Scale setting for CLS 2+1 simulations

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Scale setting: Precision lattice QCD for particle and nuclear physics

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Overview

This talk is based on

- Original 2017 publication
  [M. Bruno, T. K. and S. Schaefer,
  "*Setting the scale for the CLS 2 + 1 flavor ensembles*",
  Phys. Rev. D **95** (2017) no.7, 074504]
- 2021 update
  [B. Straßberger, M. Cè, S. Collins, A. Gérardin, G. von Hippel, P. Korcyl, T. K., D. Mohler, A. Risch, S. Schaefer, W. Söldner and R. Sommer,
  "*Scale setting for CLS 2+1 simulations*",
  PoS **LATTICE2021** (2022), 135]
- Ben's PhD thesis
  [B. Straßberger,
  "Towards Higher Precision Lattice QCD Results: Improved Scale Setting and Domain Decomposition Solvers",
  doi:10.18452/26517]
- Recent r_0 determination ($\rightarrow$ Tom Asmussen's talk)
  [T. M. B. Asmussen, R. Höllwieser, F. Knechtli and T. K.,
  "*The determination of potential scales in 2+1 flavor QCD*",
  arXiv:2412.10215 [hep-lat]]

Goal

A precise scale is necessary for all dimensionful predictions of QCD
The goals of this work are

- Determine precise values in [fm] for widely used reference scales like
  - $\sqrt{t_0}$, the original gradient flow scale
    [M. Lüscher, JHEP 08 (2010) 071]
  - w_0, a modified gradient flow scale
    [S. Borsányi, JHEP 09 (2012)]
  - r_0, the "Sommer Scale" from the static potential
    [R. Sommer, Nucl. Phys. B 411 (1994)]
  - r_1, a shorter distance cousin of r_0
    [W. Bernard et al., Phys. Rev. D 62 (2000)]

  (In continuum iso-QCD at well-defined mass points)
- Map out the relation between simulation parameter $\beta \equiv \frac{6}{g_0^2}$ and the lattice spacing a for the lattice action used by CLS

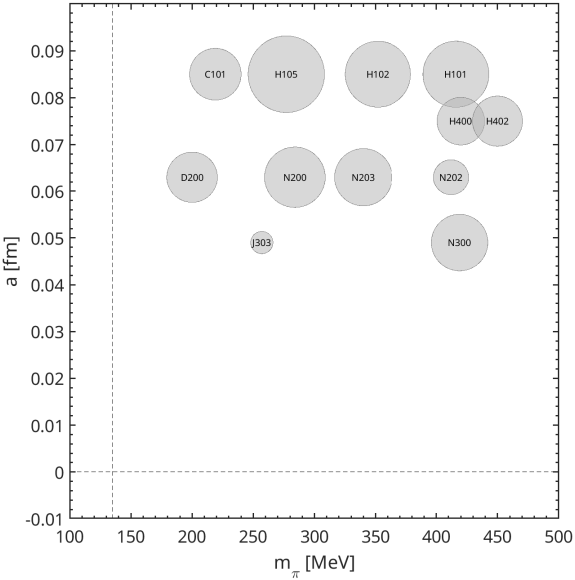
The original precision goal was modest, $1\% - 2\%$,
based on our aspiration to compute Λ_{QCD} to $\approx 3\%$

CLS

- CLS = Coordinated Lattice Simulations
  consortium of several European lattice-QCD groups
- Since 2013: generation of $N_{\mathrm{f}} = 2 + 1$ ensembles
- Action
  [M. Bruno et al. JHEP 02 (2015) 043]
  - Tree-level Symanzik $O(a^2)$ improved gauge action  , 
    [M. Lüscher, P. Weisz (1985)]
  - Non-perturbatively $O(a)$ improved Wilson fermions 
    [B. Sheikholeslami, R. Wohlert, Nucl.Phys.B 259 (1985)]
    [J. Bulava, S. Schaefer, Nucl.Phys.B 874 (2013)]
  - Open boundaries in time (mostly)
- Chiral trajectory with $\mathrm{tr}[M_q] = \mathrm{const}$
  [W. Bietenholz et al., Phys.Lett. B690 (2010)]
  Reason: with Wilson fermions
  same $\tilde{g}_0^2 = g_0^2\,(1 + b_g\; a\,\mathrm{tr}[M_q]/3) \Leftrightarrow$ same lattice spacing a up to $O(a^2)$

Ensembles and Statistics

- Area $\propto$ MDUs
- Gray = original (2017)
- Blue = update (2021)



