Scale Setting with Gradient Flow on (2+1)-flavor HISQ ensembles

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[Larsen:2025wvg]

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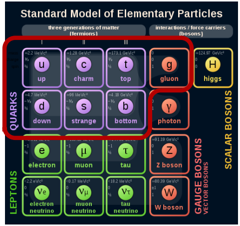
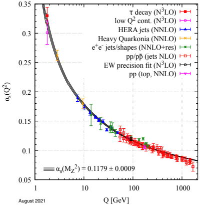
Scale setting: Precision lattice QCD for particle and nuclear physics, ECT*, Trento, Italy, 03/03/2025

QCD in the standard model

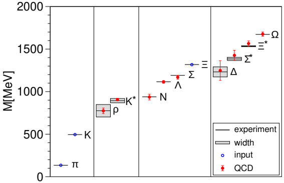
- **Standard Model of Particle Physics**: most elementary, **precisely tested** theory of nature; inconsistencies would provide hints of **New Physics**.
- **Nuclear matter** (quarks, gluons): **Quantum Chromodynamics (QCD)**.
- **Flavors**: u, d, s: **light**; c: **intermediate**; b: **heavy**; t: (irrelevant in QCD).



QCD running coupling



Light hadron spectrum



[BMW, Science 322, 2008]

- **Confinement:** neither quarks nor gluons are observable, but only **hadrons**.
- Non-perturbative QCD with **fully-controlled systematics** $\Rightarrow$ **lattice QCD**.

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Physical vs theory scales

- No explicit lattice scale, **determine** $a(g^2, \ldots)$ **in scale setting**.
- **Good: Cheap, precise, simple.** ✔
- **Bad: Cutoff, N_f, mass, volume dependence. Non-QCD effects.** ✗

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Absolute scale setting with physical scales

- **Absolute scale setting**: LQCD observable $\leftrightarrow$ **real world physical scale**.
- **Quark mass dependence or non-QCD effects** are relevant.
- **Low precision** of LQCD observables and real-world data may be an issue.

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Relative scale setting with theory scales

- **Relative scale setting**: LQCD observable $\leftrightarrow$ **another LQCD observable**.
- **Quark mass dependence or non-QCD effects** are largely irrelevant.
- **High precision** of certain LQCD observables is easy to achieve.

Two different types of physical scales

Decay constants (isospin limit)

η_s ✔	$af_{\eta_s}^{\text{meas}} = af_0 + 2sam_s^{\text{input}}$	$= 128.34(85)$ MeV	[Davies:2009tsa].
Kaon ✔	$af_K^{\text{meas}} = af_0 + s(am_l^{\text{input}} + am_s^{\text{input}})$	$= 110.10(64)$ MeV	[HotQCD:2014kol].
P4S ✔	$af_{p4s}^{\text{meas}} = af_0 + 2sa(0.4\, m_s^{\text{input}})$	$= 108.89(49)$ MeV	[MILC:2012znn].
Pion ✗	$af_\pi^{\text{meas}} = af_0 + 2sam_l^{\text{input}}$	$= 92.28(14)$ MeV	[FLAG:2024oxs].

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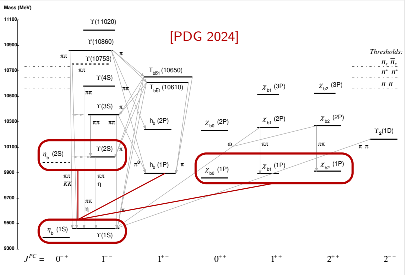
Bottomonia splittings



- Use **spin averages in** S, P**-waves**:
  $M_S = (M_\eta + 3M_\Upsilon)/4$,
  $M_{\chi_b} = (M_{\chi_{b0}} + 3M_{\chi_{b1}} + 5M_{\chi_{b2}})/9$.
- **Differences** $\Delta M_X = M_X - M_S(1S)$.
- **Splittings from** [PDG 2024].
  - $\Delta M_{h_b} = 454.5(0.9)$ MeV.
  - $\Delta M_{\chi_b} = 455.0(0.6)$ MeV.
  - $\Delta M_S(2S) = 584.5(2.3)$ MeV.
- Bottom mass $\Rightarrow$ **cutoff effects** ✗

Two different types of theory scales

Gradient flow scales

$\sqrt{t_i}$	$c_i = \tau_F^2 \langle E(\tau_F) \rangle \|_{\tau_F = t_i}$	$c_{0,2} = 0.3, 0.2$	[Luscher:2010iy].
w_i	$c_i = \tau_F \frac{\mathrm{d}}{\mathrm{d}\tau_F} \left[\tau_F^2 \langle E(\tau_F) \rangle \right] \|_{\tau_F = w_i^2}$	$c_{0,2} = 0.3, 0.2$	[BMW:2012hcm].

- For **coarse** lattices: t_1, $\sqrt{w_1}$ with $c_1 = 0.7$ introduced earlier [Borsanyi:2023wno].
- For **fine** lattices: t_2, $\sqrt{w_2}$ with $c_2 = 0.2$ introduced recently [Larsen:2025wvg].
- Gradient flow time τ_F: **smooth, arb. small increments, no cubic SB ✔**.

