(Controversial) Thermodynamics with Spin (Chemical) Potential

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Work in completion with Maxim Chernodub

— Holographic perspectives on chiral transport and spin dynamics —

Dynamical vs. Static Problems

Dynamical Problem — Spin Hydro

$S^{\mu\nu}(t, \boldsymbol{x}) \qquad \omega^{\mu\nu}(t, \boldsymbol{x})$

Pseudo-gauge Problem

$$\partial_\lambda J^{\lambda\mu\nu} = 0 \xrightarrow{\;\;?\;\;} J^{\lambda\mu\nu} = L^{\lambda\mu\nu} + S^{\lambda\mu\nu}$$

Requiring: $P(\omega)$ **such that** $\langle S \rangle = \partial P / \partial \omega$

Static Problem — Thermodynamics (EoS)

$P(\Omega)$ **Finite-T QFT in a rotating frame**

Dropping L by looking at $r = 0$

Dropping L from the operator coupled with ω

This Talk Discusses

Two approaches give different answers…

Both are physically possible…

One is very unstable:

→ **Polarization-induced Nielsen-Olesen Instability**

The other is very topological:

→ **Weyl points / CS currents**

Introduction of Rotation

In terms of the fluid language:

$$\beta_\mu p^\mu = \beta(p^0 - \underline{\boldsymbol{\Omega} \times \boldsymbol{x} \cdot \boldsymbol{p}})$$

$$= \boldsymbol{x} \times \boldsymbol{p} \cdot \boldsymbol{\Omega} = \boldsymbol{L} \cdot \boldsymbol{\Omega}$$

Can be fully relativistically generalized with $\Omega^\mu = \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma$

Cranking Hamiltonian:

$$\hat{H} \;\to\; \hat{H} - \hat{\boldsymbol{J}} \cdot \boldsymbol{\Omega}$$

Introduction of Rotation

Metric in the rotating frame:

Euclidean Cylindrical + Imaginary Rotation

$$g_{\mu\nu} = \begin{pmatrix} 1-\Omega_I^2 r^2 & -i\Omega_I r^2 & 0 & 0 \\ -i\Omega_I r^2 & r^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

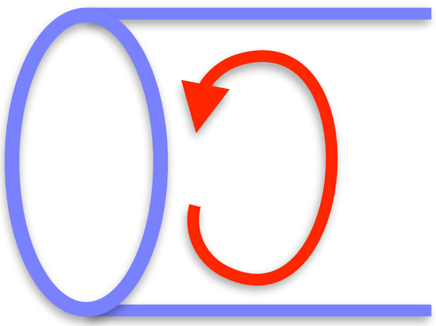
Analytical Continuation

Geometrical condition:

Period $\beta = 1/T$

$$(\tau, \theta, r, z) \;\sim\; (\tau + \beta, \theta \underline{- \beta\Omega_I}, r, z)$$

Imaginary time
× Imaginary angular velocity



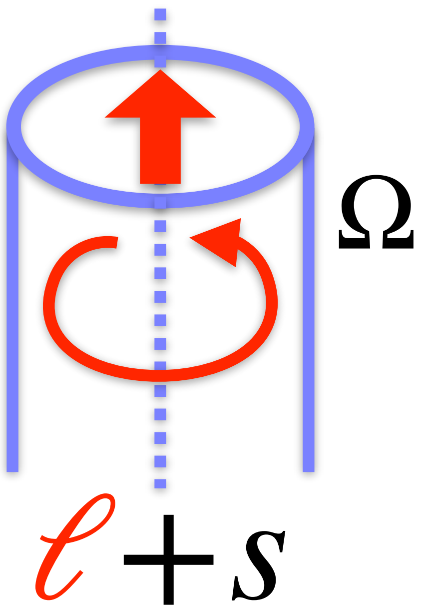
Controversy at a Quick Glance

Pressure of Rotating Fermions

$$pV = \frac{T}{8\pi^2}\int d^2r_\perp dz \int dp_r^2 dp^z \sum_{\ell=-\infty}^{\infty} J_\ell^2(p_r r)$$

$$\times \sum_{s=-1/2}^{+1/2} \ln\left[1 + e^{-(E-(\ell+s)\Omega \mp \mu)/T}\right]$$