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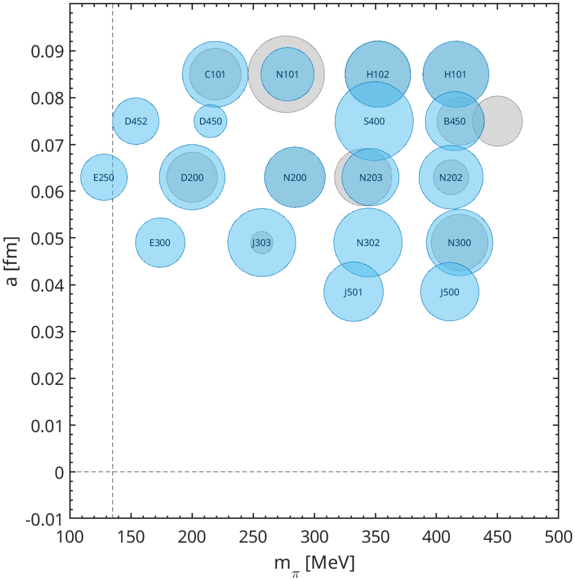


Strategy

- Inputs
- Iteration
- Results

Inputs

To fix three bare parameters $a\,m_{u,d}, a\,m_s, \beta$
we use three experimental inputs

- Pion mass m_π
- Kaon mass m_K
- Pseudo-scalar decay constant $f_{\pi K} \equiv \frac{2}{3}\left(f_K + \frac{1}{2} f_\pi\right)$

We assume that these are already “corrected” for

- QED effects (standard model $\neq$ QCD)
- $N_\mathrm{f} = 2 + 1$ is not $N_\mathrm{f} = 1 + 1 + 1 + 1 + 1 + 1$ effects

$\rightarrow$ Yesterday’s talk by Alberto Ramos

Leptonic Decays of Pseudoscalar Mesons

We want to use leptonic decay constants f_π, f_K to set the scale

Lowest order: $\Gamma^0(P \to l\nu) = \frac{G_F^2}{8\pi} {f_P}^2 m_l^2 m_P \left(1 - \frac{m_l^2}{m_P^2}\right)^2 |V_{q_1 q_2}|^2$

And $ip_\mu f_P = \langle 0|\bar{q}_1 \gamma_\mu \gamma_5 q_2 |P(p)\rangle$ is computable in QCD

- Advantages
  - Low energy quantity
  - Pseudo-scalar correlators are extremely precise (signal/noise=constant)
  - χ_{PT} to known to high orders
  - Finite volume effects are well understood
- Disadvantages
  - Indirect connection to experiment via V_{ud}, V_{us}
  - Conceptually difficult beyond QCD
  - Not a spectral quantity $\to$ needs renormalization and improvement

Iteration

We would like to

- Compute for each β: $af_{\pi K}\Big|_{\text{phys. mass}}$
- Divide by $f_{\pi K}^{\text{iso-QCD}}$ to obtain a in fm

In practice: The physical mass point is only known after scale setting.

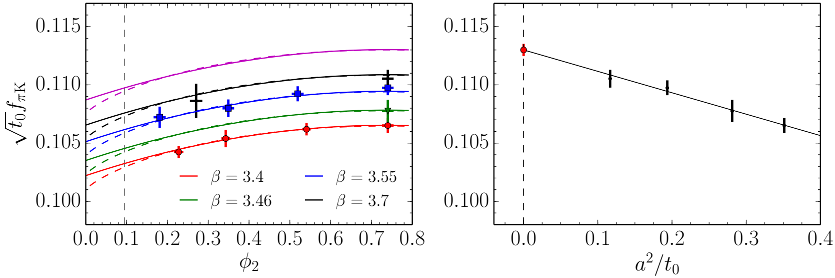
Iterate

- Take best value of $\sqrt{t_0}$ [fm]
- Express masses in t_0 units
  - $\phi_2 = 8t_0 m_\pi^2$, proxy for light quark mass
  - $\phi_4 = 8t_0 \left(m_K^2 + \frac{1}{2}m_\pi^2\right)$, proxy for $\mathrm{tr}[M^R]$
- Perform simulations on chiral trajectories that go through $\phi_2^{\text{iso-QCD}}$, $\phi_4^{\text{iso-QCD}}$
- On this set of ensembles, compute $\sqrt{t_0} f_{\pi K}$
- Continuum / chiral extrapolate to $\phi_2^{\text{iso-QCD}}$, $\phi_4^{\text{iso-QCD}}$
- Divide by $f_{\pi K}^{\text{iso-QCD}}$ to obtain $\sqrt{t_0}$ [fm]

