Chiral anomalies and hydrodynamics

Juan Hernandez

Vrije Universiteit Brussel

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Outline

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1. Introduction
   - Thermodynamics
2. Hydrostatics
   - Strong magnetic field
3. Chiral hydrostatics
   - $U(1)_A$ hydrostatics
   - $U(1)_V \times U(1)_A$ hydrostatics

Introduction

Hydrodynamics

Hydrodynamics

- Low energy description of many body systems
- Dynamics of (approximately) conserved quantities $\sim T^{\mu\nu}, J^{\mu}, (J_5^{\mu})$
- Effective degrees of freedom $\sim T, u^{\mu}, \mu, (\mu^5)$

Motivation

- Non equilibrium dynamics of many-body systems
- Conventional fluids $\sim$ water, air, ...
- Plasmas $\sim$ Quark-Gluon plasma, astrophysical plasmas
- Exotic materials $\sim$ graphene, Weyl semi-metals

Thermodynamics and chiral anomalies

System with diffeomorphism and broken $U(1)_A$ symmetry coupled to external sources $g_{\mu\nu}$, A_μ.

Due to chiral anomaly, current and stress tensor receive anomalous contribution

$$J^\mu = J_s^\mu + \xi \Omega^\mu + \xi_B B^\mu \,, \quad T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)} \,.$$

First found in holographic models [Erdmenger, Haack, Kaminski, Yarom/ Banerjee et al./Torabian, Yee]
Then in hydrodynamic framework from second law [Son, Surowka/Niemann, Oz]

These contributions can be derived directly from conservation equations in static configurations.

Thermodynamics and chiral anomalies

Constitutive relations

$$J^\mu = J_s^\mu + \xi \Omega^\mu + \xi_B B^\mu \,, \quad T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)} \,.$$

Conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu \,, \quad \nabla_\mu J^\mu = -\frac{C}{8}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = CB\cdot E \,.$$

Static configuration $\mathcal{L}_V = 0$

$$\mathcal{L}_V A = 0 \Rightarrow u^\mu \partial_\mu \left(\frac{\mu}{T}\right) = E_\mu - T\partial_\mu \left(\frac{\mu}{T}\right) = 0$$

$$\mathcal{L}_V g = 0 \Rightarrow u^\mu \partial_\mu T = \nabla_\mu u^\mu = T a_\mu + \partial_\mu T = \sigma_{\mu\nu} = 0$$

Thermodynamics and chiral anomalies

Example: CME and CMVE

Current conservation equation

$$\nabla_\mu J^\mu = \nabla_\mu J_s^\mu + \nabla_\mu \left(\xi \Omega^\mu + \xi_B B^\mu\right) = C B{\cdot}E$$

Partial differential equations

$$0 = \Omega{\cdot}a \left(2\xi - T\xi_{,T}\right) + \Omega{\cdot}E \left(\frac{\xi_{,\mu/T}}{T} - \xi_B\right)$$

$$+ B{\cdot}a \left(\xi_B - T\xi_{B,T}\right) + B{\cdot}E \left(\frac{\xi_{B,\mu/T}}{T} - C\right)$$

Solution

$$\xi_B = C\mu + c_1 T\,, \quad \xi = C\mu^2/2 + c_1 \mu T + c_2 T^2\,.$$

Thermodynamics and chiral anomalies

Example: $U(1)_V \times U(1)_A$ + diffeo sector

Constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi^T u^{(\mu}\Omega^{\nu)} + \xi_B^T u^{(\mu} B^{\nu)} + \xi_{B_5}^T u^{(\mu} B_5^{\nu)}$$

$$J^\mu = J_s^\mu + \xi\, \Omega^\mu + \xi_B B^\mu + \xi_{B_5} B_5^\mu \, .$$

$$J_5^\mu = J_s^\mu + \xi^5\, \Omega^\mu + \xi_B^5 B^\mu + \xi_{B_5}^5 B_5^\mu \, .$$

Conservation equations

$$\nabla_\nu T_A^{\mu\nu} = F^{\mu\nu} J_\nu + F_5^{\mu\nu} J_\nu^5 \, ,$$

$$\nabla_\mu J^\mu = -\frac{C}{4}\epsilon^{\mu\nu\rho\sigma} F^5_{\mu\nu} F_{\rho\sigma} = C \left(B\cdot E_5 + B_5 \cdot E \right) \, ,$$

$$\nabla_\mu J_5^\mu = -\frac{C}{8}\epsilon^{\mu\nu\rho\sigma} \left(F^5_{\mu\nu} F^5_{\rho\sigma} + F_{\mu\nu} F_{\rho\sigma} \right) = C \left(B\cdot E + B_5 \cdot E_5 \right) \, .$$