$$E = \sqrt{p_r^2 + p_z^2 + m^2}$$



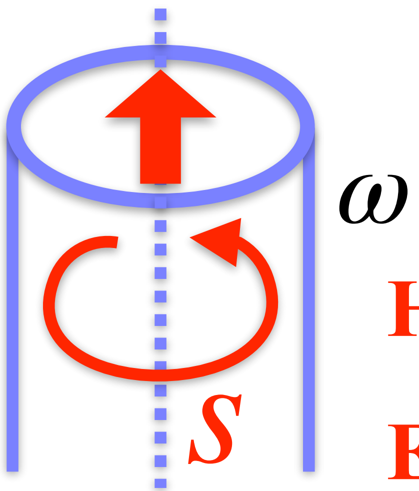
cf. Chiral Vortical Effect (by Vilenkin)

Controversy at a Quick Glance

Pressure of Fermions at Finite Spin Potential

$$pV = \frac{T}{8\pi^2} \int d^2 r_\perp dz \int dp_r^2 dp^z \boxed{\sum_{\ell=-\infty}^{\infty} J_\ell^2(p_r r)} = 1$$

$$\times \sum_{s=-1/2}^{+1/2} \ln\left[1 + e^{-(E - s\omega \mp \mu)/T}\right]$$



Homogeneous System (no centrifugal force)

Energy dispersion shifted by $\omega \cdot s$

(The same conclusion if we set $r = 0$.)

Controversy at a Quick Glance

Direct Coupling to the Spin Operator

$$J^{0\mu\nu} = \underbrace{i\psi^\dagger(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi}_{L^{\mu\nu}} + \underbrace{\psi^\dagger\Sigma^{\mu\nu}\psi}_{S^{\mu\nu}}$$

$$\psi^\dagger\mathscr{H}\psi \;\to\; \psi^\dagger(i\gamma^0\gamma^i D^i + \gamma^0 m - \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\psi$$

$$\Rightarrow\; \mathscr{L} = \bar\psi(i\gamma^\mu D_\mu - m + \underbrace{\frac{1}{2}\omega_{\mu\nu}\gamma^0\Sigma^{\mu\nu}}_{-\frac{1}{2}\omega\gamma_5\gamma^3})\psi$$

Spin ~ Axial Vector Current

Controversy at a Quick Glance

Direct Coupling to the Spin Operator

$$\mathscr{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m \ - \ \frac{1}{2}\omega\gamma_5\gamma^3)\psi$$

Energy-dispersion can be explicitly obtained as

$$\varepsilon_{\pm}^2 = p_{\perp}^2 + \left(\sqrt{p_z^2 + m^2} \pm \frac{1}{2}\omega\right)^2 \qquad ?$$

Dropping *L* in the end and dropping *L* from the beginning lead to different results!

A Quick Answer

Modified Hamiltonian with Global Rotation:

$$\hat{H} \;\to\; \hat{H} - \hat{\boldsymbol{J}} \cdot \boldsymbol{\Omega} \qquad [\hat{H}, \hat{\boldsymbol{J}}] = 0$$

$$E - (\ell + s)\Omega$$

Conserved Quantity
Simultaneous Eigenstate

Modified Hamiltonian with Spin Potential:

$$\hat{H} \;\to\; \hat{H} - \hat{\boldsymbol{S}} \cdot \boldsymbol{\omega} \qquad [\hat{H}, \hat{\boldsymbol{S}}] \neq 0$$