Quark Mass Derivatives

Instead of re-doing all simulations several times, we correct existing ones

- For every relevant observable $\mathcal{O}$, compute derivatives with respect to the bare quark masses
  $\frac{d\langle\mathcal{O}\rangle}{d\,am_u}$, $\frac{d\langle\mathcal{O}\rangle}{d\,am_d}$, $\frac{d\langle\mathcal{O}\rangle}{d\,am_s}$
- Use these to shift to slightly different chiral trajectories
  $\langle\mathcal{O}\rangle \to \langle\mathcal{O}\rangle + \sum_f \Delta_f \frac{d\langle\mathcal{O}\rangle}{d\,am_f}$
- We neglect $O(\Delta^2) \to$ shifts must be small



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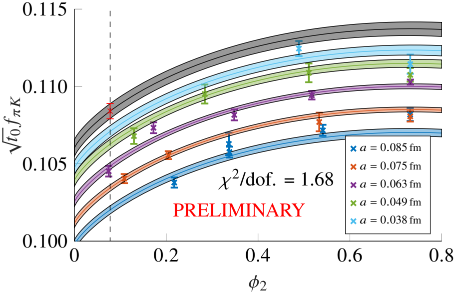


Results

$$\sqrt{t_0} = 0.1443(7)(13)\ \mathrm{fm}\,, \qquad \text{physical mass}$$
$$\sqrt{t_0^\star} = 0.1439(7)(13)\ \mathrm{fm}\,, \qquad m_u = m_d = m_s, \phi_4 = 1.11$$
$$r_0 = 0.4729(74)\ \mathrm{fm}\,, \qquad \leftarrow \text{Tom Asmussen}$$
$$r_1 = 0.3128(40)\ \mathrm{fm}$$

Lattice spacings with CLS action

β	a [fm]
3.40	0.0849(5)(8)
3.46	0.0749(4)(7)
3.55	0.0633(4)(6)
3.70	0.0491(3)(4)
3.85	0.0385(2)(3)

Details and Sources of Errors

- Systematic Errors
- Statistical Errors

Errors from Inputs

Experimentally one finds

- $m_\pi^\pm = 139.5704$ MeV, $m_\pi^0 = 134.9768$ MeV $\rightarrow$ 3.3%
- $m_K^\pm = 493.68$ MeV, $m_K^0 = 497.61$ MeV $\rightarrow$ 0.8%
- $(f_{\pi^+} - f_{\pi^0})/f_{\pi^+} \approx 0.3\%$

Spread too big $\rightarrow$ needs correction $\rightarrow$ we use as inputs

"Corrected" iso-QCD values

- $m_\pi^{\text{isoQCD}} = 134.9768(5)$ MeV
- $m_K^{\text{isoQCD}} = 497.611(13)$ MeV

  [PDG]
- $f_\pi^{\text{isoQCD}} = 130.56(02)(13)(02)$ MeV
- $f_K^{\text{isoQCD}} = 157(2)(2)(4)$ MeV

  [FLAG 2021]

This is precise enough for us. More care maybe needed in the future
$\rightarrow$ Alberto Ramos, Andreas Risch

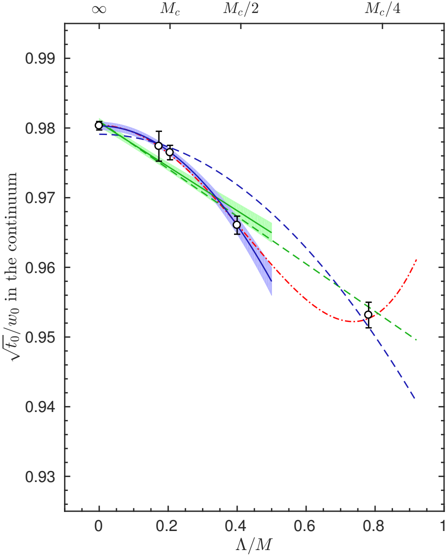
Decoupling

Can these values be used also for $N_f = 2 + 1$ QCD?
How much are dimensionless ratios like $\sqrt{t_0} f_\pi$ affected?