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Potential scales

- Static $Q\bar{Q}$ 4D-correlator via Wilson loops or gauge-fixed Wilson line corr.

  r_i $\quad c_i = (r/a)^2 \frac{d}{d(r/a)} \left[\langle aE_0(r/a) \rangle \right] |_{r = r_i}$ $\quad c_{1,2} = 1.0, 0.5$ [Sommer:1993ce].
- For **coarse** lattices: r_0 with $c_0 = 1.65$ [Sommer:1993ce].
- For **most** lattices: r_1 with $c_1 = 1.00$ [Bernard:2000gd].
- For **fine** lattices: r_2 with $c_2 = 0.50$ [Bazavov:2017dsy].
- Static $Q\bar{Q}$ separation r: **non-smooth, discrete, $O(3)$ to cubic SB ✗**.

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(2+1)-flavor HISQ ensembles

- **HotQCD ensembles**, $\frac{m_l}{m_s} = \frac{1}{20}$ or $\frac{1}{5}$ [HotQCD:2014kol],[Bazavov:2017dsy],[Altenkort:2023oms].
- Generated with **MILC** or **SIMULATeQCD** codes [MILC],[HotQCD:2023ghu].
- **Tree-level improved Lüscher-Weisz action** [Luscher:1984xn],[Luscher:1985zq].
- $(2+1)$ **flavors** with **HISQ action** [Follana:2006rc].
- Almost physical **light quarks**: $M_\pi = 161$ or 322 MeV, respectively.
- Physical **strange quarks**: unmixed $M_{\eta_s} = 695$ MeV instead of 685.8 MeV.

β	am_s	am_l	N_σ	N_τ	# conf.
7.030	0.0356	0.00178	48	48	900
7.150	0.0320	0.00160	48	64	395
7.280	0.0280	0.00142	48	64	398
7.373	0.0250	0.00125	48	64	554
7.596	0.0202	0.00101	64	64	577
7.825	0.0164	0.0082	64	64	471
8.000	0.01299	0.002598	64	64	1004
8.200	0.01071	0.002142	64	64	961
8.249	0.01011	0.002022	64	64	2241
8.400	0.00887	0.001774	64	64	2372

Overview

Formalism

- **Gradient flow** is a continuous, analytic smearing [Narayanan:2006rf], [Luscher:2009eq],

$$B_\mu(x,\tau_F=0)=A_\mu(x)\ ,\qquad \partial_{\tau_F}B_\mu(x,\tau_F)=D_\nu G_{\nu\mu}\ .$$
$$V(x,\mu)|_{\tau_F=0}=U(x,\mu)\ ,\qquad \partial_{\tau_F}V(x,\mu,\tau_F)=-g_0^2\{\partial_{x,\mu}S_G(V)\}V(x,\mu,\tau_F)\ .$$

- Realizes a **diffusive process** at tree level [Luscher:2010iy],

$$B_\mu(x,\tau_F)=\int \mathrm{d}^4y\, K_{\tau_F}(x-y)A_\mu(y)\ ,\quad K_{\tau_F}(z)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}e^{\mathrm{i}pz}e^{-\tau_F p^2}=\frac{\mathrm{e}^{-z^2/4\tau_F}}{(4\pi\tau_F)^2}\ .$$

  suppresses UV fluctuations: **reduces noise**, achieves renormalization.

- **Zeuthen flow** ($\mathcal{O}(a^2)$-improved evolution) for $\{\partial_{x,\mu}S_G(V)\}$: [Ramos:2015baa].

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- **Action density** as observable (four-volume averaged),

$$E(\tau_F,x)=-\tfrac{1}{2}\mathrm{Tr}\{F_{\mu\nu}(x,\tau_F)F_{\mu\nu}(x,\tau_F)\}\ .$$

- **Field strength** with clover or **improved discretizations** [Bilson-Thompson:2002xlt],

$$\hat{F}^{\mathrm{clo}}_{\mu\nu}(n)\ =-\frac{i}{8}\Big(Q_{\mu\nu}(n)-Q_{\nu\mu}(n)\Big)\equiv C^{(1,1)}_{\mu\nu}(n)\ ,$$
$$\hat{F}^{\mathrm{imp}}_{\mu\nu}(n)=\frac{5}{3}C^{(1,1)}_{\mu\nu}(n)-\frac{1}{3}C^{(1,2)}_{\mu\nu}(n)\ .$$

$Q_{\mu\nu}(n)$: sum of μ,ν plaquettes around site n; $C^{(a,b)}_{\mu\nu}(n)$ is generalization.

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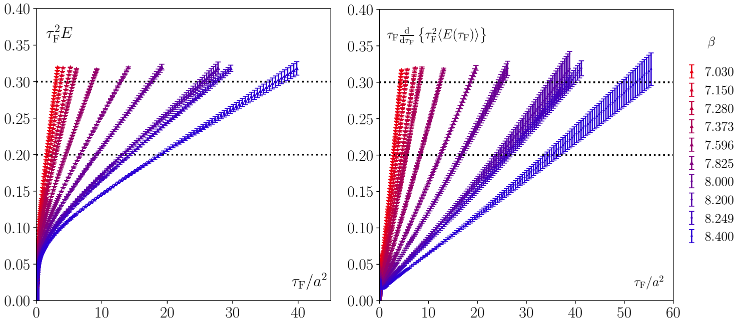
Improved flow