Thermodynamics and chiral anomalies

Example: diffeo + $U(1)_V \times U(1)_A$

Constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi^T u^{(\mu}\Omega^{\nu)} + \xi_B^T u^{(\mu}B^{\nu)} + \xi_{B_5}^T u^{(\mu}B_5^{\nu)}$$

$$J^\mu_{cov} = J^\mu_s + \xi\,\Omega^\mu + \xi_B B^\mu + \xi_{B_5} B_5^\mu\,.$$

$$J^\mu_{5,cov} = J^\mu_s + \xi^5\,\Omega^\mu + \xi_B^5 B^\mu + \xi_{B_5}^5 B_5^\mu\,.$$

Chiral conductivities

$$\xi^T = C\left(\mu^2\mu_5 + \tfrac{1}{3}\mu_5^3\right)\,,\quad \xi_{B_5}^T = \xi^5 = \tfrac{C}{2}\left(\mu^2 + \mu_5^2\right)$$

$$\xi_B^T = \xi = C\mu\mu_5\,,\quad \xi_B = \xi_{B_5}^5 = C\mu_5\,,\quad \xi_{B_5} = \xi_B^5 = C\mu\,.$$

Thermodynamics and conformal anomaly

Example, Nersnt effect and conformal anomaly

Conformal anomaly spoils tracelesness of the stress tensor

$$T^\mu_\mu \sim c_A F_{\mu\nu} F^{\mu\nu}$$

Nernst effect relates current response to magnetic field and temperature gradient

$$J^\mu \sim N_{Nernst} \epsilon^{\mu\nu\rho\sigma} u_\nu B_\rho \partial_\sigma T \ ,$$

Relation between N_{Nernst} and c_A in Dirac and Weyl semimetals [Chernodub, Cortijo, Vozmediano]

Existence of a static generating functional implies [Ammon, Grieninger, JH, Kaminski, Koirala, Leiber, Wu]

$$N_{Nernst} = \frac{2c_A}{T} \ .$$

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Thermodynamics

Generating functional and hydrodynamic equations

System with diffeomorphism and $U(1)$ symmetry coupled to external sources $g_{\mu\nu}$,A_μ. Partition function given by

$$Z[g,A] = \mathrm{Tr} e^{-\beta H[g,A]} = \int \mathcal{D}\phi e^{iS[\phi;g,A]} \,.$$

Generating functional $W[g,A] = -i \ln Z[g,A]$ for n-pt functions

$$\delta W[g,A] = \int d^d x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu \right] \,.$$

where $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$ and $J^\mu = \langle \hat{J}^\mu \rangle$.

Diffeomorphism and gauge invariance of W ensures these obey

Conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \,, \quad \nabla_\mu J^\mu = 0 \,.$$

Thermodynamics

Equilibrium constraints and derivative expansion [Jensen, Kaminski, Kovtun, Meyer, Ritz, Yarom/ Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma]

Equilibrium is given by a time-like killing vector V^μ

$$\mathcal{L}_V A_\mu = 0\,, \quad \mathcal{L}_V g_{\mu\nu} = 0\,.$$

Can expand W_s order by order in derivatives

$$W_s = \int d^d x \sqrt{-g} \left[p(T,\mu) + \sum_{i,n_i} M^{(i)}_{n_i}(T,\mu) s^{(i)}_{n_i} \right] ,$$

where

$$T = \frac{T_0}{\sqrt{-V^2}}\,, \quad \mu = \frac{V^\mu A_\mu + \Lambda_V}{\sqrt{-V^2}}\,, \quad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}\,,$$

and $s^{(i)}_{n_i}$ are $\mathcal{O}(\partial^i)$ equilibrium scalars.

Thermodynamics

Example: strong electromagnetic fields [Kovtun]

For systems with no free charges (no screening), $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$ and $E^\mu = F^{\mu\nu}u_\nu$ can be $\mathcal{O}(1)$, and

$$W_s = \int d^4x\sqrt{-g}\left[p(T,\mu,B^2,E^2,B\cdot E) + \sum_{i,n_i,B^2} M^{(i)}_{n_i}(T,\mu,B^2,E^2,B\cdot E)s^{(i)}_{n_i}\right],$$

where

$$B^2 = B_\mu B^\mu\,, \quad E^2 = E_\mu E^\mu\,, \quad B\cdot E = B_\mu E^\mu\,,$$

and $s^{(i)}_{n_i}$ are $\mathcal{O}(\partial^i)$ equilibrium scalars.

Thermodynamics

Example: strong magnetic fields [JH,Kovtun]

For systems in strong magnetic fields, $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma} = \mathcal{O}(1)$, and

$$W_s = \int d^4x\sqrt{-g}\left[p(T,\mu,B^2) + \sum_{i,n_i,B^2} M_{n_i}^{(i)}(T,\mu,B^2)s_{n_i}^{(i)}\right],$$

where

$$T = \frac{T_0}{\sqrt{-V^2}}, \quad \mu = \frac{V^\mu A_\mu + \Lambda_V}{\sqrt{-V^2}}, \quad B^2 = B_\mu B^\mu,$$

and $s_{n_i}^{(i)}$ are $\mathcal{O}(\partial^i)$ equilibrium scalars.

