ANOMALOUS TRANSPORT WITHOUT TRIANGLE DIAGRAMS

A majestically cute pygmy hippo called "Moo Deng" from Thailand (has nothing to do with this talk)

NICK POOVUTTIKUL

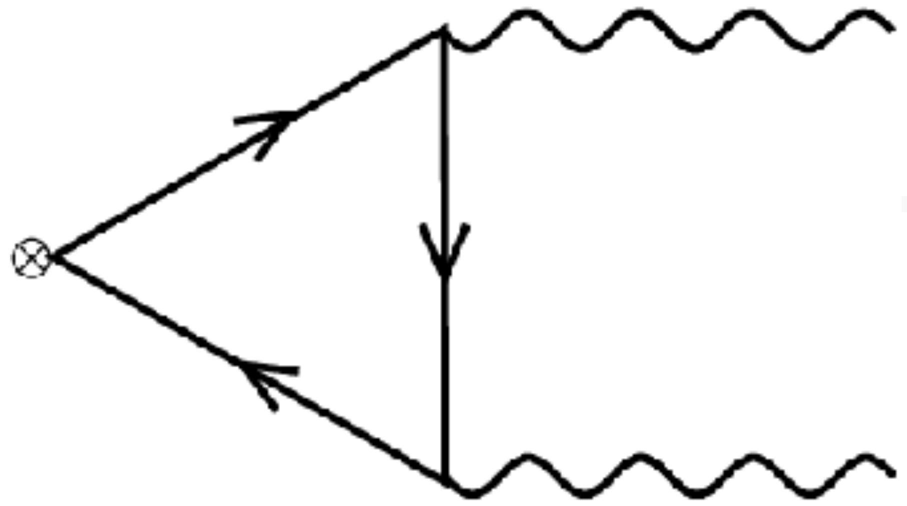
DEPARTMENT OF PHYSICS, FACULTY OF SCIENCE,
CHULALONGKORN UNIVERSITY, BANGKOK, THAILAND

BASED ON A 2311.8023 AND **2412.17650** WITH JOE DAVIGHI AND NAKARIN LOHITSIRI

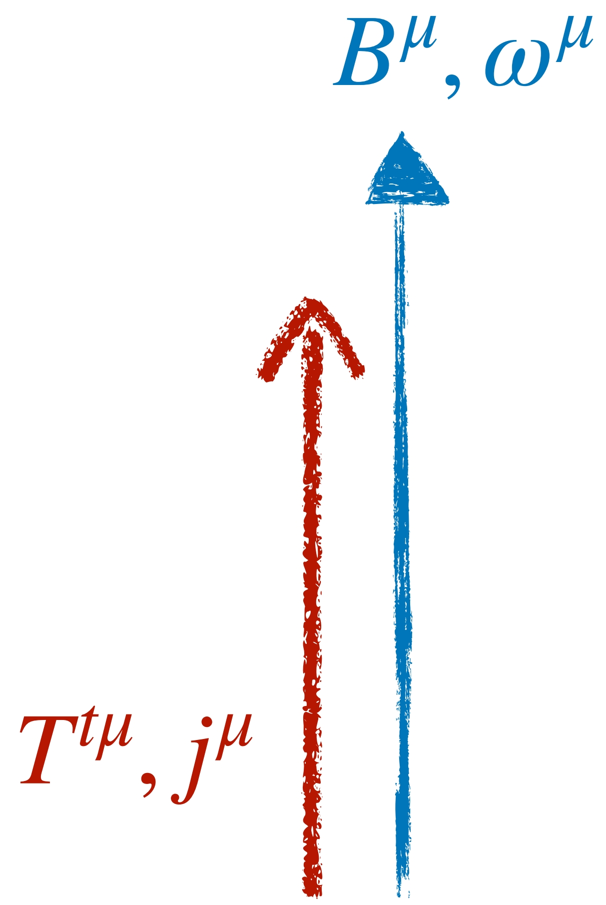
Symmetry and hydrodynamics

Conservation law is broken by anomaly

$$d \star j \sim \kappa F \wedge F$$

$\kappa \sim$ 

Anomalous transport



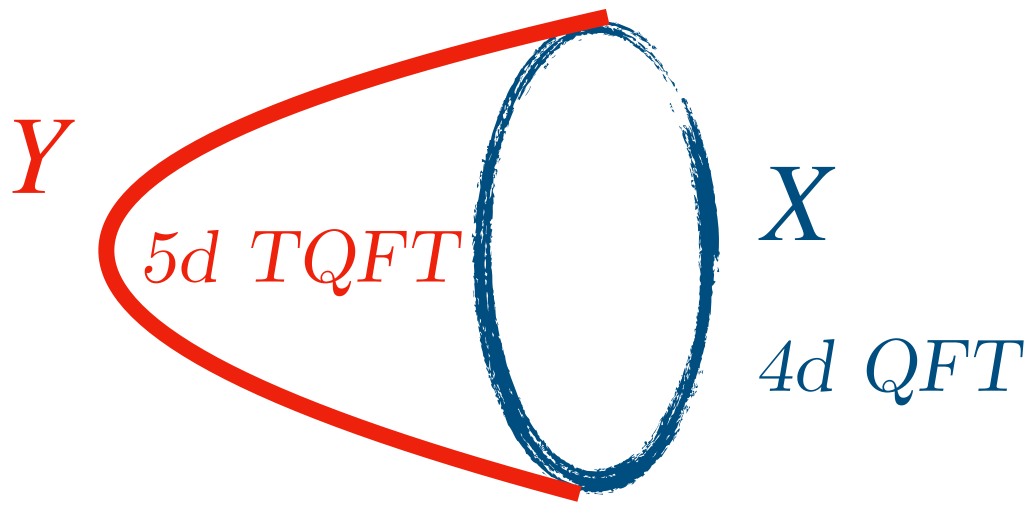
Obviously...

If triangle diagram vanishes → **No anomalous transport?**

Goal of this talk: The obvious is not true

ANOMALY $\neq$ TRIANGLE DIAGRAM

If we define 't Hooft anomaly as **an inability to gauge the global symmetry** or the ambiguity of a partition function when coupled to a background gauge field



$$Z[X;A] \quad \longrightarrow \quad Z[X;A^g] = Z[X;A]\ \exp\left(i\omega(g,A)\right)$$

where one can "cancel" the non-invariance by attaching it to **invertible TQFT** on Y with $\partial Y = X$

- Numerous examples in dim(QFT)=2+1 where the above are true without triangle diagrams (basically a whole literature on Symmetry Protected Topological (SPT) phase)

- Focusing on 3+1d, the oldest of this anomaly is Witten's SU(2) anomaly *Witten'82; Wang, Wen & Witten '18*