[F. Knechtli, T. K., B. Leder, G. Moir, Phys.Lett.B 774 (2017)]

- $N_f = 0$ vs $N_f = 2$
- The two quarks are heavy M_c, $M_c/2$, ...
- Look at dimensionless ratios (that exist in $N_f = 0$)

$\Rightarrow$ Power corrections $\approx 0.2\%$ (for one charm quark)



Other Parameters

Our calculation also relies on various renormalization and improvement factors

- $c_{sw} = \frac{1-0.1921 g_0^2 - 0.1378 g_0^4 + 0.0717 g_0^6}{1-0.3881 g_0^2}$ (non-perturbative)
  $O(a)$ improvement of the (bulk) Wilson action
  [J. Bulava, S. Schaefer, Nucl.Phys.B 874 (2013)]
- Z_A (non-perturbative) known to $\approx 0.04\%$ accuracy
  Axial current renormalization factors
  [M. Dalla Brida, T.K., S. Sint, P. Vilaseca, Eur.Phys.J.C 79 (2019)]
- $c_A = -0.006033\, g_0^2 \times \left[1 + \exp\left(9.2056 - \frac{13.9847}{g_0^2}\right)\right]$ (non-perturbative)
  Axial current improvement coefficient
  [J. Bulava, M. Della Morte, J. Heitger, C. Wittemeier, Nucl.Phys.B 896 (2015) 555-568]
- $b_A, b_P, \ldots$ (perturbative)
  Improvement coefficients that come with mass factors am_q
  Error=Difference between last known orders
  [Y. Taniguchi, A. Ukawa, Phys.Rev. D58 (1998)]

Systematic Monte-Carlo Errors - Distribution

Often it is not possible to sample exactly from a desired PDF $\propto e^{-S_A}$. Instead a similar PDF $\propto e^{-S_B}$ is used.
We correct for such deviations by reweighting

$$\langle \mathcal{O} \rangle_A = \frac{\langle \mathcal{O}\ W \rangle_B}{\langle W \rangle_B}$$
$$W = e^{S_B - S_A}$$

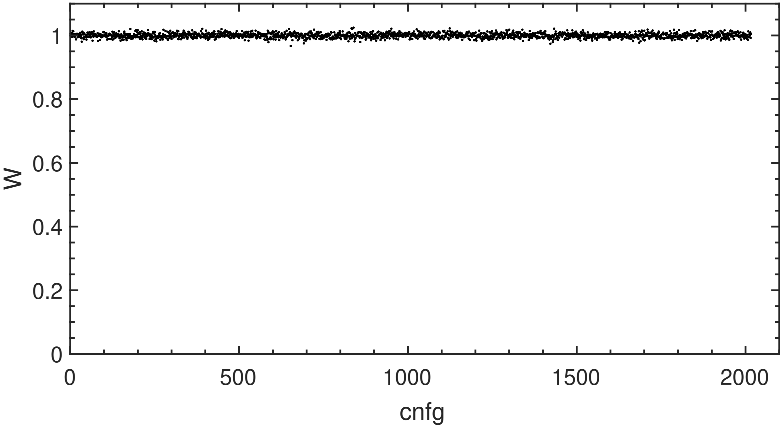
- Protect simulations from exceptionally small eigenvalues of $\hat{D}$
  $\det[\hat{D}^\dagger \hat{D}] \rightarrow \det[\hat{D}^\dagger \hat{D} + \mu^2]^2 / \det[\hat{D}^\dagger \hat{D} + 2\mu^2]$
  (twisted mass reweighting 2. kind)
  [M. Lüscher, F. Palombi, PoS (LATTICE2008) 049]
- $\det[\hat{D}] = \pm \det[\sqrt{\hat{D}^\dagger \hat{D}}] \rightarrow \pm \det[R(\hat{D}^\dagger \hat{D})]$
  - One reweighting factor for rational $R(x) \neq \sqrt{x}$
  - One reweighting factor for $\pm$

  [D. Mohler, S. Schaefer, Phys.Rev.D 102 (2020)]

$\rightarrow$ No systematic error for us

Typical Reweighting Factor

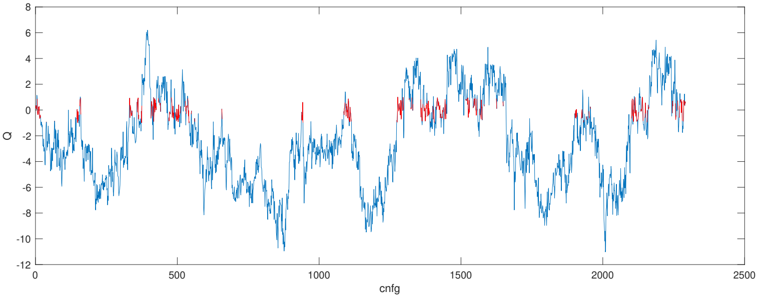
Overall reweighting factor on H100



Effect, e.g. $t_0/a^2 = 2.8468(60) \rightarrow 2.8469(60)$

Systematic Monte-Carlo Errors - Topology

Even at our smallest lattice spacings, the topology moves



J501: $a \approx 0.038$ fm, Q from GF fields at $t \approx t_0$ in the bulk

- Topology sampling is not more problematic than other quantities
- Ordinary critical slowing down $\propto a^{-2}$ is bad enough
- What would happen if we were stuck in a "sector" with $|Q| < 1$?