Thermodynamics

Example, Nersnt effect and conformal anomaly [Ammon, Grieninger, JH, Kaminski, Koirala, Leiber, Wu]

Consider the leading term in the generating functional

$$W_s \sim \int d^4x \sqrt{-g}\, p(T, \mu, B^2)\,.$$

We can find the conformal anomaly coefficient and the Nernst coefficient

$$T^\mu_\mu \sim c_A F_{\mu\nu}F^{\mu\nu} \approx 2c_A B^2\,, \quad J^\mu \sim N_{Nernst}\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho \partial_\sigma T\,,$$

where

$$N_{Nernst} = \frac{2c_A}{T} = -\chi_{B,T} - \mu\frac{\chi_{B,\mu}}{T}\,,$$

and $\chi_B = 2p_{,B^2}$ is the magnetic susceptibility.

Thermodynamics

Example, magneto-vortical susceptibility M_5 [JH, Kovtun]

$$W_s \sim \int d^4x \sqrt{-g}\, M_5 B\cdot\Omega$$

Einstein-de Haas, Barnett effects

$$\mathcal{Q}^\mu \sim \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho B_\sigma \,, \quad m^\mu \sim \Omega^\mu$$

Momentum Nernst effect

$$\mathcal{Q}^\mu \sim \epsilon^{\mu\nu\rho\sigma} u_\nu B_\rho \partial_\sigma T$$

Response to Poynting vetor

$$\mathcal{Q}^\mu \sim \epsilon^{\mu\nu\rho\sigma} u_\nu E_\rho B_\sigma$$

Thermodynamics

Example, perpendicular magnetic vorticity susceptibility M_2 [Ammon, Grieninger, JH, Kaminski, Koirala, Leiber, Wu]

$$W_s \sim \int d^4x \sqrt{-g} M_2 \epsilon^{\mu\nu\rho\sigma} u_\mu B_\nu \partial_\rho B_\sigma$$

Magnetic Nernst effect

$$m^\mu \sim \epsilon^{\mu\nu\rho\sigma} u_\nu B_\rho \partial_\sigma T$$

Response to magnetic vorticity

$$m^\mu \sim \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho B_\sigma$$

Response to Poynting vector

$$\mathcal{T}^{\mu\nu} \sim B^{<\mu} \epsilon^{\nu>\rho\sigma\alpha} u_\rho E_\sigma B_\alpha$$

Outline

Chiral hydrodynamics

Consistent generating functional [Jensen, Kovtun, Ritz]

Generating functional W not $U(1)$ invariant

$$\delta_\alpha W = \frac{C}{24}\int d^4x\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\alpha F_{\mu\nu}F_{\rho\sigma}\,.$$

Consistent generating functional

$$W_{cons} = W_s + \frac{C}{6}\int d^4x\sqrt{-g}\mu(\mu\Omega\cdot A + 2B\cdot A)\,,$$

$$\delta W_{cons} = \int d^4x\sqrt{-g}(J^\mu_{cons}\delta A_\mu + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu})\,.$$

Because $\delta_\alpha W_{cons} \neq 0$

$$\delta_\alpha J^\mu_{cons} \neq 0\,.$$

But since $\delta_\alpha W_{cons}$ is independent of the metric

$$\delta_\alpha T^{\mu\nu} = 0\,.$$

Chiral hydrodynamics

Hydrodynamic equations, constitutive relations and Kubo formulas

From W_{cons} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cons} - A^\nu \nabla_\mu J^\mu_{cons}\,, \quad \nabla_\mu J^\mu_{cons} = \frac{C}{3} B\cdot E\,.$$

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)}$$

$$J^\mu_{cons} = J^\mu_s + \tfrac{1}{3} C B\cdot A u^\mu + \xi\,\Omega^\mu + \left(\xi_B - \tfrac{1}{3} C A\cdot u\right) B^\mu + \tfrac{1}{3} C \epsilon^{\mu\nu\rho\sigma} A_\nu u_\rho E_\sigma\,.$$

Chiral conductivities

$$\xi = \tfrac{1}{2} C\mu^2\,, \quad \xi_B = C\mu\,,$$

$$\xi_T = \tfrac{1}{3} C\mu^3\,, \quad \xi_{TB} = \tfrac{1}{2} C\mu^2\,.$$

Chiral hydrodynamics

Anomaly inflow and covariant generating functional [Callan, Harvey/Jensen, Loganayagam, Yarom/ Haehl, Loganayagam, Rangamani/Ammon, Grieninger, JH, Kaminski, Koirala, Leiber, Wu]

Consider an auxiliary manifold $\mathcal{M}$ whose boundary $\partial\mathcal{M}$ is where the chiral fluid lives.

