Guiding Center Dynamics

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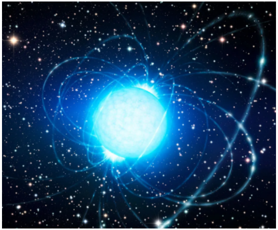
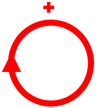
Charge in magnetic field – guiding center approximation

- Matter in magnetic fields – many important contexts and applications.
- Scale separation: fast gyro motion, slower guiding center motion
- Effective theory of this slower motion is a very useful tool

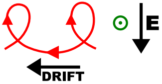


- Simple case $\boldsymbol{B} = \text{const}$ and $\boldsymbol{E} \parallel \boldsymbol{B}$: guiding center moves along a magnetic field line — 1+1d motion.
- But what if $\boldsymbol{E} \nparallel \boldsymbol{B}$ and/or $\boldsymbol{B} \neq \text{const}$?

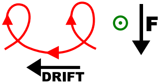
Drift



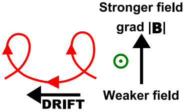
⊙ Magnetic field upwards



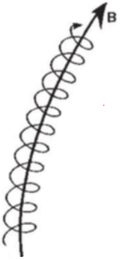
ExB (Hall)



gravity/inertia or B-curvature



B-gradient



- Important problem, e.g., for thermonuclear plasma confinement.

Guiding-center approximation

- In (special) relativistic context, averaging over fast gyromotion, Vandervoort '60 finds a rather long and non-covariant formula:

$$\dot{\boldsymbol{R}}_\perp = \frac{\hat{\mathbf{e}}_1}{B(1 - E_\perp{}^2/B^2)} \times \left\{ -\left(1 - \frac{E_\perp{}^2}{B^2}\right) c\boldsymbol{E} + \frac{M_r c}{\gamma e} \nabla \left[B \left(1 - \frac{E_\perp{}^2}{B^2}\right)^{1/2} \right] + \frac{m_0 c \gamma}{e} \left(v_\parallel \frac{d\hat{\mathbf{e}}_1}{dt} + \frac{d\boldsymbol{u}_E}{dt} \right) + \frac{v_\parallel E_\parallel}{c} \boldsymbol{u}_E + \frac{M_r}{\gamma e} \frac{\boldsymbol{u}_E}{c} \frac{\partial}{\partial t} \left[B \left(1 - \frac{E_\perp{}^2}{B^2}\right)^{1/2} \right] \right\} + \mathcal{O}(\epsilon^2)$$

  $\epsilon = a/L$ (orbit radius over characteristic length of B variation).

- We shall begin by rederiving this formula using simpler and manifestly Lorentz (and general coordinate) covariant formalism.

$\boldsymbol{E} \parallel \boldsymbol{B}$ frame

- The motion is simplest in a (family of) frame(s) where $\boldsymbol{E} \parallel \boldsymbol{B}$.
  In such a frame we define $B_* \equiv |\boldsymbol{B}|$ and $E_* \equiv \boldsymbol{E} \cdot \boldsymbol{B}/|\boldsymbol{B}|$.
  B_* and E_* are invariants of the EM field. more
- Cyclotron radius $a = p_\perp / B_*$.
  Treat both a/L and E_*/B_* as small.
- Upon averaging over $\boldsymbol{p}_\perp$ only $\boldsymbol{p} \parallel \boldsymbol{B}$ remains, i.e., $\boldsymbol{p} \times \boldsymbol{B} = 0$.
- Effectively $1+1$d motion. With effective mass
  $\tilde{m}^2 = m^2 + p_\perp^2 \equiv m^2 + JB_*$
  where $J = ap_\perp = p_\perp^2 / B_*$ – adiabatic invariant.
  In quantum case $J = \hbar(2n + 1 + gs_z)$.