$$\text{All sectors:} \quad t_0/a^2 = 14.01(7) \quad w_0/a = 4.46(2)$$
$$\text{Stuck:} \quad t_0/a^2 = 14.06 \quad w_0/a = 4.46$$

$\rightarrow$ Effect not very clear, needs a dedicated study

Systematic Monte-Carlo Errors - Plateaus

- Hadron correlators usually suffer from a signal/noise problem
- But not Pions nor Kaons!
  $\rightarrow$ We observe long and precise plateaus even without quark smearing
- Fixed source time-slice (at $x_0 = a$) + correlations
  $\rightarrow$ “waves” in effective masses

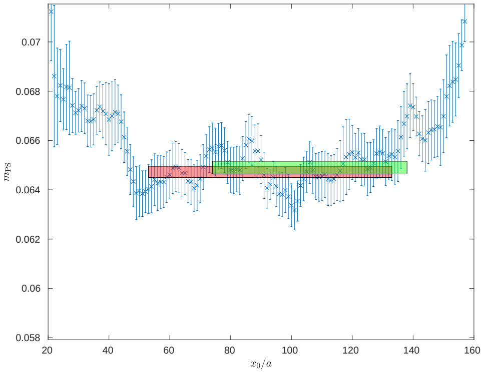
$$f_P^{r,s}(x_0, y_0) = -\frac{a^6}{L^3} \sum_{\mathbf{x},\mathbf{y}} \langle P^{r,s}(x) P^{s,r}(y) \rangle$$

$$f_A^{r,s}(x_0, y_0) = -\frac{a^6}{L^3} \sum_{\mathbf{x},\mathbf{y}} \langle A_0^{r,s}(x) P^{s,r}(y) \rangle$$

$$m_{\mathrm{PS}} = \log\left[f_P(a, y_0)/f_P(a, y_0 + a)\right]$$

with local

- Pseudo-scalar density P
- Axial current A_μ



typical case (here J303)

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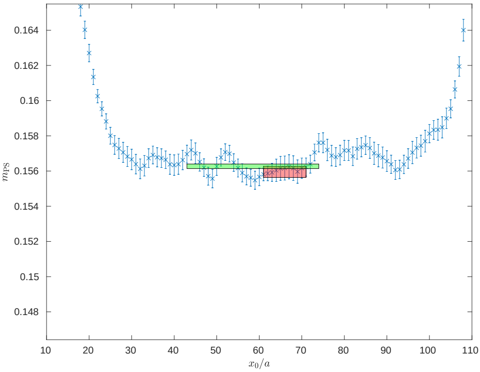


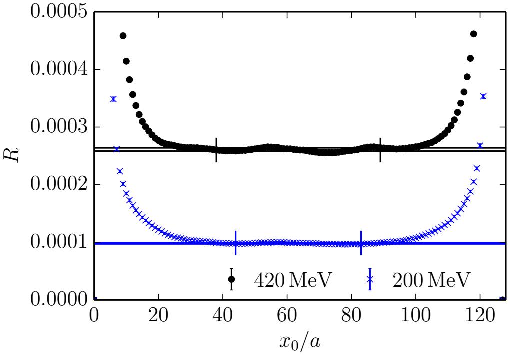
difficult case (here D200)

Systematic Monte-Carlo Errors - Plateaus

Matrix elements are isolated from ratios

$$R(x_0, y_0) = \sqrt{\frac{f_A^{\mathrm{imp}}(x_0, y_0) f_A^{\mathrm{imp}}(x_0, T - y_0)}{f_P(T - y_0, y_0)}}$$

$$f_{PS} = Z_A[1 + \bar{b}_A \; a\,\mathrm{tr} M_q + \tilde{b}_A \; a m_{\mathrm{PCAC}}] \sqrt{2/m_{PS}} \; R^{\mathrm{aver}}$$



Systematic Monte-Carlo Errors - Mass Shifts

We deal with reweighted "primary" observables

$$o = \langle \mathcal{O} \rangle_{QCD} = \frac{\langle \mathcal{O}\, W \rangle}{\langle W \rangle} \, .$$

As well as with "derived" observables

$$f(\langle \mathcal{O}_1 \rangle_{QCD}, \ldots, \langle \mathcal{O}_n \rangle_{QCD}, m)$$