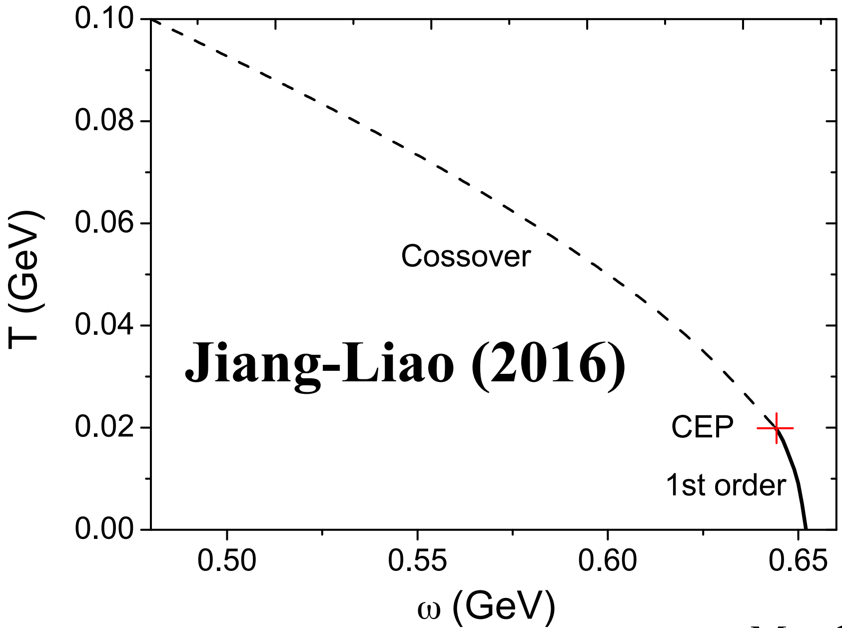
Removing $\hat{L}$ is not equivalent to removing ℓ …

cf. Pseudo-gauge ambiguity in Spin Hydro

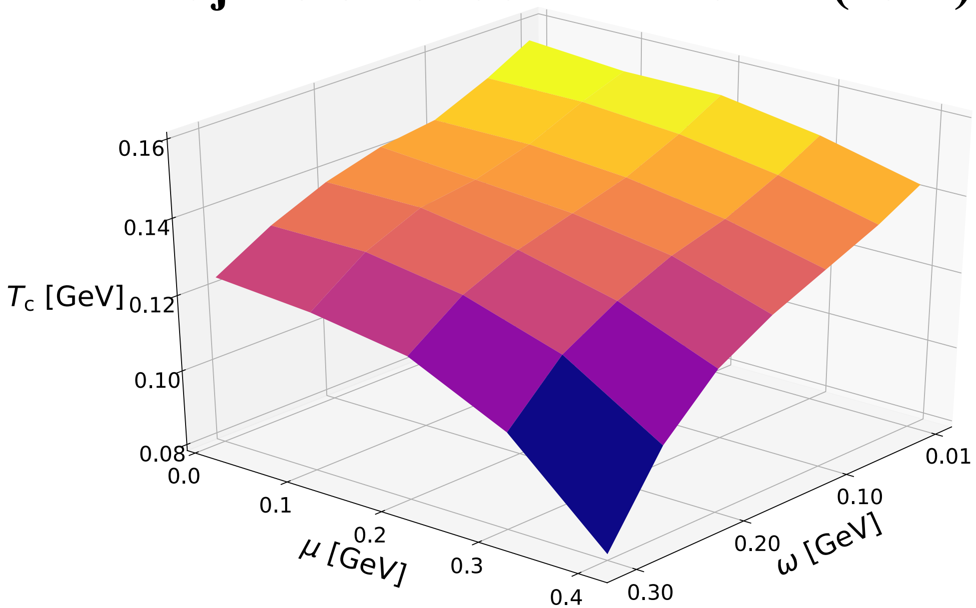
Removing ℓ

The spin (chemical) potential is introduced for the conserved total angular momentum and then the orbital part is assumed to dissipate.

This treatment is assumed in most of preceding works (r = 0).

