DETERMINATION OF A GRADIENT FLOW SCALE

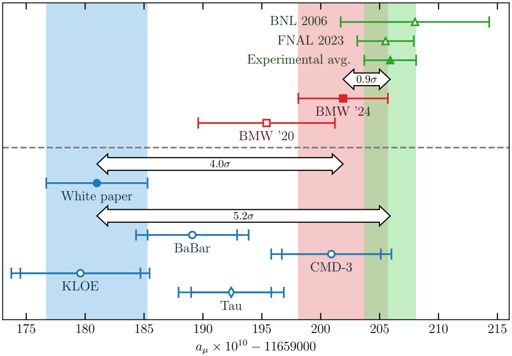
Kalman Szabo

together with Fabian Frech, Finn Stokes, Gen Wang for
Budapest-Marseille-Wuppertal collaboration

1. our use case
2. w_0 from Omega baryon mass [2407.10913]
3. w_0 from pion decay rate

Scale precision for HVP

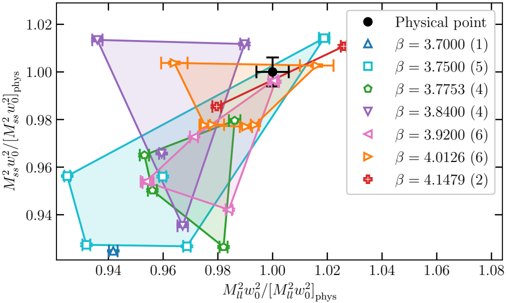
- rule of thumb: $\delta a/a = 1\% \to \delta a_\mu/a_\mu = 2\%$ → sub-percent needs scale determination with few per-mill



arxiv: 2407.10913

- $a_\mu^{\mathrm{LO-HVP}} = 714.1(2.2)(2.5)[3.3]$ using omega mass
- $w_0 = 0.17245(22)(46)[51]$ fm used in intermediate step

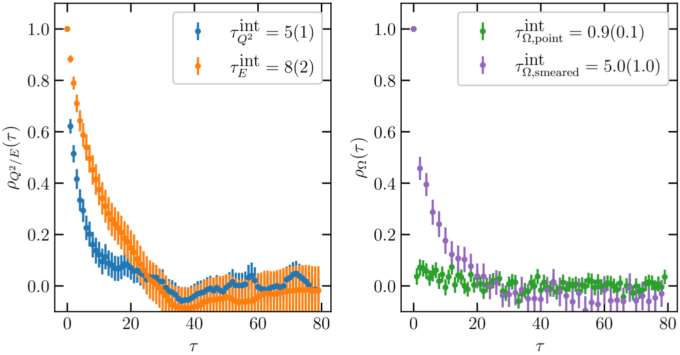
4stout simulations



β	a[fm]	#conf
3.7000	0.1315	900
3.7500	0.1191	2100
3.7753	0.1116	1900
3.8400	0.0952	3100
3.9200	0.0787	4300
4.0126	0.0640	7000
4.1479	0.0483	5000

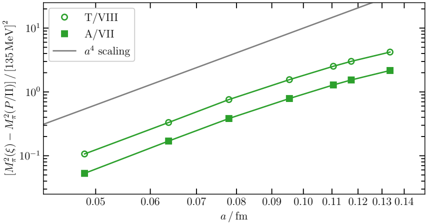
- `4stout` smeared, 2+1+1 flavor staggered, Symanzik improved gauge
- m_l, m_s bracketing the physical point, $m_c/m_s = 11.85$ within one per-cent of FLAG value
- $L \approx 6$ fm spatial, $T \approx 9$ fm temporal extent

4stout autocorrelations



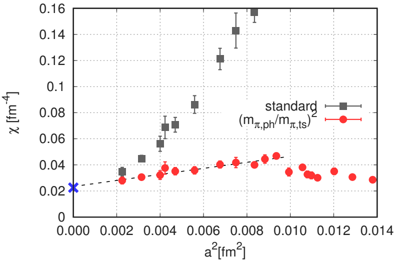
- normalized autocorrelation function and integrated autocorrelation times for finest lattice in units of configs (10 RHMC trajectories)
- $\tau^{\rm int} \leq 10$ for considered observables
- jackknife resampling 48 blocks $\rightarrow B \approx 5\tau^{\rm int}$ on finest