Covariant description – action principle

- Boost $\boldsymbol{B}$ and $\boldsymbol{E}$ fields separately back to Lab frame: $F_{\mu\nu} = B_{\mu\nu} + E_{\mu\nu}$. Then $\boldsymbol{p} \times \boldsymbol{B} = 0$ becomes $B^{\mu\nu} p_\nu = 0$.

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- Consider Lagrangian with additional constraint $B^{\mu\nu} p_\nu = 0$:

$$L = -\dot{x} \cdot (p + A) + \frac{\alpha}{2}(p^2 - \tilde{m}^2) + \lambda_\mu B^{\mu\nu} p_\nu.$$

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- EOMs:

$$\dot{x}^\mu = \alpha p^\mu + \lambda_\alpha B^{\alpha\mu} \equiv \alpha p^\mu + v_{\mathrm{D}}^\mu ,$$

$$\dot{p}_\mu = F_{\mu\nu}\dot{x}^\nu + \frac{\alpha}{2} J \partial_\mu B_* - \lambda_\alpha p_\beta \partial_\mu B^{\alpha\beta} ,$$

and two constraints: $p^2 - \tilde{m}^2 = 0$, $B^{\mu\nu} p_\nu = 0$.

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  and two constraints: $p^2 - \tilde{m}^2 = 0$, $B^{\mu\nu} p_\nu = 0$.
- Substitute $\dot{x}$ into eq. for $\dot{p}$, use $B^{\mu\nu} p_\nu = 0$ and obtain:

$$v_{\mathrm{D}}^\mu = \alpha B_*^{-2} \left((p \cdot \partial) B^{\mu\nu} p_\nu + \frac{J}{2} B^{\mu\nu} \partial_\nu B_* \right) .$$

Manifestly GR covariant drift formula

$$v^\mu_{\mathrm{D}} = \alpha B_*^{-2}\Big(\underbrace{(p\cdot\nabla)B^{\mu\nu}p_\nu}_{\substack{B\text{-curvature,}\\ \text{gravity/inertia}}} + \underbrace{\frac{J}{2}B^{\mu\nu}\nabla_\nu B_*}_{\text{gradient drift}}\Big).$$

written in components contains all of this

$$\begin{aligned}\dot{\mathbf{R}}_\perp = \frac{\hat{\mathbf{e}}_1}{B(1-E_\perp{}^2/B^2)} \times \Big\{ &-\Big(1-\frac{E_\perp{}^2}{B^2}\Big)c\mathbf{E} \\ &+ \frac{M_r c}{\gamma e}\nabla\Big[B\Big(1-\frac{E_\perp{}^2}{B^2}\Big)^{1/2}\Big] + \frac{m_0 c\gamma}{e}\Big(v_\parallel\frac{d\hat{\mathbf{e}}_1}{dt} + \frac{d\mathbf{u}_E}{dt}\Big) \\ &+ \frac{v_\parallel E_\parallel}{c}\mathbf{u}_E + \frac{M_r}{\gamma e}\frac{\mathbf{u}_E}{c}\frac{\partial}{\partial t}\Big[B\Big(1-\frac{E_\perp{}^2}{B^2}\Big)^{1/2}\Big]\Big\} + \mathcal{O}(\epsilon^2)\end{aligned}$$

but also gravity effects.

Mainfestly general coordinate covariant and much shorter.

Guiding center kinetic theory

$$\mathcal{L}[f] \equiv \dot{x} \cdot \frac{\partial f}{\partial x} + \dot{p} \cdot \frac{\partial f}{\partial p} = \mathcal{C}[f]\,,$$

$$\begin{aligned}
\dot{x}^\mu &= \alpha p^\mu + \lambda_\alpha B^{\alpha\mu} \equiv \alpha p^\mu + v_{\rm D}^\mu, \\
\dot{p}_\mu &= F_{\mu\nu}\dot{x}^\nu + \frac{\alpha}{2} J \partial_\mu B_* - \lambda_\alpha p_\beta \partial_\mu B^{\alpha\beta},
\end{aligned}$$