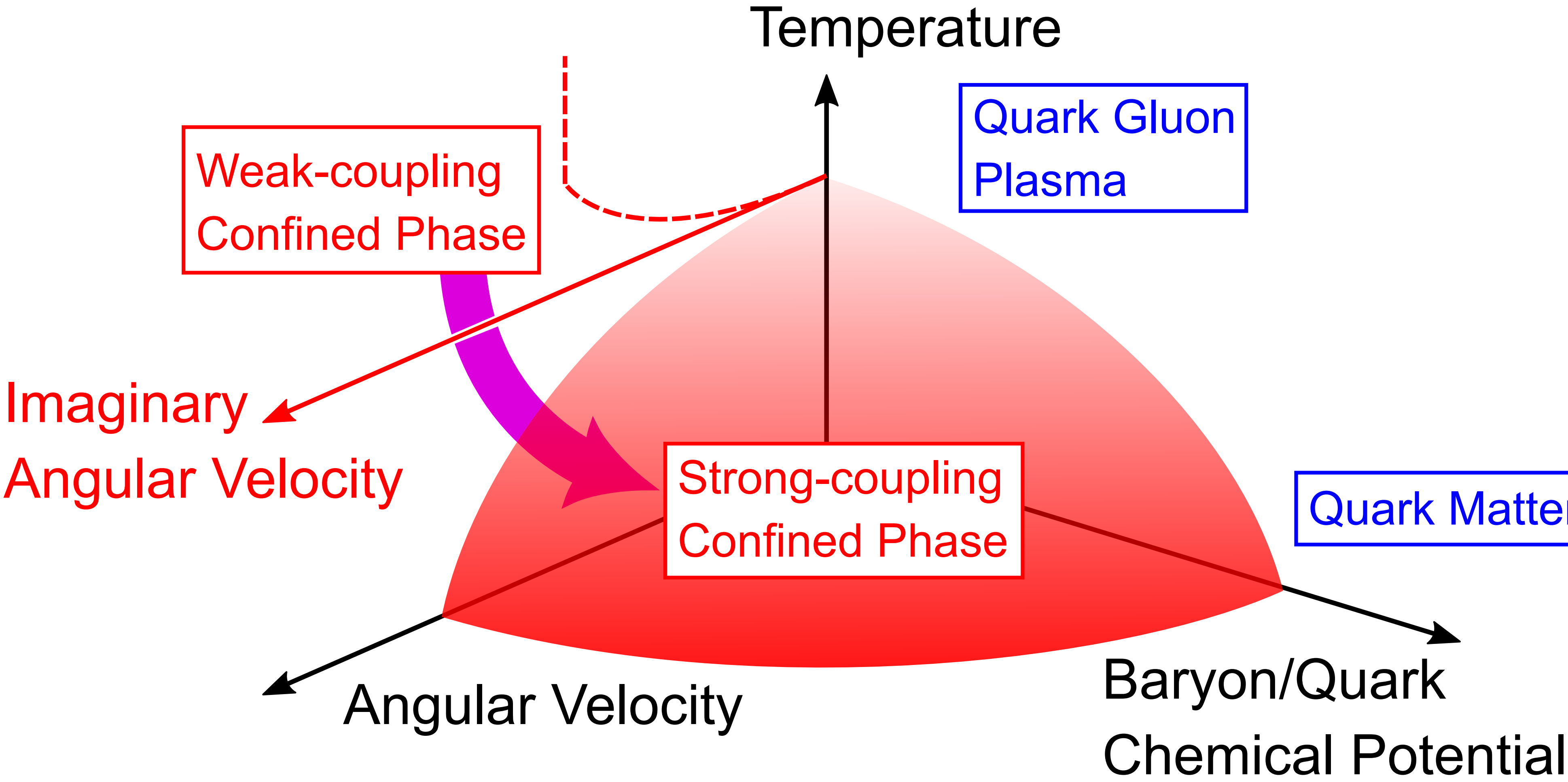


Fujimoto-Fukushima-Hidaka (2021)



Removing ℓ

Chen-Fukushima-Shimada (2023)



Removing ℓ

Polarization-induced Nielsen-Olesen Instability

Gluon pressure in the presence of the spin potential:

$$p = -\frac{(N_c^2-1)T}{8\pi^2}\int dp_r^2 dp^z \sum_{s=-1}^{+1} \ln\left[1 + e^{-(|\boldsymbol{p}|-s\omega)/T}\right]$$

Usually, $s\omega$ does not exceed $|\boldsymbol{p}|$ due to the boundary and the causality condition.
($|\boldsymbol{p}|$ has a gap $\sim 1/R > \omega$)

However, the spin-polarized system is homogeneous, not requiring any boundary (not rotating), so this is a physical instability!