Covariant generating functional

$$W_{cov} = W_{cons} - \frac{C}{24}\int_{\mathcal{M}} d^5x\sqrt{-G}\epsilon^{mnopq}A_m F_{no} F_{pq}\,,$$

$$\delta W_{cov} = \int_{\partial\mathcal{M}} d^4x\sqrt{-g}(J^\mu_{cov}\delta A_\mu + \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}) + \int_{\mathcal{M}} d^5x\sqrt{-G}J^m_H\delta A_m\,.$$

Where $J^\mu_{cov} = J^\mu_{cons} - \frac{C}{6}\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$, and $J^m_H = -\frac{C}{8}\epsilon^{mnopq}F_{no}F_{pq}$
Since $\delta_\alpha W_{cov} = 0$

$$\delta_\alpha J^\mu_{cov} = 0\,.$$

Chiral hydrodynamics

Constitutive relations and Kubo formulas

From W_{cov} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cov} \,, \quad \nabla_\mu J^\mu_{cov} = J^\rho_H = C B\cdot E \,.$$

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)} \,, \quad J^\mu_{cov} = J^\mu_s + \xi\,\Omega^\mu + \xi_B B^\mu \,.$$

Thermodynamic Kubo formulas

$$\langle J^x_{cov}(\mathrm{k}) T^{ty}(-\mathrm{k})\rangle = -i\xi\, k_z \,, \quad \langle J^x_{cov}(\mathrm{k}) J^y_{cons}(-\mathrm{k}))\rangle = -i\xi_B\, k_z \,,$$

$$\langle T^{tx}(\mathrm{k}) T^{ty}(-\mathrm{k})\rangle = -i\xi_T\, k_z \,, \quad \langle T^{tx}(\mathrm{k}) J^y_{cons}(-\mathrm{k})\rangle = -i\xi_{TB}\, k_z \,.$$

Chiral hydrodynamics

Constitutive relations and Kubo formulas

From W_{cov} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cov} \,, \quad \nabla_\mu J^\mu_{cov} = J^\rho_H = C B\cdot E \,.$$

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)} \,, \quad J^\mu_{cov} = J^\mu_s + \xi\, \Omega^\mu + \xi_B B^\mu \,.$$

Thermodynamic Kubo formulas

$$\langle J^x_{cov}(\mathrm{k}) T^{tz}(-\mathrm{k})\rangle = -i\xi\, k_y \,, \quad \langle J^x_{cov}(\mathrm{k}) J^z_{cons}(-\mathrm{k}))\rangle = -i\xi_B\, k_y \,,$$
$$\langle T^{tx}(\mathrm{k}) T^{tz}(-\mathrm{k})\rangle = -i\xi_T\, k_y \,, \quad \langle T^{tx}(\mathrm{k}) J^z_{cons}(-\mathrm{k})\rangle = -i\xi_{TB}\, k_y \,.$$

$U(1)_V \times U(1)_A$ hydrodynamics

Consistent generating functional

Generating functional W not $U(1)_A$ invariant

$$\delta_{\alpha,\alpha_5} W = \frac{C}{24} \int d^4x \sqrt{-g}\, \epsilon^{\mu\nu\rho\sigma} \alpha_5 \left(F^5_{\mu\nu} F^5_{\rho\sigma} + 3 F_{\mu\nu} F_{\rho\sigma} \right) .$$

Consistent generating functional

$$W_{cons} = W_s + \int d^4x \sqrt{-g} \Big[\tfrac{C}{3} \left(\mu B^\mu + \tfrac{1}{2} \mu^2 \Omega^\mu \right) A^5_\mu + C \left(\mu_5 B^\mu_5 + \tfrac{1}{2} \mu_5^2 \Omega^\mu \right) A^5_\mu \Big] ,$$

$$\delta W_{cons} = \int d^4x \sqrt{-g} (J^\mu_{cons} \delta A_\mu + J^\mu_{5,cons} \delta A^5_\mu + \tfrac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}) .$$

Because $\delta_{\alpha_5} W_{cons} \neq 0$, but metric independent

$$\delta_{\alpha_5} J^\mu_{cons} \,, \delta_{\alpha_5} J^\mu_{5,cons} \neq 0 \,, \quad \delta_{\alpha_5} T^{\mu\nu} = 0 \,.$$

$U(1)_V \times U(1)_A$ hydrodynamics

Anomaly inflow and covariant generating functional

Covariant generating functional

$$W_{cov} = W_{cons} - \frac{C}{24}\int_{\mathcal{M}} d^5x\sqrt{-G}\epsilon^{mnopq}A^5_m\left(F^5_{no}F^5_{pq} + 3F_{no}F_{pq}\right) ,$$

$$\delta W_{cov} = \int_{\partial\mathcal{M}} d^4x\sqrt{-g}(J^\mu_{cov}\delta A_\mu + J^\mu_{5,cov}\delta A^5_\mu + \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu})$$
$$+ \int_{\mathcal{M}} d^5x\sqrt{-G}\left(J^m_H\delta A_m + J^m_{5,H}\delta A^5_m\right) .$$