4stout taste violation



- mass of pion tastes
  $M_\pi^2(\xi) = M_{ll}^2 + \Delta_{KS}(\xi)$
- decreases faster than naive $\alpha_s a^2$ - more like $\alpha_s^3 a^2 \approx a^4$
- on finest lattice RMS is 5% larger than physical

- topological susceptibility highly sensitive to taste violation
- on finest lattice $\chi = 0.0358(29)$ and $\chi_{\text{imp}} = 0.0299(24)$
- cont. extrap is consistent with LO XPT 0.0224(12)



Omega operators

7 tastes degenerate in continuum, 2 operators [Golterman'84]

$$\Omega_{\rm VI}(t) = \sum_{x_k \rm even} \epsilon_{abc} \left[S_1 \chi_a S_{12} \chi_b S_{13} \chi_c - S_2 \chi_a S_{21} \chi_b S_{23} \chi_c + S_3 \chi_a S_{31} \chi_b S_{32} \chi_c \right] (x),$$

$$\Omega_{\rm XI}(t) = \sum_{x_k \rm even} \epsilon_{abc} [S_1 \chi_a S_2 \chi_b S_3 \chi_c](x).$$

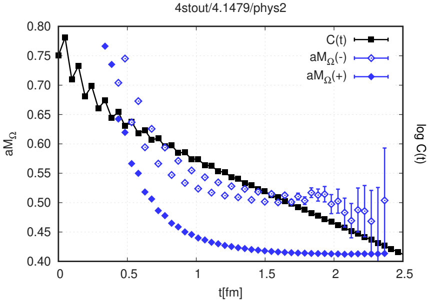
both couple to two tastes, disentangling by cross-correlations

can also construct operator coupling to a single taste [Bailey'06]

$$\Omega_{\rm Ba}(t) = \left[2\delta_{\alpha 1}\delta_{\beta 2}\delta_{\gamma 3} - \delta_{\alpha 3}\delta_{\beta 1}\delta_{\gamma 2} - \delta_{\alpha 2}\delta_{\beta 3}\delta_{\gamma 1} + (\ldots \beta \leftrightarrow \gamma \ldots) \right] \cdot$$
$$\cdot \sum_{x_k \rm even} \epsilon_{abc} \left[S_1 \chi_{a\alpha} S_{12} \chi_{b\beta} S_{13} \chi_{c\gamma} - S_2 \chi_{a\alpha} S_{21} \chi_{b\beta} S_{23} \chi_{c\gamma} + S_3 \chi_{a\alpha} S_{31} \chi_{b\beta} S_{32} \chi_{c\gamma} \right] (x)$$

by working with multi-flavor operators

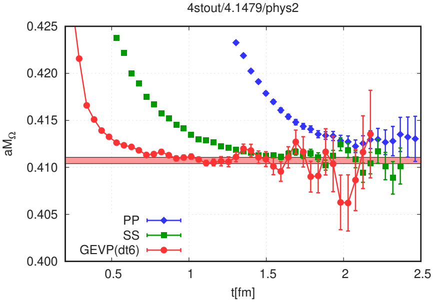
Omega propagators



- beside positive parity oscillating negative parity

$$C(t) = A_0 e^{-M_0 t} + (-1)^t A_1 e^{-M_1 t} + \ldots$$

- supress excited state contamination
- 3d gauge-link and quark-field smearing [Gusken'89]
- Z_3 grid source



GEVP/GPoF methods

cross-correlator of point and smeared operators

$$\begin{pmatrix} P(t)P(0) & P(t)S(0) \\ S(t)P(0) & S(t)S(0) \end{pmatrix}$$

cross-correlator time-shift operators $P_1 = e^{-H} P e^{H}$ [Aubin,Orginos'11]

$$\begin{pmatrix} P_0(t)P_0(0) & P_0(t)P_1(0) \\ P_1(t)P_0(0) & P_1(t)P_1(0) \end{pmatrix} = \begin{pmatrix} C_{PP}(t) & C_{PP}(t+1) \\ C_{PP}(t+1) & C_{PP}(t+2) \end{pmatrix}$$

$\rightarrow$ efficiently projects the oscillating states

combine the two techniques and solve Generalized Eigenvalue Problem [Blossier et al '09, ...]