β	7.03	7.15	7.28	7.373	7.596	7.825	8.000	8.200	8.249	8.400
$\frac{\sqrt{t_0}}{a}$ (c)	1.8259(5)	2.0217(8)	2.2636(13)	2.4549(19)	2.9847(28)	3.634(9)	4.228(33)	5.083(79)	5.262(50)	6.076(108)
$\frac{\sqrt{t_0}}{a}$ (i)	1.7298(5)	1.9334(7)	2.1831(13)	2.3799(19)	2.9218(30)	3.586(7)	4.184(33)	5.044(78)	5.225(50)	6.045(106)
$\frac{\sqrt{t_2}}{a}$ (c)	1.3891(2)	1.5276(4)	1.6076(5)	1.8320(7)	2.2048(12)	2.669(3)	3.096(12)	3.688(21)	3.837(26)	4.399(32)
$\frac{\sqrt{t_2}}{a}$ (i)	1.2593(2)	1.4056(3)	1.5841(4)	1.7244(7)	2.1116(12)	2.591(3)	3.027(11)	3.629(20)	3.780(25)	4.347(29)
$\frac{w_0}{a}$ (c)	2.0730(13)	2.3159(21)	2.6203(36)	2.8575(48)	3.5194(73)	4.292(24)	4.978(83)	6.156(345)	6.233(81)	7.293(330)
$\frac{w_0}{a}$ (i)	2.0674(13)	2.3121(22)	2.6178(37)	2.8558(50)	3.5193(72)	4.304(19)	4.978(83)	6.156(345)	6.233(81)	7.297(333)
$\frac{w_2}{a}$ (c)	1.6832(8)	1.8826(13)	2.1302(15)	2.3258(27)	2.8653(40)	3.510(13)	4.083(47)	4.999(188)	5.109(60)	5.954(199)
$\frac{w_2}{a}$ (i)	1.6883(8)	1.8893(14)	2.1380(16)	2.3341(27)	2.8740(44)	3.524(11)	4.089(47)	5.006(189)	5.115(60)	5.962(201)



Beta function and scale ratios

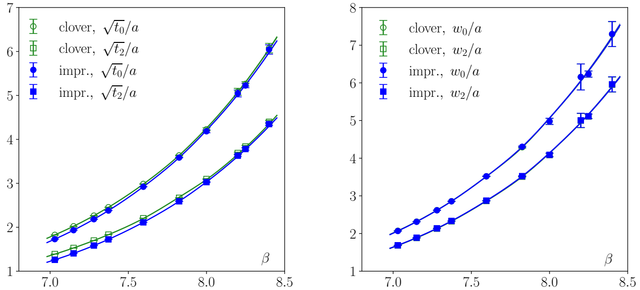
Beta function (Allton fits)

- **Beta function** with Allton-type Ansatz [Allton:1996dn],

$$\ln\left(\frac{\sqrt{t_i}}{a}\right) = \frac{1+d_{2i}\left(\frac{10}{\beta}\right)f^2(\beta)}{c_{0i}f(\beta)+c_{2i}\left(\frac{10}{\beta}\right)f^3(\beta)}\ , \quad \ln\left(\frac{w_i}{a}\right) = \frac{1+d'_{2i}\left(\frac{10}{\beta}\right)f^2(\beta)}{c'_{0i}f(\beta)+c'_{2i}\left(\frac{10}{\beta}\right)f^3(\beta)}\ ,$$

$$f(\beta) = \left(\frac{10b_0}{\beta}\right)^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{\beta}{20b_0}\right)\ , \quad \beta = \frac{10}{g_0^2}\ , \quad b_0 = \frac{9}{(4\pi)^2}\ , \quad b_1 = \frac{1}{(4\pi^4)} \ @\ N_f = 3\ .$$

- Two **different LCPs due to** m_l: include extra factor $(1 + c_{mi}(am_s + 2am_l))$.
- Allton fits at red. $\chi^2 = 0.8 \ldots 1.9$, except $\frac{\sqrt{t_2}}{a}$(clov) at red. $\chi^2 = 15.5$.



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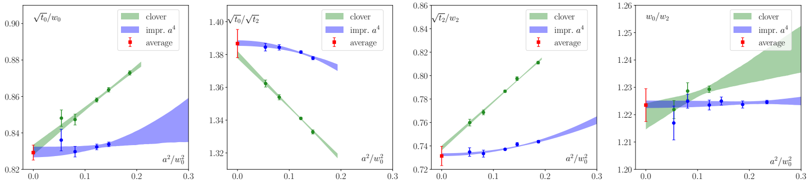
Flow scale ratios

Scale ratios

- $\frac{m_l}{m_s} = \frac{1}{20}$ ensembles: **ratios of flow scales** $\frac{\sqrt{t_0}}{w_0}, \frac{\sqrt{t_2}}{w_2}, \frac{\sqrt{t_0}}{\sqrt{t_2}}, \frac{w_0}{w_2}$, via clov./imp.
- Binning/bootstrapping determined by **larger auto-correlation time**.
- Clov. as a^2, imp. as $\alpha_s a^2$, a^2, a^4 via 3...6 betas, red. $\chi^2 \simeq 0.3 \ldots 1.9$.