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The invariant phase volume is $W d^4p$, where

$$W = \underbrace{\theta(p_0)\delta(p^2 - \tilde{m}^2)}_{\text{mass shell}}\underbrace{\delta^2\left(\Delta^\mu_\nu p^\nu\right)}_{\boldsymbol{p}\,\parallel\,\boldsymbol{B}}\ \underbrace{\frac{B_*}{2\pi^2\alpha}}_{\text{LL density}}\ \left(1 + \frac{p_\mu \partial_\nu B^{\mu\nu}}{B_*^2}\right).$$

$\Delta^\mu_\nu \equiv B_*^{-2} B^{\mu\alpha} B_{\nu\alpha}$ — projector onto $\perp \boldsymbol{B}$ plane (in $\boldsymbol{E} \parallel \boldsymbol{B}$ frame).

The phase space has only one momentum direction: $\boldsymbol{p} \parallel \boldsymbol{B}$.

$\boldsymbol{B}$-field line curvature correction is important.

more $T^{\mu\nu}$

Ideal hydrodynamics

- To lowest non-trivial order in a/L and E_*/B_*:

$$\partial_\mu \tilde{N}^\mu = 0 \quad \text{— charge conservation,}$$

$$\tilde{\Delta}^\lambda_\nu \partial_\mu T^\mu_{\ \lambda} = E_{\nu\lambda} \tilde{N}^\lambda \quad \text{— 1+1 energy-momentum conservation.}$$

where $\tilde{\Delta}^\mu_\nu = \delta^\mu_\nu - \Delta^\mu_\nu$ projector onto 1+1 $\boldsymbol{E} \parallel \boldsymbol{B}$ space.

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- Constitutive equations:

$$\tilde{N}^\mu = n u^\mu\,,$$
$$T^{\mu\nu} = \underbrace{(\epsilon + P)u^\mu u^\nu - P\tilde{\Delta}^{\mu\nu}}_{\tilde{T}^{\mu\nu}\text{ – 1+1d tensor}} - P_\perp \Delta^{\mu\nu}\,,$$

and $\Delta^\mu_\nu(x)u^\nu(x) = 0$ — i.e., fluid rest frame is an $\boldsymbol{E} \parallel \boldsymbol{B}$ frame.

- The flow is (locally) 1+1-dimensional (along $\boldsymbol{B}$).

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- The flow is (locally) 1+1-dimensional (along $\boldsymbol{B}$).
- But there is transverse pressure $P_\perp$ (due to gyromotion). more

Full EM current

The full (unprojected) divergence of the full T^μ_ν:

$$\partial_\mu T^\mu_{\ \nu} = F_{\nu\mu} J^\mu, \qquad \text{with} \qquad J^\mu \equiv N^\mu + \partial_\lambda M^{\lambda\mu} \,.$$

Transport current $N^\mu = \tilde{N}^\mu + N^\mu_{\mathsf{D}}$ contains "longitudinal" ($\parallel u$) + "drift"

$$N^\mu_{\mathsf{D}} \equiv \int_p W v^\mu_{\mathsf{D}} f = B_*^{-2} (\tilde{T}^\lambda_\nu \partial_\lambda B^{\mu\nu} - M B^{\mu\nu} \partial_\nu B_*) \,.$$

currents. The "magnetization current" is $\partial_\lambda M^{\lambda\mu}$, where

$$M^{\lambda\mu} = M B^{\lambda\mu} / B_*$$

is the magnetization density due to the fast transverse gyro motion.

The drift and the magnetization currents conspire to make the 3+1 energy-momentum conservation equations linearly dependent, reducible to 1+1 (projected) equations.

