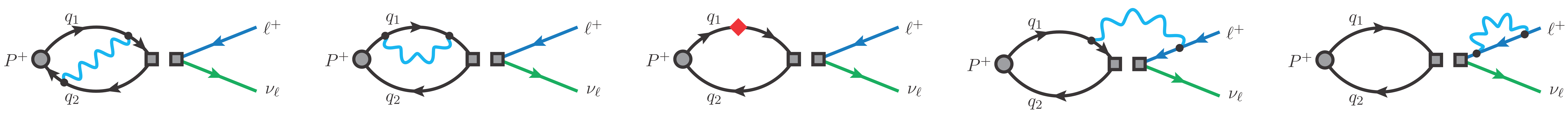


Isospin breaking effects in the leptonic decays of light mesons

Matteo Di Carlo

6th March 2025

Funded by
the European Union

Scale setting: precision lattice QCD for particle and nuclear physics

ECT*, Trento, 3-7 March 2025

Outline of the talk

1. Why are **isospin-breaking** and **QED** corrections **relevant**?

2. How are these effects **included in lattice calculations**?

3. What has been done for **light-meson leptonic decays**?

4. Where do we stand and where do we go?

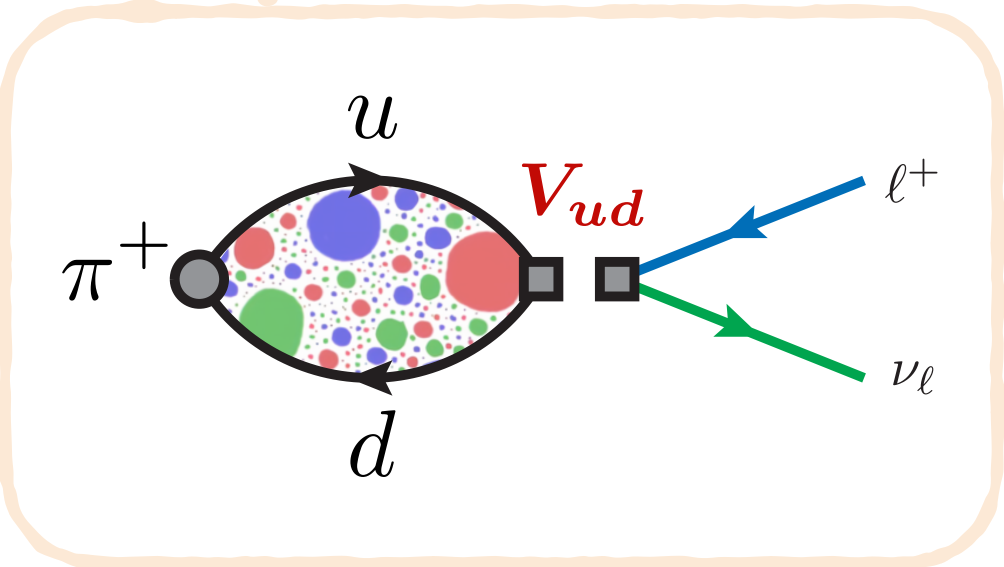
Testing the Standard Model with flavour physics

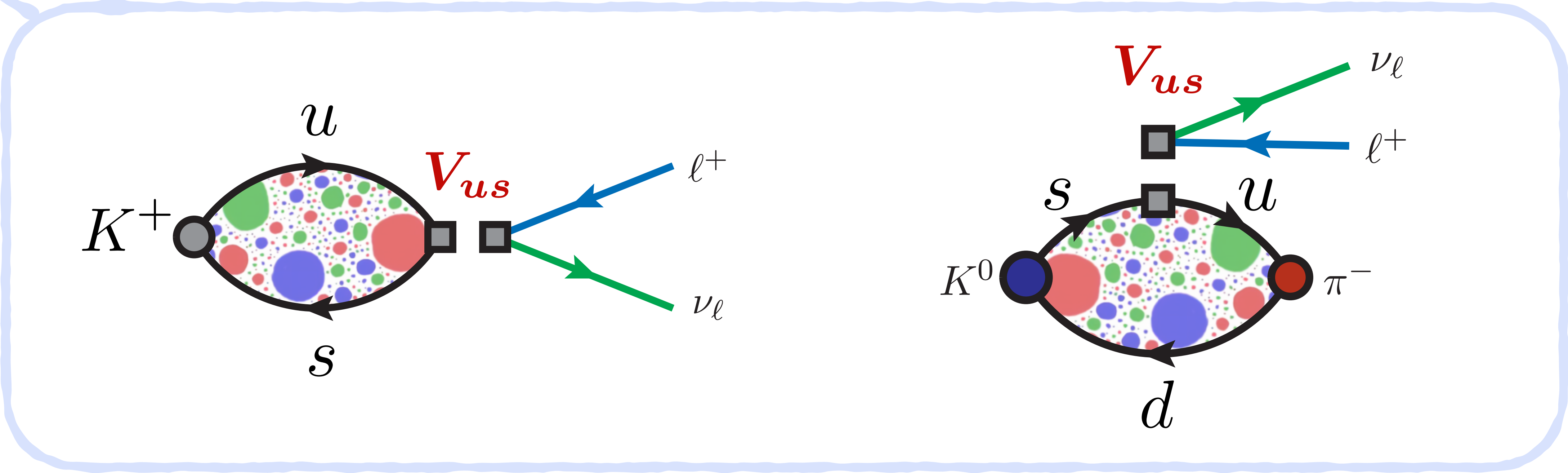
Unitarity of the CKM matrix $\Longleftrightarrow$ test the validity of the Standard Model

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

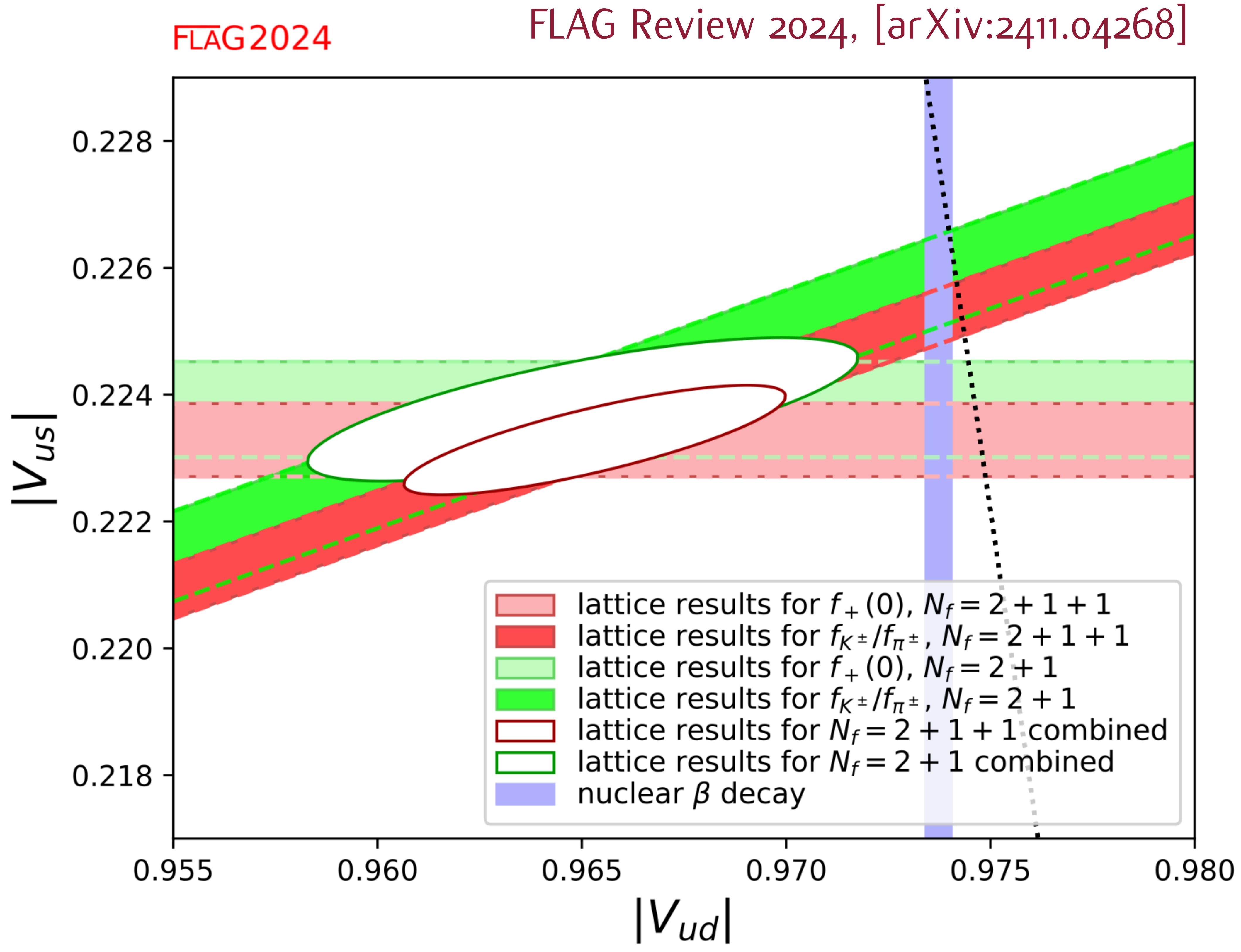
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$





First-row CKM unitarity tests (Cabibbo anomaly)

FLAG Review 2024, [arXiv:2411.04268]



$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(41)$$

$$|V_{us}|\,|f_+^{K^0\pi^-}(0)| = 0.21654(41)$$

M.Moulson, PoS CKM2016 (2017)
PDG, PTET **2022** (2022)

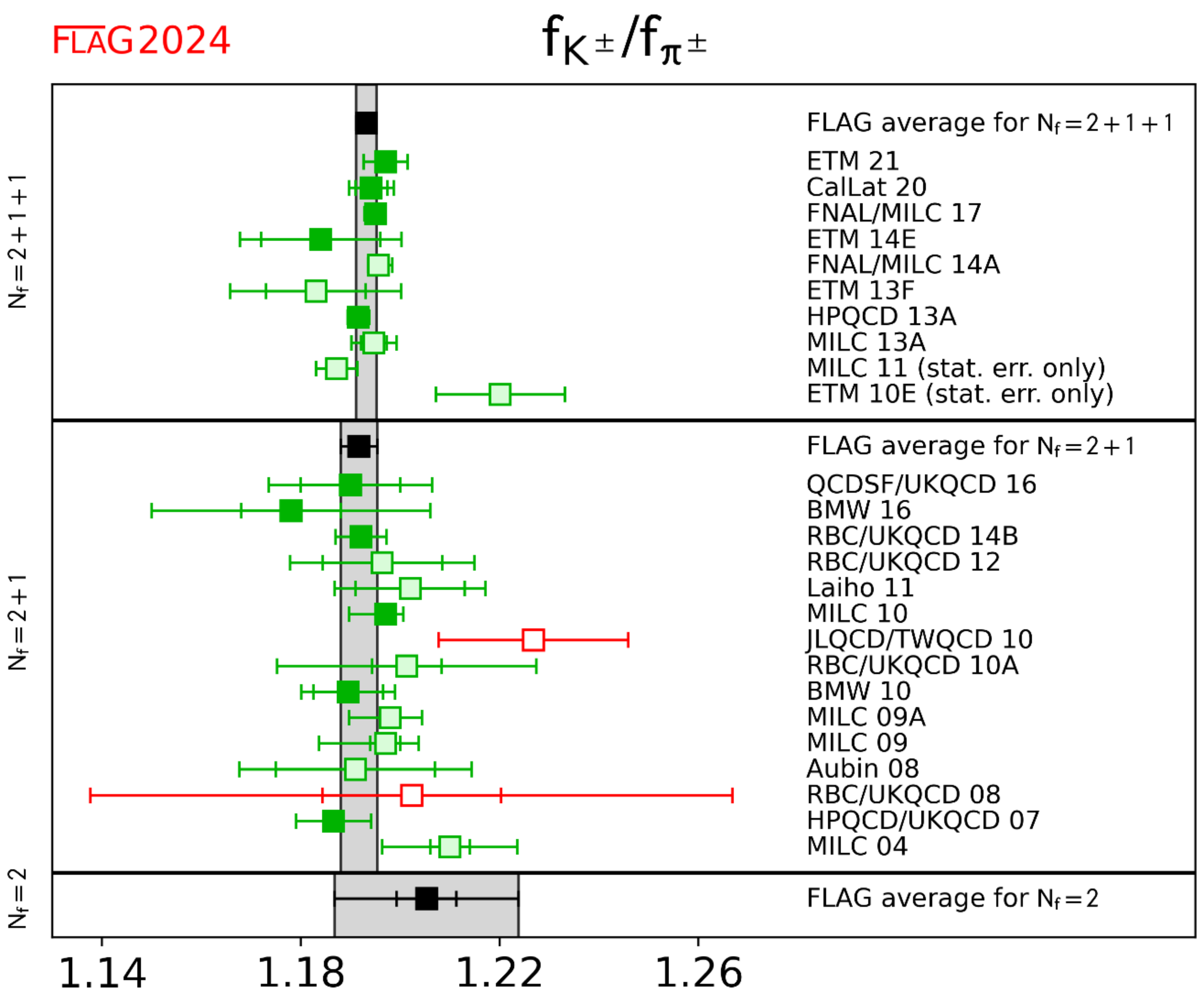
Different **tensions** in the V_{us}-V_{ud} plane:

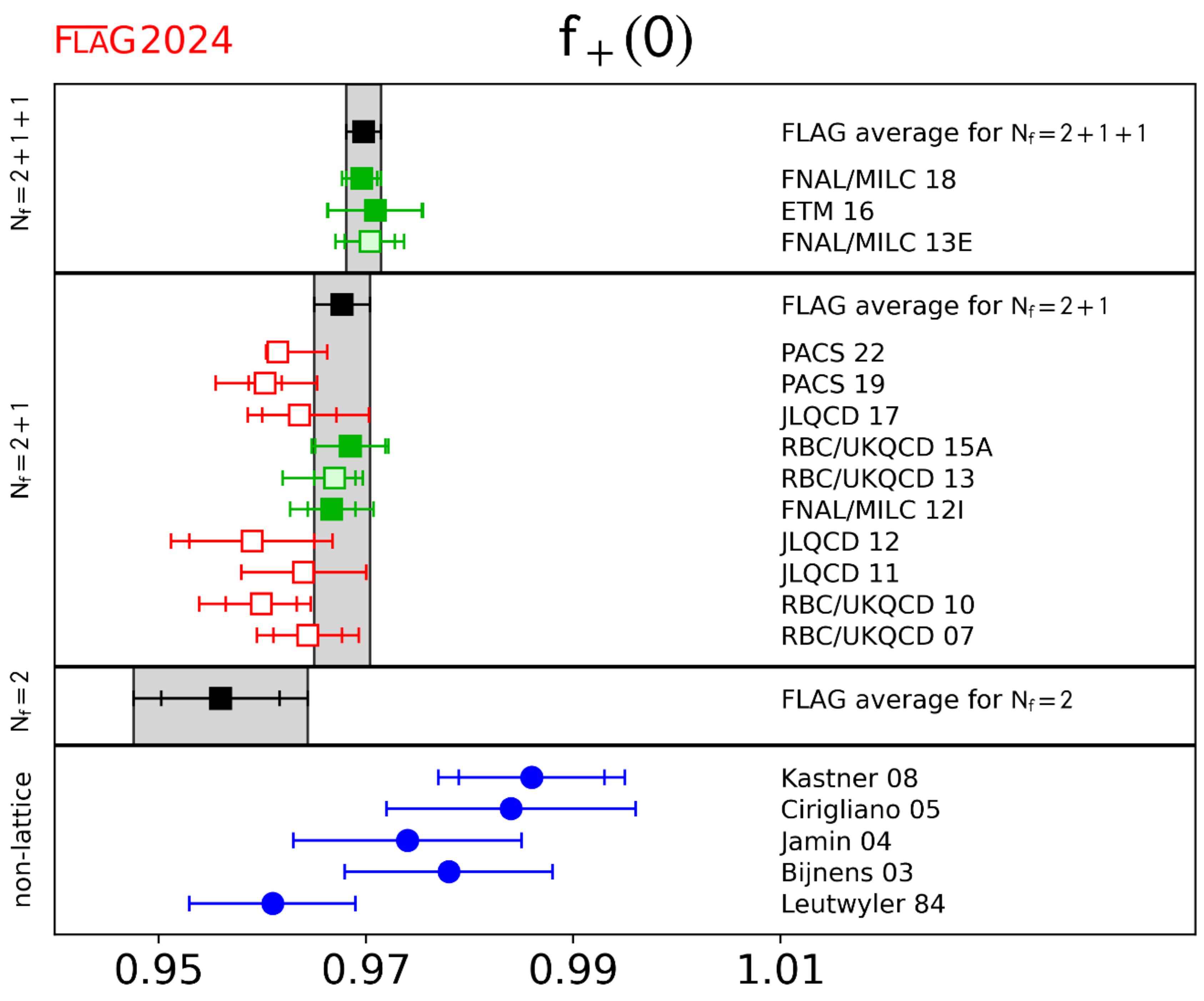
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and **theoretical** control of these quantities is of crucial importance to solve the issue

Lattice QCD inputs





$$f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19)$$

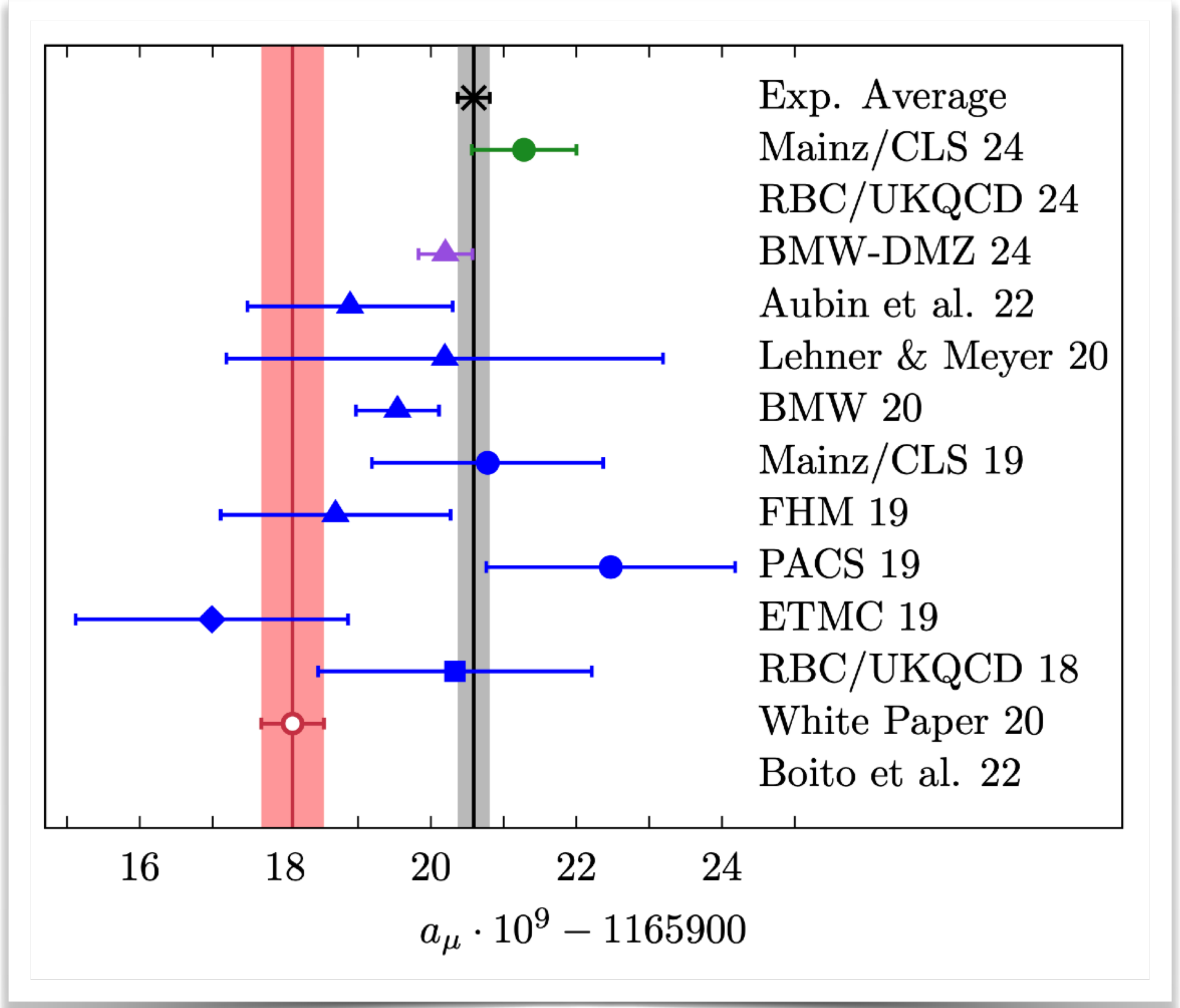
$$f_+^{K\pi}(0) = 0.9698\,(17)$$

FLAG 2024 Flavour Lattice Averaging Group

f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with **sub percent precision!**