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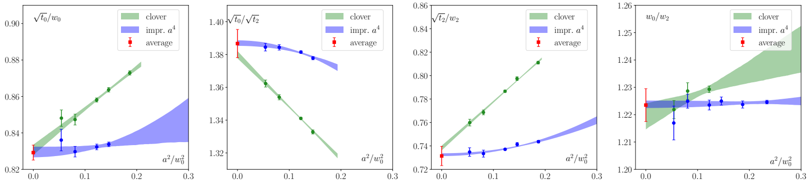

- Stat. error from wgt. mean of best fits (red. $\chi^2 \simeq 1.0$) of each form.
- Syst. error due to max deviation from weighted mean, symmetrized.

$$\frac{\sqrt{t_0}}{w_0} = 0.8292(26)(15)\,, \quad \frac{\sqrt{t_0}}{\sqrt{t_2}} = 1.3867(18)(68)\,, \quad \frac{\sqrt{t_2}}{w_2} = 0.7314(15)(67)\,, \quad \frac{w_0}{w_2} = 1.2235(19)(41)\,.$$

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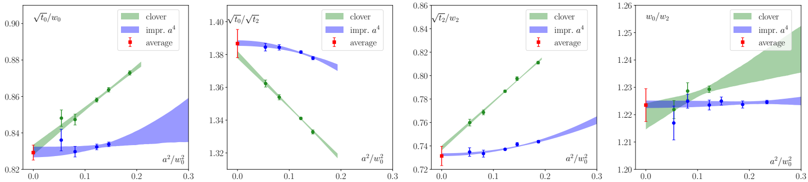

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- **Consistent** w. $N_f = 2+1+1$ estimates of ETM or HPQCD Collaborations:

$$\frac{\sqrt{t_0}}{w_0} = 0.82930(65) \qquad [\textit{ExtendedTwistedMass} : 2021\textit{qui}]\,,$$

$$\frac{\sqrt{t_0}}{w_0} = 0.835(8) \qquad [\textit{Dowdall} : 2013\textit{rya}]\,.$$

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Ratios with potential scales

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- $\frac{m_l}{m_s} = \frac{1}{20}$ ensembles: **ratios** $\frac{\sqrt{t_0}}{r_1}, \frac{w_0}{r_1}$, via clov./imp.
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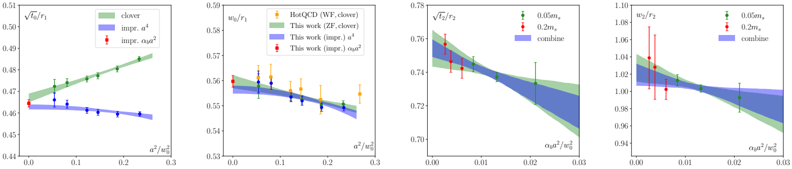

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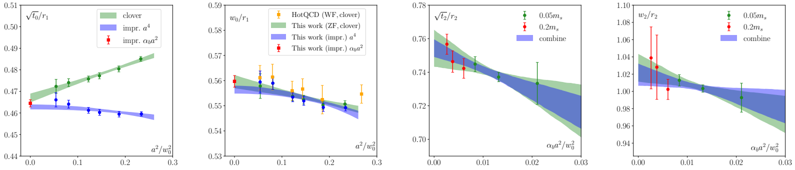

$$\frac{r_1}{\sqrt{t_0}} = 2.1530(78)\ , \quad \frac{r_1}{w_0} = 1.7866(74)\ , \quad \frac{\sqrt{t_2}}{r_2} = 0.7544(52)\ , \quad \frac{w_2}{r_2} = 1.0265(142)\ .$$

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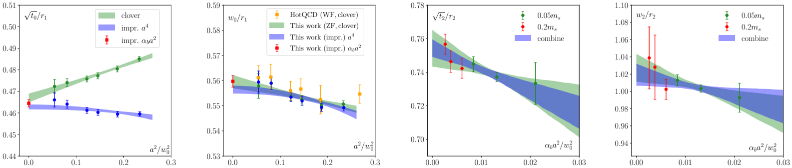

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- **Consistent** w. HotQCD, HPQCD, or combined TUMQCD & MILC:

$$\frac{r_1}{w_0} = 1.7797(67) \quad @\ N_f{=}2{+}1 \quad [HotQCD : 2014kol]\ ,$$
$$\frac{r_1}{w_0} = 1.789(26) \quad @\ N_f{=}2{+}1{+}1 \quad [Dowdall : 2013rya]\ ,$$
$$\frac{r_1}{\sqrt{t_0}} = 2.1601(83) \quad @\ N_f{=}2{+}1{+}1 \quad [MILC : 2015tqx]\ ,$$
$$\frac{r_1}{w_0} = 1.7749(96) \quad @\ N_f{=}2{+}1{+}1 \quad [Brambilla : 2022het]\ .$$

Extrapolation to physical point

- Pseudoscalar decay constants from **conserved axial current** with HISQ.
- Avoid via **strange quarks** much **taste-breaking or noise** (due to light q.).

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- Avoid via **strange quarks** much **taste-breaking or noise** (due to light q.).
- **Extrapolate** from $M_{\eta_s}^{\mathrm{LCP}} = 695$ MeV to $M_{\eta_s}^{\mathrm{phys}} = 685.8$ MeV, and $\frac{m_l}{m_s} = \frac{1}{27.3}$,

$$af_{\eta_s}^{\mathrm{phys}}(\beta) = af_{\eta_s}^{\mathrm{meas}}(\beta) + 2s \cdot \left[am_s^{\mathrm{LCP}}(\beta) \left(\frac{m_{\eta_s}^{\mathrm{phys}}}{m_{\eta_s}^{\mathrm{LCP}}} \right)^2 - am_s^{\mathrm{input}}(\beta) \right],$$

$$af_{K}^{\mathrm{phys}}(\beta) = af_{K}^{\mathrm{meas}}(\beta) + s \cdot \left[am_s^{\mathrm{LCP}}(\beta) \left(\frac{m_{\eta_s}^{\mathrm{phys}}}{m_{\eta_s}^{\mathrm{LCP}}} \right)^2 \cdot \frac{28.3}{27.3} - am_s^{\mathrm{input}}(\beta) \cdot \frac{21}{20} \right].$$