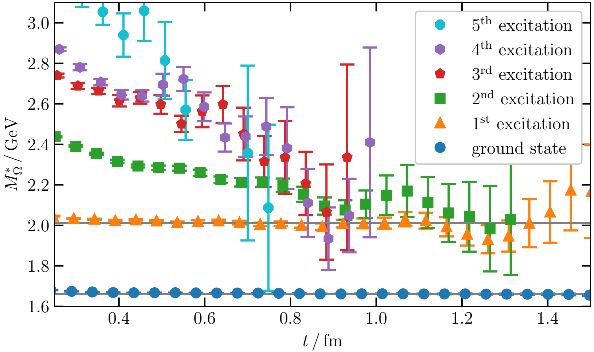
$$\mathbf{C}(t_a)v(t_a,t_b) = \lambda(t_a,t_b)\mathbf{C}(t_b)v(t_a,t_b)$$

propagator (v, λ) eigenstate is given in fixed-eigenvector approach

$$P(t;t_a,t_b) = v^{\dagger}(t_a,t_b)\mathbf{C}(t)v(t_a,t_b)$$

Omega GEVP spectrum

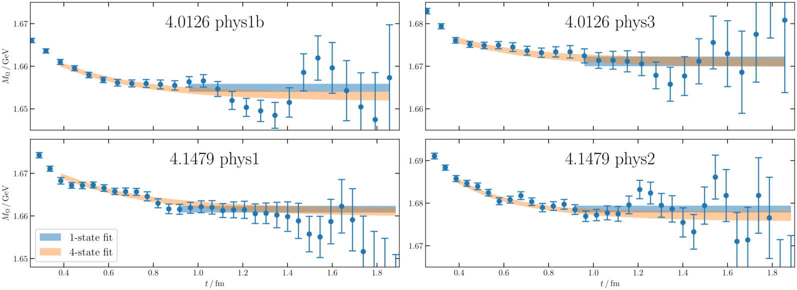
- build 6×6 cross-correlator of $\{P_0, P_1, S_0, S_1, S_2, S_3\}$



- ground and first excited states have nice plateaus - consistent across changing GEVP parameters (t_a, t_b)

- excited states compatible with PDG states $\Omega(2012, ***), \Omega(2250, ***), \Omega(2380, **), \Omega(2470, **)$

Omega GEVP ground state

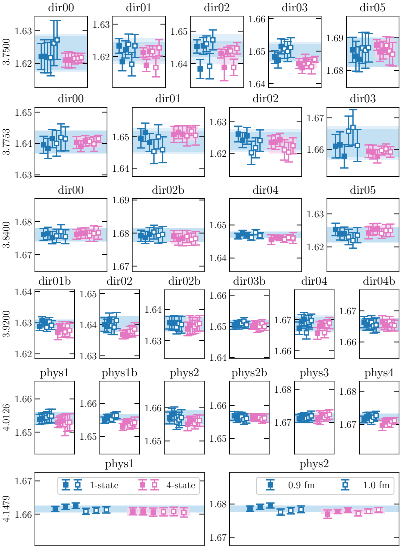


- single-exponential fit to the propagator to extract mass
- multi-exponential fits to check the single-exponential fits

$$A_0 e^{-M_0 t} + \cancel{A_1(-1)^t e^{-M_1 t}} + A_2 e^{-M_2 t} + A_3(-1)^t e^{-M_3 t} + A_4 e^{-M_4 t}$$