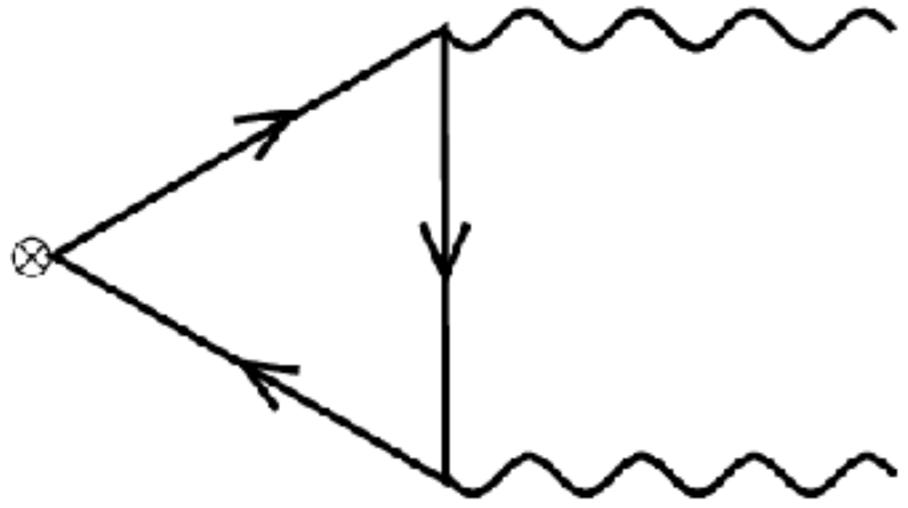
Massless Weyl fermion in SU(2) flavour doublet

or any reps with odd Dynkin label

$$T(j) = \frac{2}{3} j(j+1)(2j+1)$$

$+$

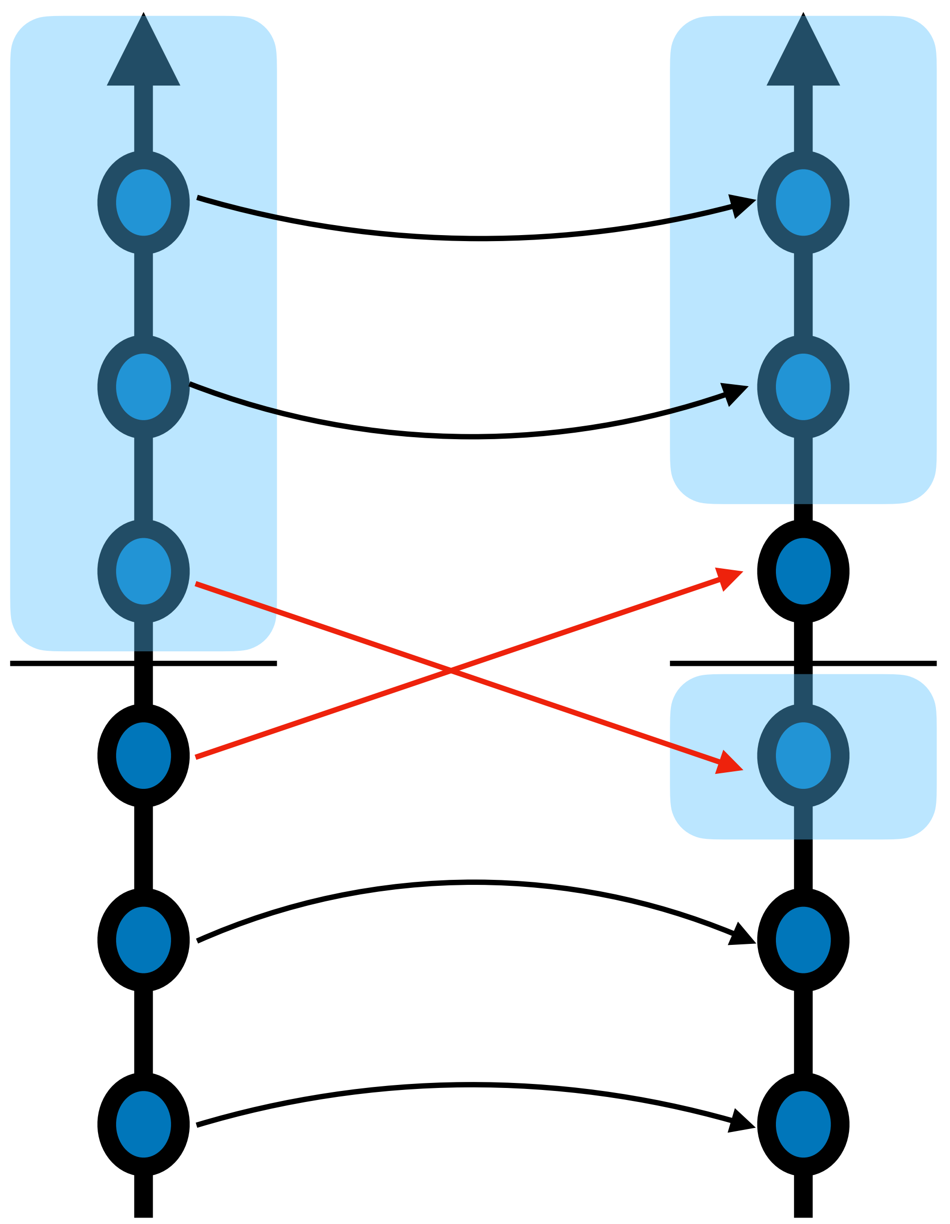
SU(2) preserving interactions



$$\sim \mathrm{tr}\left(\sigma^{(a}\sigma^{b}\sigma^{c)}\right) = 0$$

No way to build nontrivial 6d anomaly polynomial

Why is it an anomaly



In free theory, the partition $Z[A]$ " = " $\det(i\not{D})^{1/2}$

If we transformed $A \to A^g$ such that a pair of eigenvalues cross $\lambda = 0$, then $Z[A] \to Z[A]^g = -Z[A]$. This can happen when

- Put theory on S^4 and take nontrivial $[g(\tau, x)] \in \pi_4(SU(2)) \cong \mathbb{Z}_2$ *Witten'82;*
- Put theory on S^4 with nontrivial instanton in $\int \mathrm{tr}\,(F \wedge F)$ that comes with a zero mode and perform centre $Z(SU(2)) = \mathbb{Z}_2 \cong (-1)^F$ transformation *Wang, Wen & Witten '18*
- Put theory on $S^1 \times S^3$ with $\mathrm{tr}\left(\exp\ i\oint_{S^1} A\right) = -1$ and take $[g(x)] \in \pi_3(SU(2))$ *Davighi, Lohitsiri & NP '24*