Where

$$J^\mu_{cov} = J^\mu_{cons} - \tfrac{C}{2}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F_{\rho\sigma} , \quad J^\mu_{5,cov} = J^\mu_{5,cons} - \tfrac{C}{6}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F^5_{\rho\sigma}$$

$$J^m_H = -\tfrac{C}{4}\epsilon^{mnopq}F_{no}F^5_{pq} , \quad J^m_{5,H} = -\tfrac{C}{8}\epsilon^{mnopq}\left(F_{no}F_{pq} + F^5_{no}F^5_{pq}\right)$$

Since $\delta_{\alpha_5}W_{cov} = 0$

$$\delta_{\alpha_5}J^\mu_{cov} = \delta_{\alpha_5}J^\mu_{5,cov} = 0 .$$

$U(1)_V \times U(1)_A$ hydrodynamics

Hydrodynamic equations

From W_{cons} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cons} + F_5^{\mu\nu} J_\nu^{5,cons} - A_5^\nu \nabla_\mu J^\mu_{cons}$$

$$\nabla_\mu J^\mu_{cons} = 0\,, \quad \nabla_\mu J^\mu_{5,cons} = -\frac{C}{24}\epsilon^{\mu\nu\rho\sigma}\left(F^5_{\mu\nu}F^5_{\rho\sigma} + 3F_{\mu\nu}F_{\rho\sigma}\right)\,.$$

From W_{cov} we find

Hydrodynamic equations

$$\nabla_\nu T_A^{\mu\nu} = F^{\mu\nu} J_\nu^{cov} + F_5^{\mu\nu} J_\nu^{5,cov}\,,$$

$$\nabla_\mu J^\mu_{cov} = -\frac{C}{4}\epsilon^{\mu\nu\rho\sigma}F^5_{\mu\nu}F_{\rho\sigma}\,,$$

$$\nabla_\mu J^\mu_{5,cov} = -\frac{C}{8}\epsilon^{\mu\nu\rho\sigma}\left(F^5_{\mu\nu}F^5_{\rho\sigma} + F_{\mu\nu}F_{\rho\sigma}\right)\,.$$

$U(1)_V \times U(1)_A$ hydrodynamics

Constitutive relations and Kubo formulas

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi^T u^{(\mu}\Omega^{\nu)} + \xi_B^T u^{(\mu} B^{\nu)} + \xi_{B_5}^T u^{(\mu} B_5^{\nu)}$$

$$J^\mu_{cons} = J^\mu_s + CB\cdot A^5 u^\mu + \xi\,\Omega^\mu + \left(\xi_B - CA^5\cdot u\right) B^\mu + \xi_{B_5} B_5^\mu + C\epsilon^{\mu\nu\rho\sigma} A^5_\nu u_\rho E_\sigma\,.$$

$$J^\mu_{5,cons} = J^\mu_s + \frac{C}{3} B_5\cdot A^5 u^\mu + \xi^5\,\Omega^\mu + \xi^5_B B^\mu + \left(\xi^5_{B_5} - \frac{C}{3} A^5\cdot u\right) B_5^\mu + \frac{C}{3}\epsilon^{\mu\nu\rho\sigma} A^5_\nu u_\rho E^5_\sigma\,.$$

$$J^\mu_{cov} = J^\mu_s + \xi\,\Omega^\mu + \xi_B B^\mu + \xi_{B_5} B_5^\mu\,.$$

$$J^\mu_{5,cov} = J^\mu_s + \xi^5\,\Omega^\mu + \xi^5_B B^\mu + \xi^5_{B_5} B_5^\mu\,.$$

Chiral conductivities

$$\xi^T = C\left(\mu^2\mu_5 + \tfrac{1}{3}\mu_5^3\right)\,, \quad \xi^T_{B_5} = \xi^5 = \frac{C}{2}\left(\mu^2 + \mu_5^2\right)$$

$$\xi^T_B = \xi = C\mu\mu_5\,, \quad \xi_B = \xi^5_{B_5} = C\mu_5\,, \quad \xi_{B_5} = \xi^5_B = C\mu\,.$$

Recap

We overviewed

- Equilibrium generating functional at strong magnetic field
- Anomaly inflow and consistent vs covariant W and J
- Hydro with $U(1)_V \times U(1)_A$ symmetry
- Equilibrium constitutive relations and Kubo formulas

More to the story

- Non-equilibrium constitutive relations
- Kubo formulas, eigenmodes
- Weak gauging of $U(1)_V$ for chiral MHD

Thermodynamics

Strong magnetic field and derivative expansion

For $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma} \sim \mathcal{O}(1)$, the generating functional is

$$W_s = \int d^4x \sqrt{-g} \left[\tilde{p}(T, \mu, B^2) + \sum_{i, n_i} \tilde{M}^{(i)}_{n_i}(T, \mu, B^2) \tilde{s}^{(i)}_{n_i} \right] ,$$

where, for example

$$\tilde{p}(T, \mu, B^2) = p(T, \mu) + M^{(2)}_{B^2}(T, \mu) B^2 + M^{(4)}_{B^4}(T, \mu) B^4 + \cdots ,$$

and

$$\tilde{s}^{(i)}_{n_i} = s^{(j)}_{n_j} ,$$

for some n_j with $j \geq i$.

