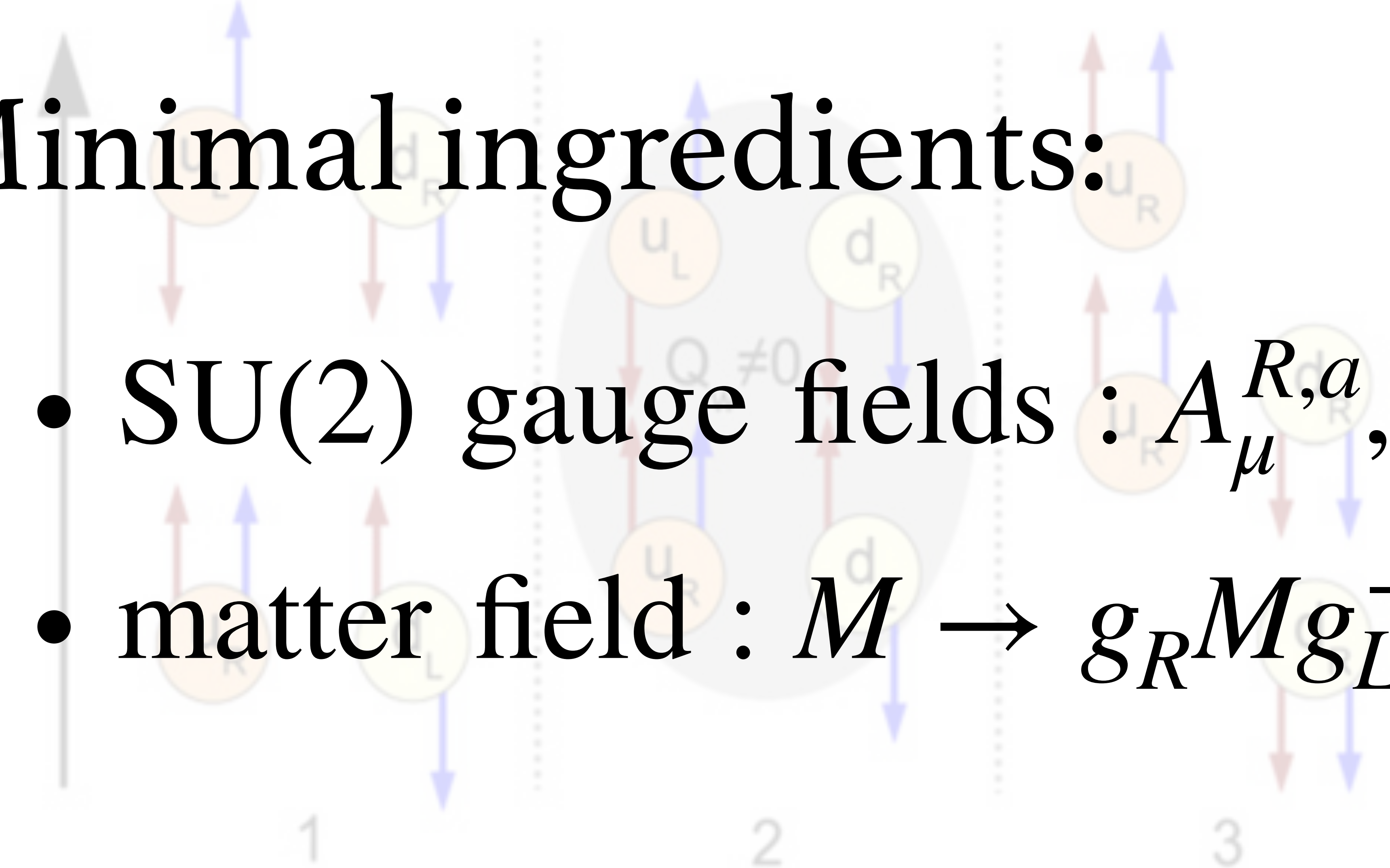
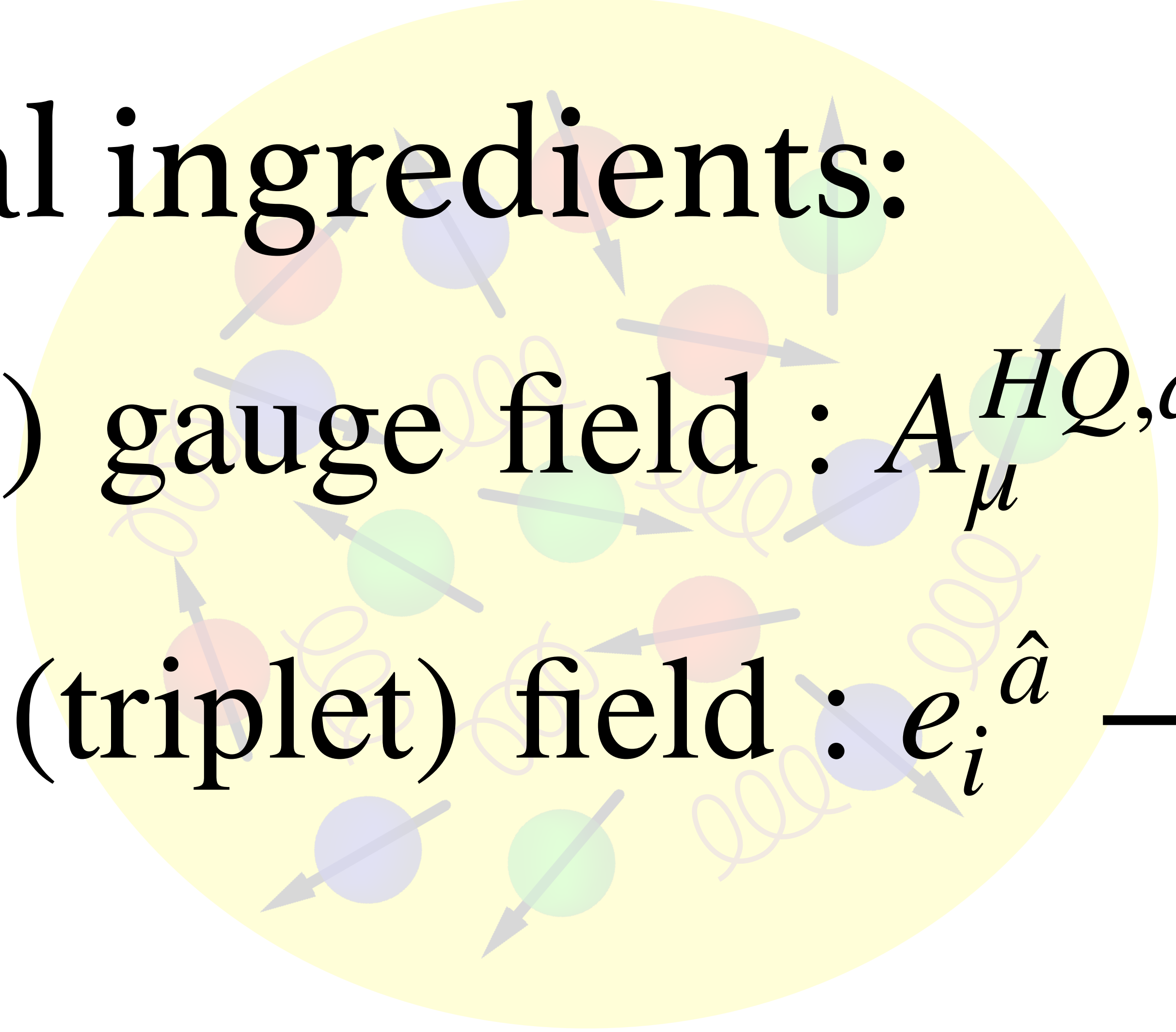


Minimal ingredients:

- SU(2) gauge fields : $A_\mu^{R,a}, A_\mu^{L,a}$
- matter field : $M \to g_R M g_L^{-1}$

$SU(2)_L \times SU(2)_R$-sym at $m \simeq 0$

Spin dynamics



Minimal ingredients:

- SU(2) gauge field : $A_\mu^{HQ,a}$
- triad (triplet) field : $e_i{}^{\hat a} \to R^{\hat a}{}_{\hat b}\, e_i{}^{\hat b}$

$SU(2)_{HQ}$-sym at $M \to \infty$

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