***** Partition function remains invariant under infinitesimal SU(2) transformation**

Anomaly inflow

Carefully looking $Z[A]$, not just $\delta_g Z[A]$, one finds that ambiguity of Weyl fermion can be cancelled by

$$Z_{\mathrm{I}}[Y] = \exp\left(-2\pi i \eta[Y]\right)$$

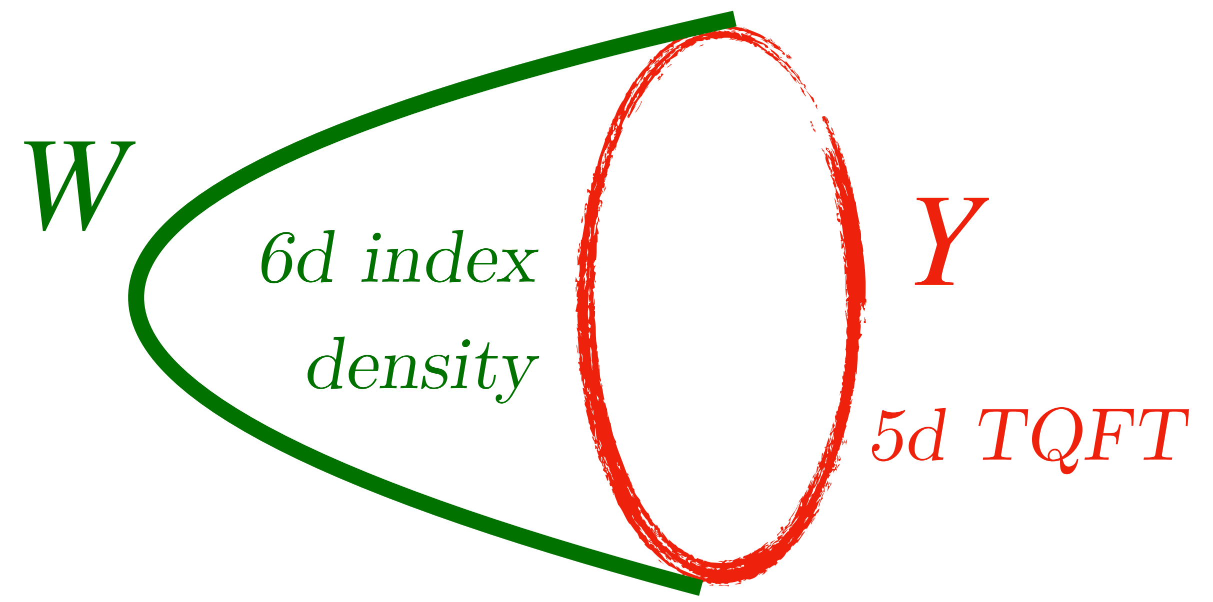
Dai & Freed '94 ; See also Yonekura '16 for "physicsist" proof

not just Chern-Simons but APS η-invariant $\eta = \lim_{\epsilon\to 0^+} \sum_k e^{-\epsilon|\lambda_k|}\mathrm{sign}(\lambda_k)/2 \sim \arg\det(i\not{D}_A)$

Atiyah,Patodi & Singer '75-76 ;

But it has a neat property (APS index theorem) **IF** $Y = \partial W$,

$$-2\pi i\eta[Y] = \text{Index density}[W] = \int_W \hat{A}(R)\exp\left(\frac{F}{2\pi}\right)$$



But η-invariant **can still be nontrivial** when Index density $= 0$

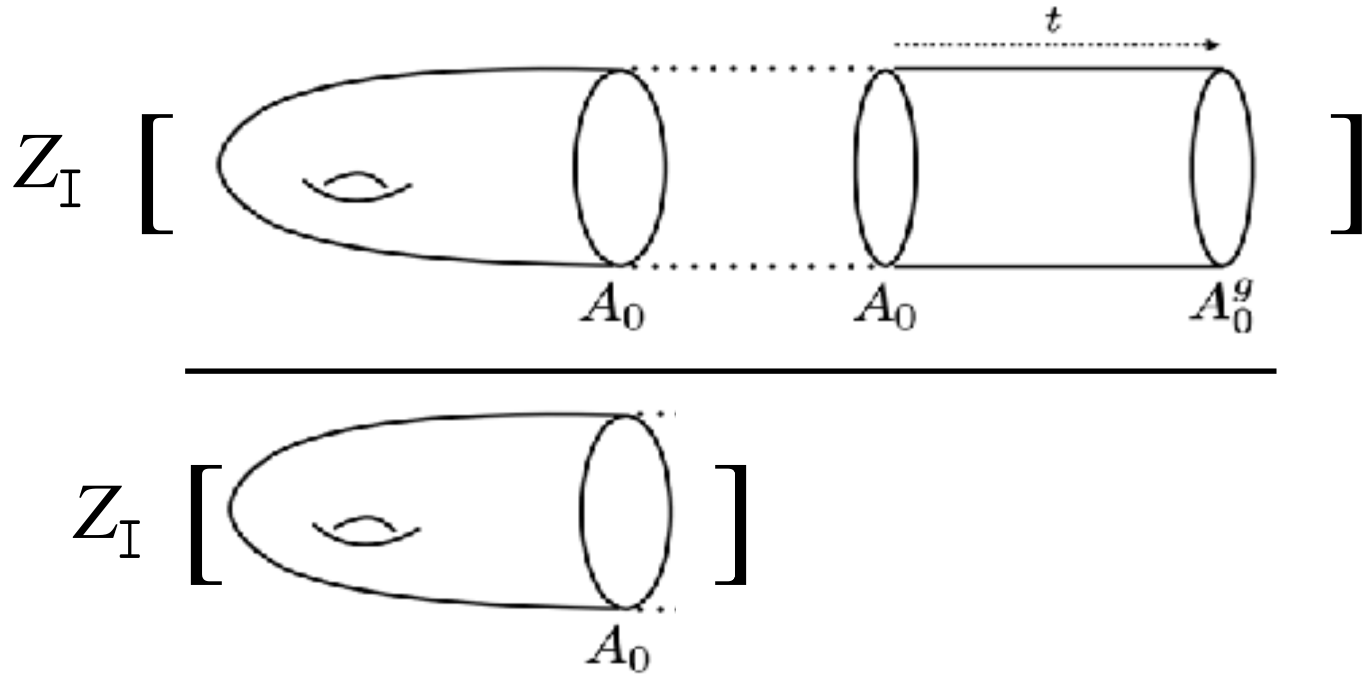
IF there is no extension of Y (with its G-bundle and spin structure) to W such that $\partial W = Y$.