Thermodynamics

Constitutive relations, strong magnetic fields

For $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma} \sim \mathcal{O}(1)$, five $\mathcal{O}(\partial)$ equilibrium scalars s_n.

n	1	2	3	4	5
s_n	$B^\mu\partial_\mu(\frac{B^2}{T^4})$	$\epsilon^{\mu\nu\rho\sigma}u_\mu B_\nu\partial_\rho B_\sigma$	$B\cdot a$	$B\cdot E$	$B\cdot\Omega$
$(C,P,T)_{\rm vector}$	$-/-/-$	$+/-/+$	$-/-/-$	$+/-/-$	$-/+/+$
$(C,P,T)_{\rm axial}$	$+/+/-$	$+/-/+$	$+/+/-$	$+/-/-$	$+/-/+$
W	3	5	n/a	4	3

where $a^\mu = u^\nu\nabla_\nu u^\mu\,, \quad E^\mu = F^{\mu\nu}u_\nu\,, \quad \Omega^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma\,.$

Equilibrium constitutive relations

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}\,,$$

$$J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu\,.$$

Second order variations of W_s give thermodynamic two point functions $(\omega = 0, k \ll T)$

$$\delta W_s = \int d^d x \sqrt{-g} \left[G_{JJ} \delta A \delta A + \frac{1}{2} G_{TJ} \delta g \delta A + \frac{1}{4} G_{TT} \delta g \delta g \right] .$$

This allows us to derive

Thermodynamic Kubo formulas ($g_{\mu\nu} = \eta_{\mu\nu}\,,\ B^\mu = B_0 \delta^\mu_z$)

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{T^{xz}T^{yz}}(\omega = 0, k\hat{z}) = -2\, B_0^2\, M_2\,,$$

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{T^{tx}T^{yz}}(\omega = 0, k\hat{z}) = -B_0\, M_5\,.$$

Thermodynamics

Thermodynamic Kubo formulas

Second order variations of W_s give thermodynamic two point functions ($\omega = 0, k \ll T$)

$$\delta W_s = \int d^d x \sqrt{-g} \left[G_{JJ} \delta A \delta A + \tfrac{1}{2} G_{TJ} \delta g \delta A + \tfrac{1}{4} G_{TT} \delta g \delta g \right] .$$

This allows us to derive

Thermodynamic Kubo formulas

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{J^t T^{xx}}(\omega = 0, k\hat{z}) = -\frac{2\, B_0^3}{T_0^4} \frac{\partial M_1}{\partial \mu} ,$$

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{J^t J^t}(\omega = 0, k\hat{z}) = B_0 \frac{\partial M_4}{\partial \mu} ,$$

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{J^t T^{tt}}(\omega = 0, k\hat{z}) = -B_0 \left(\frac{\partial M_3}{\partial \mu} + \frac{4 B_0^2}{T_0^4} \frac{\partial M_1}{\partial \mu} \right) .$$

Thermodynamics

$U(1)_V \times U(1)_A$ symmetry

Thermodynamic variables

$$T, \mu, \mu^5, u^\mu$$

Generating functional

$$W_s = \int d^4x \sqrt{-g}\, p(T, \mu, \mu^5) + \cdots$$

Constitutive relations

$$\delta W_s = \int d^4x \sqrt{-g} \left[\tfrac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu + J_5^\mu \delta A_\mu^5 \right]$$

Thermodynamic correlation functions and Kubo formulas

$$\delta W_s = \int d^dx \sqrt{-g} \left[G_{J_a J_b} \delta A^a \delta A^b + \tfrac{1}{2} G_{T J_a} \delta g \delta A^a + \tfrac{1}{4} G_{TT} \delta g \delta g \right] .$$

Thermodynamics

$U(1)_V \times U(1)_A$ symmetry, strong magnetic field

Thermodynamic variables

$$T, \mu, \mu^5, B^2, u^\mu$$

Generating functional

$$W_s = \int d^4x\sqrt{-g}\,\tilde{p}(T, \mu, \mu^5, B^2) + \sum_1^7 \tilde{M}_n(T, \mu, \mu^5, B^2)s_n + \cdots$$

$$\delta W_s = \int d^4x\sqrt{-g}\left[\tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + J^\mu \delta A_\mu + J_5^\mu \delta A_\mu^5\right]$$

Hydrodynamics

Non-equilibrium constitutive relations

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p - \zeta \nabla \cdot u)\, \Delta^{\mu\nu} - \eta \sigma^{\mu\nu}\,,$$

$$J^\mu = n u^\mu + \sigma \left(E^\mu - T \Delta^{\mu\nu} \partial_\nu \frac{\mu}{T} \right)\,.$$

where $\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d-1}\Delta_{\alpha\beta}\nabla\cdot u\right)$.