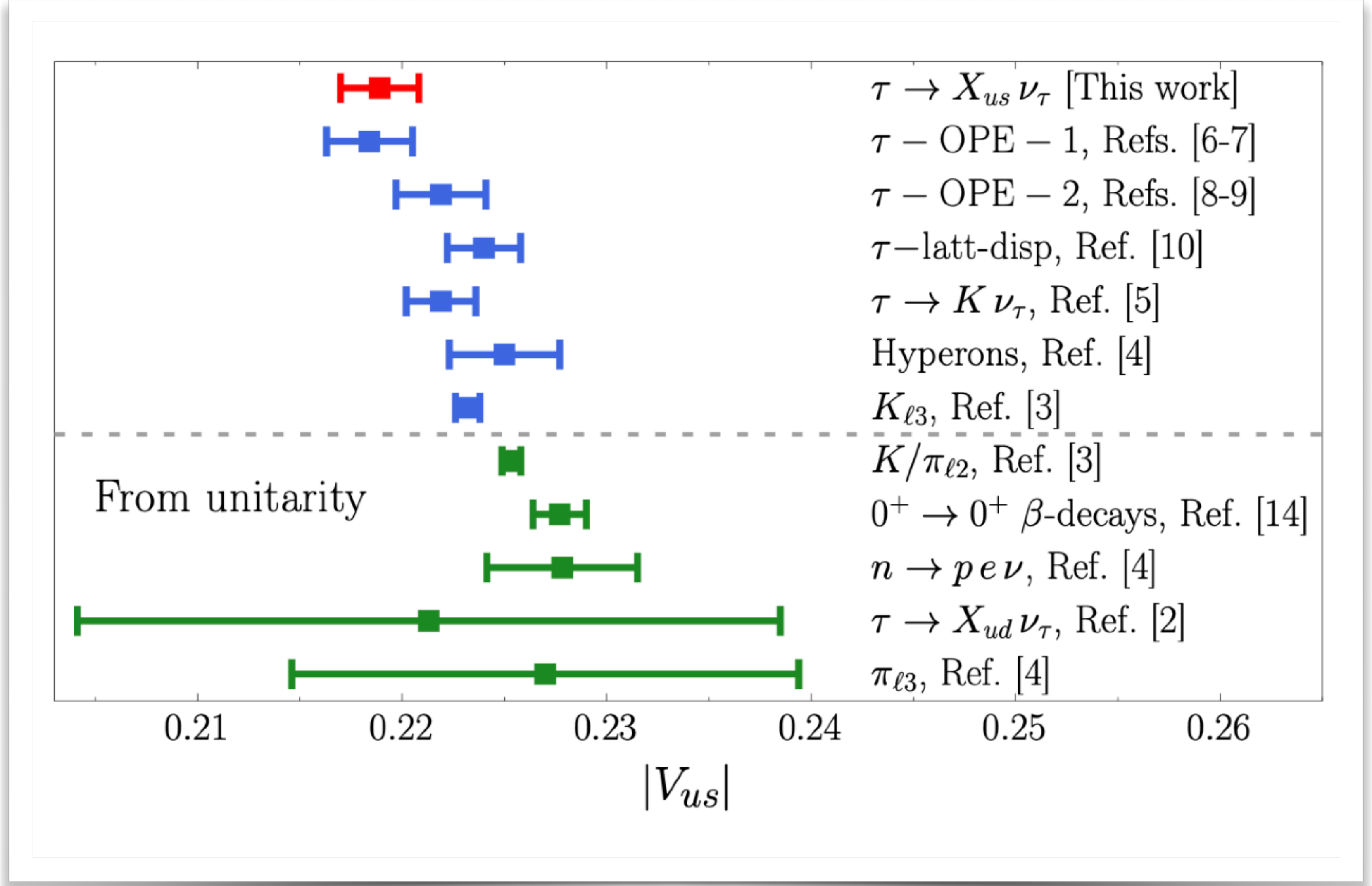
Some other motivations…

D.Djukanovic et al. [Mainz/CLS], 2411.07969



HVP contribution to muon g-2

ETMC, PRL **132** (2024)



Inclusive hadronic decay of τ lepton

Setting the scale

MDC et al., PRD **100** (2019)

Studying **radiative corrections to leptonic decays** is also relevant as **decay rates** can be used to set the scale in lattice calculations:

$$\mathcal{F}_P^2 \equiv \frac{\Gamma(P^\pm \to \ell^\pm \bar{\nu}_\ell[\gamma])}{\frac{G_F^2}{8\pi}|V_{q_1 q_2}|^2 m_\ell^2 \left(1-\frac{m_\ell^2}{M_P^2}\right)^2 M_P} = [f_P^{(0)}]^2 (1+\delta R_P)$$

$$R_1(aN; g_s, e, \mathbf{m}) = \frac{aM_{\pi^0}}{a\mathcal{F}_\pi}(aN; g_s, e, \mathbf{m}),$$

$$R_2(aN; g_s, e, \mathbf{m}) = \frac{aM_{K^0}}{a\mathcal{F}_\pi}(aN; g_s, e, \mathbf{m}),$$

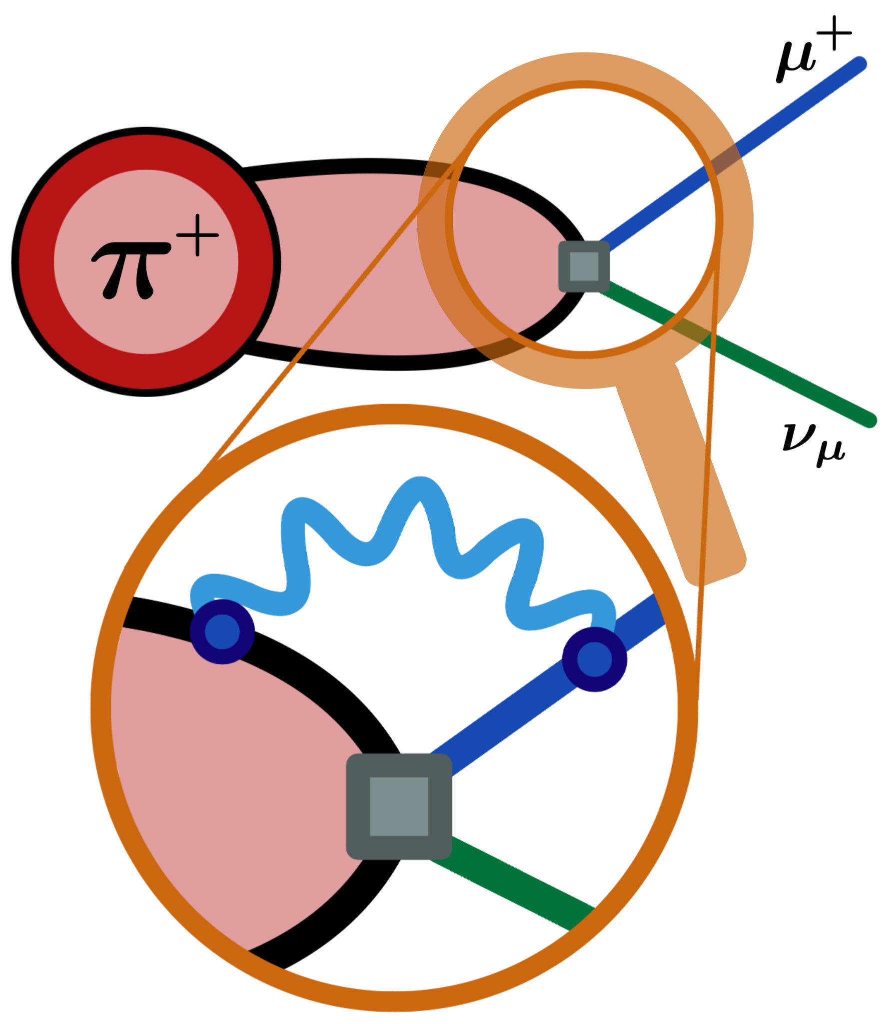
$$R_3(aN; g_s, e, \mathbf{m}) = \frac{aM_{D_s}}{a\mathcal{F}_\pi}(aN; g_s, e, \mathbf{m}),$$

$$R_4(aN; g_s, e, \mathbf{m}) = \frac{aM_{K^+} - aM_{K^0}}{a\mathcal{F}_\pi}(aN; g_s, e, \mathbf{m}).$$

But... is it **convenient**?

- it prevents us from being able to **predict** $|V_{q_1 q_2}|$ (it is a required input)
- it assumes that the decay is free from **new physics** effects

QED and isospin-breaking effects



Current level of precision requires the inclusion of isospin-breaking corrections due to

- strong effects $[m_u - m_d]_{\text{QCD}} \neq 0$
- electromagnetic effects $\alpha \neq 0$

$\sim \mathcal{O}(1\%)$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |V_{us}|^2 \, |f_+^{K\pi}(0)|^2 \, (1 + \delta R^\ell_{K\pi})$$

- results from χPT currently quoted in the PDG

V.Cirigliano & H.Neufeld, PLB **700** (2011)

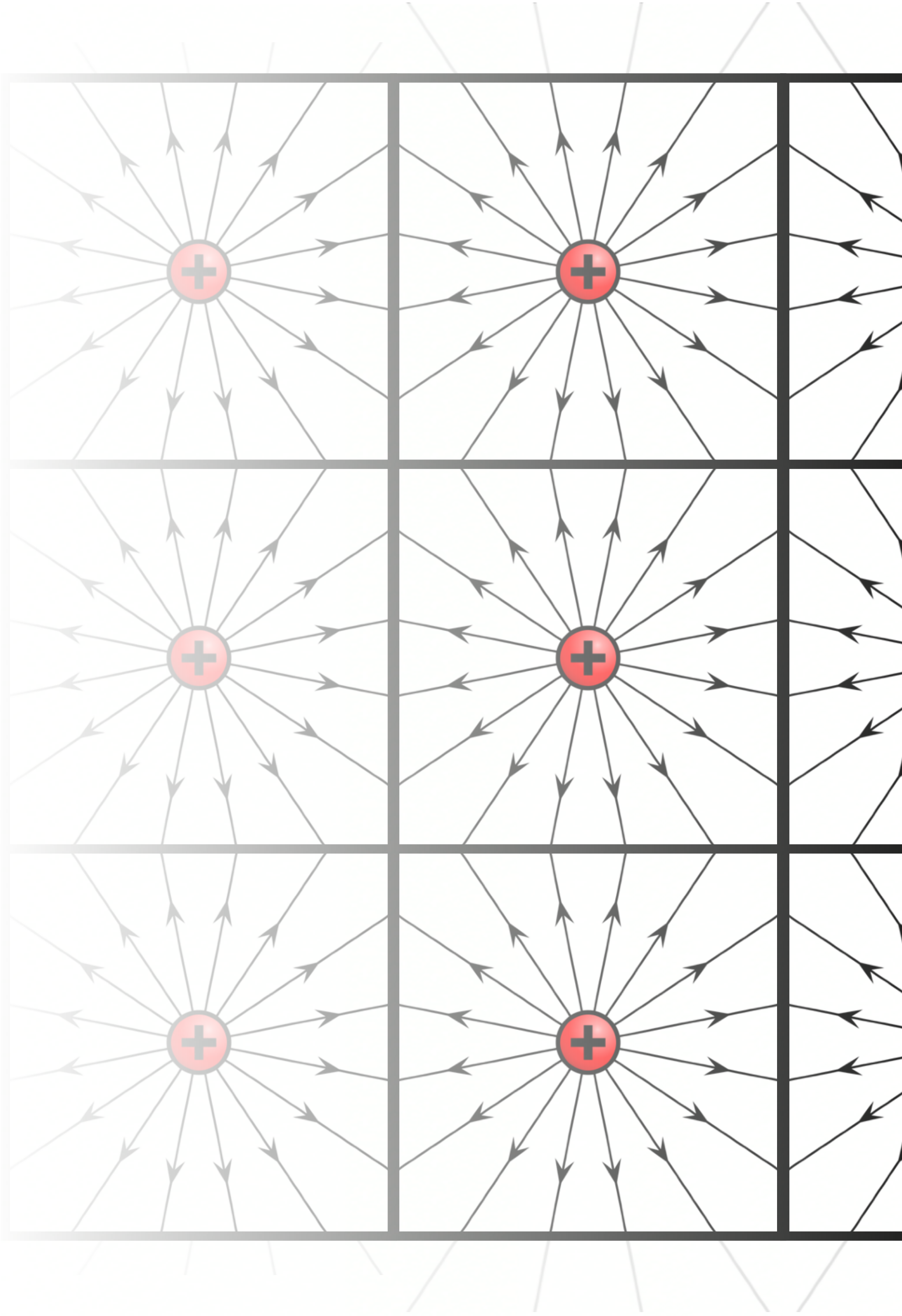
- fully **non-perturbative** (i.e. structure dependent)
- can be obtained through first-principle **lattice calculations**

2. How: Lattice QCD + QED

A conceptual challenge: how to define QED in a finite periodic box?

- need to circumvent **Gauss' law:** no charged states in a periodic box
- **finite-volume effects** can be sizeable and power-like
- logarithmic **infrared divergences** arise when studying decays

Problems well studied. Different lattice QED formulations proposed and used.



RM123 approach:

G.M.de Divitiis et al. [RM123], PRD 87 (2013)

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi\, \mathcal{O}\, e^{-S_{\text{iso}} - \Delta S} = \langle \mathcal{O} \rangle_{\text{iso}} + \langle \Delta S\, \mathcal{O} \rangle_{\text{iso}} + \ldots$$

$$\text{“iso”} = \begin{cases} m_{\mathrm{u}} = m_{\mathrm{d}} \\ \alpha_{\mathrm{em}} = 0 \end{cases}$$

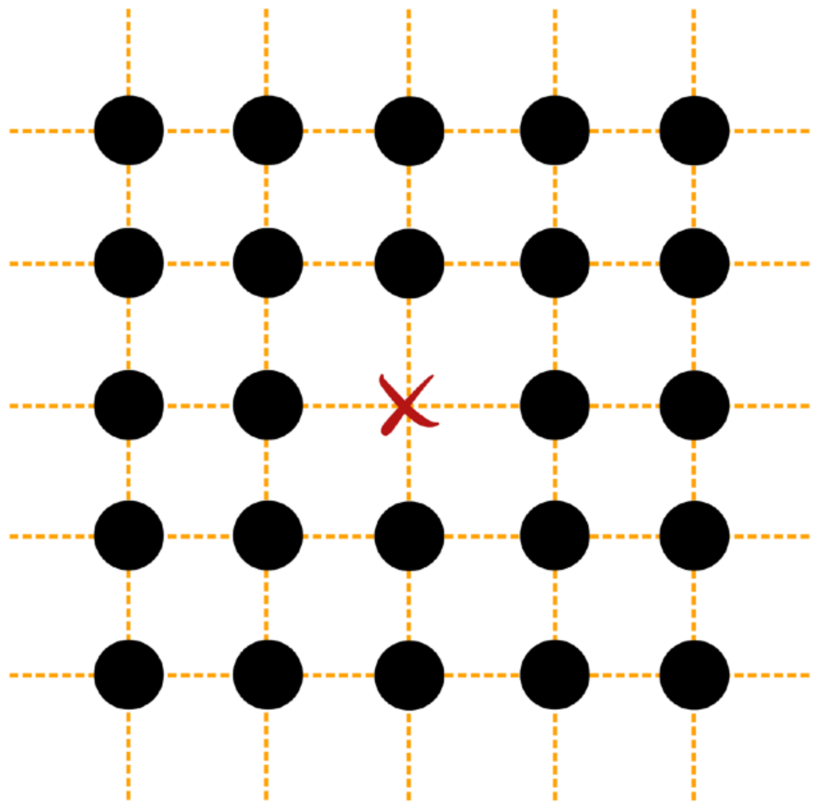
Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\mathrm{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\mathrm{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Possible solutions:

$\mathrm{QED_L}$

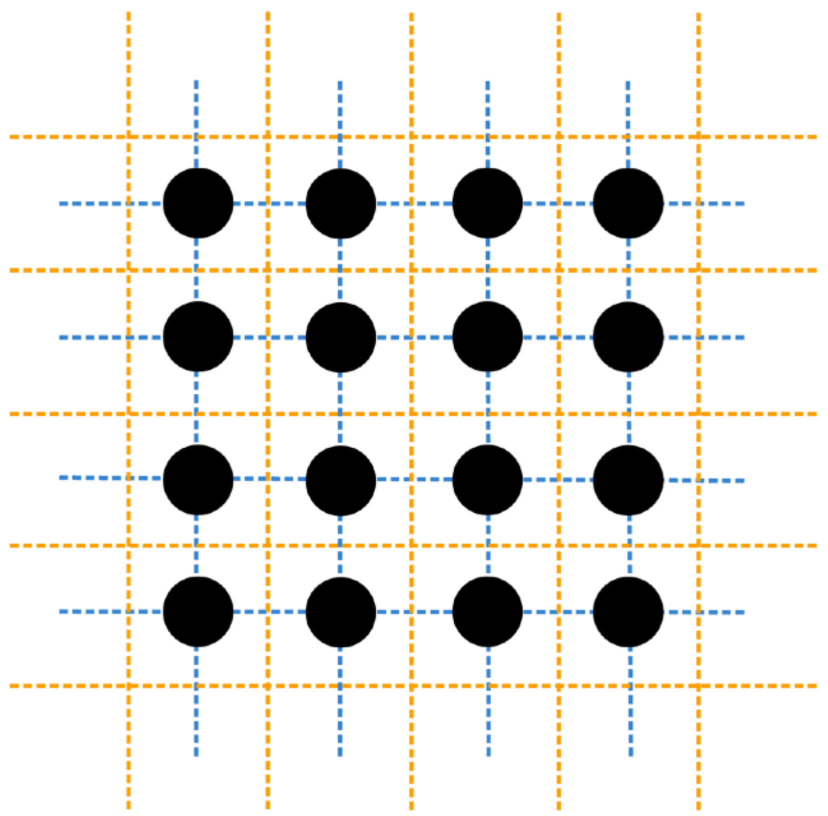


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

$\mathrm{QED_{C^*}}$

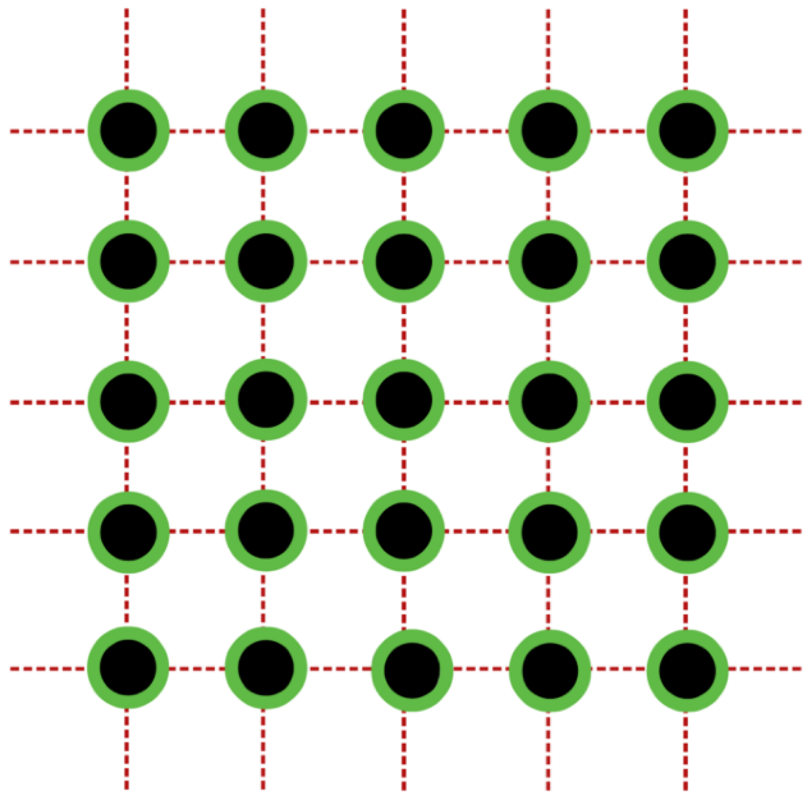


$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB **357** (1991)
B.Lucini et al., JHEP **02** (2016)