The bare quark mass derivative of the latter is given by

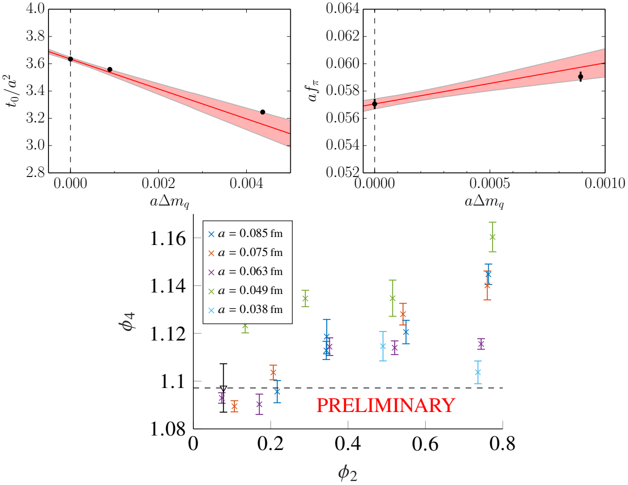
$$\begin{aligned} \frac{df}{dm} &= \frac{\partial f}{\partial o_i} \left\langle \frac{\partial \mathcal{O}_i}{\partial m} \right\rangle_{QCD} - \frac{\partial f}{\partial o_i} \left(\left\langle \mathcal{O}_i \frac{\partial S}{\partial m} \right\rangle_{QCD} - o_i \left\langle \frac{\partial S}{\partial m} \right\rangle_{QCD} \right) + \frac{\partial f}{\partial m} \\ &= \frac{\partial f}{\partial o_i} \frac{\left\langle \frac{\partial \mathcal{O}_i}{\partial m} W \right\rangle}{\langle W \rangle} - \frac{\partial f}{\partial o_i} \left(\frac{\left\langle \mathcal{O}_i \frac{\partial S}{\partial m} W \right\rangle}{\langle W \rangle} - \frac{\langle \mathcal{O}_i W \rangle \left\langle \frac{\partial S}{\partial m} W \right\rangle}{\langle W \rangle^2} \right) + \frac{\partial f}{\partial m} \, . \end{aligned}$$

Systematic Monte-Carlo Errors - Mass Shifts

- All observables will need: sea quark mass dependence
  $\frac{\partial S}{\partial m} \to \mathrm{tr}[D^{-1}] \to$ stochastic estimation, 1 solve per noise
- Fermionic observables also: valence quark mass dependence
  E.g. pion correlator ($\bar{u}\gamma_5 d$):
  $\mathcal{O} = -\sum_{\mathbf{x},\mathbf{y}} \mathrm{tr}[\gamma_5 D^{-1}(x,y)\gamma_5 D^{-1}(y,x)]$
  estimated stochastically
  $\mathcal{O} = -\sum_{\mathbf{y}} \langle \xi^\dagger(y)\xi(y)\rangle_{\text{noise}}$
  with $\xi = D^{-1}\eta$ and $\eta =$ vector with noise on x_0.
  (1 solve per noise)
  Each mass derivative needs an additional inversion
  $\frac{\partial \mathcal{O}}{\partial m_u} = -\sum_{\mathbf{y}} \langle \xi^\dagger(y) \frac{\partial \xi(y)}{\partial m_u} \rangle_{\text{noise}}$
  $\frac{\partial \xi}{\partial m_u} = -D^{-2}\eta$

Systematic Monte-Carlo Errors - Mass Shifts

Two $m_{u,d} = m_s$ ensembles with 7% difference in ϕ_4

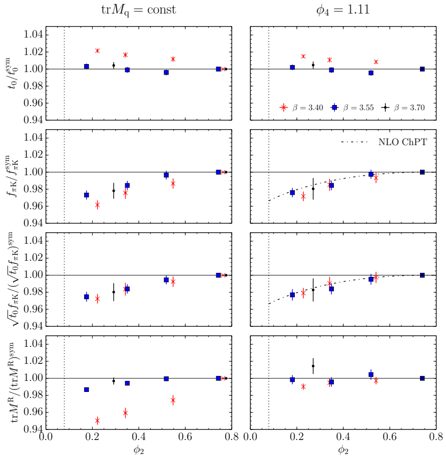


Largest shift: 5%, majority $\approx$ 2% $\rightarrow$ We assume that $O(\Delta^2)$ is negligible

Systematic Monte-Carlo Errors - Mass Shifts

Since we are already changing the chiral trajectories: one more change:

- Instead of trajectories with tr$[M_q]$ = const.
- Create trajectories with ϕ_4 = const. $\sim$ tr$[M^R]$