Constitutive relations

Can be obtained in three different ways, with the same result:

$$n = \frac{\partial P}{\partial \mu}\,, \quad \epsilon = T\frac{\partial P}{\partial T} + \mu\frac{\partial P}{\partial \mu} - P\,, \quad P_\perp = P - B_*\frac{\partial P}{\partial B_*}\,.$$

- Directly from kinetic theory in terms of f_{eq}:

$$\{n, \epsilon, P, P_\perp\} = \frac{B_*}{2\pi}\int\frac{dp_z}{2\pi}\, f_{\text{eq}}(\beta u\cdot p - \alpha)\left\{1, p_0, \frac{p_z^2}{p_0}, \underbrace{\frac{JB_*}{2p_0}}_{JB_* = p_\perp^2}\right\}.$$

Pressure depends on f_{eq}. Relations between P, n, ϵ, $P_\perp$ do not.

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- By applying the 2nd law of thermodynamics.
- From the partition function $\ln Z = \int d^4x\,\sqrt{-g}\,P(\mu, T, B_*)$ in external gauge field and metric backgrounds.

Beyond kinetic theory

- With u^μ defined via $B_{\mu\nu}u^\nu = 0$ and $\tilde{\Delta}^\mu_\lambda T^\lambda_{\ \nu} u^\nu = \epsilon u^\mu$ (1+1 projected Landau condition), the constitutive relations

$$\tilde{N}^\mu = n u^\mu \,,$$
$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P\tilde{\Delta}^{\mu\nu} - P_\perp \Delta^{\mu\nu},$$

$$n = \frac{\partial P}{\partial \mu}\,,\ \epsilon = T\frac{\partial P}{\partial T} + \mu\frac{\partial P}{\partial \mu} - P\,,\ P_\perp = P - B_*\frac{\partial P}{\partial B_*}\,,$$

are the most general covariant relations at zeroth order in derivatives satisfying second law constraints.

- Thus, such hydrodynamic description would also apply to strongly coupled QGP (as found in, e.g., Hattori et al ’22).
- The condition $B_{\mu\nu}u^\nu = 0$ eliminates non-hydro modes relaxing fast: $\Gamma_\perp = \sigma B^2/(\epsilon + P)$ – flow $\perp$ to $\boldsymbol{B}$ generates ohmic heat.

Conclusions and outlook

- Action principle with a momentum constraint offers a simple and manifestly covariant derivation of the guiding-center dynamics.
- This approach leads to a number of new results such as the mainfestly covariant expression for v_{D}, invariant phase space volume W, covariant kinetic theory and hydrodynamics.
- More work is needed to incorporate collisions and dissipation.
- Did not consider feedback of the particles onto the EM fields. But we found the full EM current (longitudinal, drift and magnetization) – a crucial ingredient for MHD simulations.

More

Properties of the $\boldsymbol{E} \parallel \boldsymbol{B}$ frame(s)

Frame velocity u^μ is an eigenvector: $(F^{\mu\lambda}F_{\lambda\nu})u^\nu = E_*^2 u^\mu$.

In such a frame $\boldsymbol{E} \times \boldsymbol{B} = 0$.

$\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant under boosts along $\boldsymbol{E}$ and $\boldsymbol{B}$.

$2(B_*^2 - E_*^2) = F_{\mu\nu}F^{\mu\nu}$ and $-4E_*B_* = F_{\mu\nu}\tilde{F}^{\mu\nu}$.

Unique decompostion $F_{\mu\nu} = E_{\mu\nu} + B_{\mu\nu}$, where

$$B_{\mu\nu} = \frac{F_{\mu\nu} - \epsilon_*\tilde{F}_{\mu\nu}}{1+\epsilon_*^2}, \quad E_{\mu\nu} = \epsilon_*\frac{\tilde{F}_{\mu\nu} + \epsilon_* F_{\mu\nu}}{1+\epsilon_*^2}, \qquad \epsilon_* = E_*/B_*.$$

such that $B_{\mu\nu}u^\nu = 0$, $E^{\mu\nu}B_{\lambda\nu} = 0$, $B^{\mu\nu}B_{\mu\nu} = 2B_*^2$, $E^{\mu\nu}E_{\mu\nu} = -2E_*^2$.