- LCP strange quark mass at each beta via [HotQCD:2014kol]

$$\frac{r_1}{a} \cdot am_s^{\mathrm{LCP}} = \frac{r_1}{a} \cdot am^{\mathrm{RGI}} \left(\frac{20 b_0}{\beta} \right)^{4/9} \frac{1 + m_1 \frac{10}{\beta} f^2(\beta) + m_2 (\frac{10}{\beta})^2 f^2(\beta) + m_3 \frac{10}{\beta} f^4(\beta)}{1 + dm_1 \frac{10}{\beta} f^2(\beta)}$$

- $\frac{r_1}{a}, am^{\mathrm{RGI}}, m_1, m_2, m_3, dm_1$, prop. errors of $\frac{r_1}{a}, am^{\mathrm{RGI}}$ via **Gaussian bootstrap**.

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Extrapolation to physical point

- Pseudoscalar decay constants from **conserved axial current** with HISQ.
- Avoid via **strange quarks** much **taste-breaking or noise** (due to light q.).
- **Extrapolate** from $M_{\eta_s}^{\mathrm{LCP}} = 695$ MeV to $M_{\eta_s}^{\mathrm{phys}} = 685.8$ MeV, and $\frac{m_l}{m_s} = \frac{1}{27.3}$,

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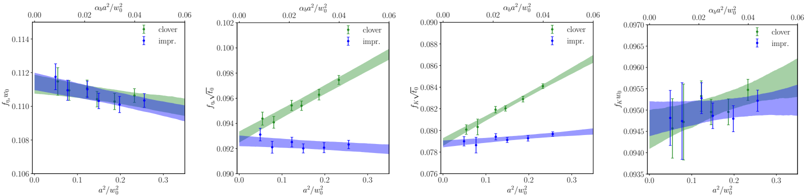
- $\frac{r_1}{a}, am^{\mathrm{RGI}}, m_1, m_2, m_3, dm_1$, prop. errors of $\frac{r_1}{a}, am^{\mathrm{RGI}}$ via **Gaussian bootstrap**.
- Assume **theory scales** $\frac{\sqrt{t_0}}{a}$, $\frac{w_0}{a}$, $\frac{r_1}{a}$ are **independent of LCP/mistuning**.

Flow scales from decay constants

Scale ratios

- $\frac{m_l}{m_s} = \frac{1}{20}$ ensembles: products $w_0 f_{\eta_s}, \sqrt{t_0} f_{\eta_s}, \sqrt{t_0} f_K, w_0 f_K$, via clov./imp.
- **Mixed bootstrap** due to extrapolation of f_{K,η_s} to physical point.
- a^2 for clov., $\alpha_s a^2$ or a^4 for imp.; via 6 betas, red. $\chi^2 \simeq 0.9 \ldots 1.1$.



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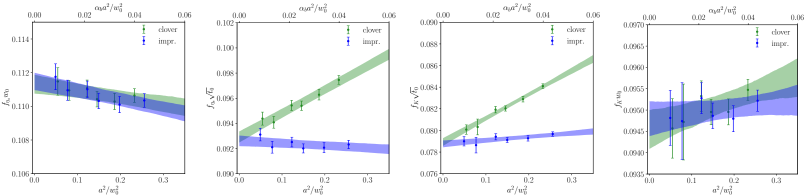


- All extrapolations agree, quoting $\alpha_s a^2$ **and imp.**:

	$f_{\eta_s}^{\text{meas}} w_0$	$f_{\eta_s}^{\text{meas}} \sqrt{t_0}$	$f_K^{\text{meas}} \sqrt{t_0}$	$f_K^{\text{meas}} w_0$	$\frac{r_1}{\sqrt{t_0}}$	$\frac{r_1}{w_0}$
clover	0.11132(56)	0.09292(44)	0.07893(36)	0.09455(45)	2.1530(78)	1.7866(74)
impr.	0.11148(52)	0.09261(42)	0.07876(32)	0.09480(41)	2.1506(86)	17835(80)

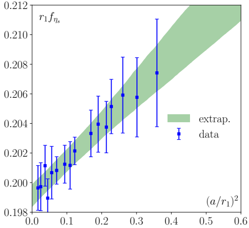
$\sqrt{t_0} = 0.14239(65)(94)$ fm , $w_0 = 0.17139(80)(113)$ fm , via f_{η_s} ,

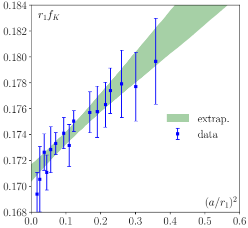
$\sqrt{t_0} = 0.14133(57)(82)$ fm , $w_0 = 0.17012(74)(98)$ fm , via f_K .

Potential scales from decay constants

Potential scales from decay constants

- Products of r_1 with $\sqrt{t_0}$, w_0, f_{η_s}, f_K .
- Continuum extrapolation of $r_1 f_{K,\eta_s}$ with **more, coarser lattices** [HotQCD:2014kol].