Hydrodynamic transport coefficients

$$\zeta\,,\ \eta\,,\ \sigma\,.$$

Entropy production constraints

$$S^\mu|_{\mathrm{eq.}} = s u^\mu \ \text{s.t.}\ \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \zeta \geqslant 0\,,\ \eta \geqslant 0\,,\ \sigma \geqslant 0\,.$$

Hydrodynamics

Non-equilibrium constitutive relations, $B^\mu \sim \mathcal{O}(1)$

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \cdots \,,$$

$$J^\mu = \cdots \,.$$

Hydrodynamic transport coefficients

$$\eta_\perp\,,\ \eta_\parallel\,,\ \eta_1\,,\ \eta_2\,,\ \zeta_1\,,\ \zeta_2\,,\ \sigma_\parallel\,,\ \sigma_\perp\,,\ \tilde{\eta}_\perp\,,\ \tilde{\eta}_\parallel\,,\ \tilde{\sigma}_\perp\,.$$

Entropy production constraints

$$S^\mu|_{\text{eq.}} = su^\mu \text{ s.t. } \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \text{linear} + \text{quadratic}$$

Hydrodynamics

Non-equilibrium constitutive relations, $B^\mu \sim \mathcal{O}(1)$, parity violating

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \cdots \,, \quad J^\mu = \cdots \,.$$

Hydrodynamic transport coefficients

$$\eta_\perp \,,\; \eta_\| \,,\; \eta_1 \,,\; \eta_2 \,,\; \zeta_1 \,,\; \zeta_2 \,,\; \sigma_\| \,,\; \sigma_\perp \,,\; \tilde{\eta}_\perp \,,\; \tilde{\eta}_\| \,,\; \tilde{\sigma}_\perp \,,$$
$$\sigma_1^V \,,\; \sigma_2^V \,,\; \sigma_3^V \,,\; \sigma_4^V \,,\; \zeta^V \,,\; \eta^V \,,\; \eta_\|^V \,,\; \tilde{\eta}_\|^V \,.$$

Entropy production constraints

$$S^\mu|_{\text{eq.}} = s u^\mu \text{ s.t. } \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \text{linear} + \text{quadratic} + \text{qubic}$$

Hydrodynamics

Non-equilibrium constitutive relations, $B^\mu \sim \mathcal{O}(1)$, parity violating, $U(1)_V \times U(1)_A$

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \cdots\,, \quad J^\mu_V = \cdots\,, \quad J^\mu_A = \cdots\,.$$

Hydrodynamic transport coefficients

$$\eta_\perp\,,\ \eta_\parallel\,,\ \eta_1\,,\ \eta_2\,,\ \zeta_1\,,\ \zeta_2\,,\ \sigma^{ab}_\parallel\,,\ \sigma^{ab}_\perp\,,\ \tilde\eta_\perp\,,\ \tilde\eta_\parallel\,,\ \tilde\sigma^{ab}_\perp\,,$$
$$\sigma^a_1\,,\ \sigma^a_2\,,\ \sigma^a_3\,,\ \sigma^a_4\,,\ \zeta^a\,,\ \eta^a\,,\ \eta^a_\parallel\,,\ \tilde\eta^a_\parallel\,.$$

Entropy production constraints

$$S^\mu|_{\text{eq.}} = s u^\mu \text{ s.t. } \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \text{linear} + \text{quadratic} + \text{qubic}+?$$

Hydrodynamics

Eigenmodes and Kubo formulas

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \,, \quad \nabla_\mu J^\mu = 0 \,.$$

Linearize hydrodynamic equations for $\delta A_\mu \,, \delta g_{\mu\nu} \propto \exp(-i\,(\omega t - \mathrm{k}\cdot\mathrm{x}))$

Eigenmodes

$$\omega = \omega(k)\,.$$

Varying on-shell $T^{\mu\nu}[g, A]$ and $J^\mu[g, A]$ (limit $k \to 0$ first, $\omega \to 0$ second)

Hydrodynamic Kubo formulas

$$\lim_{\omega\to 0} \mathrm{Im}\, G_{\mathcal{O}\mathcal{O}}(\omega, k = 0) \sim \zeta_i \,,\ \eta_i \,,\ \sigma_i \,,\ c_i \,.$$