Obstruction to the extension is measured by **Bordism group** $\Omega_5^{\mathrm{Spin}}(BG)$

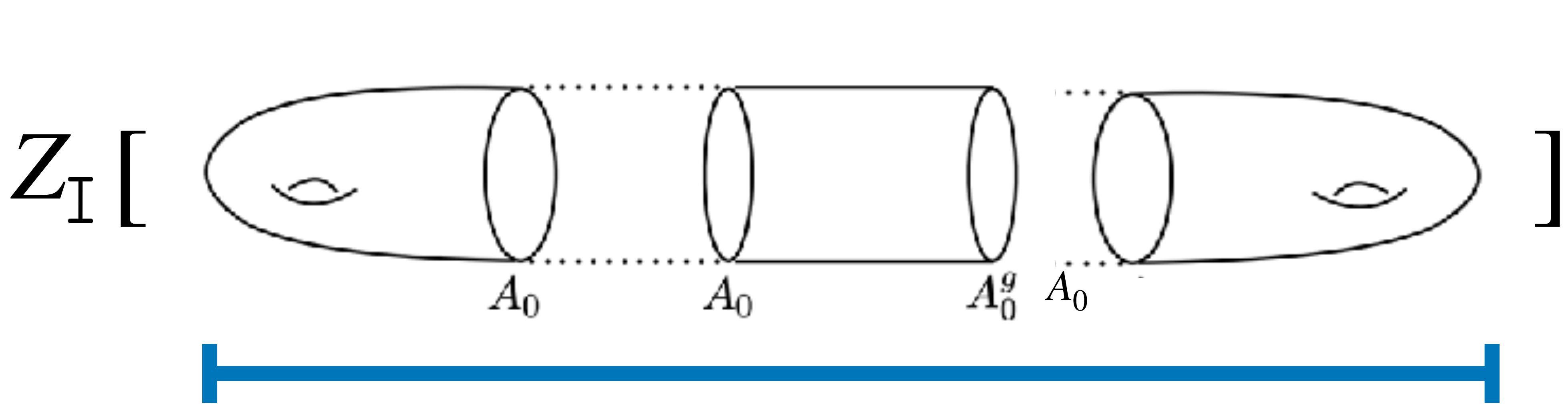
Kapustin '14; Kapustin, Thorngren, Turzillo & Wang '14'; Freed & Hopkins '16

Ambiguity as TQFT

Now we can compute the phase ambiguity of $Z[A^g]/Z[A]$ on $X = \partial Y$ from $Z_{\mathrm{I}}[Y]$ by "cutting & gluing"



=



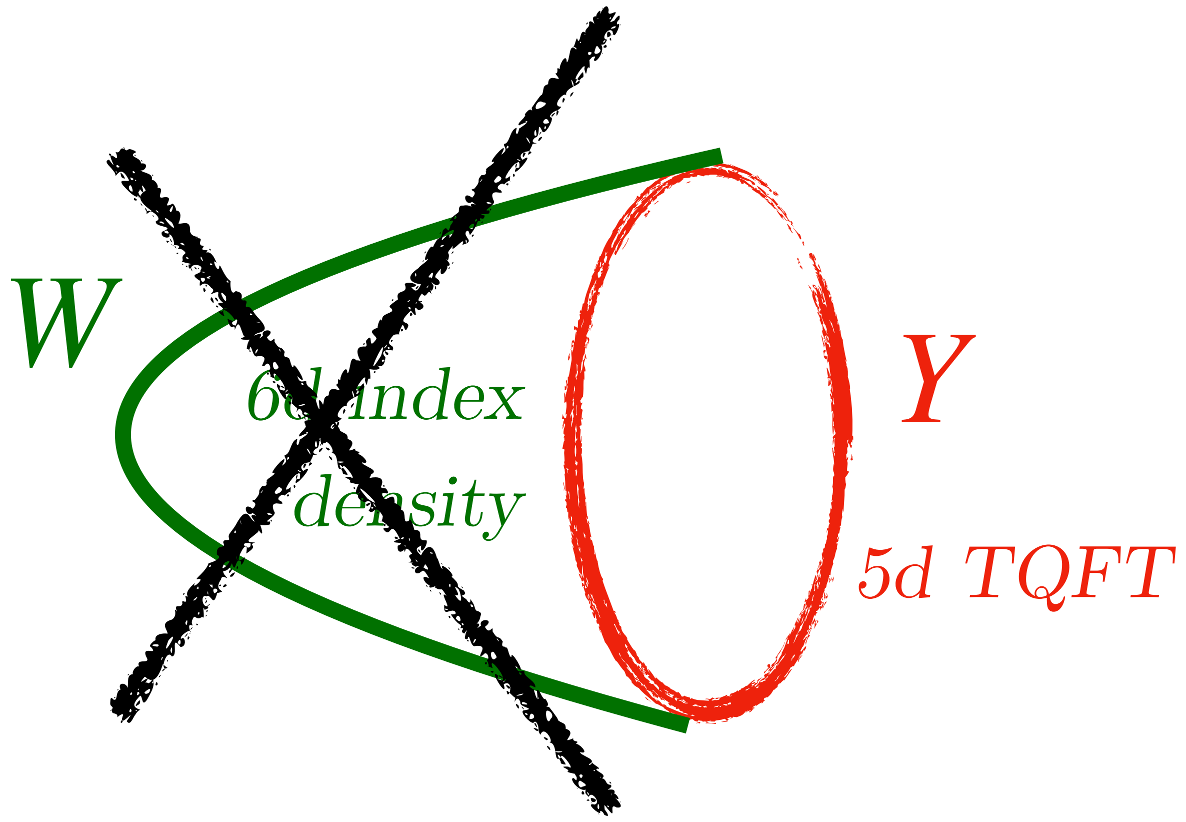
Witten '15; Yonekura '16; Yonekura & Witten '19

Now this Y_{tot} is a closed manifold.
We can hope to use APS index theorem!

- For $G = U(1)$ in and $Z_{\mathrm{I}}[Y] = e^{-2\pi i\eta} \sim \exp\left(\frac{i}{4\pi}\int_Y AdA\right)$, this configuration yields the usual

$$\log Z_I[Y_{tot}] = \frac{i}{2\pi}\int_{X=\partial Y} \lambda dA \quad \text{for any } A^g = A + d\lambda$$

- But, in SU(2) case, for Y_{tot} s.t. $Z_{\mathrm{I}}[Y_{tot}] = -1$, there is no W with $Y_{tot} = \partial W$. Then $\int_W \mathrm{tr}_{su(2)}(F \wedge F \wedge F) = 0$ doesn't not implies absence of anomaly



Anomaly in thermal equilibrium

BBJMS '12
JKKMY '12

Let's put the anomalous theory in **thermal equilibrium** and by thermal equilibrium, I mean put theory on $X = S^1_\beta \times M$, or $S^1_\beta \to X \to M$ with G-holonomy and field strength

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\sigma(x)}(d\tau + \alpha_i dx^i)^2 + \gamma_{ij}dx^i dx^j \,, \quad \text{and} \quad A = -\mu u + \mathcal{A}\,,$$