For $u = \gamma(1,0,0,v)$, $E_{0z} = E_*$ and $B_{xy} = -B_*$.

back

Invariant phase-space volume

Invariant phase-space volume = flux of the state density current

$$J^M = (\dot{x}^\mu, \dot{p}_\nu)W\,,$$

in phase space through a given equal-time surface:

$$d^7\Gamma = J^M d^7\Sigma_M = (n\cdot\dot{x})W d^3x d^4p.$$

Solving conservation equation $\partial_M J^M = 0$ we find

$$W = \frac{B_* + B_*^{-1} p_\mu \partial_\nu B^{\mu\nu}}{2\pi^2\alpha}\theta(p_0)\delta(p^2 - \tilde{m}^2)\delta^2(\Delta^{\mu\nu}p_\nu).$$

$\Delta^\mu_\nu = B_*^{-2} B^{\mu\alpha} B_{\nu\alpha}$ – projector onto the plane orthogonal to $\boldsymbol{B}$.

The overall (quantum) coefficient is fixed by Landau-level degeneracy.

back

From kinetic theory to hydrodynamics

Using conservation of charge and energy/momentum in collisions:

$$\int_p W\{1, p_\nu\}\mathcal{L}[f] = \{\partial_\mu N^\mu, \partial_\mu \tilde{T}^\mu{}_\nu - F_\nu\} = 0\,.$$

where current, energy-momentum and force density:

$$\{N^\mu, \tilde{T}^\mu{}_\nu, F_\nu\} = \int_p W\{\dot{x}^\mu, \dot{x}^\mu p_\nu, \dot{p}_\nu\} f\,.$$

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Using equations of motion, to leading order:

$$\tilde{T}^{\mu\nu} = \int_p W\alpha\, p^\mu p^\nu f,$$

$$F_\nu = F_{\nu\lambda}N^\lambda - M\partial_\nu B_*, \quad M \equiv -\int_p W\frac{\alpha J}{2}f.$$

Charge transport current

"Transport" current $N^\mu = \tilde{N}^\mu + N^\mu_{\mathsf{D}}$ consists of "longitudinal"

$$\tilde{N}^\mu \equiv \int_p W \alpha p^\mu f \,,$$

and "drift" components (higher order in a/L, due to ∂s)

$$N^\mu_{\mathsf{D}} \equiv \int_p W v^\mu_{\mathsf{D}} f = B_*^{-2} (\tilde{T}^\lambda_\nu \partial_\lambda B^{\mu\nu} - M B^{\mu\nu} \partial_\nu B_*) \,.$$

N_{D} is parametrically negligible in current conservation equation.

Hence $\partial \cdot \tilde{N} = 0$ to leading order.

Stress tensor

But N_{D} is *not* negligible in energy-momentum (non-)conservation (because leading term $B_{\nu\lambda}\tilde{N}^{\lambda} = 0$).
We can rewrite

$$\partial_{\mu}\tilde{T}^{\mu}{}_{\nu} = F_{\nu\lambda}N^{\lambda} - M\partial_{\nu}B_{*}$$

in the final form (1+1 projected equations)

$$\tilde{\Delta}^{\lambda}_{\nu}\partial_{\mu}T^{\mu}_{\lambda} = E_{\nu\lambda}\tilde{N}^{\lambda}$$

if we define full stress tensor as

$$T^{\mu\nu} \equiv \tilde{T}^{\mu\nu} + T^{\mu\nu}_{\perp}, \quad \text{where} \quad T^{\mu\nu}_{\perp} = MB_{*}\Delta^{\mu}_{\nu}\,.$$

Since

$$T^{\mu\nu}_{\perp} \equiv \int_{p} W\alpha\,\overline{p^{\mu}_{\perp}p^{\nu}_{\perp}}\,f.$$

this is the contribution to pressure from the fast transverse gyro motion.

back