Chiral hydrodynamics

Eigenmodes, gapped

There are two gapped modes

$$\omega = \pm \frac{B_0^2}{w_0}\sigma_{12} - \frac{iB_0^2}{w_0}\sigma_{11} + v_{gap\pm} k \cos\theta - iD_c(\theta)k^2 \,,$$

where $w_0 = \epsilon_0 + p_0$ and $\sigma_{ab} = \delta_{ab}\sigma_\perp + \epsilon_{ab}\left(\tilde{\sigma}_\perp + \frac{n_0}{|B_0|}\right)$. The gapped modes have velocity

$$v_{gap\pm} = \frac{B_0^2 C\mu_0^3}{3w_0^2}(\sigma_{12} \pm i\sigma_{11}) \,,$$

and damping coefficient $D_c(\theta) = D_c(\theta)|_{C=0} \pm i\frac{C\mu_0^3}{3}\frac{v_{gap\pm}}{w_0}\cos^2\theta \,.$

Chiral hydrodynamics

Eigenmodes, gappless parallel

There are three gapless eigenmodes. For k ∥ B, they are

$$\omega = kv_\pm - i\frac{\Gamma_\parallel}{2}k^2\,, \quad \omega = kv_0 - iD_\parallel k^2\,,$$

where

$$v_0 = \frac{B_0 C}{\det(\chi)}\left(\frac{s_0 T_0}{v_s}\right)^2\,,$$

$$v_\pm = \pm v_s - \frac{v_0}{2} + \frac{B_0 C}{2}\gamma\,.$$

The speed of sound is given by

$$v_s^2 = \frac{n_0^2\chi_{11} + w_0^2\chi_{33} - 2n_0 w_0\chi_{13}}{\det(\chi)}\,,$$

and $\chi_{ab} = \frac{\delta\langle\varphi_a\rangle}{\delta\lambda^b}$ is the susceptibility matrix. Here, $\varphi_a = (T^{tt}, T^{ti}, J^t)$, and $\delta\lambda^a = (\delta T/T, \delta u^i, T\delta\frac{\mu}{T})$.

Chiral hydrodynamics

Eigenmodes, gappless parallel

There are three gapless eigenmodes. For k $\parallel$ B, they are

$$\omega = kv_\pm - i\frac{\Gamma_\parallel}{2}k^2\,, \quad \omega = kv_0 - iD_\parallel k^2\,,$$

where

$$\Gamma_\parallel = \frac{3\zeta_1 + 10\eta_1 + 6\eta_2}{3w_0} + \frac{v_s^2\chi_{11} - w_0}{\det(\chi)}\frac{w_0}{v_s^2}\sigma_\parallel$$
$$+ CB_0\left(\Sigma_\eta(3\zeta_1 + 10\eta_1 + 6\eta_2) + \Sigma_\parallel\sigma_\parallel + \Sigma_\perp\sigma_\perp\right) + \mathcal{O}\left(B_0^2C^2\right)\,,$$
$$D_\parallel = \frac{w_0^2\sigma_\parallel}{v_s^2\det(\chi)} + \mathcal{O}\left(B_0^2C^2\right)\,.$$

Σ_η, $\Sigma_\parallel$ and $\Sigma_\perp$ are functions of the susceptibilities, other thermodynamic derivatives of the pressure, the chemical potential and the temperature.

Chiral hydrodynamics

Eigenmmode, gapless non-orthogonal

For modes propagating at an angle $\theta \neq \pi/2$ with respect to B_0

$$\omega = k v_{\pm} \cos\theta - \frac{i}{2}\Gamma(\theta) k^2 \,, \quad \omega = k v_0 \cos\theta - i D(\theta) k^2 \,,$$

where

$$D(\theta) = D_{\|} \cos^2\theta + \left(\frac{n_0^2\, w_0^2\, \rho_\perp}{B_0^2 v_s^2 \det(\chi)^2} + \mathcal{O}(B_0 C) \right) \sin^2\theta \,,$$

$$\Gamma(\theta) = \Gamma_{\|} \cos^2\theta + \left(\frac{\eta_{\|}}{w_0} + \frac{(n_0 \chi_{13} - w_0 \chi_{33})^2 w_0^3}{B_0^2 \det(\chi)^2} \rho_\perp + \mathcal{O}(B_0 C) \right) \sin^2\theta \,.$$

Chiral hydrodynamics

Eigenmodes, gapless perpendicular

For k $\perp$ B, two diffusive modes

$$\omega = -i\left(\frac{w_0^3\chi_{33}\rho_\perp}{\det(\chi)B_0^2} + \mathcal{O}\left(B_0^2C^2\right)\right)k^2\,,$$

$$\omega = i\left(\frac{\eta_\parallel}{w_0} + \mathcal{O}\left(B_0^2C^2\right)\right)k^2\,,$$

and one subdiffusive mode

$$\omega = -i\frac{\eta_\perp k^4}{B_0^2\,\chi_{33}}\,.$$