$\mathrm{QED_m}$

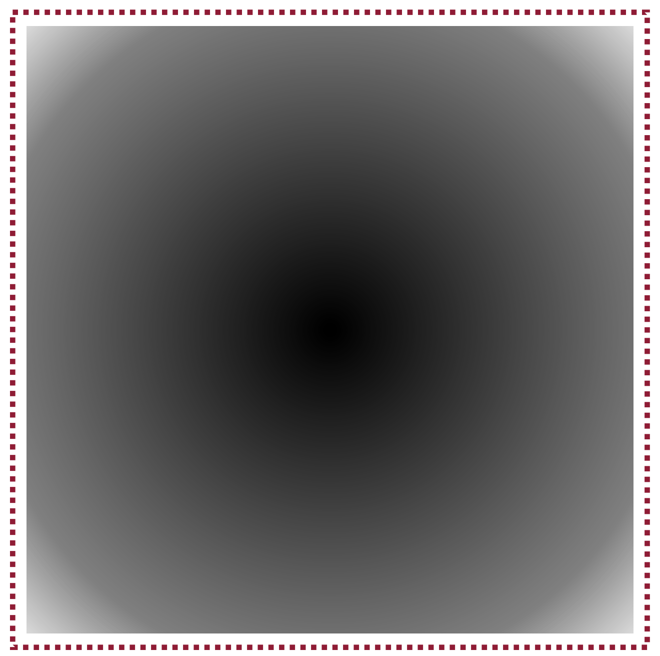


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

$\mathrm{QED_\infty}$



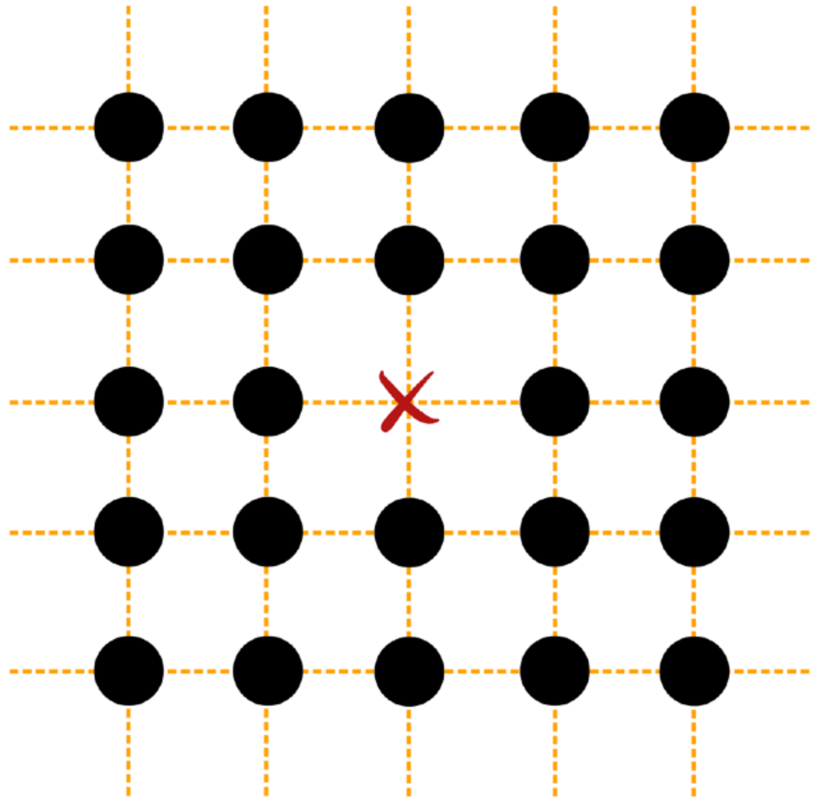
$\Omega_4 = \mathbb{R}^4$

infinite-volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

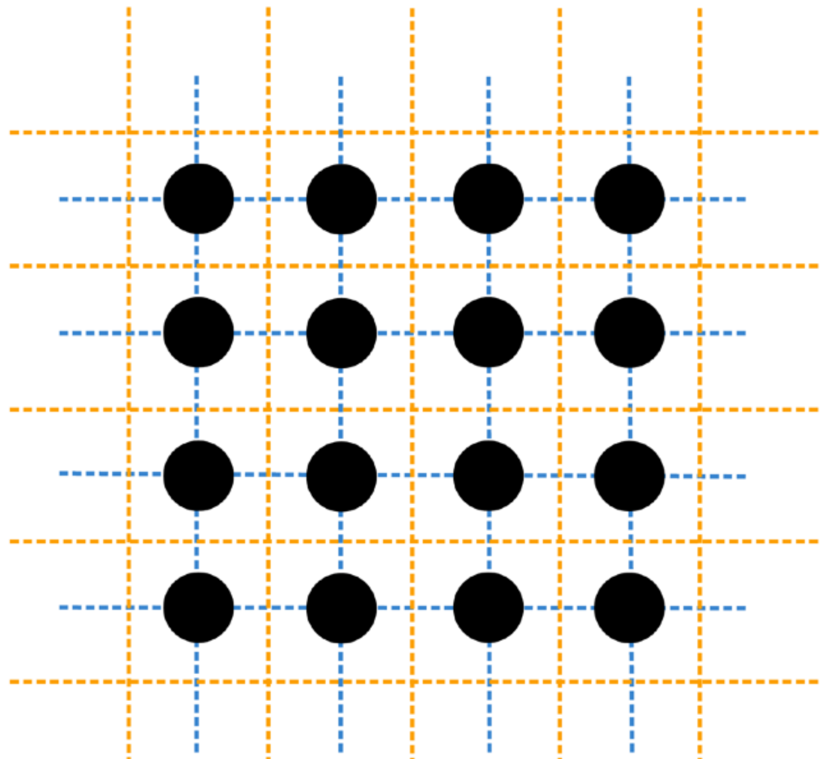
Charged states in a finite box

$\mathrm{QED_L}$



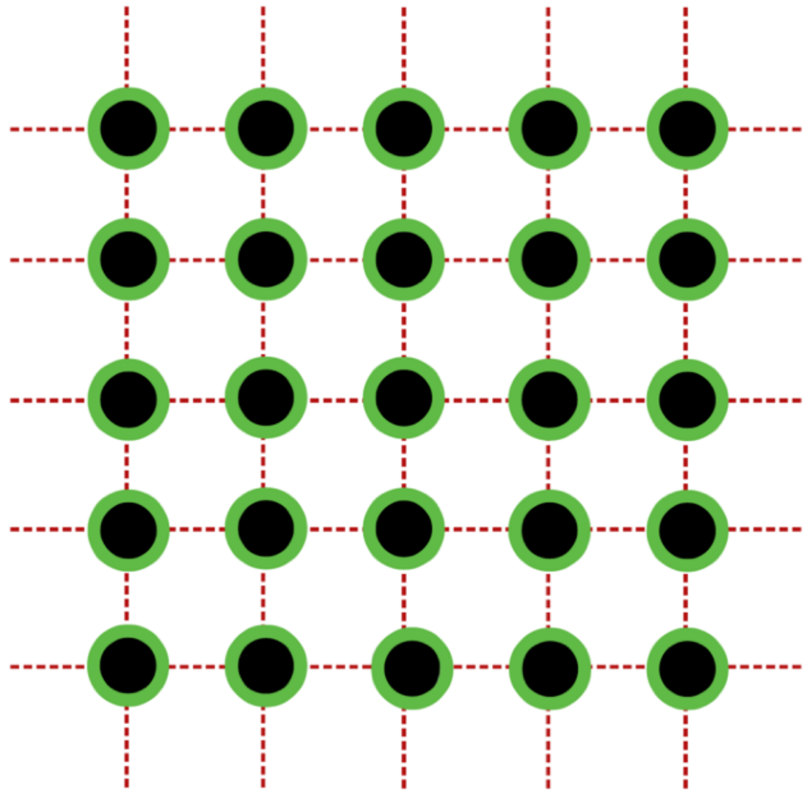
$\Omega_3 = 2\pi\mathbb{Z}^3/L$

$\mathrm{QED_{C^*}}$



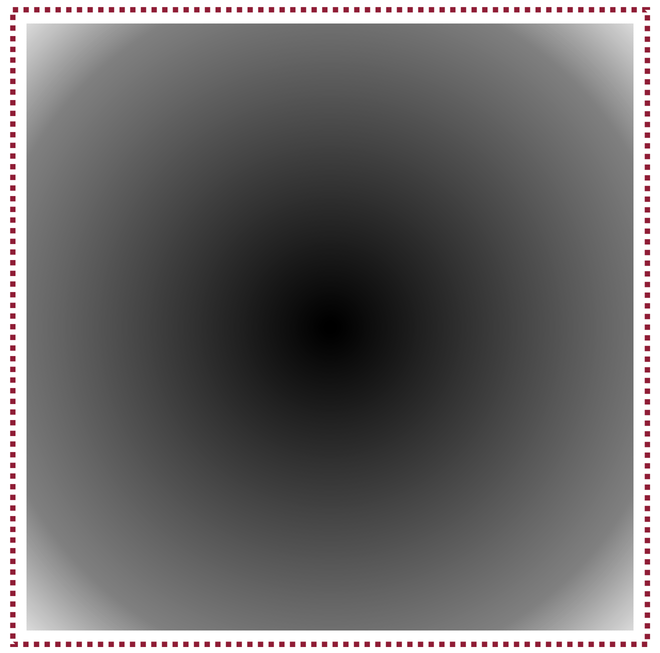
$\Omega_3 = 2\pi\mathbb{Z}^3/L \quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

$\mathrm{QED_m}$



$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

$\mathrm{QED_\infty}$



$\Omega_4 = \mathbb{R}^4$

finite-volume photon

non-local

local

∞-volume photon

power-like finite-volume effects

exponential finite-volume effects

UV / IR mixing

dedicated ensembles

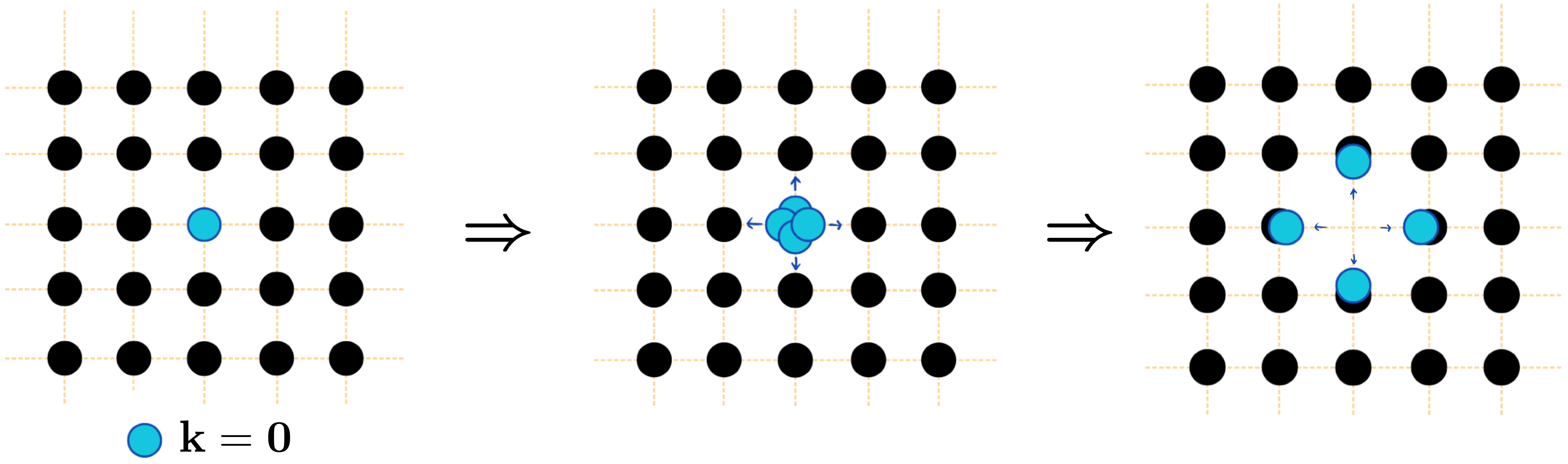
two IR regulators

observable-dependent

QEDr regularization

Special case of "IR-improvement"

Z.Davoudi et al., PRD 99 (2019)
MDC, PoS LATTICE2023 (2024) [2401.07666]
MDC et al., [2501.07936]



The spatial zero mode is <u>not</u> **removed** but **redistributed over the neighbouring modes** on a shell of radius $|\mathbf{p}| = \frac{2\pi}{L}|\mathbf{r}| \quad (\mathbf{r} \in \mathbb{Z}^3)$

QED$_\mathrm{L}$: $D_L^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + \mathbf{k}^2}$ $\Rightarrow$ **QED$_\mathrm{r}$:** $D_{\mathbf{p}}^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + \mathbf{k}^2} + \frac{\delta_{\mathbf{k}^2,\mathbf{p}^2}}{n(\mathbf{p}^2)} \frac{\delta^{\mu\nu}}{k_0^2 + \mathbf{p}^2}$

3. What

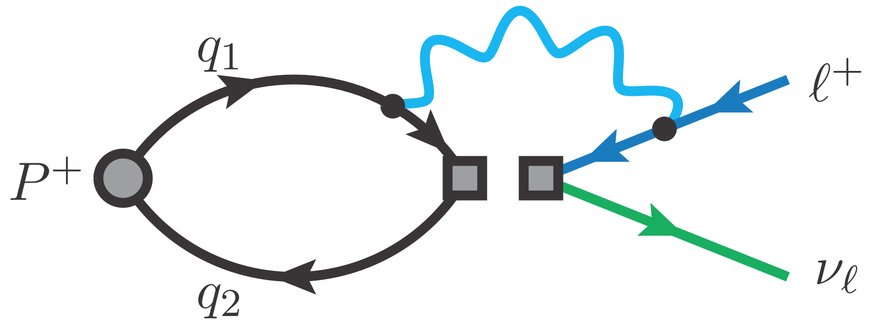
Lattice QCD+QED calculations can provide IB corrections for several hadronic observables:

- hadron masses & quark masses
- HVP contribution to muon g-2
- leptonic & semileptonic weak decay rates
- inclusive hadronic τ decays
- CP violation parameters
- ...

As hadronic uncertainties **decrease**, such corrections become more and more **relevant**!

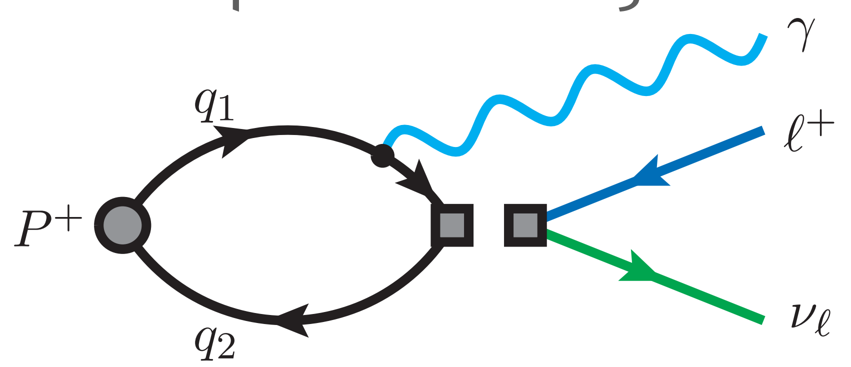
Weak decays — some recent works

leptonic decays



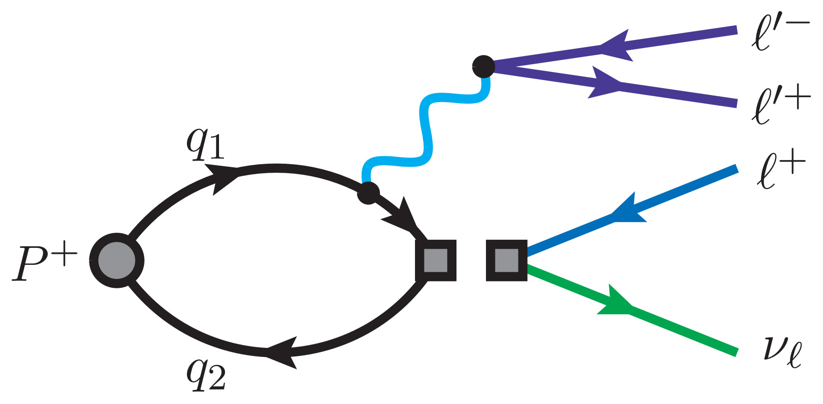
N.Carrasco et al., PRD 91 (2015)
V.Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199v2]
D.Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)
MDC et al., PRD 105 (2022)
P.Boyle, MDC et al., JHEP 02 (2023)
N.Christ et al., [2304.08026]

real photon emission
leptonic decays



G.M.de Divitiis et al., [1908.10160]
C.Kane et al., [1907.00279 & 2110.13196]
R.Frezzotti et al., PRD 103 (2021)
A.Desiderio et al., PRD 102 (2021)
D.Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

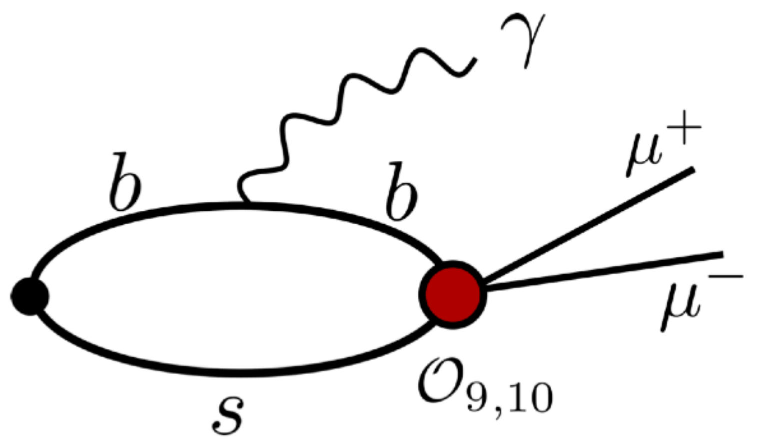
virtual photon emission
leptonic decays



G.Gagliardi et al., Phys. Rev. D 105 (2022)
R.Frezzotti et al., [2306.07228]

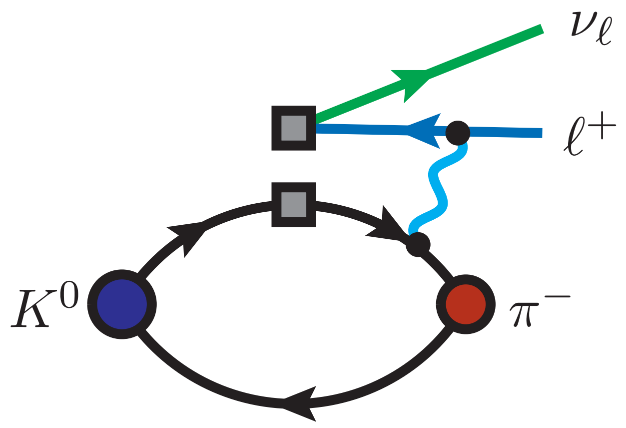
R.Frezzotti et al., [2402.03262]

rare FCNC decays



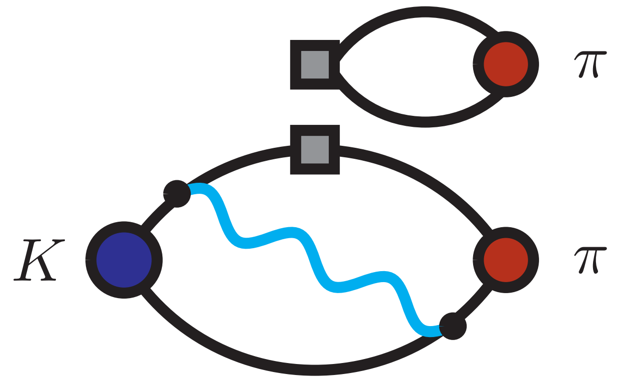
C.Sachrajda et al., [1910.07342]
N.Christ et al., PRD 108 (2023)
N.Christ et al., [2402.08915]

semileptonic decays



R.Abbott et al., PRD 102 (2020)
Z.Bai et al., PRL 115 (2015)
N.Christ et al., PRD 106 (2022)
N.Christ & X.Feng, EPJ Web Conf. 175 (2018)
Y.Cai & Z.Davoudi, [1812.11015]

hadronic decays



Leptonic decays of pseudoscalar mesons

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by

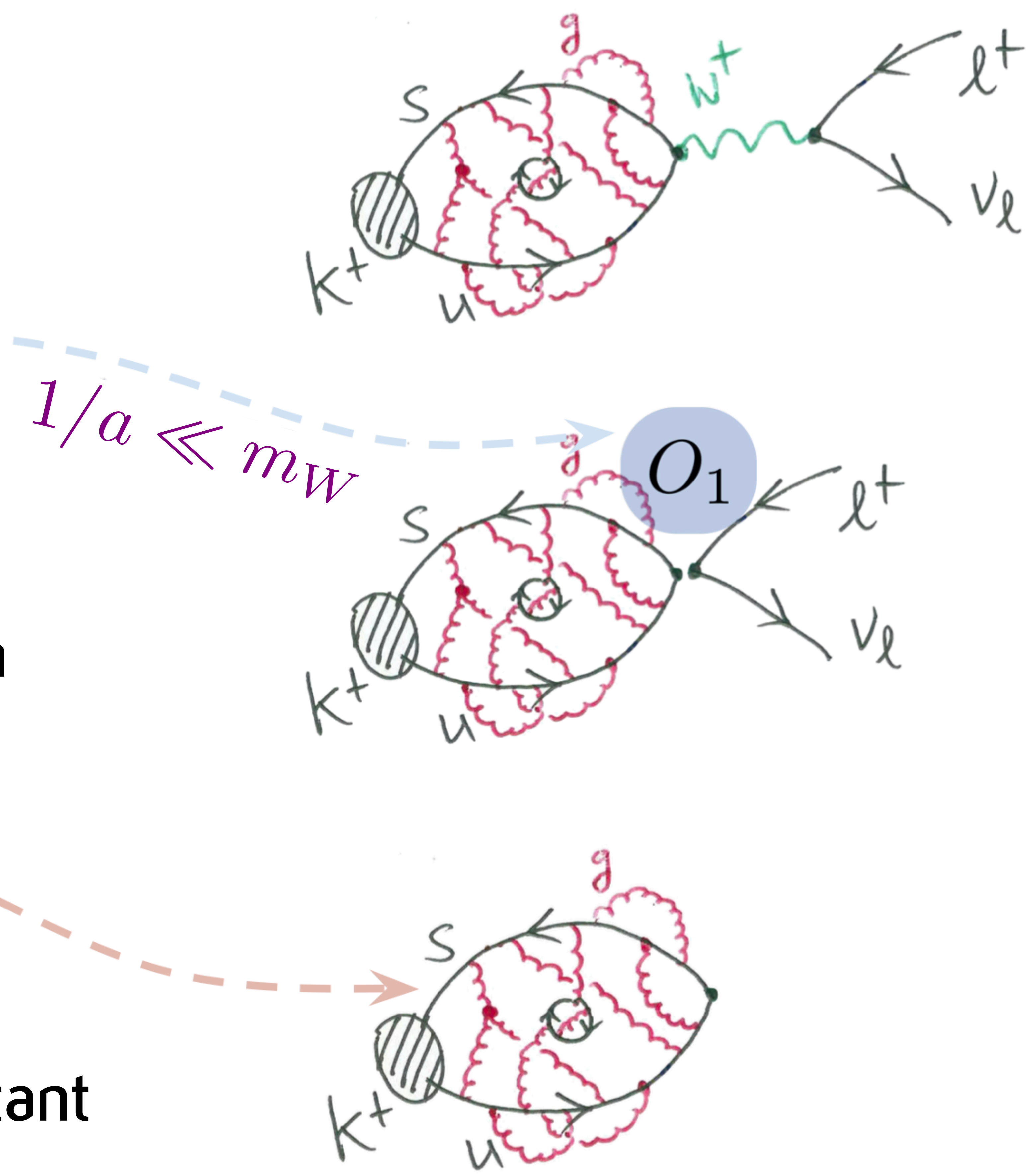
$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$