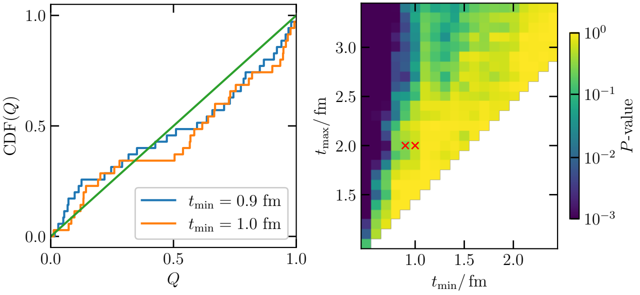
with (wide) priors from PDG values

Omega results



- 30k gauge configurations
- 10M propagator measurements
- GPU code based on QUDA and Qlattice
- correlated/uncorrelated fits are consistent
- different taste operators are consistent
- multi-exponential fits are consistent with single-exponential

Omega fitrange



- correlated fit with regulated small eigenvalues [Michael '94, ...]
- take improved estimator for fit-quality [Bruno,Sommer'22]
- perform Kolmogorov-Smirnov test across all ensembles → for $t_{\min} = 0.9/1.0$ fm pvalue is $0.27/0.57$

Omega scattering states

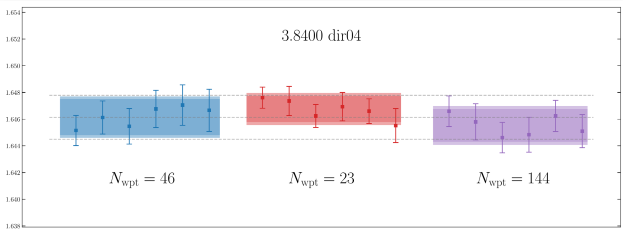
- scattering excited states might be possible $\Xi + K$, $\Xi + K + \pi$, lowest (1860) might spoil the ground state extraction
- point three quark operators are possibly weakly coupled
- how about smeared operators?

scattering states are spread out over the volume, change coupling by

1. changing smearing size
2. changing volume

Omega smearing study

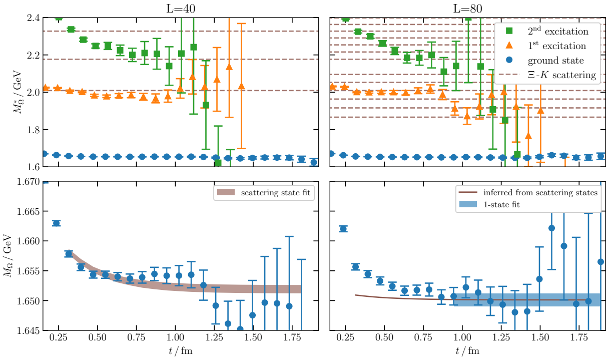
- changing smearing size changes localization radius of the operator
- larger smearing should couple more to scattering states



- $23 \to 46 \to 144$ increases operator eff. volume by factor ≈ 16
- ground state seems to be unaffected

Omega volume study

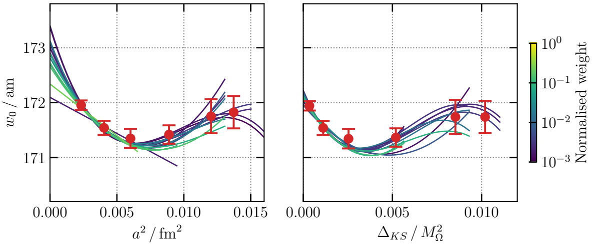
- do GEVP in two volumes: $a = 0.0793$ fm, $L = 40$ and 80
- simple volume scaling of masses and amplitudes, $M^2 + (2\pi/L)^2$ A/L^3, gives very little contamination on the large volume



- more elaborate estimate requires $\Omega - Xi - K$ coupling, might be extracted from the finite-size effect of ground state mass?

$w_0 M_\Omega$ cont. extrap.