Size of thermal circle → $e^{2\sigma(x)}$

KK gauge field ~ fluid velocity → α_i

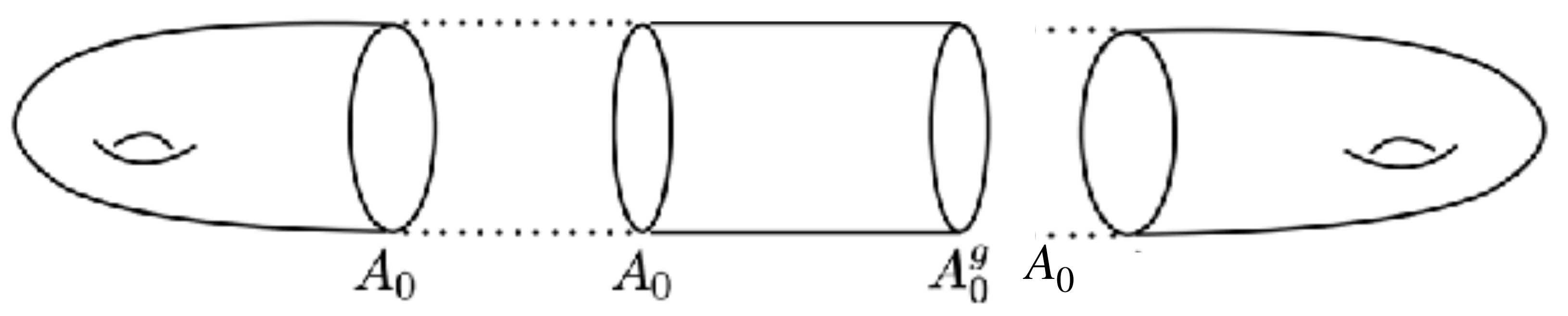
"Spatial component" of gauge field → $\mathcal{A}$

And the object of interest is the thermal partition function with symmetry operator inserted

$$Z[X;A] = \mathrm{tr}\left[U\, e^{-\beta H}\right] \qquad U \sim \exp(i\int \star j \wedge A) \in G$$

Build Y_{tot} out of $(X, A) \to (X', A^g)$ with X, X' has the same temperature

$$\frac{Z[X';A^g]}{Z[X;A]} = Z_{\mathrm{I}}[$$



$$]$$

The partition must transform according to anomaly

** Such thermal equilibrium-compatible Y_{tot} may not exist for every nontrivial $[M] \in \Omega^{\mathrm{Spin}}_{d+1}(BG)$

Procedure streamline

Want to find hydro for this type of anomaly?
Suppose you know a theory in $d+1$ dimensions
and a symmetry group G

- Know your invertible theory $Z_{\mathrm{I}}[Y]$. If you don't, or $Z_{\mathrm{I}} = \exp(-2\pi i\eta)$, find bordism group
  If $\Omega_5^{\mathrm{Spin}}(BG) = 0$, then $\eta[Y] \sim CS_5[Y]$ and no global anomaly

look it up in e.g.
Kapustin, Thorngren, Turzillo & Wang '14;
Garcia-Etxebarria & Montero '18;
Zheyan Wan & Juven Wang '19
Davighi & Lohitsiri '20-'24

- Find 'mapping torus' Y_{tot} that is build from $X = S^1_\beta \times M$ such that $Z_{\mathrm{I}} \neq 1$
  and, if possible, generates bordism group.

A neat trick to compute η-invariant in by embedding inside G'
with only perturbative anomaly in e.g. Elitzur & Nair '84; Davighi & Lohitsiri '20

- Build (thermally compactified) EFT for $\log Z[g, A]$ out of KK variables
  and find appropriate non-invariant terms that capture $Z_{\mathrm{I}}[Y_{tot}]$

$$g_{\mu\nu} dx^\mu dx^\nu = e^{2\sigma(x)}(d\tau + \alpha_i dx^i)^2 + \gamma_{ij} dx^i dx^j \,, \quad \text{and} \quad A = -\mu u + \mathcal{A}\,,$$

For this type of applications
Golkar & Sethi '15
Chowdhury & David '16
Glorioso, Liu & Rajagopal '17

- Vary $\log Z[g, A]$ w.r.t. g, A to get stress-energy tensor and conserved currents

Procedure streamline

Want to find hydro for this type of anomaly?
Suppose you know a theory in $d+1$ dimensions
and a symmetry group G

For this type of applications
Golkar & Sethi '15
Chowdhury & David '16
Glorioso, Liu & Rajagopal '17

* Build (thermally compactified) EFT for $\log Z[g, A]$ out of KK variables and find appropriate non-invariant terms that capture $Z_{\mathrm{I}}[Y_{tot}]$

$$g_{\mu\nu}dx^\mu dx^\nu = e^{2\sigma(x)}(d\tau + \alpha_i dx^i)^2 + \gamma_{ij}dx^i dx^j\,, \quad \text{and} \quad A = -\mu u + \mathcal{A}\,,$$

For G=U(1) with triangle anomaly, the most general expansions is known

BBJMS '12
Jensen, Loganayagam & Yarom '12

$$\log Z_{QFT} = \beta \int d^3x \sqrt{\gamma}\, p(T, \mu) + W_{inv} + W_{anom}$$

$$W_{inv} = \frac{C_1}{4\pi}\int \mathcal{A}\wedge d\mathcal{A} + \frac{C_2}{2\pi\beta}\int \alpha\wedge d\mathcal{A} + \frac{C_3}{4\pi\beta^2}\int \alpha\wedge d\alpha$$

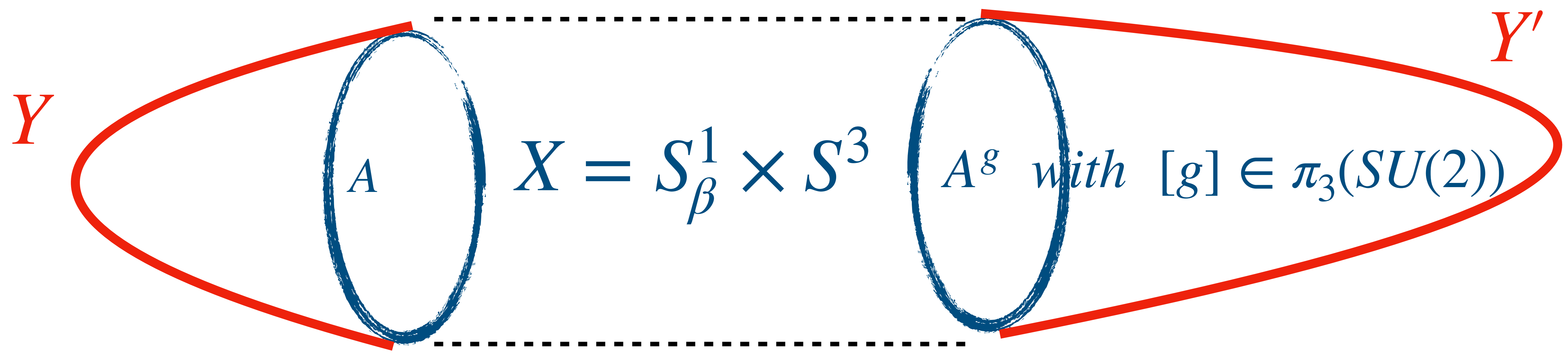
Invariant under small gauge transformation
If C_1, C_2, C_3 are integer, invariant under all gauge transf.