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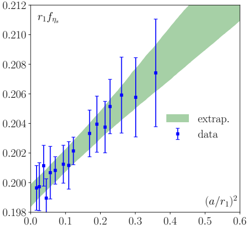

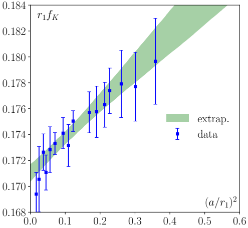

	$\sqrt{t_0}$	w_0	f_{η_s}	f_K
r_1	0.3053(37) fm	0.3050(38) fm	0.3062(23) fm	0.3065(22) fm

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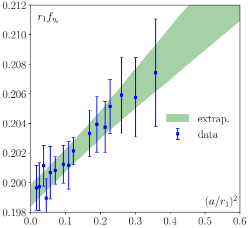

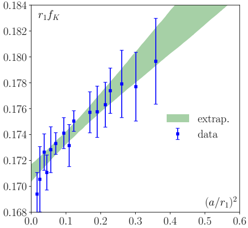

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- **Consistent** w. TUMQCD, **lower** than MILC, RBC, or HPQCD:

$r_1 = 0.3037(26)$ fm @ N_f=2+1+1 [*Brambilla* : 2022*het*] ,
$r_1 = 0.3106(8)(14)(4)$ fm @ N_f=2+1 [*MILC* : 2010*hzw*] ,
$r_1 = 0.3209(26)$ fm @ N_f=2+1 [*RBC* : 2010*qam*] ,
$r_1 = 0.333(9)$ fm @ N_f=2+1+1 [*Dowdall* : 2013*rya*] .

Lattice NRQCD

- **Bottomonia splittings** for scale setting [Gray:2005ur], [Davies:2009tsa], [HPQCD:2011qwj].
- Heavy quarks via $\mathcal{O}(v^6)$-**improved lattice NRQCD** [Meinel:2009rd], [Meinel:2010pv].
- Bottomonia levels computed on HotQCD ensembles [Larsen:2019zqv], [Ding:2025fvo].

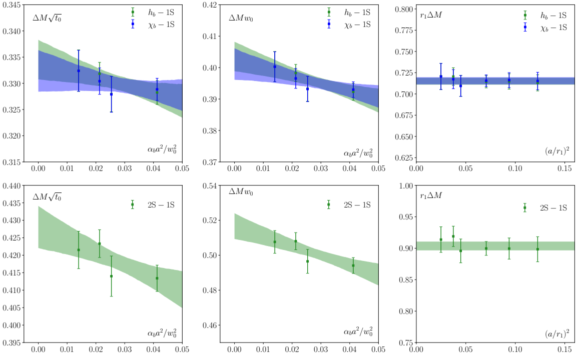
state	ΔM [MeV]	$\Delta M(PDG)$ [MeV]
$\Upsilon(3S)$	906.0(25.0)(5.2)	910.3(0.7)
$h_b(2P)$	804.4(35.8)(4.7)	814.9(1.3)
$\chi_{b2}(2P)$	809.2(36.2)(4.7)	823.8(0.9)
$\chi_{b1}(2P)$	802.2(34.9)(4.7)	810.6(0.7)
$\chi_{b0}(2P)$	786.8(32.7)(4.6)	787.6(0.8)
$\Upsilon(2S)$	582.7(9.8)(3.4)	578.4(0.6)
$h_b(1P)$	454.5(4.7)(2.6)	454.4(0.9)
$\chi_{b2}(1P)$	463.3(4.8)(2.7)	467.3(0.6)
$\chi_{b1}(1P)$	448.9(4.6)(2.6)	447.9(0.6)
$\chi_{b0}(1P)$	421.3(4.7)(2.4)	414.5(0.7)
hyperfine(3S)	13.4(6.2)(0.1)	NA
hyperfine(2S)	24.1(1.0)(0.1)	24.5(4.5)

- Lattice NRQCD levels **deviate from PDG** by 0.2-1.2% .

Scales from bottomonia splittings

Ratios with potential scales

- $\frac{m_l}{m_s} = \frac{1}{20}$ ensembles: extrapolate **products** of ΔM_X with $\sqrt{t_0}, w_0, r_1$ via imp.?
- Due to lattice NRQCD levels **estimate 1% systematic error** on ΔM_X .



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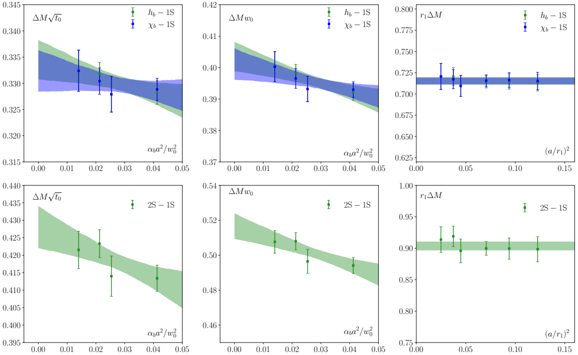

	$\Delta M\sqrt{t_0}$	$\Delta M w_0$	$\Delta M r_1$	$\sqrt{t_0}$ [fm]	w_0 [fm]	r_1 [fm]
$h_b - 1S$	0.3330(53)	0.4024(63)	0.7150 (41)	0.1453(17)(15)	0.1752(21)(18)	0.3105(18)(31)
$\chi_b - 1S$	0.3325(52)	0.4018(62)	0.7155(42)	0.1441(17)(14)	0.1739(21)(17)	0.3104(18)(31)
$2S - 1S$	0.4243(76)	0.5128(91)	0.9036(70)	0.1477(40)(15)	0.1783(49)(18)	0.3119(76)(31)

Systematically higher than $\sqrt{t_0}, w_0, r_1$ from decay constants, agrees w. MILC.