Removing ℓ

Polarization-induced Nielsen-Olesen Instability

Is this really such unstable? For Imaginary $\omega_I = -i\omega$:

$$p = -\frac{(N_c^2-1)\pi^2 T^4}{3}\sum_{s=-1}^{+1} B_4\left(\left(\frac{s\omega_I}{2\pi}\right)_{\mathrm{mod}1}\right)$$

$$B_4(x) = \underline{x^4} - 2x^3 + \underline{x^2} - 1/30$$

Contribution from Gluon Condensation with Opposite Sign

Moment of Inertia of Perturbative Gluons

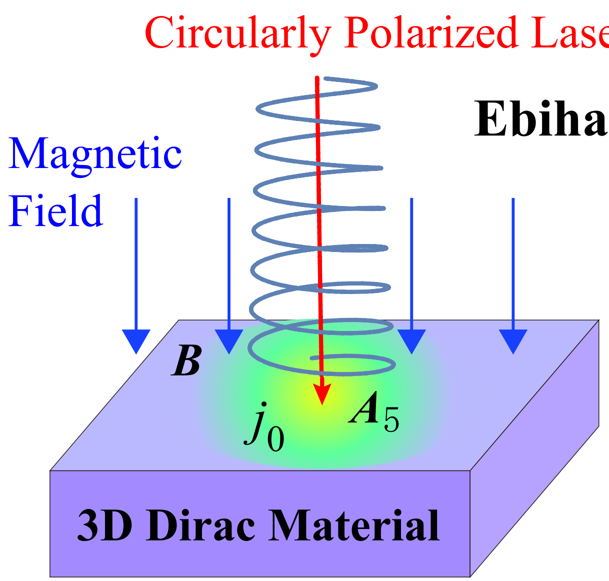
Removing $\hat{L}$

For Fermions

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m - \underline{\tfrac{1}{2}\omega\gamma_5\gamma^3})\psi$$

$$\omega \sim A_5^z$$

Spin potential = Spatial component of the axial gauge field



Ebihara-Fukushima-Oka (2015)

$$\varepsilon_\pm(p) = \sqrt{p_x^2 + p_y^2 + (\sqrt{p_z^2 + m^2} \pm \beta)^2}$$

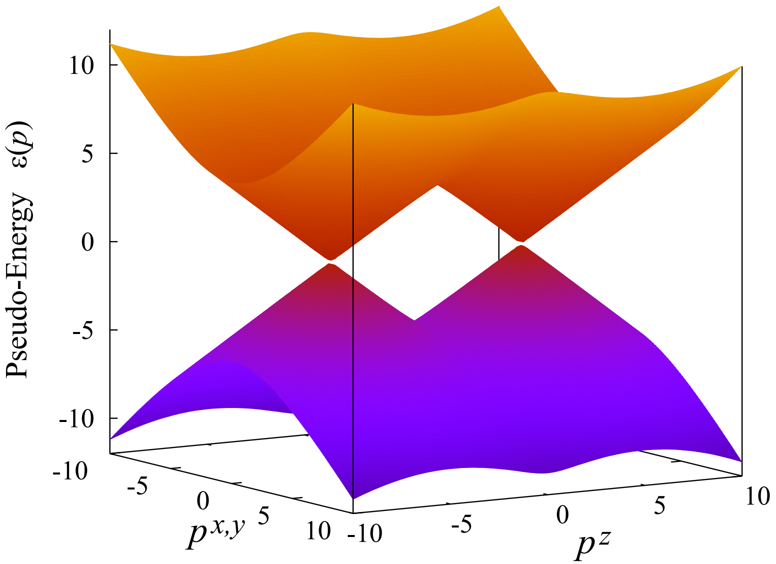
$$\beta \equiv (eE)^2/\Omega^3$$

Removing $\hat{L}$

For Fermions   Two Weyl cones appears!

C. Anomalies and magnetotransport in Weyl semi-metals

Ebihara-Fukushima-Oka (2015)



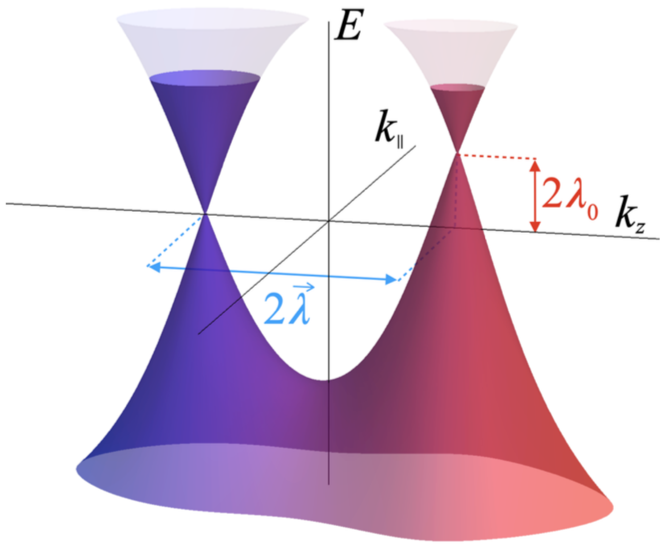


FIG. 4. A schematic view of a Weyl semimetal's band structure.