Most will be killed by 3+1d CPT

$$W_{anom} = \frac{C}{2}\int\left(\frac{\beta\mu}{3}\mathcal{A}\wedge d\mathcal{A} + \frac{\beta\mu^2}{6}\mathcal{A}\wedge d\alpha\right)$$

CS term with spatially dependent coupling
⇒ Produce the non-invariance under small gauge transformation

HYDRODYNAMICS OF $G = SU(2)$ IN $4d$,



*Davighi, Lohitsiri & **NP** '24*

* Build EFT for $\log Z[g, A]$ out of KK variables such that

  - When place on $S^3 \times S^1_\beta$ **with** $\mathbb{Z}_2 \subset SU(2)$ holonomy around thermal cycle

$$-\frac{i}{2\pi}\log Z[g,A] \to -\frac{i}{2\pi}\log Z[g,A] + \frac{1}{2} \mod 1 \qquad \text{under } A \to A^g \text{ with odd degree of } [g] \in \pi_3(SU(2))$$

* A term responsible for this is (for theory with fermion in isospin $j = 1/2$)

$$-i\log Z[g,A] \supset \frac{1}{4\pi}(\mathbb{Z}+j)\int_M \mathrm{tr}\left[\mathcal{A}\wedge d\mathcal{A} + \frac{3}{2}\mathcal{A}^3\right] \overset{\text{Variations}}{\Longrightarrow} \begin{pmatrix} J^i_a \\ T^i_t \end{pmatrix} = \frac{1}{4\pi}(\mathbb{Z}+j)\begin{pmatrix} T\delta_{ab} & \mu_a T \\ \mu_b T & (\mu_a\mu^a)T \end{pmatrix}\begin{pmatrix} B^i_b \\ \omega^i \end{pmatrix}$$

More on protected fractional part: Closset, Dumitrescu, Festuccia, Komargodski & Seiberg '12

Summary of the result

- There is a recipe on how to extract transport coeffcients for a large class of theories with global anomalies. In fact, among $SU(n), SO(n), USp(2n), E_{6,7,8}, F_4, G_2$ non-abelian fluids, only $USp(2n)$ family has **half-integer** transport coefficient **when holonomy of the $\mathbb{Z}_2$ centre is inserted on the thermal cycle**

$$Z_{\textbf{thermal}} = \textbf{tr}\left[(-1)^F e^{-\beta H}\right] \qquad (-1)^F \sim \mathcal{Z}(SU(2)) \cong \mathbb{Z}_2$$

- There are anomaly induced transports like CME, CVE etc. with conductivities fixed by thermodynamic variables and **fractional number**, even *without* triangle diagram.

$$\begin{pmatrix} J^i_a \\ T^i_t \end{pmatrix} = \frac{1}{4\pi}(\mathbb{Z}+j)\begin{pmatrix} T\delta_{ab} & \mu_a T \\ \mu_b T & (\mu_a\mu^a)T \end{pmatrix}\begin{pmatrix} B^i_b \\ \omega^i \end{pmatrix}$$

For a generic isospin j reps, replace j by Dynkin label mod 2

$$T^i{}_t = -\left(\frac{1}{3}C_{bcd}\mu^b\mu^c\mu^d + 2\beta_b\mu^b T^2\right)\omega^i + \left(\frac{1}{2}C_{abc}\mu^b\mu^c + \beta_a T\right)B^{ai},$$

$$J^i_a = \left(\frac{1}{2}C_{abc}\mu^b\mu^c + \beta_a T^2\right)\omega^i + C_{abc}\mu^b B^{ci}$$

Here is Neiman & Oz '10 result for comparison

- Question for the audience: How to deal with this eta-invariant in holography?

Remarks and (anomalous)dreams

- Any idea how to prepare this? $Z_{\textbf{thermal}} = \mathbf{tr}\left[(-1)^F e^{-\beta H}\right]$ $(-1)^F \sim \mathcal{Z}(SU(2)) \cong \mathbb{Z}_2$

- Can there be anything more than $Z[A^g] = -Z[A]$? Depending on $\Omega_5^{\text{Spin}}(BG)$: no for SU(2) but yes for other G *See literature on "cobordism classification" of anomaly*

- Finding Y_{tot} is really painful and luck-dependent, can we have a general procedure like WZW term?
Lee, Ohmori & Tachikawa '20 & Yonekura '20

- Signature of global anomaly in other EFT?
Preskill '90; See also e.g. Shimizu & Yonekura '18; Furusawa, Tanizaki & Itou '20 for QCD context

- Gauging non-anomalous subgroup ⇒ New symmetry structure that remembers its anomalous parents

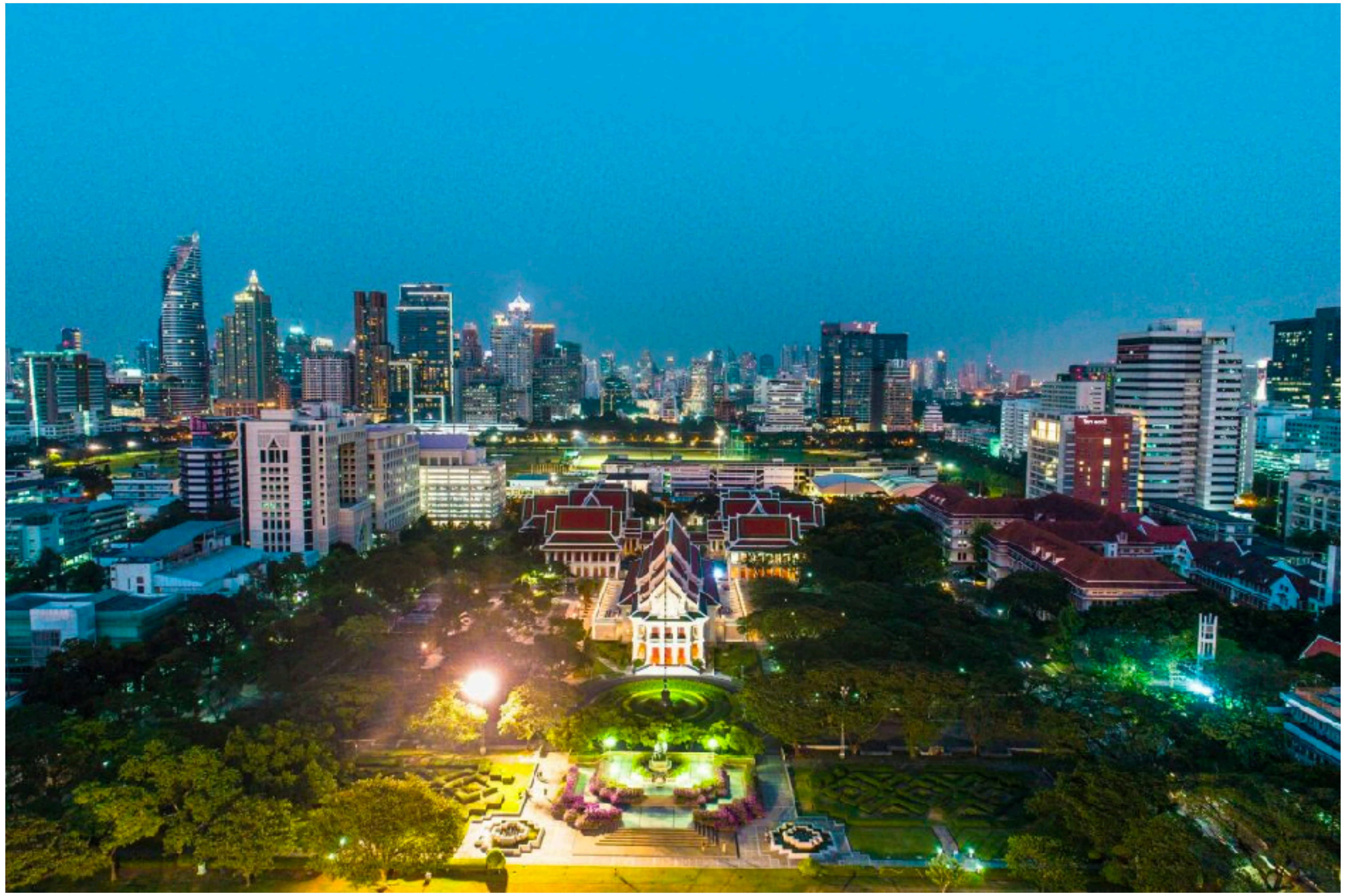
"Categorical symmetry" literature see e.g.
Iqbal & NP '20; Brauner '20; Davighi & Lohitsiri '24
Das, Iqbal & NP '21-'22; Hsin, Kobayashi & Zhang '24 ; Hsin & Gomis '24