Comparison à la FLAG

- Choice of the physical scale matters at one sigma level!

	$\sqrt{t_0}$ [fm]	w_0 [fm]	r_1 [fm]
f_{η_s}	0.14239(65)(94)	0.17139(80)(113)	0.3062(23)
f_K	0.14133(57)(82)	0.17012(74)(98)	0.3065(22)
$h_b - 1\mathrm{S}$	0.1453(17)(15)	0.1752(21)(18)	0.3105(18)(31)
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$2\mathrm{S} - 1\mathrm{S}$	0.1477(40)(15)	0.1783(49)(18)	0.3119(76)(31)
wgt. avg.	0.14229 (98)	0.17190 (140)	0.3072 (22)

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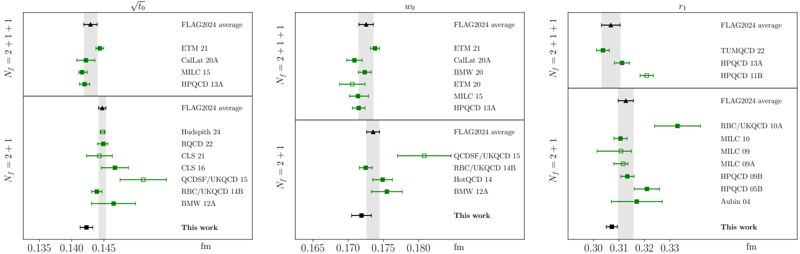
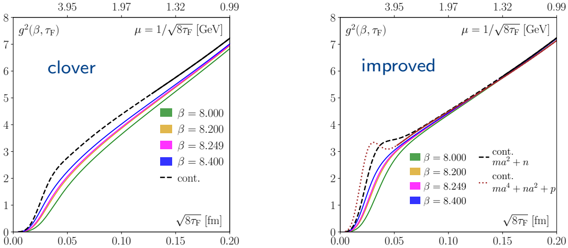


- $N_\mathrm{f} = 2+1$ or $N_\mathrm{f} = 2+1+1$ results are **consistent due to decoupling**.
- Our $N_\mathrm{f} = 2+1$ results are **lower than FLAG averages**.
- No obvious link between **staggered ensembles** and values of scales.

Gradient flow coupling

- Define **gradient flow coupling** $g^2_{\text{flow}} = \frac{128\pi^2\tau_F^2}{3(N_c^2-1)}\langle E\rangle$ [Fodor:2012td], [Hasenfratz:2019hpg], associated with a flow scale $\mu_{\text{flow}} \simeq \sqrt{\frac{1}{\tau_F}}$ (usually $\frac{1}{\sqrt{8\tau_F}}$).



- Solid lines: extrapolations w. red. $\chi^2 < 1.5$. Focus on imp/quartic.

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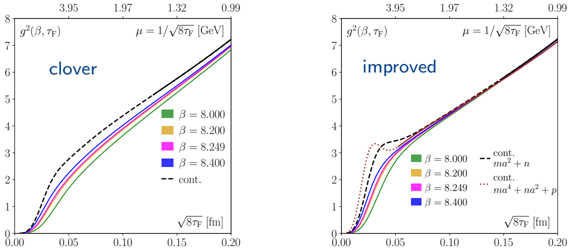
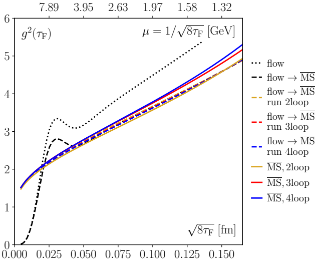


- Solid lines: extrapolations w. red. $\chi^2 < 1.5$. Focus on imp/quartic.
- **Scheme change** from gradient flow to $\overline{\text{MS}}$ known at NNLO [Harlander:2021esn]

$$\alpha_{\text{flow}} = \alpha_{\overline{\text{MS}}}\left(1 + k_1\alpha_{\overline{\text{MS}}} + k_2\alpha^2_{\overline{\text{MS}}}\right) \quad \text{for} \quad \mu^2_{\overline{\text{MS}}} = \mu^2_{\text{flow}} \quad ,$$

$$k_1 = 1.098 + 0.008N_f \ , \quad k_2 = -0.982 - 0.070N_f + 0.002N_f^2 \ .$$

Lambda extraction



- To control both **continuum extrapolation** and **scheme conversion** choose a **reference point** above $\mu_{\mathrm{flow}} \geq 1.278$ GeV (i.e. $\sqrt{8\tau_F} \leq 0.1543$ fm).
- g^2_{flow} from gradient flow to $\overline{\mathrm{MS}}$, ran up from ref. point $\mu_{\mathrm{flow}} \geq 1.278$ GeV vs $g^2_{\overline{\mathrm{MS}}}$ ran down from $g^2_{\overline{\mathrm{MS}}} = 0$ using $\Lambda_{\overline{\mathrm{MS}}} = 338$ MeV, at different loop orders.