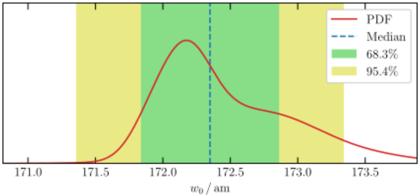
- w_0 from logarithmic derivative of gauge action density along gradient-flow time



- use fit functions like $Y = A_0 + A_1 a^2 + A_2 a^4 + A_3 a^6$
- in general leading behavior is $a^2 \alpha^n(a)$ [Husung et al '19]
  taste violation seems to scale with $n \approx 3$
  also use $a^2 \to \Delta_{KS}(a)$

$w_0 M_\Omega$ histogram

- estimate stat/syst errors and make error budgets



Median	172.35 am	
Total error	0.51 am	0.30 %
Statistical error	0.22 am	0.13 %
Systematic error	0.46 am	0.27 %
Pseudoscalar fit range	0.01 am	< 0.01 %
Omega baryon fit range	0.24 am	0.14 %
Physical value of M_Ω	0.06 am	0.03 %
Lattice spacing cuts	0.09 am	0.05 %
Order of fit polynomials	0.17 am	0.10 %
Continuum parameter (Δ_{KS} or a^2)	0.30 am	0.17 %

- apply flat weighting for systematic ingredients like omega operator, omega/pseudoscalar fit range

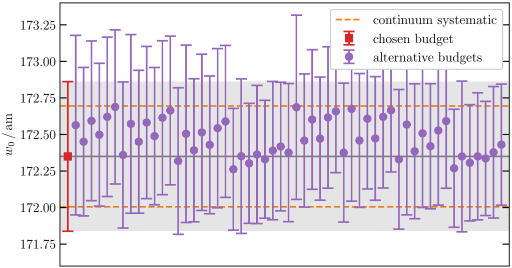
- apply AIC weighting for order of continuum extrapolation, a^2 vs Δ_{KS}, beta cuts

$$\exp\left(-\chi^2/2 + n_{\rm par} - n_{\rm data}/2\right)$$ [Akaike]

- build histogram 8640 different fits $\rightarrow$ median, $1\sigma/2\sigma$ bands

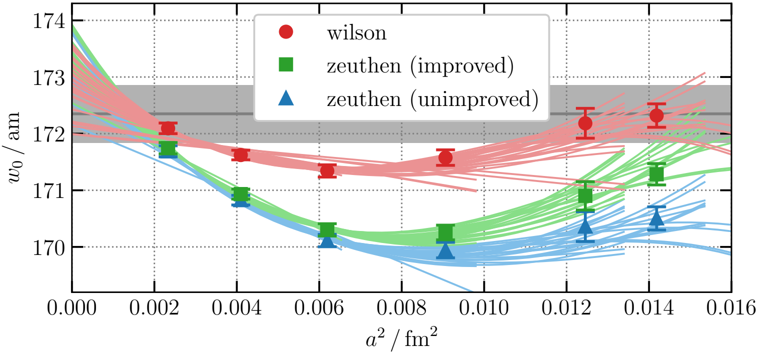
$w_0 M_\Omega$ alternative analyses

- mixed a^2 and Δ_{KS}, $\alpha_s(a)^n a^2$ instead of Δ_{KS}, only a^2
- drop 0, 1, 2 or 3 coarsest lattice completely
- apply fits only upto quadratic



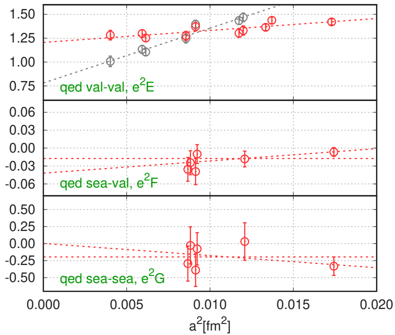
- consistent within our systematical error