- Entire 'universe' for anomaly in 2+1 dimensions at finite temperature?

Spare slides

My university

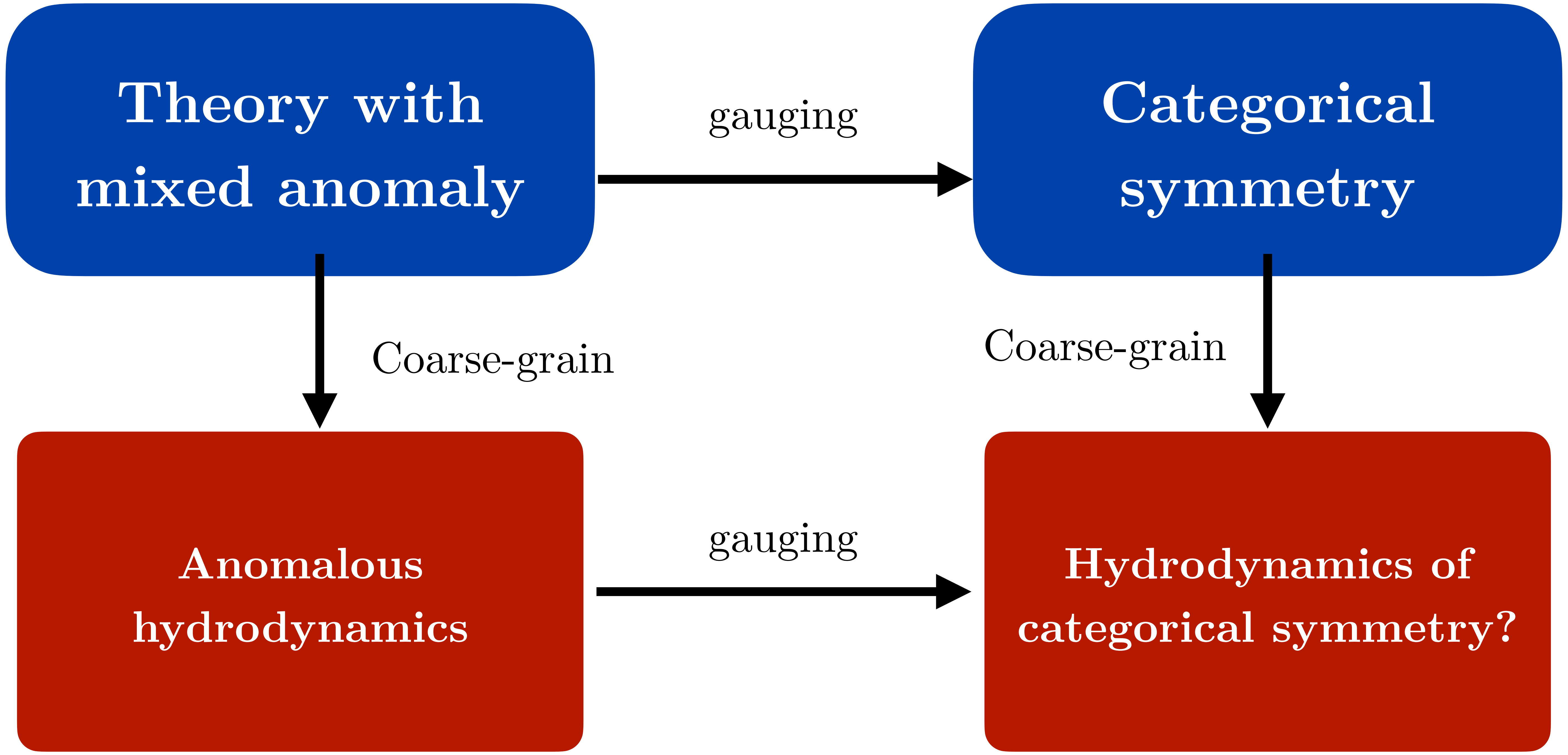
Chula
Chulalongkorn University



A possible way to get categorical symmetry

Most examples involves discrete groups

No triangle diagram/
violation of Ward identity
for discrete group or
odd $d + 1$