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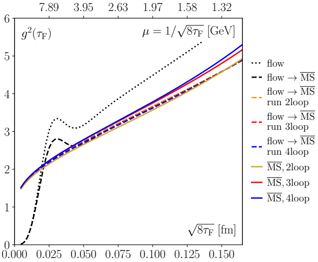

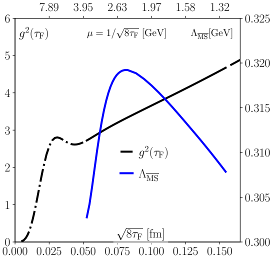

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- **Strong dependence of $\Lambda_{\overline{\text{MS}}}$ on τ_F** in conversion, below $\Lambda_{\overline{\text{MS}}} = 338$ MeV:

$$\Lambda_{\overline{\text{MS}}}^{N_f=3} = 311.0^{+0.7\,+4.8\,+34.0}_{-0.7\,-4.8\,-11.7} \text{ MeV} , \quad \alpha_{\overline{\text{MS}}}^{(N_f=5)}(M_Z) = 0.1162^{+0.0023}_{-0.0009} .$$

- Stat. error via bootstrap; symmetrized **cont. error** (leaving out $\beta = 8.0$).
- **Truncation error estimate** by including $\alpha^3_{\overline{\text{MS}}}$ with coefficient $k_3 = \pm 2k_2$.

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Summary

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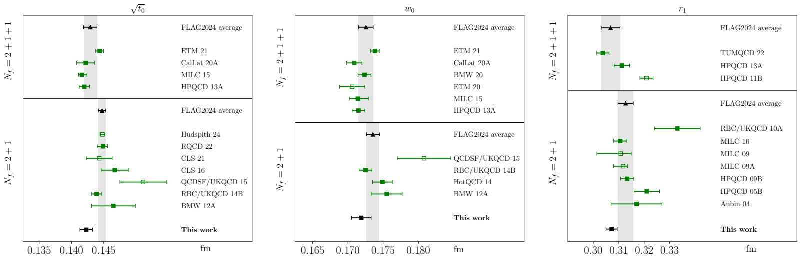

- We obtain **gradient flow and potential scales** in $(2+1)$-flavor QCD.
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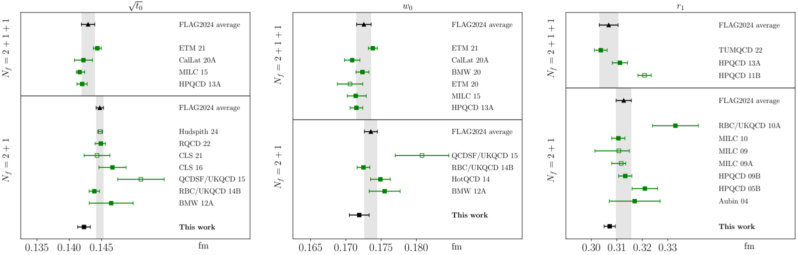

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Thank you for your attention!

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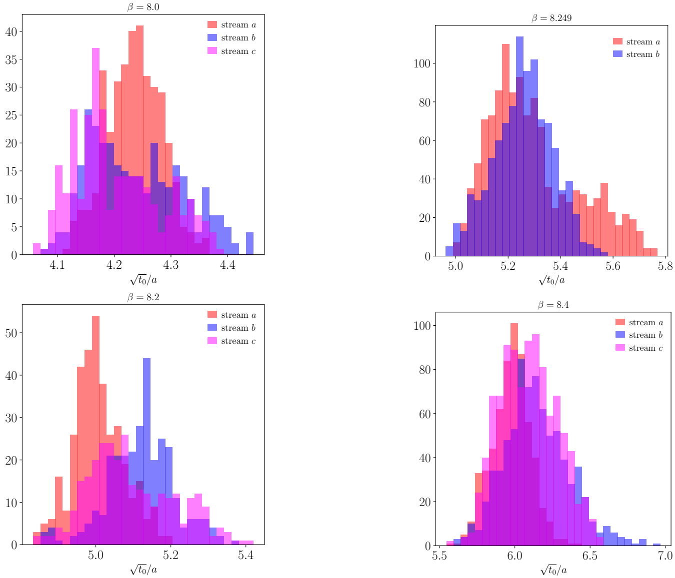
Effects of toplogy on $\frac{\sqrt{t_0}}{a}$



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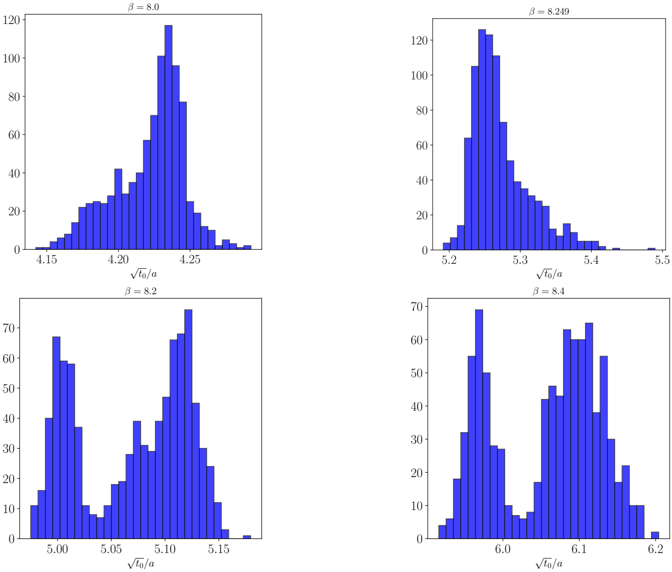

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