In the **PDG convention**, the tree-level decay rate takes the form

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$

with the non-perturbative dynamic encoded in the **decay constant**

$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$

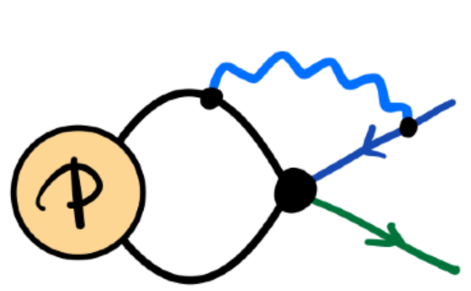
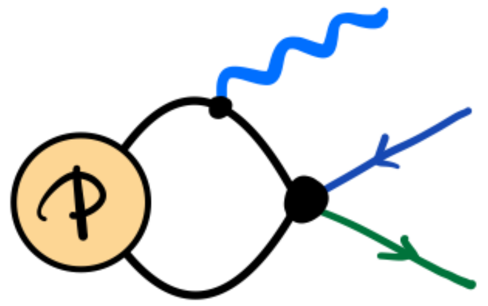


Leptonic decay rate at $\mathcal{O}(\alpha)$

- The **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity

- **IR divergences** appear in intermediate steps of the calculation

F. Bloch & A. Nordsieck, PR 52 (1937) 54

$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Big\{$  $+$  $\Big\}$

IR finite — IR divergent — IR divergent

- **UV divergences**: need to include QED corrections to the renormalization of the weak Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(1 + \frac{\alpha_{\rm em}}{\pi} \ln\left(\frac{M_Z}{M_W}\right)\right) \underbrace{[\,\bar{q}_2\,\gamma_\mu(1-\gamma_5)\,q_1\,]\,[\,\bar{\nu}_\ell\,\gamma^\mu(1-\gamma_5)\,\ell\,]}_{O_1^{\text{W-reg}}(M_W)}$$

A.Sirlin, NPB **196** (1982)

E.Braaten & C.S.Li, PRD **42** (1990)

$$O_1^{\text{W-reg}}(M_W) = Z^{\text{W-S}}\left(\frac{M_W}{\mu}, \alpha_s(\mu), \alpha_{\rm em}\right) O_1^{\rm S}(\mu)$$

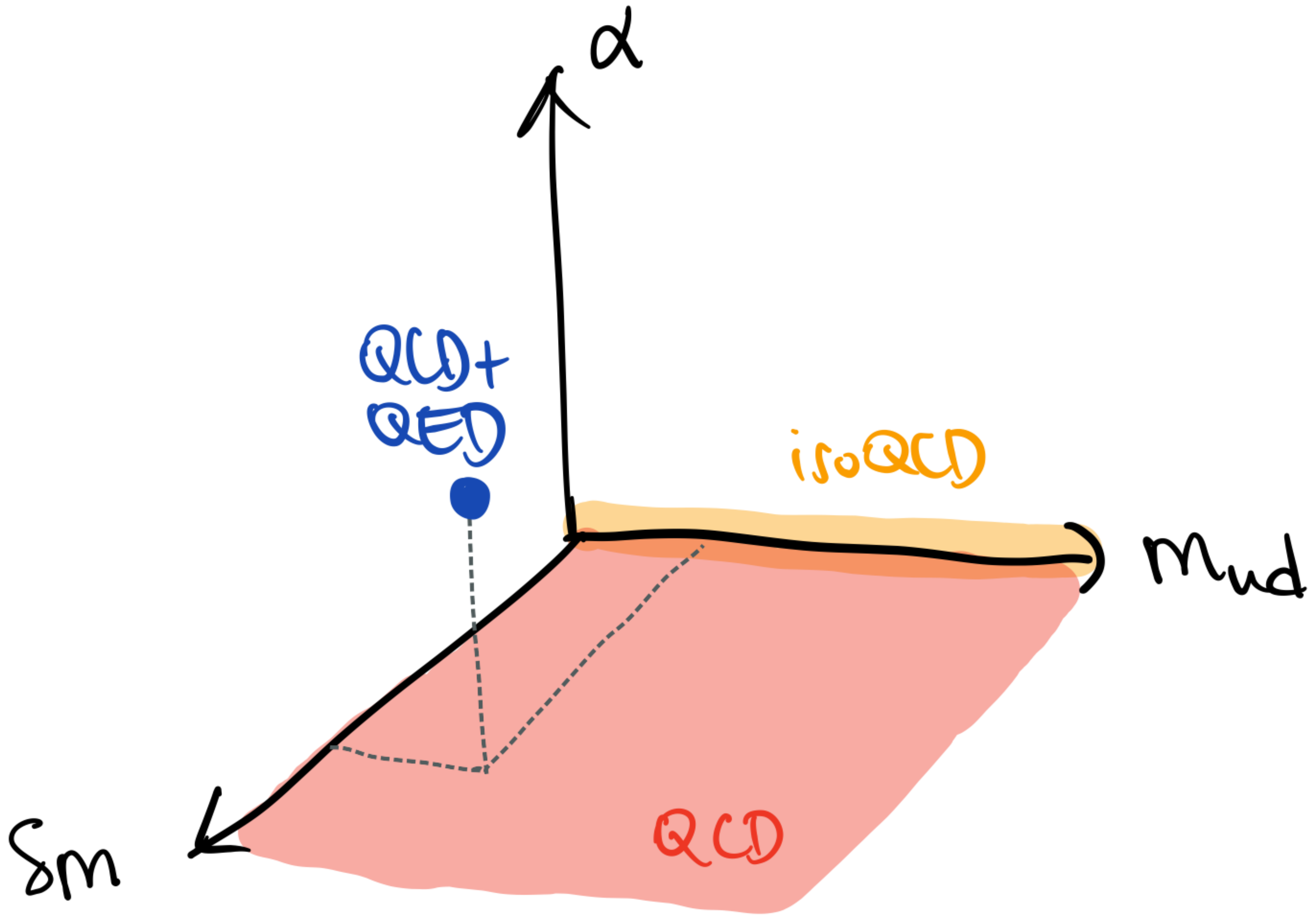
- perturbative @ 2 loops in QCD+QED
- non-perturbative in lattice QCD+QED

MDC et al., PRD **100** (2019)

Leptonic decay rate at $\mathcal{O}(\alpha)$

see talk by A.Ramos

Defining the isospin symmetric world



- The full QCD+QED theory is unambiguously defined after **matching** a set of observables to the real world

$$\left[\frac{\hat{\mathrm{M}}_j}{\hat{\Lambda}}\right]^2(g, e^\phi, \hat{\mathbf{m}}^\phi) = \left(\frac{\mathrm{M}_j^\phi}{\Lambda^\phi}\right)^2 \longrightarrow \hat{\mathbf{m}}^\phi(g)$$

$$j = 1, \ldots, N_f$$

- The definition of QCD or isoQCD requires a prescription, i.e. some renormalization conditions to **fix the bare parameters** of the action

$$\boldsymbol{\sigma}^{\mathrm{QCD}} = (g^{\mathrm{QCD}}, 0, \hat{\mathbf{m}}^{\mathrm{QCD}}) \qquad \hat{\mathbf{m}}^{\mathrm{QCD}} = (\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}}, \ldots)$$

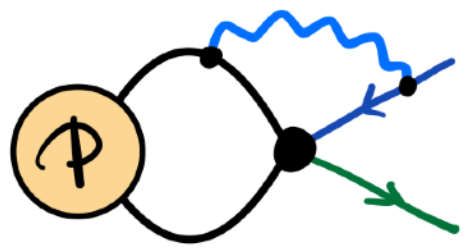
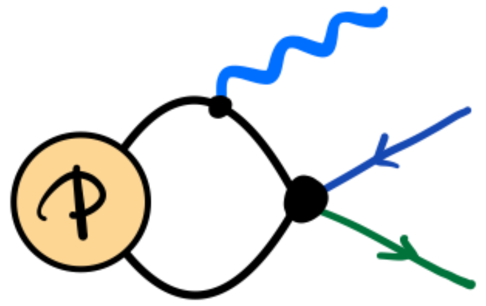
$$\boldsymbol{\sigma}^{(0)} = (g^{(0)}, 0, \hat{\mathbf{m}}^{(0)}) \qquad \hat{\mathbf{m}}^{(0)} = (\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)}, \ldots)$$

FLAG 2024 now includes a discussion on this topic where a reference scheme ("Edinburgh consensus") is proposed.

Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}}\to 0} \Big\{$$  $$+$$  $$\Big\}$$

IR finite — IR divergent — IR divergent

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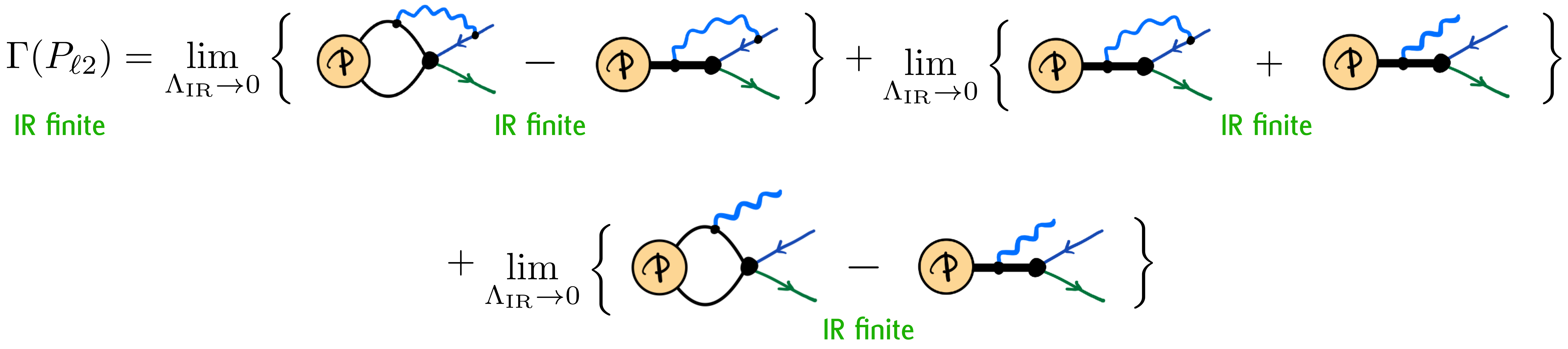

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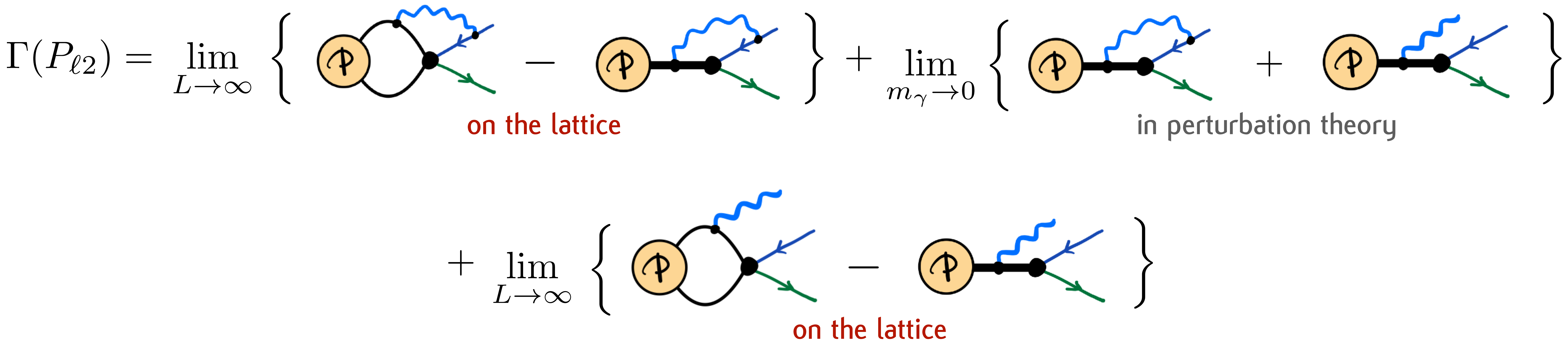

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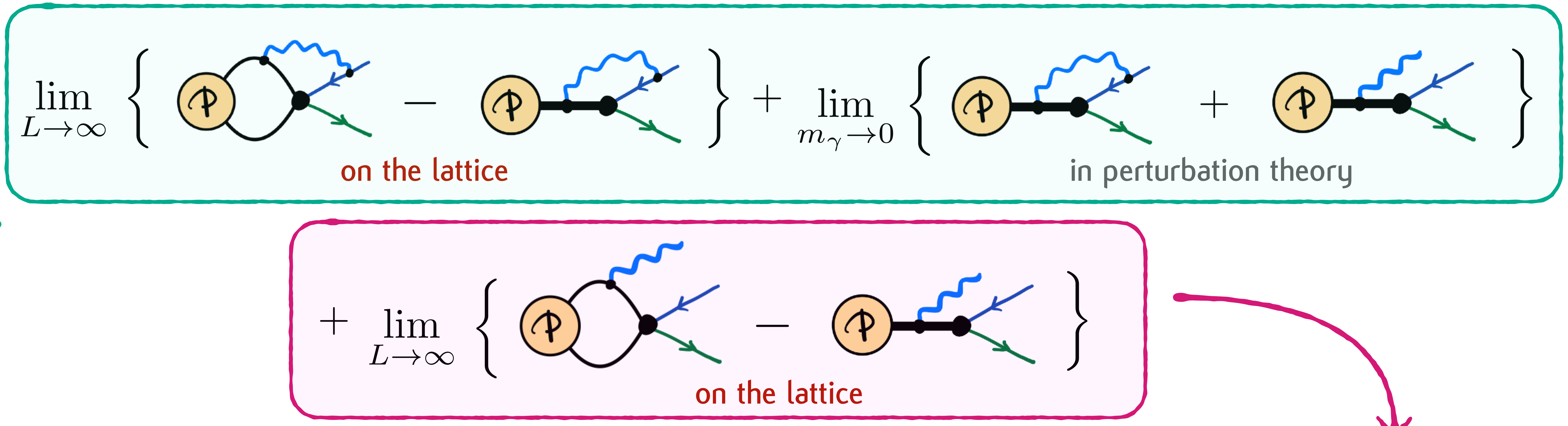
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) =$



enough for $K_{\mu 2}$ and $\pi_{\mu 2}$

leading **finite-volume scaling** well studied

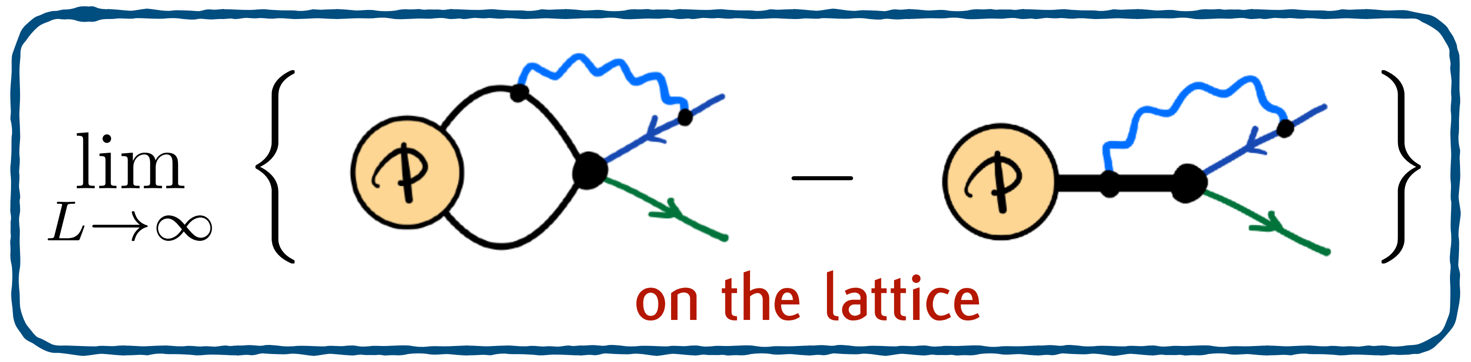
V.Lubicz et al., PRD **95** (2017) N.Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

relevant for K_{e2} and π_{e2}
& decays of heavier mesons

G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD **103** (2021)
A. Desiderio et al., PRD **102** (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

Leptonic decay rate at $\mathcal{O}(\alpha)$

Virtual decay rate



$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}(1 + \delta R_P) \quad \blacktriangleright \quad \Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2 \quad \blacktriangleright \quad \delta R_P = 2\left(\frac{\delta \mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta \mathcal{Z}}{\mathcal{Z}_0}\right)$$