$w_0 M_\Omega$ along Zeuthen flow



- using improved Symanzik and unimproved clover energy densites

$w_0 M_\Omega$ e.m. effects

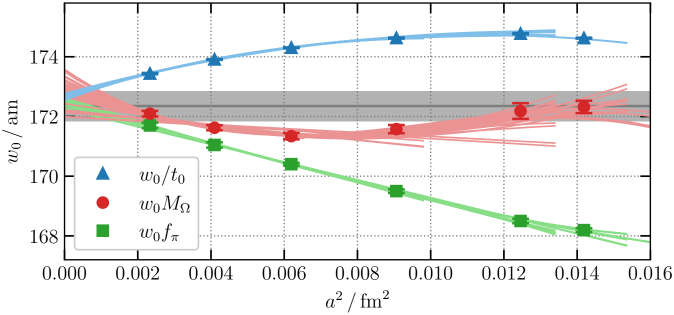


- add photon background and compute derivative $\partial/\partial e^2$ numerically or analytically
- photon can connect valence-valence, sea-valence and sea-sea quarks, w_0 has only ss contribution only, M_Ω all three
- there are also contributions from e.m. corrections on the quark masses
- $\to$ 0.00010 fm effect [BMW'21]

Final Result $w_0 = 0.17245(22)(46)[51]$ fm [BMW'24]

w_0 from other inputs

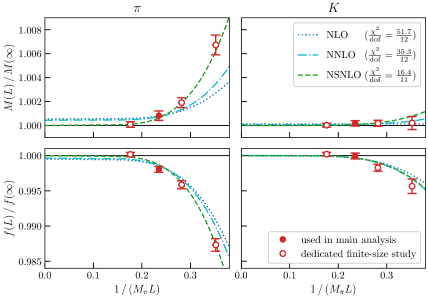
- consider alternatives to M_Ω



- use $f_\pi = 130.5$ MeV from [FLAG]
- use $\sqrt{t_0} = 0.1416(+8/-5)$ fm from [MILC'15] (from f_π)
- continuum extrapolated values are consistent (different isospin schemes, but we expect only small differences)

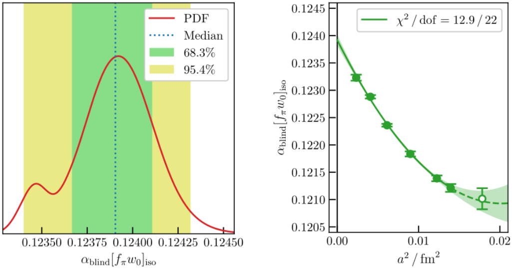
Decay constant f.v. effects

- aiming for per-mill precision → have to take many effects into account, eg. finite-volume



- finite-size effects are 0.1% in the continuum
- staggered artefacts considerably increase finite-size effects (pions become heavier, etas become lighter)
- use SXPT to control them [Aubin,Bernard,Bailey,. . .]

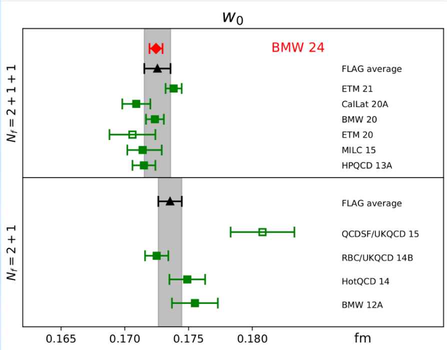
$w_0 f_\pi$ cont. extrap.



	median	0.123912	
1.	$[w_0]_{\rm phys}$ error	0.000034	0.027 %
2.	$[M_{ss}]_{\rm phys}$ error	0.000001	< 0.001 %
3.	Plateau region	0.000013	0.011 %
4.	Δ in effective mass	0.000006	0.005 %
5.	Effective mass form	0.000010	0.008 %
6.	Finite size correction	0.000013	0.010 %
7.	Hairpin parameters	0.000003	0.002 %
8.	Number of lattice spacings included	0.000040	0.032 %
9.	Order of A polynomial	0.000108	0.088 %
10.	Order of A' polynomial	0.000077	0.062 %
11.	Order of B polynomial	0.000006	0.005 %
12.	Order of C polynomial	0.000005	0.004 %

- aim for 0.1% precision on isospin symmetric analysis
- add e.m. effects in quenched QED [Giusti et al] and also in dynamical QED

Summary



- $w_0 = 0.17245(22)(46)[51]$ fm from omega 0.3%
- w_0 from pion decay under construction