Chernodub-Ferreiros-Grushin-
-Landsteiner-Vozmediano, Phys.Rept. (2022)

Removing $\hat{L}$

For Gauge Particles

In QED in the Jaffe-Manohar decomposition:

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E}\times\boldsymbol{A}}_{\Delta G} - \underbrace{i\psi^\dagger(\boldsymbol{x}\times\nabla)\psi}_{L^q} + \underbrace{\boldsymbol{E}(\boldsymbol{x}\times\nabla)\boldsymbol{A}}_{L^g}$$

$$\boldsymbol{E}\cdot\boldsymbol{B} = \partial_t(\boldsymbol{A}\cdot\boldsymbol{B}) + \nabla\cdot(A_0\boldsymbol{B} + \boldsymbol{E}\times\boldsymbol{A}) \sim \partial_\mu K^\mu$$

We are proposing the gauge-spin coupling as

$$\boxed{\Delta\mathscr{L} = N\int d^4x\,\boldsymbol{\omega}\cdot\boldsymbol{K}}$$

Removing $\hat{L}$

For Gauge Particles

This is the spatial version of the chiral chemical pot.:

On chiral magnetic effect and holography

V.A. Rubakov

arXiv:1005.1888 [hep-ph]

Institute for Nuclear Research of the Russian Academy of Sciences,
60th October Anniversary Prospect, 7a, 117312 Moscow, Russia

$$\boxed{S_{eff} = 3\kappa\mu_A \int d^4x\, \epsilon^{ijk} A_i^V F_{jk}^V}$$

Removing $\hat{L}$

For Gauge Particles

More generally:

$$\Delta \mathscr{L} = N \int d^4x \, \underbrace{\boxed{(\partial_\mu \theta)}}_{\omega_\mu} K^\mu$$

$$= -N \int d^4x \, \theta \, \partial_\mu K^\mu$$

Introducing the spin potential in the gauge sector is equivalent to introducing inhomogeneous θ angle!

See: Vazifeh-Franz (2013) for applications in cond-mat.

Removing $\hat{L}$

For Gauge Particles

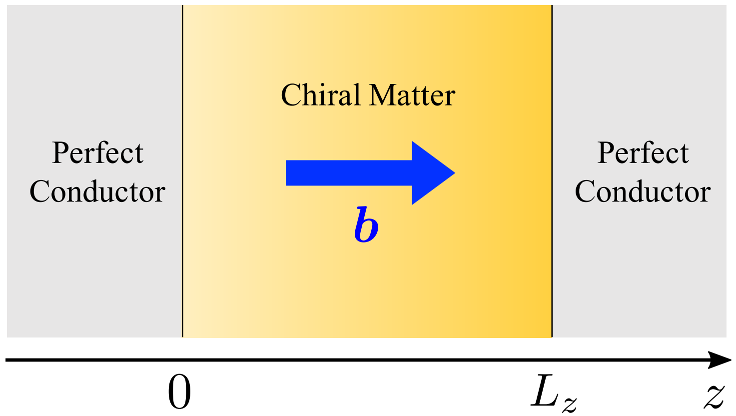
Qiu-Cao-Huang (2016)

We already know the answer: $b = -\partial_z\theta$

$$\omega_{1,2}^2 = \boldsymbol{k}^2 \qquad \omega_\pm^2 = k_x^2 + k_y^2 + \left(\sqrt{k_z^2 + \frac{b^2}{4}} \pm \frac{b}{2}\right)^2$$

For the application to the Casimir force, see: Fukushima-Imaki-Qiu (2019)

Not only the attractive but also the repulsive force!



Summary

Spin Potential

- □ Removing the orbital part after/before diagonalization leads to different results.
- □ Which corresponds to which physical setup?

Removing-ℓ

- □ Polarization-induced Nielsen-Olesen instability.
- □ Thermodynamics looks non-singular.

Removing-$\hat{L}$

- □ Spin-potential couples the topological CS current…???