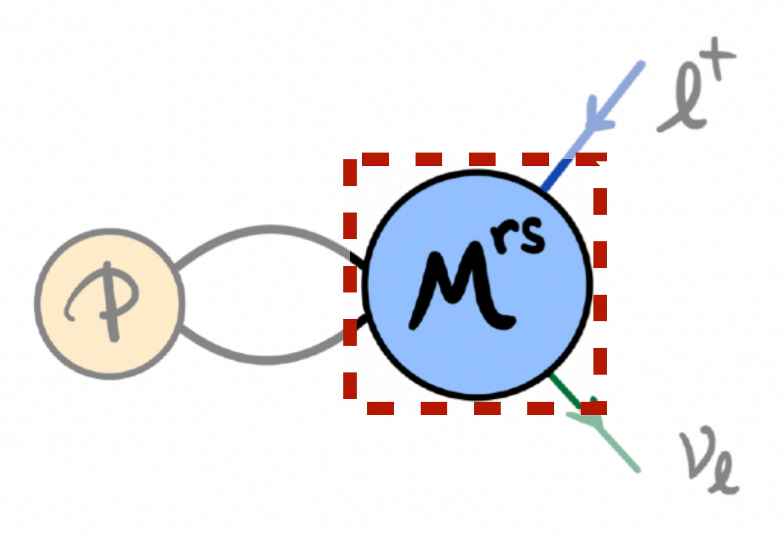
PDG convention

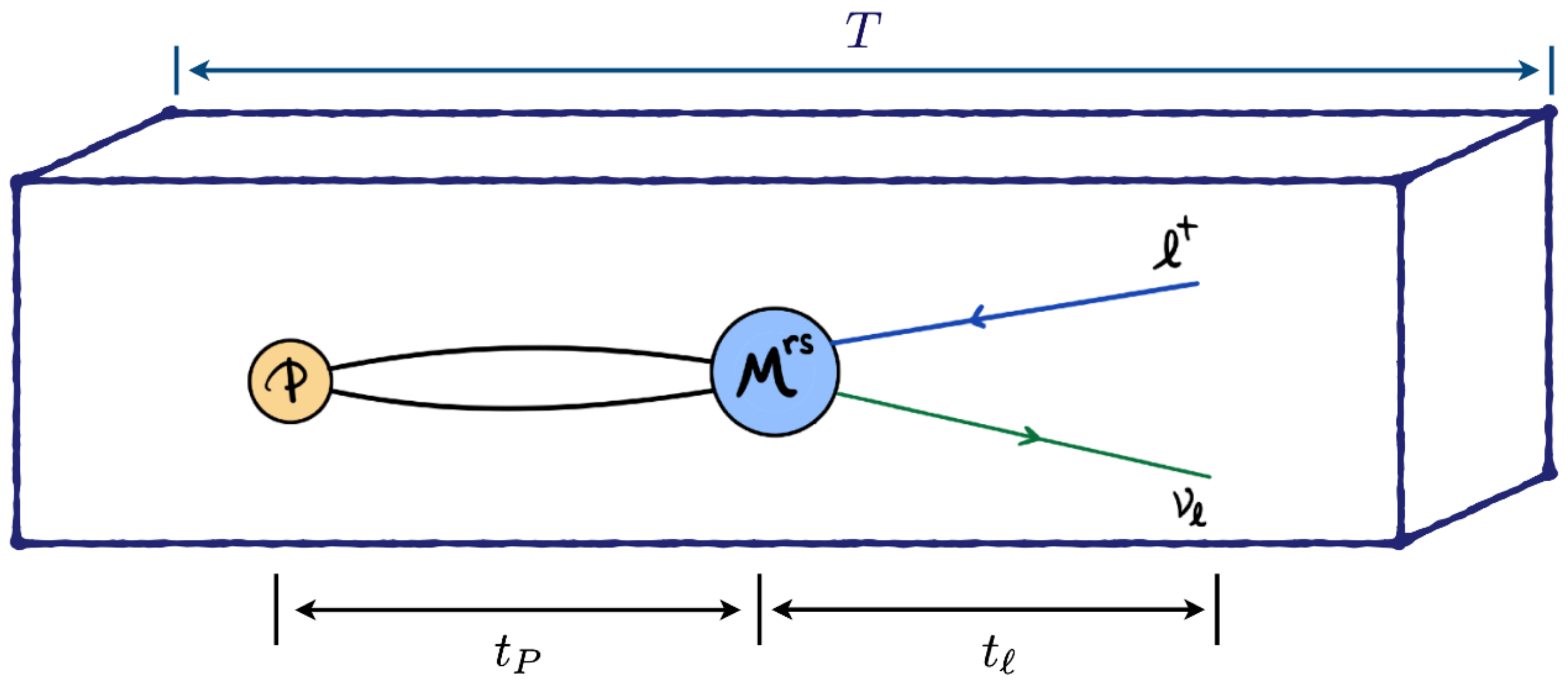
- $\boldsymbol{\delta \mathcal{A}_P}$ from the correction to the (bare) matrix element $\mathcal{M}_P^{rs}(\mathbf{p}_\ell) = \langle \ell^+, r, \mathbf{p}_\ell; \nu_\ell, s, \mathbf{p}_\nu | O_W | P^+, \mathbf{0}\rangle$
- $\boldsymbol{\delta m_P}$ correction to the meson mass
- $\boldsymbol{\delta \mathcal{Z}}$ correction to the renormalization of the weak operator O_W

MDC et al., PRD **100** (2019)

$$\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})} \rightarrow \delta R_{K\pi} = 2\left(\frac{\delta \mathcal{A}_K}{\mathcal{A}_{K,0}} - \frac{\delta m_K}{m_{K,0}}\right) - 2\left(\frac{\delta \mathcal{A}_\pi}{\mathcal{A}_{\pi,0}} - \frac{\delta m_\pi}{m_{\pi,0}}\right)$$

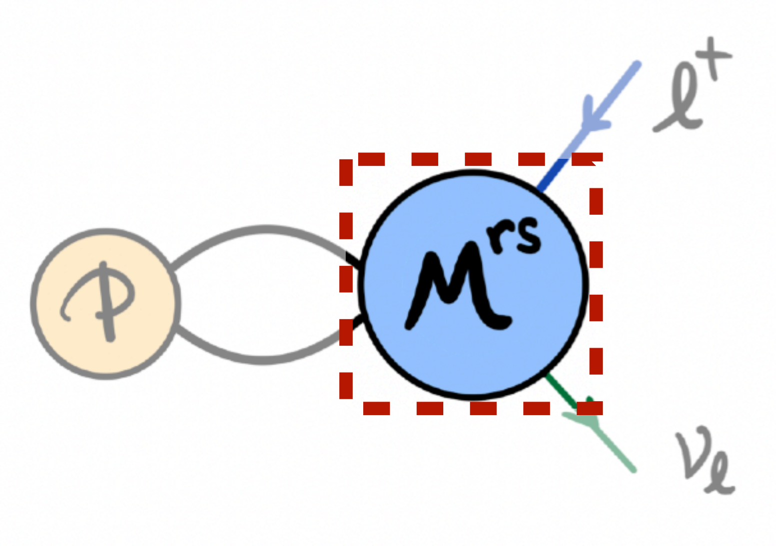
From correlators to matrix elements

Our goal: 

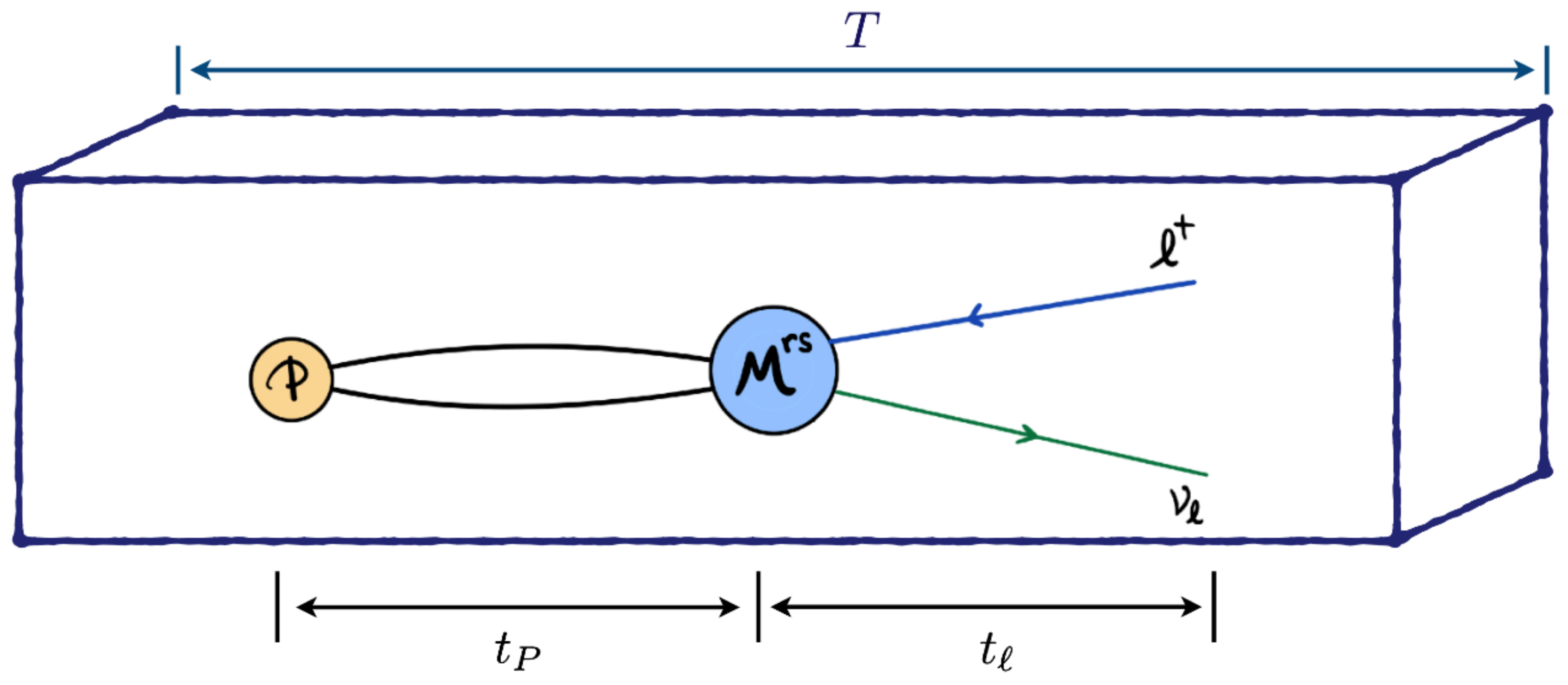
How we realise it: 

From correlators to matrix elements

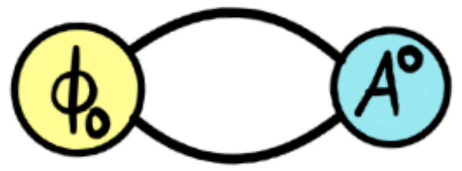
Our goal:



How we realise it:



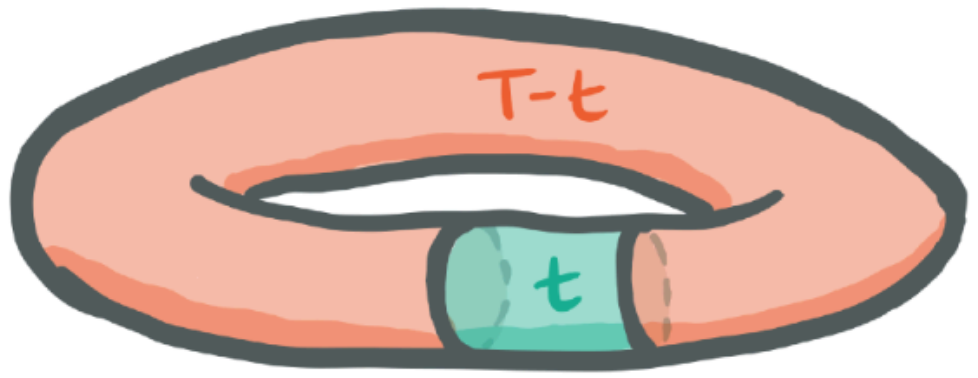
Tree-level decay amplitude: $|\mathcal{M}_0(\mathbf{p}_\ell)|^2 = |\mathcal{A}_{P,0}|^2 |\mathcal{L}_0(\mathbf{p}_\ell)|^2$ $\quad \mathcal{A}_{P,0} = \langle 0|A^0|P\rangle_0 = i m_{P,0}\,[f_{P,0}]$



$$= \langle 0|A^0(0)\phi^\dagger(-t)|0\rangle_T \to \frac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}}\left\{\mathrm{e}^{-m_{P,0}t} - \mathrm{e}^{-m_{P,0}(T-t)}\right\}$$

$$Z_{P,0} = \langle P, \mathbf{p} = \mathbf{0}|\phi^\dagger|0\rangle_0$$

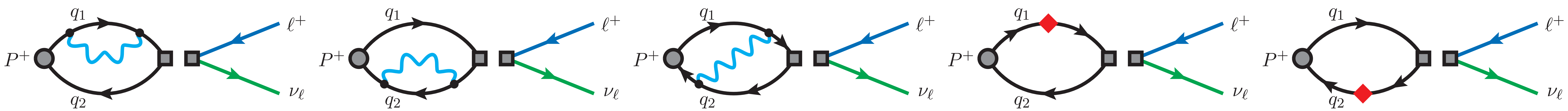
$$= \langle 0|\phi(0)\phi^\dagger(-t)|0\rangle_T \to \frac{Z_{P,0}^2}{2m_{P,0}}\left\{\mathrm{e}^{-m_{P,0}t} + \mathrm{e}^{-m_{P,0}(T-t)}\right\}$$

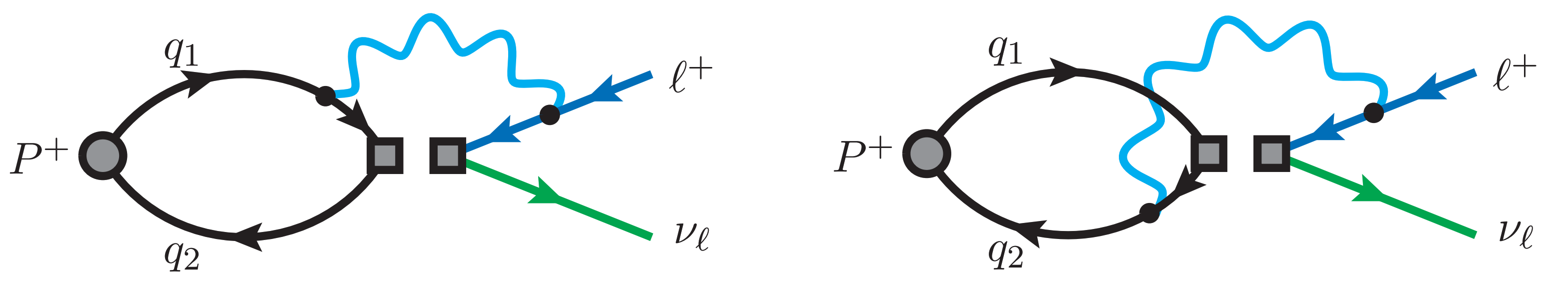


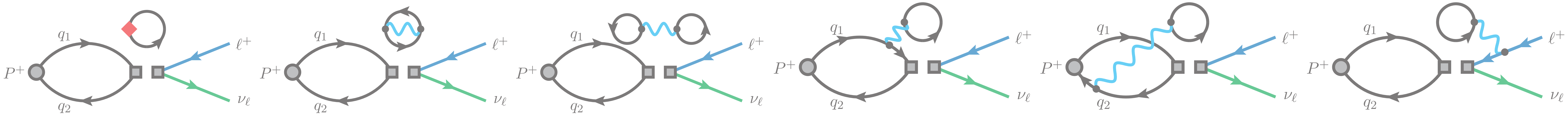
IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_u - m_d = 0$







Current calculations have been performed in the **electro-quenched approximation (sea quarks electrically neutral).**
Work is in progress to compute the remaining diagrams.

IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD 87 (2013)

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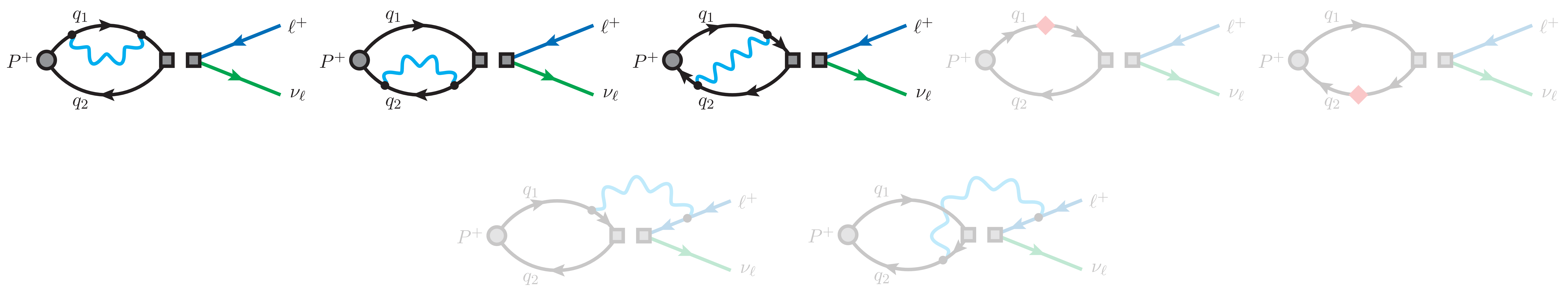
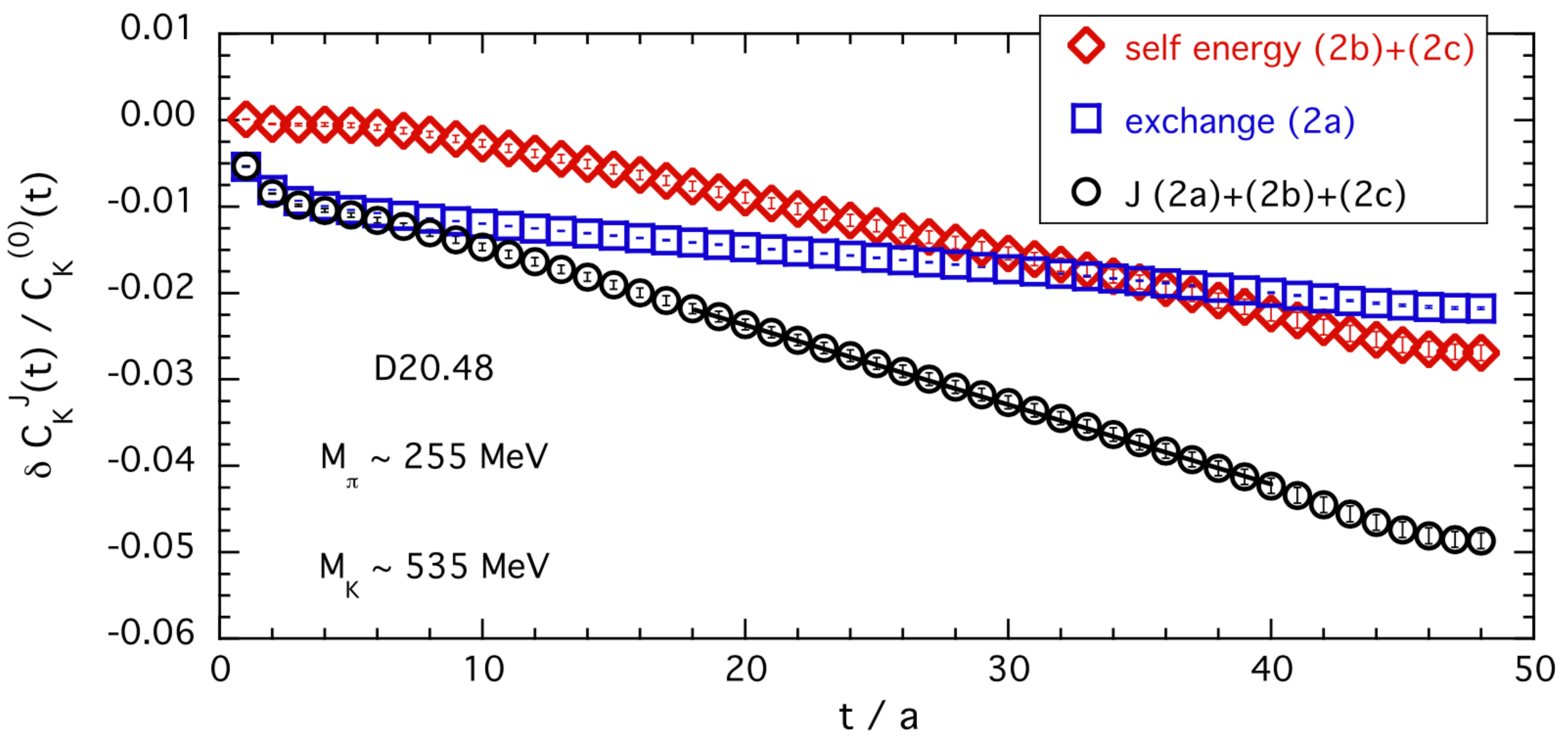
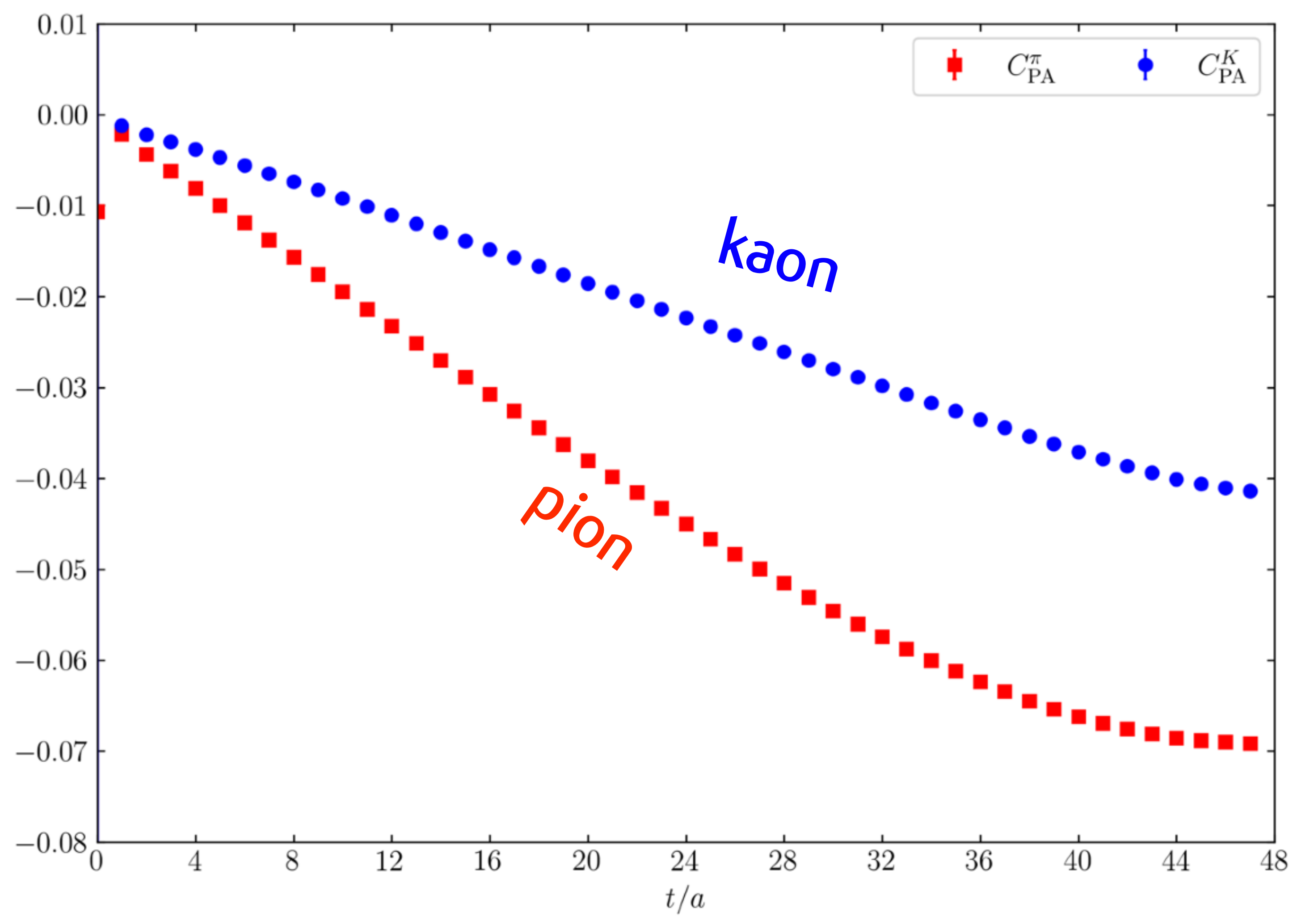