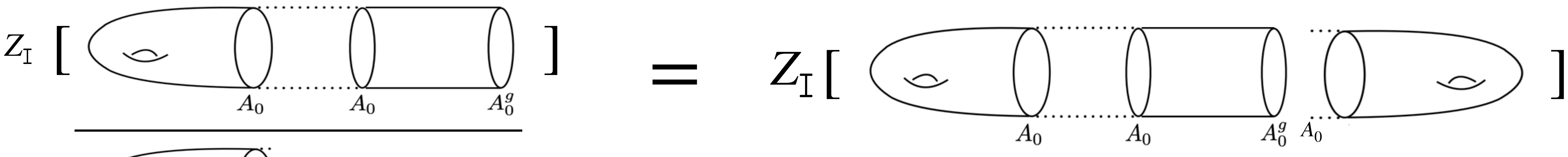


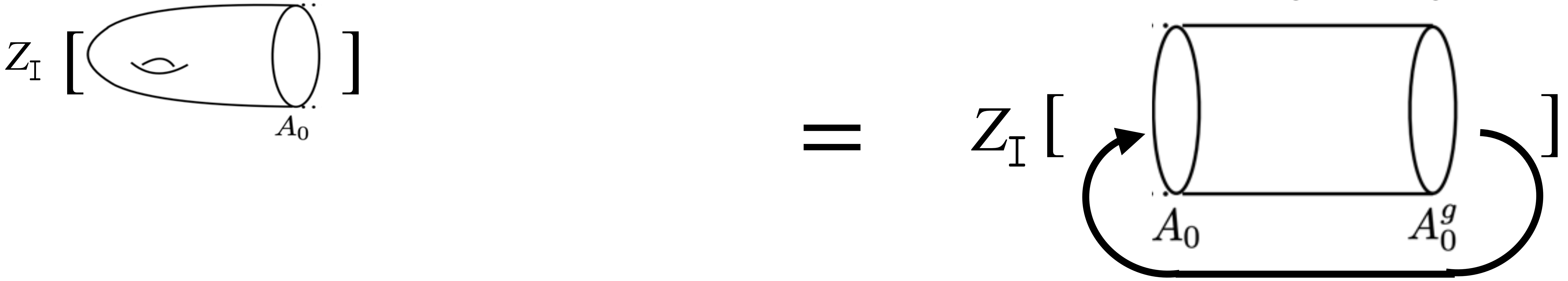
How to turn
categorical data
into hydrodynamics?

How will this EFT
knows about anomalous
symmetry involving
discrete group?

What is anomalous transf?

Now we can compute the phase ambiguity $Z[X;A] \longrightarrow Z[X;A^g] = Z[X;A]\ \exp\big(i\omega(g,A)\big)$





- A configuration called **Mapping torus**

Say, for $X = S^2$, $Z_I[Y] = \exp\left[\frac{i}{4\pi}\int_Y AdA\right]$, and $A^g = A + d\lambda$

this produce the familiar phase ambiguity $\omega(\lambda, A) \sim \int_X \lambda F$

Notice: when $F = 0$, then $Z_I[MP] = 1$

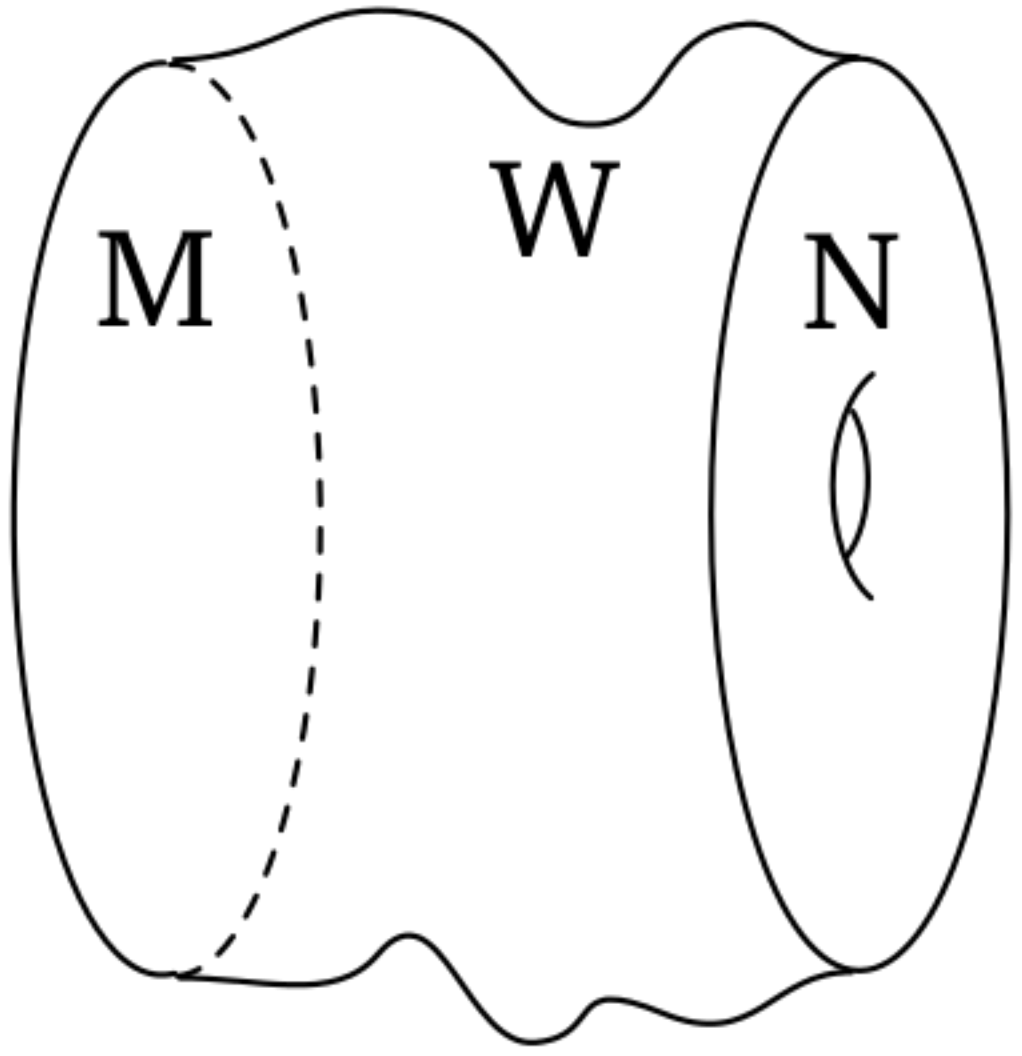
In general, one need to activate "F" on X and particular $A \to A^g$ for Z_I to be activated

Bordism group & Cobordism classification

* What is this bordism group represent ?
  Suppose you have d-dimensional manifold M and N equipped with, say, **spin structure** and principal G-bundle

  If they can be "connected" via some $(d+1)$-dim space W then, we say $M \sim N$ or **bordant**

  The bordism group count a class of M that are not bordant to one another



$$\Omega_d^{\mathrm{Spin}}(BG) \cong \frac{\text{All } M \text{ with spin structure with G-bundle}}{\text{Equivalent relation}}$$

Kapustin, Thorngren, Turzillo & Wang '14
Freed & Hopkins '16

Classes of mapping tori

E.g. if $\Omega(BG) = \mathbb{Z}_2$, then there are two classes $[M_0]$ and $[M_1]$ that are not bordant and only M_0 can be extended to W with $\partial W = M_0$. Then if Index density $= 0$, we have

$$\eta[M_0] = 0 \mod 1 \quad \& \quad 2\eta_{M_1} = \eta[M_0] \mod 1 \quad \Rightarrow \quad \eta[M_1] = \frac{1}{2} \mod 1$$