Systematic Monte Carlo Errors - Finite Volume

Our volumes are finite, but large

- Smallest $m_\pi L \approx 3.8$ on D452
  (only case below 4)
- Largest $m_\pi L \approx 6.4$ on N202
- Smallest $L \approx 2.35$ fm

Finite volume effects in $f_{\pi,K}$ and $m_{\pi,K}$ have been worked out in χ_{PT}

[G. Colangelo, S. Dürr, Nucl. Phys. B 721 (2005)]

We apply these corrections

- Average correction: 0.1%
- Biggest correction: 0.6%

Uncanceled finite size effects are assumed to be negligible

Statistical Monte-Carlo Errors

- To calculate the statistical errors of a MC estimate

$$\sigma_f = \sqrt{\frac{\mathrm{var}(f)}{N/(2\tau_{\mathrm{int},f})}}$$

We use the Γ-method, i.e. explicitly compute $\tau_{\mathrm{int},f}$ from the data

[U. Wolff, Comput.Phys.Commun. 156 (2004)]

- We use rough estimates of τ_{exp} to attach exponential tails to Γ

[S. Schaefer, R. Sommer, F. Virotta, Nucl.Phys.B 845 (2011)]

- Projected Observables

[A. Ramos, Comput.Phys.Commun. 238 (2019)]

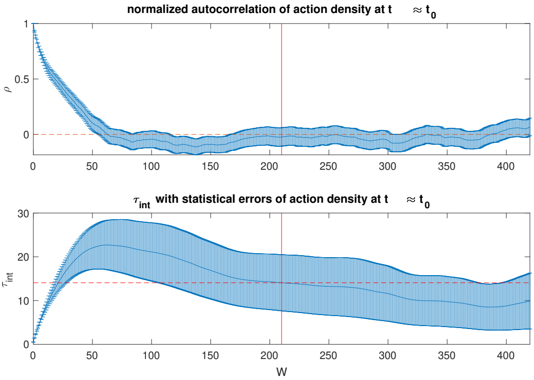
For "derived" observables $f(\langle a_1\rangle, \ldots, \langle a_n\rangle)$, store a projected time series

$$f^i_{\mathrm{proj}} = f(\langle a_1\rangle, \ldots, \langle a_n\rangle) + \sum_\alpha \frac{\partial f}{\partial\langle a_\alpha\rangle}(a^i_\alpha - \langle a_\alpha\rangle), \qquad i = 1, \ldots, N$$

Analyzed like a "primary" observable, this yields the correct error

The Γ-Method

Example: action density at flow time $t \approx t_0$ on $N = 4032$ measurements on H101

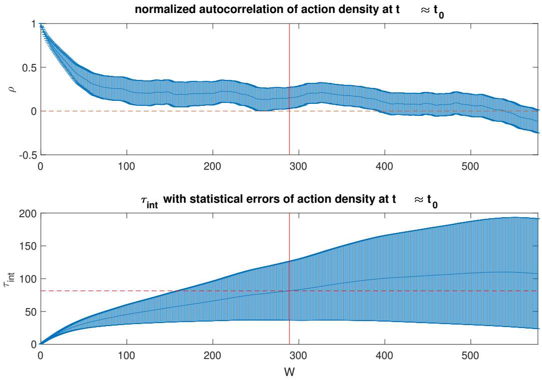


$\rightarrow \tau_{\text{int}} = 14 \pm 6$ measurements

$N/(2\tau_{\text{int}}) \approx 287$

The Γ-Method

Example: action density at flow time $t \approx t_0$ on $N = 3804$ measurements of N202



$\rightarrow \tau_{\text{int}} = 70 \pm 28$ measurements

$N/(2\tau_{\text{int}}) \approx 54$

How Much Statistics is Necessary?

Experiment: let us use different fractions of H101 statistics

N	$N/14$	τ_{int}
4032	287	14 ± 6
1400	100	22 ± 8
700	50	9 ± 6
350	25	6 ± 2

$\rightarrow$ statistics of at least $N/\tau_{\text{exp}} > 50$ is necessary

Using full statistics and attaching a tail with $\tau_{\text{exp}} = 13.3$, we obtain

$$\tau_{\text{int}} = 24 \pm 6$$

Chiral / Continuum Extrapolations

Lattice results $\sqrt{t_0} f_{\pi K}$ vs ϕ_2 at $\phi_4 = \text{const.}$ is fitted globally:

- Continuum part of the fit
  - 1. Order Taylor in $\phi_2 - \phi_2^{\text{sym}}$
  - 2. Order Taylor in $\phi_2 - \phi_2^{\text{sym}}$
  - $SU(3)$ χPT
  - $SU(2)$ χPT for f_π

    only in [B.Straßberger PhD thesis]
- Lattice artifacts
  - Factor $(1 + c_1 \frac{a^2}{t_0^{\text{sym}}})$
  - As above $+c_2\, a^2 m_\pi^2$
- Cuts
  - neglect coarse ensembles
  - neglect heavy ensembles
  - both

Variation = leading source of (systematic) error!

type	cut	χ^2/dof.	$\sqrt{t_0} f_\pi$	$\sqrt{t_0} f_{\pi K}$
Taylor	-	2.05		0.1083(3)
Taylor	$\beta > 3.4$	2.07		0.1088(4)
Taylor	$\beta > 3.5$	2.68		0.1084(5)
Taylor	$\phi_2 < 0.6$	1.77		0.1086(3)
Taylor	$\phi_2 < 0.4$	2.16		0.1086(4)
Taylor	$\beta > 3.4, \phi_2 < 0.6$	2.03		0.1090(5)
Taylor(4)	-	1.98		0.1081(3)
Taylor(4)	$\beta > 3.4$	1.69		0.1083(4)
Taylor(4)	$\beta > 3.5$	2.26		0.1078(5)
Taylor(4)	$\phi_2 < 0.6$	1.64		0.1084(4)
Taylor(4)	$\phi_2 < 0.4$	2.03		0.1083(4)
Taylor(4)	$\beta > 3.4, \phi_2 < 0.6$	1.43		0.1086(5)
SU(3) χPT	-	1.84		0.1081(3)
SU(3) χPT	$\beta > 3.4$	1.63		0.1085(4)
SU(3) χPT	$\beta > 3.5$	2.09		0.1081(5)
SU(3) χPT	$\phi_2 < 0.6$	1.50		0.1084(3)
SU(3) χPT	$\phi_2 < 0.4$	1.86		0.1084(4)
SU(3) χPT	$\beta > 3.4, \phi_2 < 0.6$	1.48		0.1088(5)
SU(3) χPT + $a^2 m_\pi^2$	-	1.82		0.1085(4)
SU(3) χPT + $a^2 m_\pi^2$	$\beta > 3.4$	1.77		0.1084(6)
SU(3) χPT + $a^2 m_\pi^2$	$\beta > 3.5$	2.39		0.1080(8)
SU(3) χPT + $a^2 m_\pi^2$	$\phi_2 < 0.6$	1.63		0.1085(5)
SU(3) χPT + $a^2 m_\pi^2$	$\phi_2 < 0.4$	2.16		0.1086(7)
SU(3) χPT + $a^2 m_\pi^2$	$\beta > 3.4, \phi_2 < 0.6$	1.18		0.1078(7)
SU(2) χPT	-	1.82	0.0933(4)	0.1074(3)
SU(2) χPT	$\beta > 3.4$	1.58	0.0937(4)	0.1079(4)
SU(2) χPT	$\beta > 3.5$	1.94	0.0933(6)	0.1075(5)
SU(2) χPT	$\phi_2 < 0.6$	1.26	0.0941(4)	0.1078(3)
SU(2) χPT	$\phi_2 < 0.4$	1.30	0.0945(5)	0.1079(4)
SU(2) χPT	$\beta > 3.4, \phi_2 < 0.6$	0.96	0.0947(6)	0.1085(5)

[B.Straßberger PhD thesis]

Lattice Artifacts Beyond $O(a^2)$

- Our fits assume that we see only leading corrections to scaling, i.e. pure $O(a^2)$
- Other cutoff effects that could be expected
  - $O(a^3)$, next order of the Symanzik action
  - $O(a^2 \log(a)^{\Gamma})$, taking the leading order more seriously.
    There are $O(10)$ such terms with largely known Γ
    [N. Husung, P. Marquard, R. Sommer, Phys.Lett.B 829 (2022)]
  - $O(a)$ from uncanceled am_q
- With “only” 5 lattice spacings it is difficult to fit more than 1-2 lattice-artifact terms.
  Practical approach: use only pure $O(a^2)$, but study cuts and χ^2 values.

Extrapolations are a major source of uncertainty

Conclusions

Conclusions

- We obtained $\sqrt{t_0}$ to $\approx 1\%$ accuracy
- At this level of precision the errors are dominated by
  - Statistical error
  - Systematic error due to extrapolations
- Negligible sources of errors
  - External parameters
  - Charm quark effects
  - Mass shifts beyond linear approximation
  - Finite Volume

Wish-list / Possible Improvements

- Longer MC chains
- $m_s = \text{const.}$ trajectory
- Improved flow