MDC et al., PRD 100 (2019)

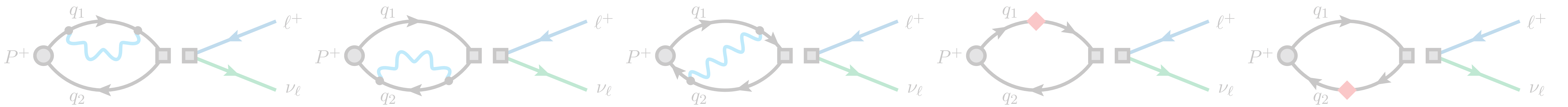


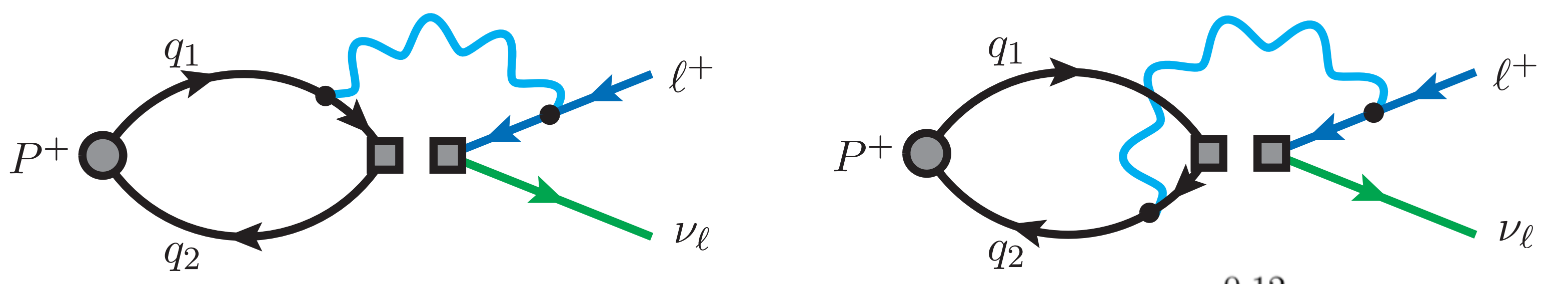
P.Boyle, MDC et al., JHEP 02 (2023)

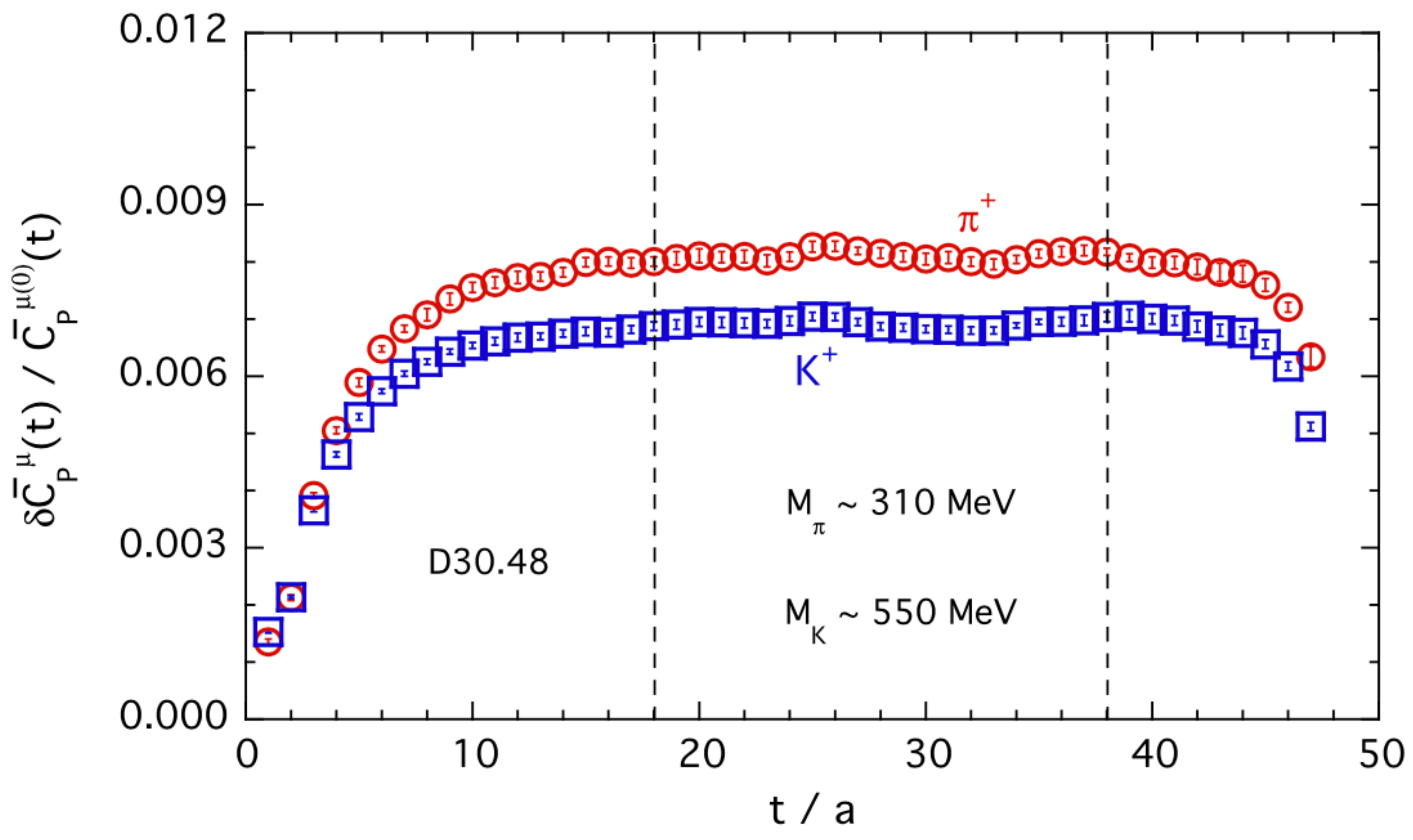
IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

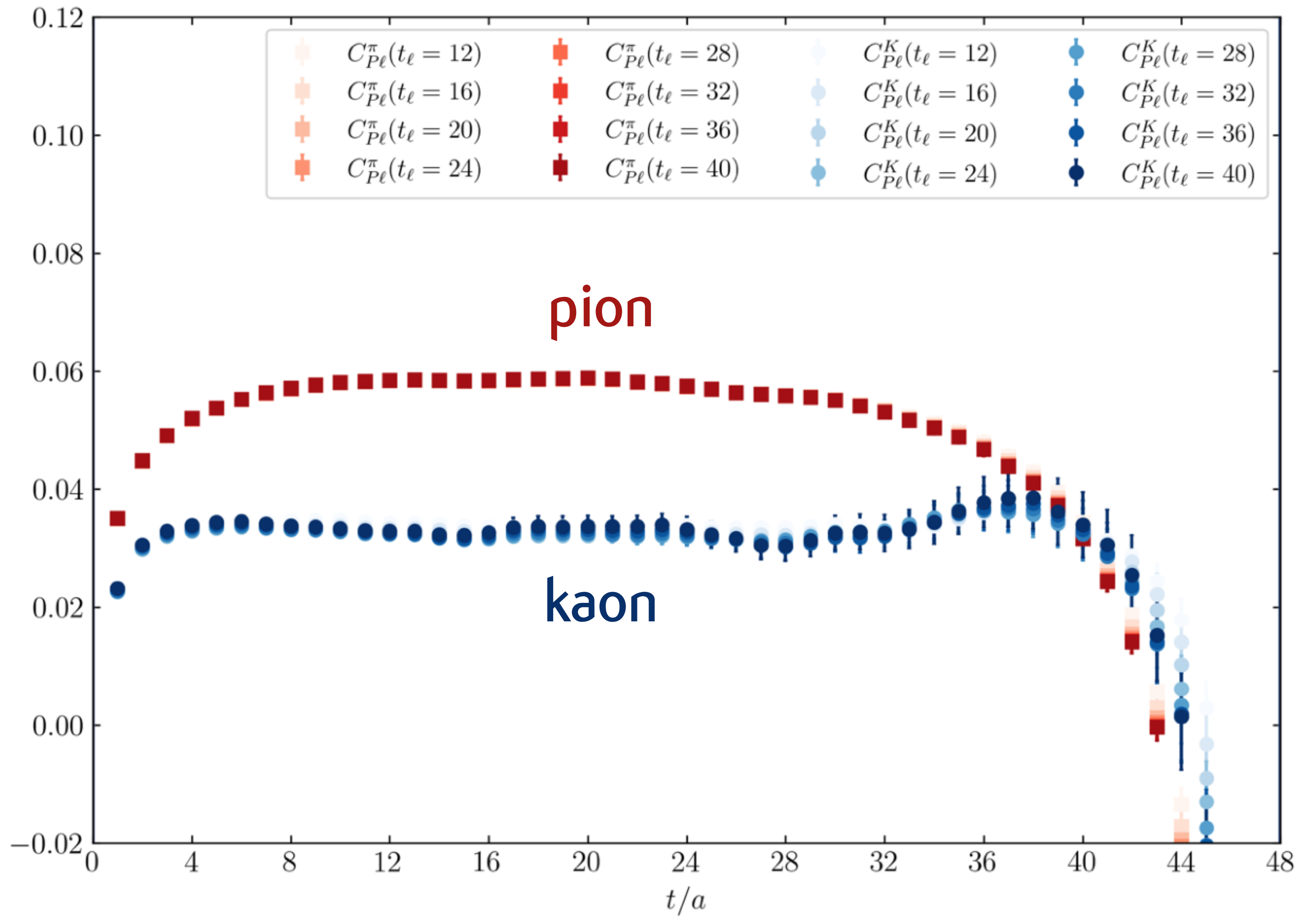
RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_u - m_d = 0$







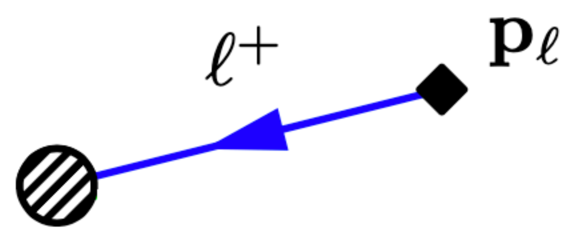
MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

Non-factorisable QED corrections

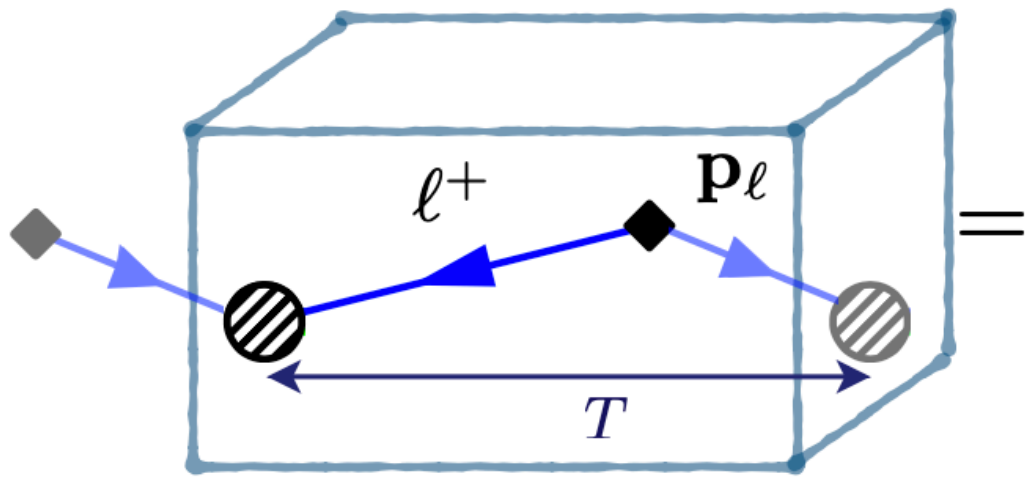
The lepton in a finite volume



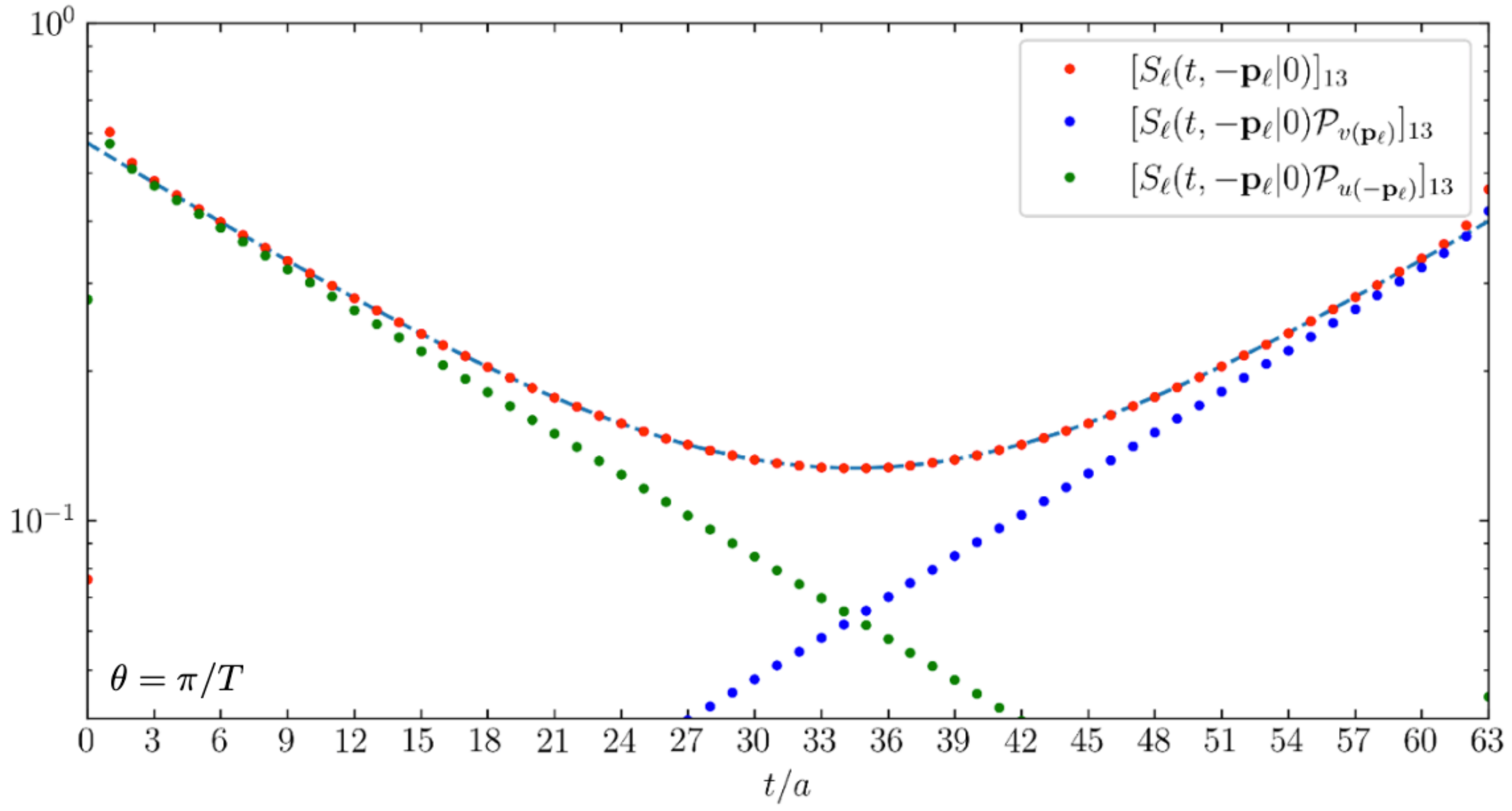
$$= \quad S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} \right\}$$

Non-factorisable QED corrections

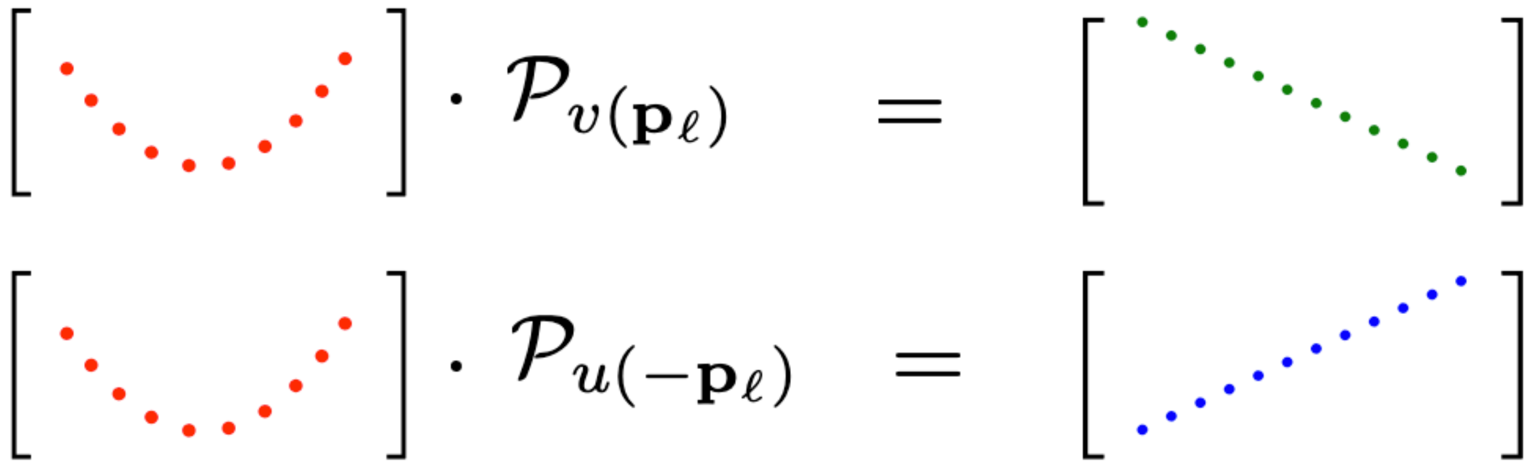
The lepton in a finite volume



$$= \; S(0|t, \mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} + \mathrm{e}^{\mathrm{i}\theta T} \mathrm{e}^{-(T-t)E_\ell} \frac{u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)}{2\Omega_\ell} \right\} \times \frac{1}{1 - \mathrm{e}^{-TE_\ell} \mathrm{e}^{\mathrm{i}\theta T}}$$

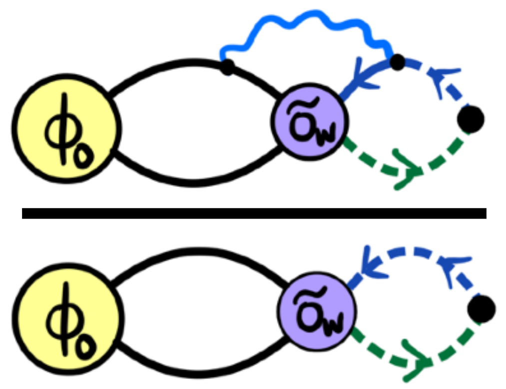


We can select specific components using projectors:

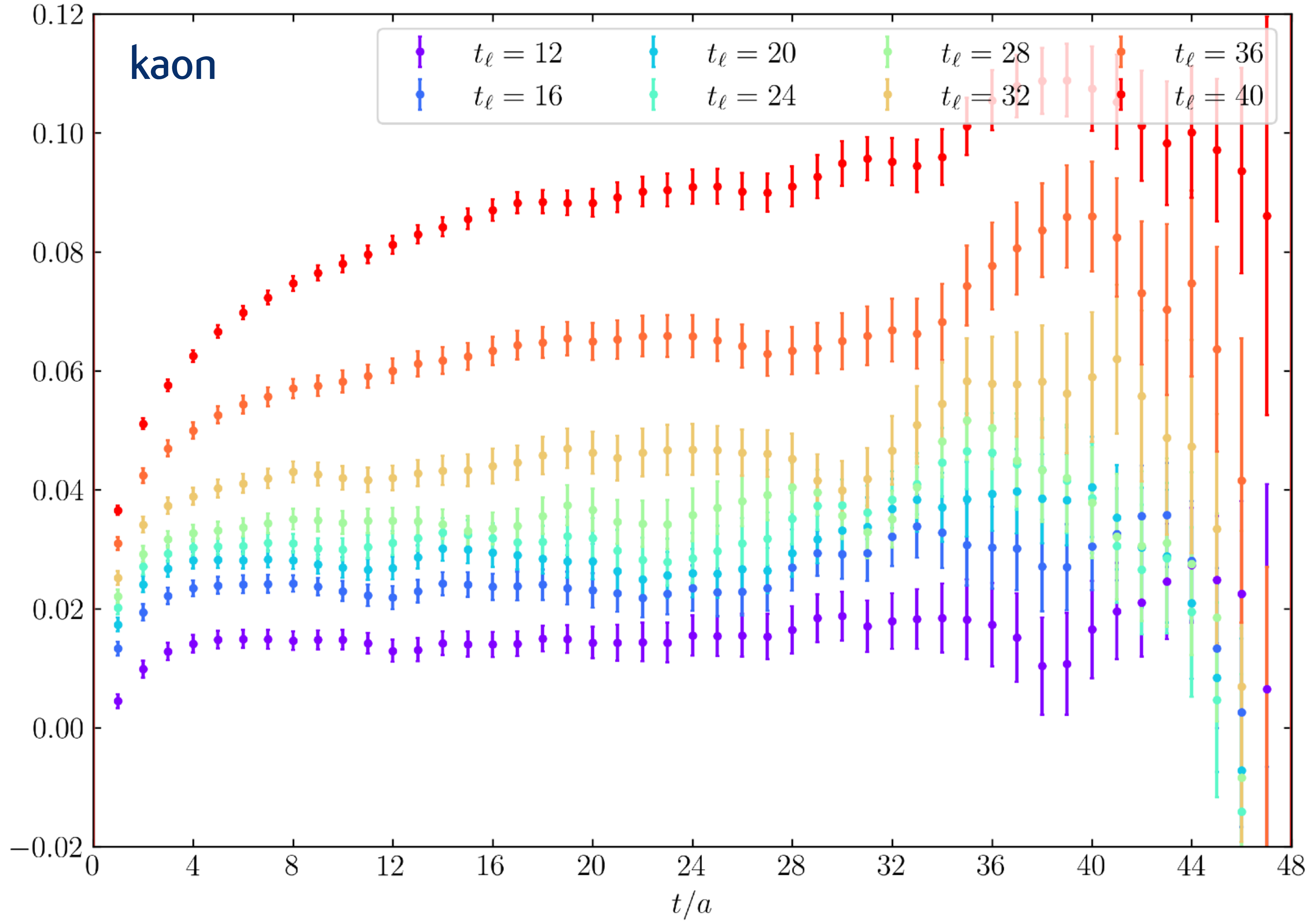


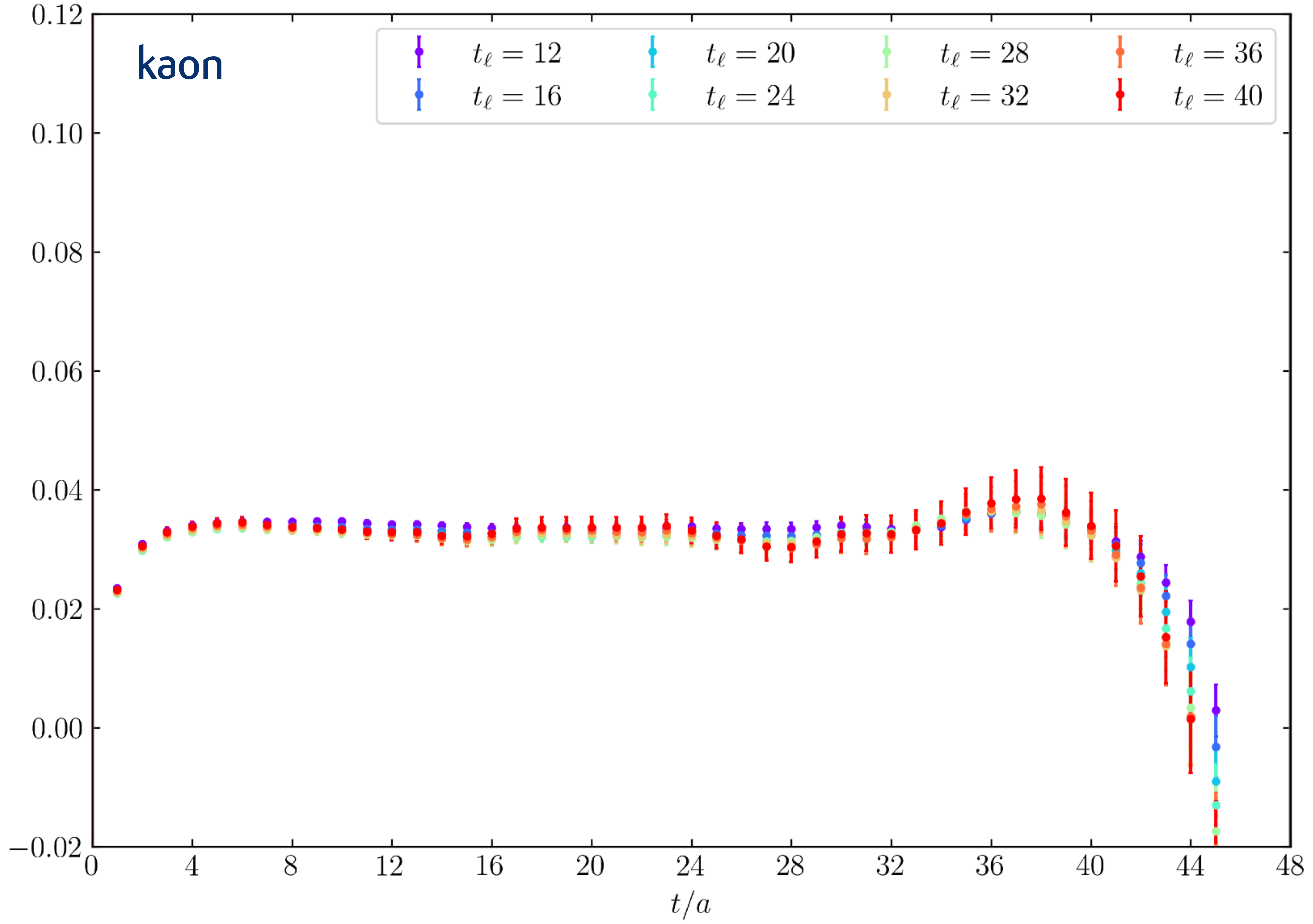
$$\mathcal{P}_{v(\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1} [v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)]$$
$$\mathcal{P}_{u(-\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1} [u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)]$$

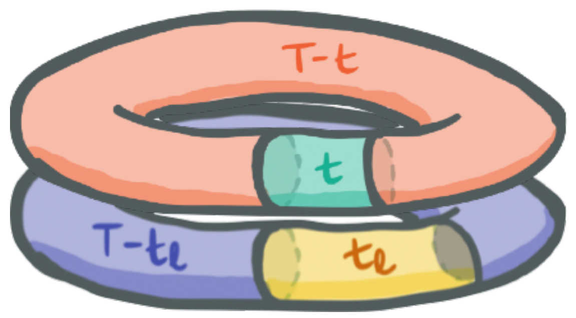
Non-factorisable QED corrections



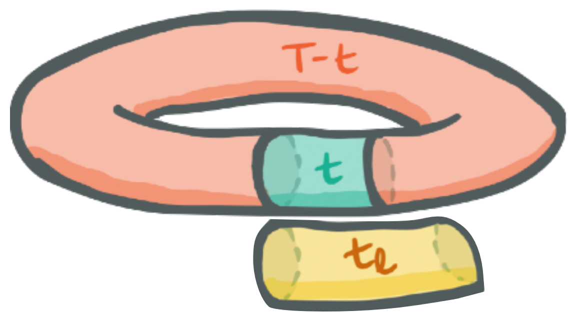
$$\to \frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} \; f_{P\ell}(t,T)$$





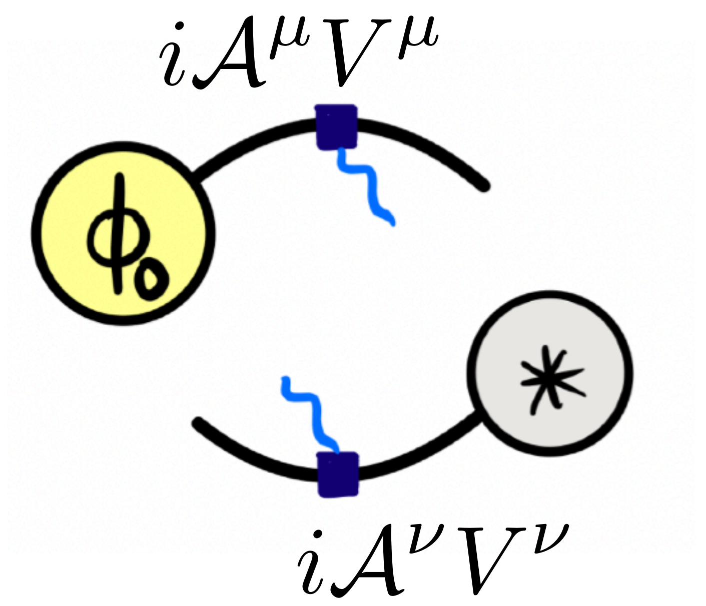


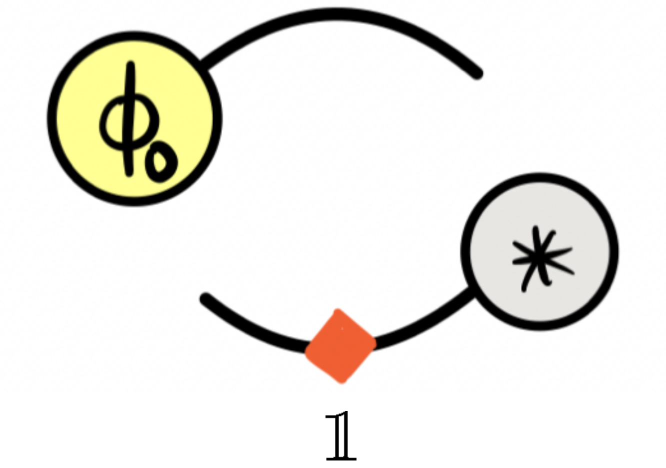
without projection

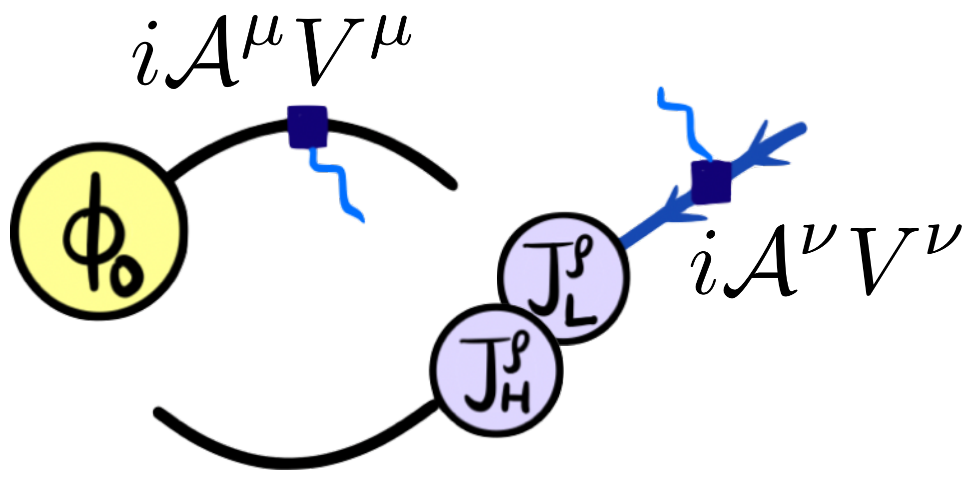


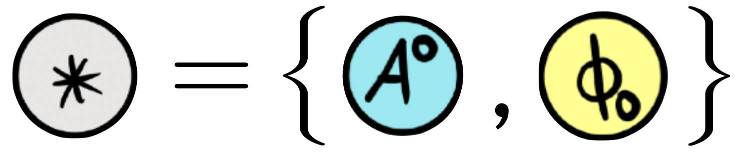
with projection

Numerical implementation of correlators







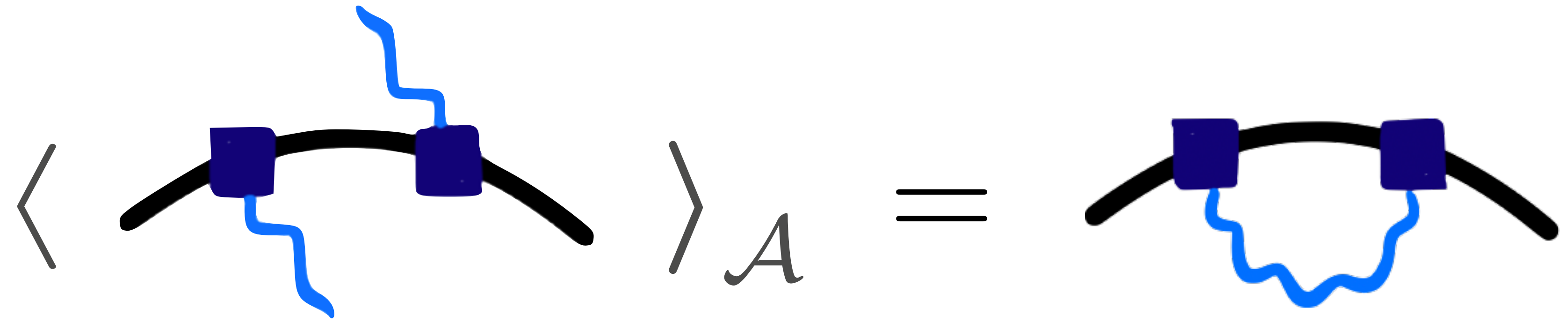


- Correlators created using **sequential propagators**

- **Muon momentum** $\mathbf{p}_\ell \propto \{1,1,1\}$ fixed by energy conservation & injected via twisted boundary conditions

- **Photon fields** sampled from Gaussian distribution (QED_L)

$$P(\tilde{A})\mathrm{d}\tilde{A} \propto \mathrm{e}^{-S_\gamma[\tilde{A}]} \qquad S_\gamma^{\mathrm{Feyn.}}[\tilde{A}] = \frac{1}{2V} \sum_{k_0,\mathbf{k}\neq\mathbf{0}} \hat{k}^2 \sum_\mu |\tilde{A}_\mu(k)|^2$$



- **Sources** : point (RM123S) / Coulomb gauge-fixed wall (RBC-UKQCD)

- **Electromagnetic current:** conserved (RM123S) / local (RBC-UKQCD)

A general comparison of the calculations

	RBC/UKQCD	RM123+Soton
physical masses	✓ physical point simulations	extrapolation needed
chiral symmetry	✓ at finite lattice spacing	recovered in the continuum
fermionic action	Domain Wall	Twisted Mass
continuum limit	single lattice spacing	✓ continuum limit (3)
infinite volume limit	single volume	✓ multiple volumes
QED prescription	QED_L	QED_L
sea effects	electro-quenching	electro-quenching
IB scheme	BMW [a]	GRS [b]

[a] BMW, PRL 111 (2013); BMW, PRL 117 (2016)

[b] Gasser, Rusetsky & Scimemi, EPJC 32 (2003); RM123, PRD 87 (2013)

Defining the iso-symmetric theory

BMW, PRL 111 (2013)
BMW, PRL 117 (2016)

RBC/UKQCD (2023): BMW scheme with N_f=2+1 flavours

QCD+QED

$(\hat{m}_{ud}^{\phi}, \delta\hat{m}^{\phi}, \hat{m}_s^{\phi} | g, \alpha^{\phi})$

$$\left(\frac{\hat{m}_{\pi^+}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{m}_{K^+}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{m}_{K^0}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\phi}} = \left(\frac{m_{\pi^+}^2}{m_{\Omega^-}^2}, \frac{m_{K^+}^2}{m_{\Omega^-}^2}, \frac{m_{K^0}^2}{m_{\Omega^-}^2}\right)_{\mathrm{PDG}}$$

QCD

$(\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}} | g, 0)$

$$\left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\Delta\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\mathrm{QCD}}} = \left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\Delta\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\phi}}$$

iso-QCD

$(\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)} | g, 0)$

$$\left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\Delta\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{(0)}} = \left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, 0, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\phi}}$$

BMW mesons: $M_{\mathrm{ud}}^2 = \frac{1}{2}\left(M_{\bar{\mathrm{u}}\mathrm{u}}^2 + M_{\bar{\mathrm{d}}\mathrm{d}}^2\right)$ $\quad M_{K\chi}^2 = \frac{1}{2}\left(M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2\right)$ $\quad \Delta M_{\mathrm{ud}}^2 = M_{\bar{\mathrm{u}}\mathrm{u}}^2 - M_{\bar{\mathrm{d}}\mathrm{d}}^2$

Defining the iso-symmetric theory

Gasser, Rusetsky & Scimemi, EPJC 32 (2003)
RM123, PRD 87 (2013)

RM123S (2019): "GRS" scheme (electroquenched)

QCD+QED

$(\hat{m}_{ud}^{\phi}, \delta\hat{m}^{\phi}, \hat{m}_s^{\phi}, \hat{m}_c^{\phi} | g, \alpha^{\phi})$

$$\left(\frac{\hat{m}_{\pi^0}^2}{\hat{\mathcal{F}}_\pi^2}, \frac{\hat{m}_{K^0}^2}{\hat{\mathcal{F}}_\pi^2}, \frac{\hat{m}_{D_s}^2}{\hat{\mathcal{F}}_\pi^2}, \frac{\hat{m}_{K^+}^2 - \hat{m}_{K^0}^2}{\hat{\mathcal{F}}_\pi^2} \right)_{\boldsymbol{\sigma}^\phi} = \left(\frac{\hat{m}_{\pi^0}^2}{\hat{\mathcal{F}}_\pi^2}, \frac{\hat{m}_{K^0}^2}{\hat{\mathcal{F}}_\pi^2}, \frac{\hat{m}_{D_s}^2}{\hat{\mathcal{F}}_\pi^2}, \frac{\hat{m}_{K^+}^2 - \hat{m}_{K^0}^2}{\hat{\mathcal{F}}_\pi^2} \right)_{\mathrm{PDG}}$$

QCD

$(\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}}, \hat{m}_c^{\mathrm{QCD}} | g_0, 0)$

$$a^{\mathrm{QCD}} = a^{\phi}$$

$$m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{\mathrm{QCD}} \equiv m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{\phi}$$
$$f = \{u, d, s, c\}$$

iso-QCD

$(\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)}, \hat{m}_c^{(0)} | g_0, 0)$

$$a^{(0)} = a^{\phi}$$

$$m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{(0)} \equiv m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{\phi}$$
$$f = \{ud, s, c\}$$

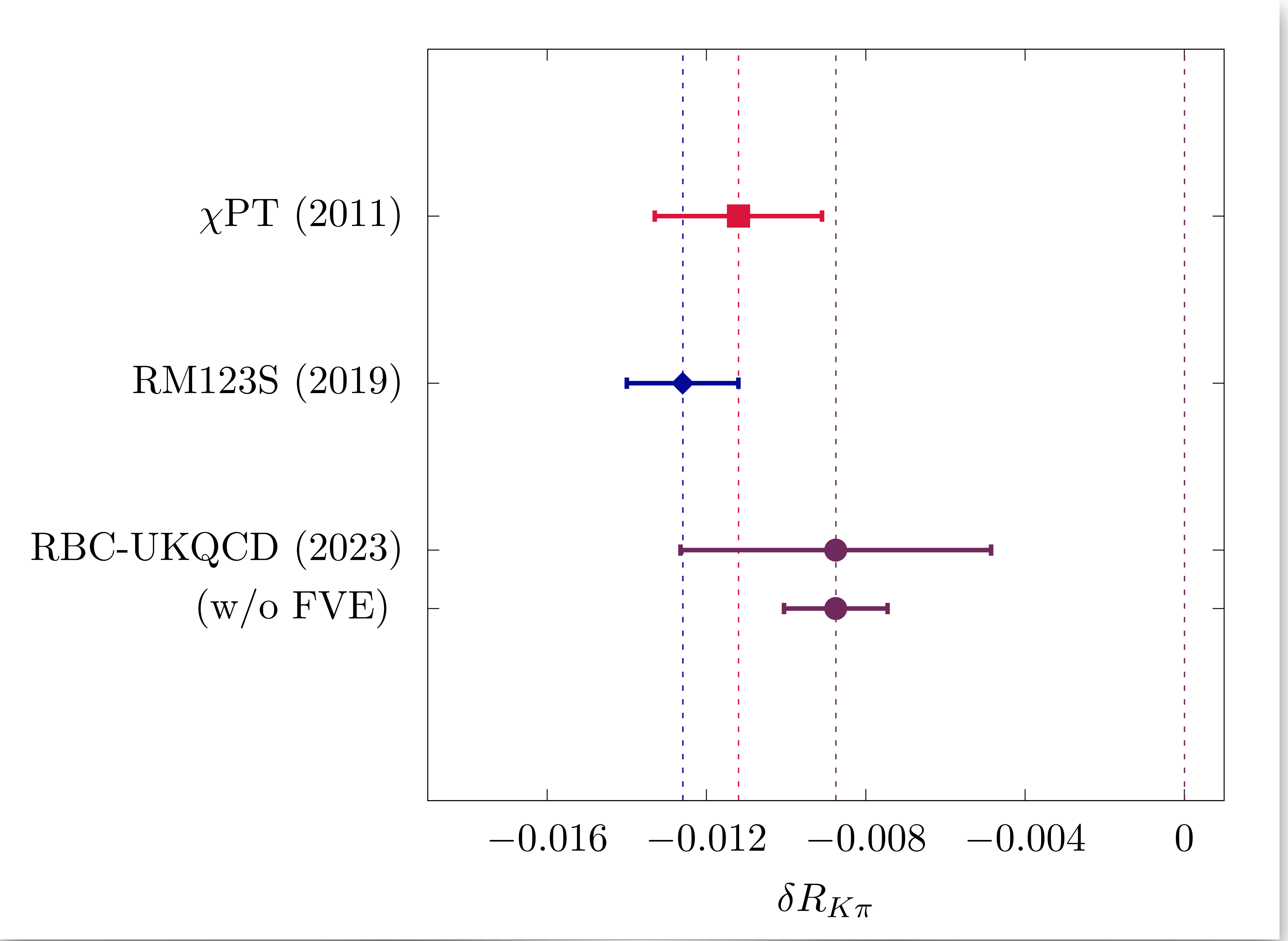
In practice, the renormalization condition on the strong coupling $g^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV}) \equiv g_0^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})$ is neglected in the "**electroquenched approximation**"

Results for $\delta R_{K\pi}$

V. Cirigliano et al., PLB **700** (2011)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

- ■ $\delta R_{K\pi} = -0.0112\,(21)$
- ◆ $\delta R_{K\pi} = -0.0126\,(14)$
- ● $\delta R_{K\pi} = -0.0086\,(13)(39)_{\mathrm{vol.}}$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$



- Strong evidence that $\delta R_{K\pi}$ **can be computed from first principles non-perturbatively on the lattice!**
- **RBC-UKQCD error** dominated by a large systematic uncertainty related to **finite-volume effects**
- **Errors on** $|V_{us}|/|V_{ud}|$ from **theoretical** inputs can become comparable with those from **experiments**

QED finite-volume effects

In **finite-volume** (massless) QED the photon zero modes require a regularisation

$$\Delta g(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathcal{G}(k_0, \mathbf{k})}{k_0^2 + |\mathbf{k}|^2} \qquad D^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1}{k_0^2 + |\mathbf{k}|^2}$$

M. Hayakawa & S. Uno, PTP 120 (2008) ↓ QED_L

$$\Delta' g(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathcal{G}(k_0, \mathbf{k})}{k_0^2 + |\mathbf{k}|^2} \qquad D_L^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + |\mathbf{k}|^2}$$

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\Delta_{\mathbf{k}} \frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|, \mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|, \mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

QED finite-volume effects

Hadron masses

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

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$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

QED finite-volume effects

Hadron masses

using the notation of
B.Lucini et al., JHEP 1602 (2016)

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universal terms fixed by Ward identities — structure + multi-particle dependence

QED finite-volume effects

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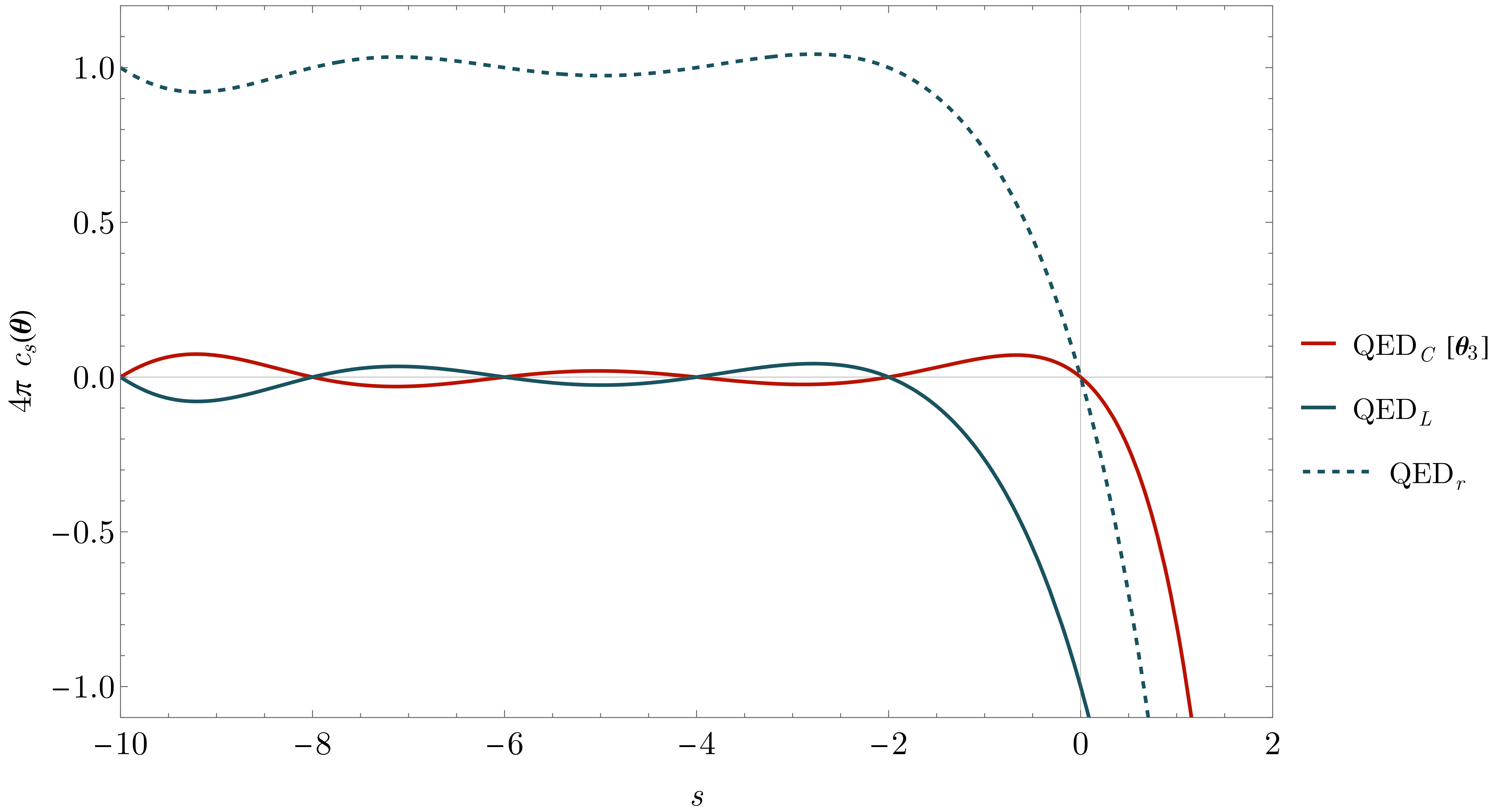


QED finite-volume effects

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]
MDC et al., [2501.07936]

Leptonic decays

$$\Delta Y(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$
$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$
$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$
$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$
$$+ \cdots$$

$$C_\ell^{(2)} = \frac{32\pi^2}{f_P\, m_P^2\,(1-r_\ell^4)}\left[F_V^P - F_A^P + 2\, r_\ell^2\,\frac{\partial F_A^P}{\partial x_\gamma}\right]$$

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- Collinear divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

Next steps...

Analytical

- We still need to quantitatively answer the question "how large should the volume be?"
- **Working** on the derivation of FV effects to **all-orders in $1/L$**

  —› Understand asymptotic behaviour of the $1/L$ series

  —› Put bounds on neglected higher orders

Numerical

- **QEDr** action implemented in **Hadrons**
- Ongoing runs with **QEDr** action for leptonic decays
- Working on a numerical **comparison** between **QEDL** and **QEDr**

Hadrons

https://github.com/aportelli/Hadrons

Infinite volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)

QED$_\infty$

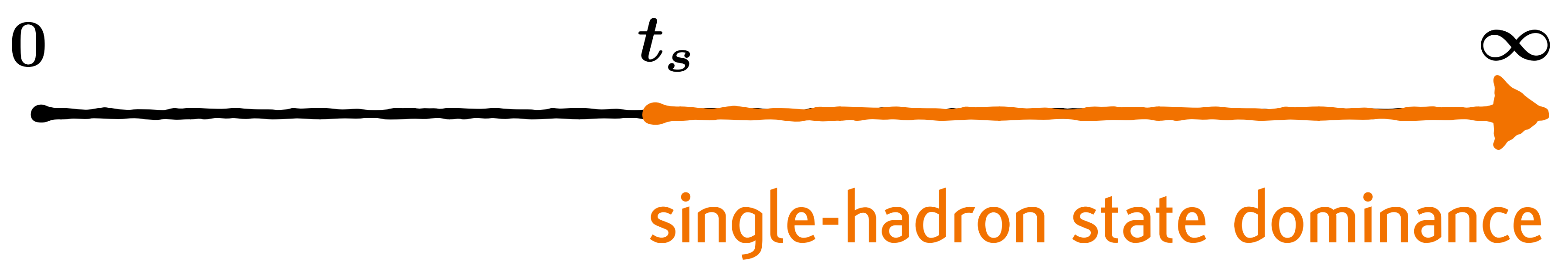
- An alternative approach is to compute radiative corrections as a **convolution** of **hadronic correlators** with **infinite-volume QED kernels**

$$\Delta\mathcal{O} = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x})\, f_{\mathrm{QED}}(t,\mathbf{x}) = \Delta\mathcal{O}^{(s)} + \Delta\mathcal{O}^{(l)}$$

Separate correlator into **short** and **long** distance parts:

$$\Delta\mathcal{O}^{(s)} \approx \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x})$$

$$\Delta\mathcal{O}^{(l)} \approx \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t_s,\mathbf{x})\, \mathcal{F}_{\mathrm{QED}}(t_s,\mathbf{x})$$

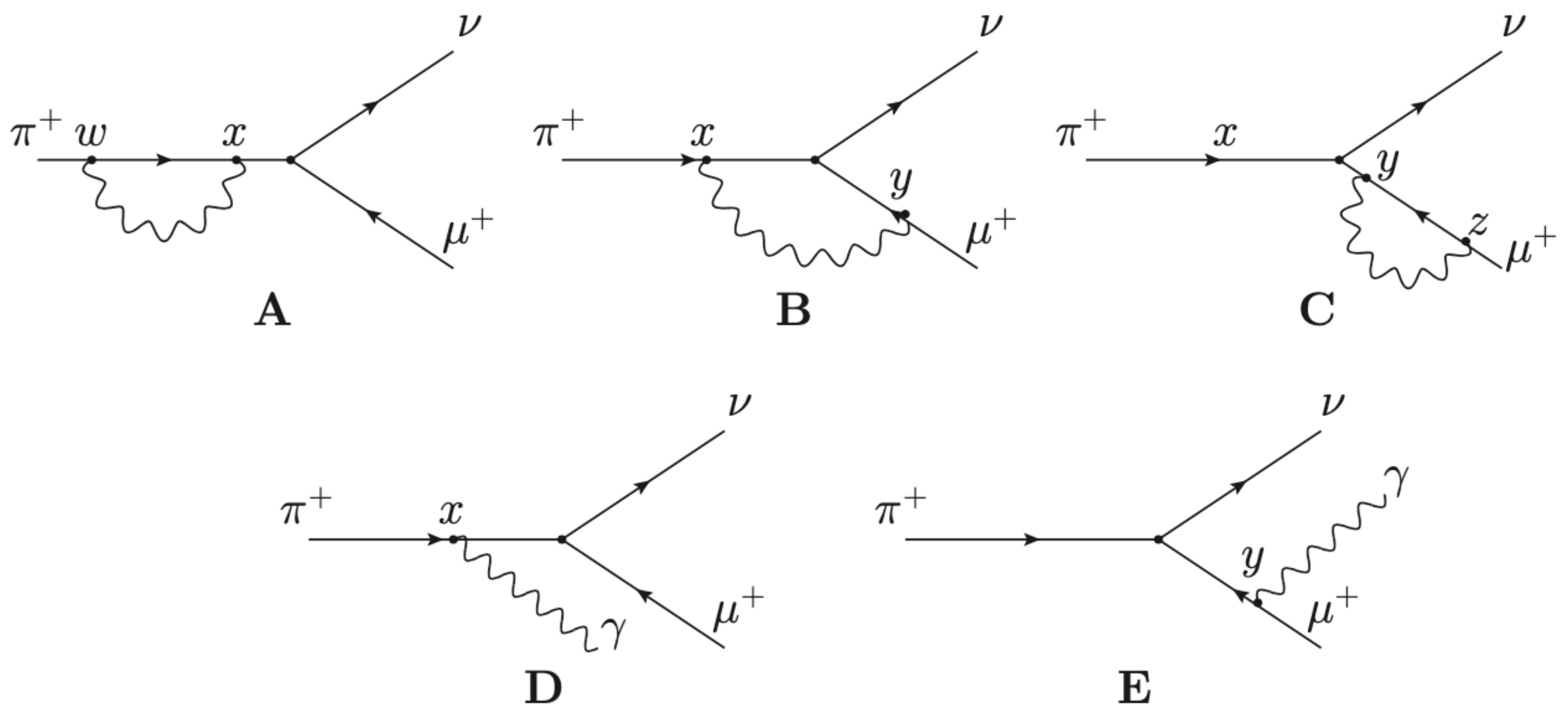


Exponentially suppressed (a) finite-volume effects (b) contributions of states with higher energy

Infinite volume reconstruction

N.Christ et al., [2304.08026]

QED$_\infty$



- Diagram A:

$$H^{(2)}_{\mu,\rho,\sigma}(t_1, t_2, \vec{x}) = \int d^3\vec{w}\, \langle 0|T\{J^W_\mu(0) J^{EM}_\rho(t_1, \vec{w}+\vec{x}) J^{EM}_\sigma(t_2, \vec{w})\}|\pi(\vec{0})\rangle$$

- Diagram B and D:

$$H^{(1)}_{\mu,\rho}(x) = H^{(1)}_{\mu,\rho}(x_t, \vec{x}) = \langle 0|T\{J^W_\mu(0) J^{EM}_\rho(x)\}|\pi(\vec{0})\rangle$$

- Diagram C and E ($f_\pi \approx 130$ MeV):

$$H^{(0)}_\mu = H^{(0)}_t \delta_{\mu,t} = \langle 0|J^W_\mu(0)|\pi(\vec{0})\rangle = -i m_\pi f_\pi \delta_{\mu,t}$$

from Luchang Jin's talk @ Edinburgh May 30, 2023

Strategy proposed for **leptonic decay rates**:

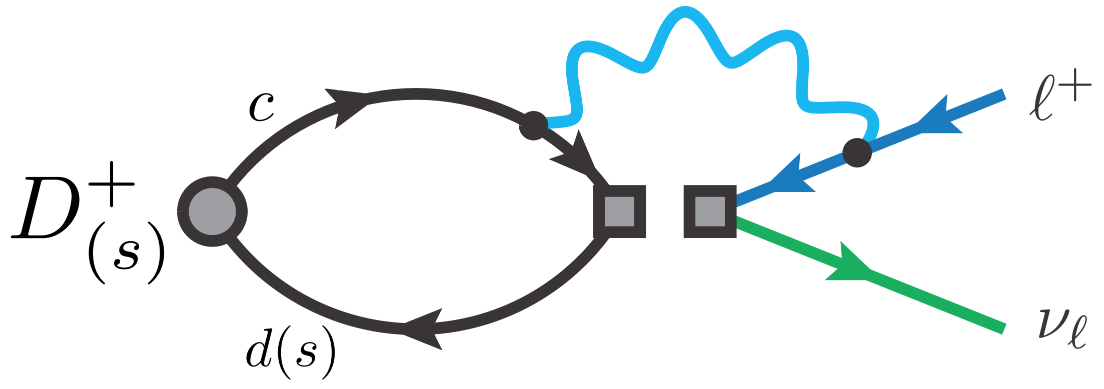
- Logarithmic IR divergences appear
- **but they cancel analytically between diagrams**
- Numerical calculation still **ongoing**...

... systematics under control?

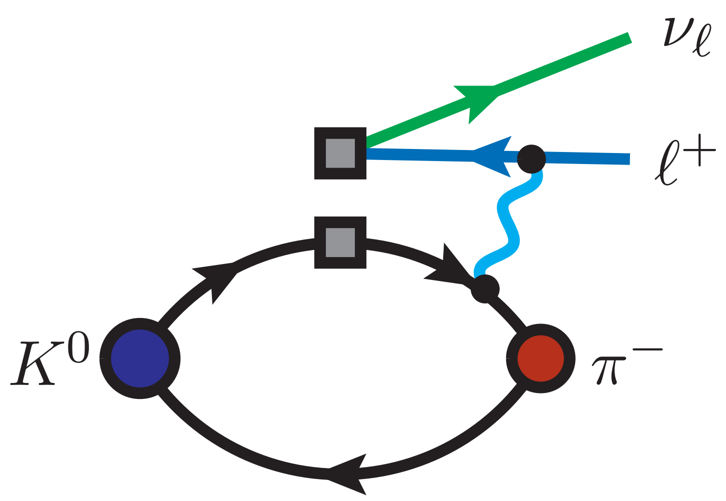
4. Where do we stand and where to go?

- **Two lattice calculations** of IB and QED corrections to **light-meson leptonic decay rates**
- Work in progress on independent calculation using **QED$_\infty$** by a third group
- Current main **challenges**: **electro-unquenching** & **finite volume QED effects**

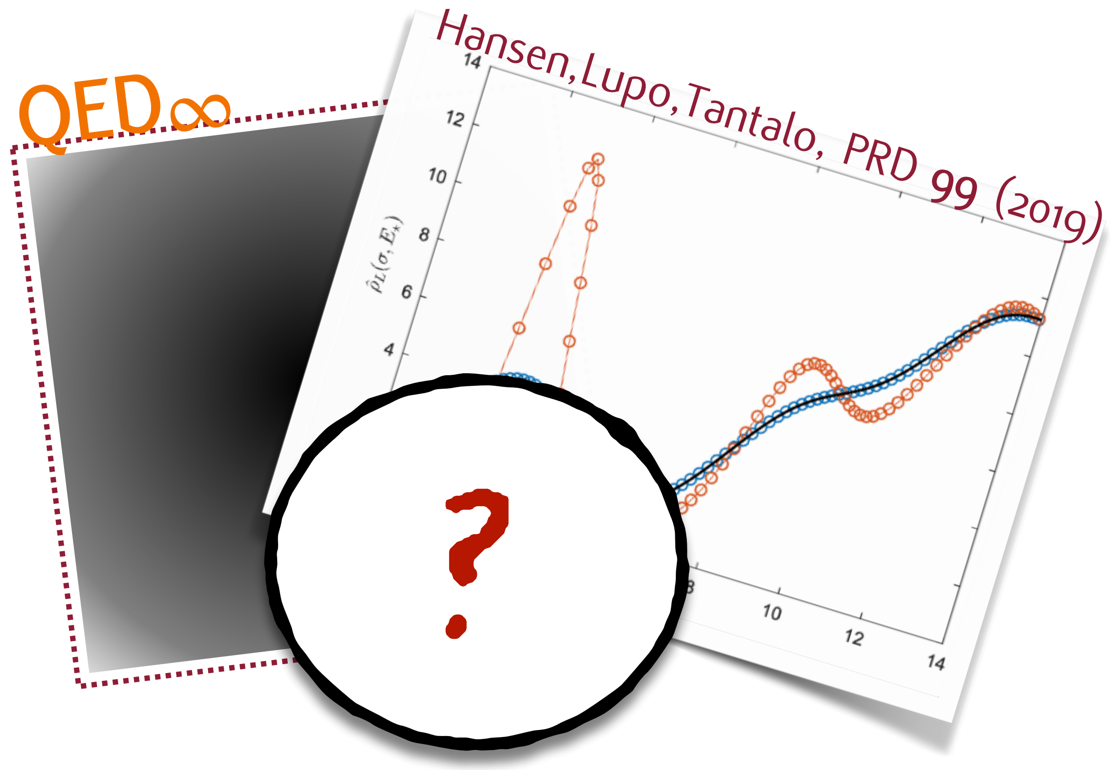
Other directions...



extend calculation to heavier mesons



tackle different weak processes



develop and apply new techniques

Thank you

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