Scale setting: motivation and FLAG overview

Alexei Bazavov, Sara Collins, Simon Kuberski,
Rainer Sommer

Trento, 3 March 2025

Sources of information

FLAG 2024 review

Some simple considerations

Compilations / review of g−2 by S. Kuberski

Lattice gauge theory for low energy processes

Framework

low energy effective theory

QCD + QED

Approximation

QCD

renormalizable QFTs (perturbative in $\alpha \equiv \alpha_{em}$)

"scale setting" = part of renormalization

Renormalization schemes

Perturbative schemes

$\alpha_s \equiv \alpha_{\overline{MS}}(m_Z), \bar{m}_j(x\ GeV)$ (+ α_{em})

impractical

need RGI's $\Lambda_{\overline{MS}}, M_j^{\mathrm{RGI}}$ for precise definitions/values

—> step scaling or faith

often gauge fixing

Hadronic renormalization schemes

(+ α_{em})

Natural + common practice + theoretically sound

—> we only consider those

hadron masses/ properties/ scales —> bare $g_0, m_{0,i}$ parameters —> predictions

Hadronic renormalization scheme, scale setting

Renormalization conditions

$$\frac{M_i(g_0,\{am_{0,\mathrm{j}}\})}{M_1(g_0,\{am_{0,\mathrm{j}}\})} = \frac{M_i^{\mathrm{exp}}}{M_1^{\mathrm{exp}}}\,, \quad i = 2\ldots N_\mathrm{f}+1\,, \quad j = 1\ldots N_\mathrm{f}\,. \qquad (+\ \alpha_{em})$$

(with hadron masses M_i) define LCP

$$am_{0,\mathrm{j}} = \mu_j(g_0)$$

Observable $\mathcal{O}$ with dimension $[\mathcal{O}] = d_\mathcal{O}$ is then predicted as

$$\mathcal{O}^{\mathrm{cont}} = \left(M_1^{\mathrm{exp}}\right)^{d_\mathcal{O}} \lim_{aM_1\to 0} \hat{\mathcal{O}}(aM_1) \quad \text{with} \quad \hat{\mathcal{O}}(aM_1) = \frac{\mathcal{O}}{M_1^{d_\mathcal{O}}}\Bigg|_{am_{0,j}=\mu_j(g_0)}$$

“The” scale: $M_1 \equiv \mathcal{S}$.

Useful to replace M_1 by theory scales, $\mathcal{S}^{-1} = r_0,\, r_1,\, \ldots\,,\, \sqrt{t_0},\, w_0$ because of

— excited state corrections
— extrapolation to physical point
— statistical precision

theory scales predicted as above

Impact / relevance 0

Predictions for phenomenology do depend on the precision of the LCP in general. Scale + other quantities “setting the quark masses”.

Often the uncertainty of the scale may be dominant /relevant.

Examples follow

Impact / relevance 1

Lambda-parameter $\Lambda = \Lambda_{\overline{MS}}$ (or $\alpha_s(m_z)$)

recent ALPHA-result: $\Lambda = 343.9(8.4)\,\text{MeV}$, $\frac{\delta\Lambda}{\Lambda} = 2.4\,\%$ (most precise result) [2501.06633]

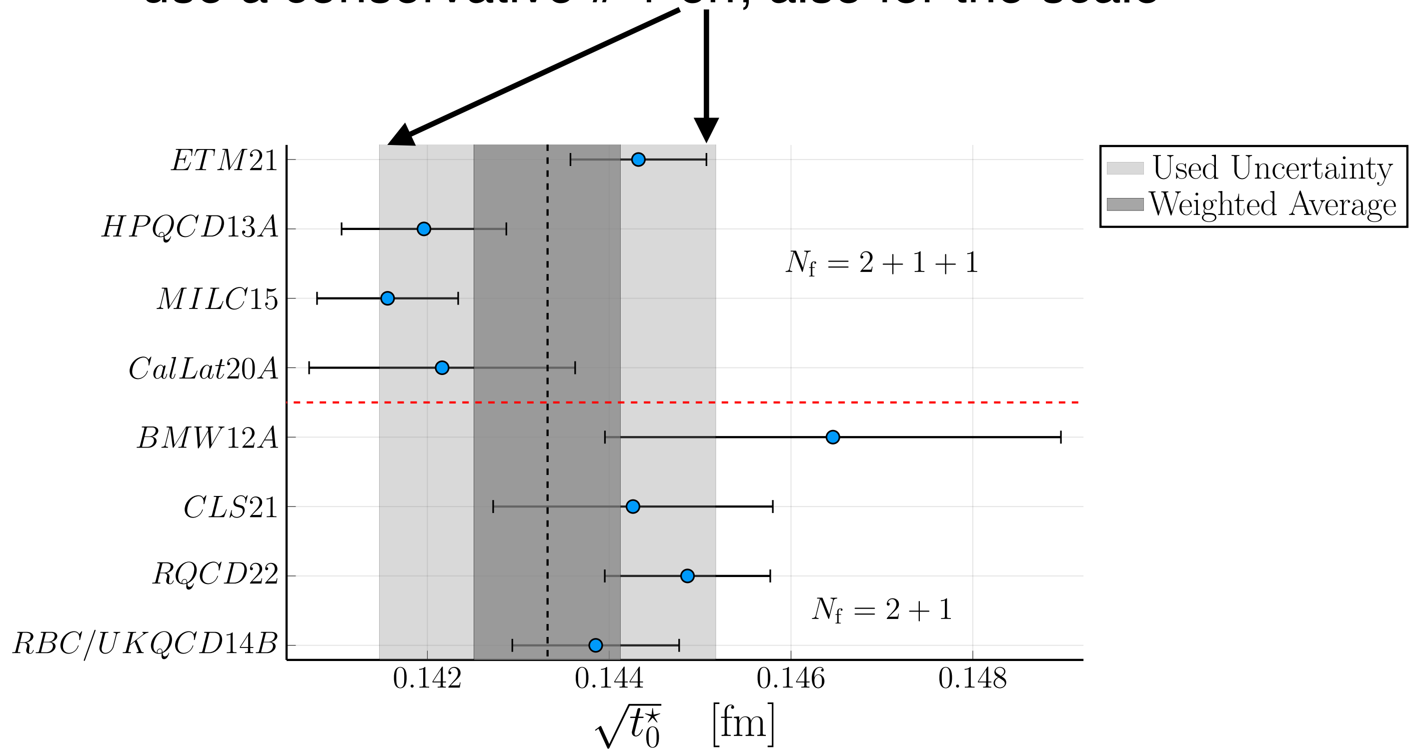
$$\left(\frac{\delta\Lambda}{\Lambda}\right)^2 = \ldots + \left(\frac{\delta t_0}{2t_0}\right)^2, \quad \frac{\delta t_0}{2t_0} = 1.3\,\%$$ —> unpleasant 25% in squared error

note: FLAG 2024, 2+1: $\sqrt{t_0} = 0.1447(\ \ 6)$ (0.4%) difference:

2+1+1: $\sqrt{t_0} = 0.1429(10)$ (0.7%) 1.3%

The result is very relevant: to be used in the analysis of LHC data etc.

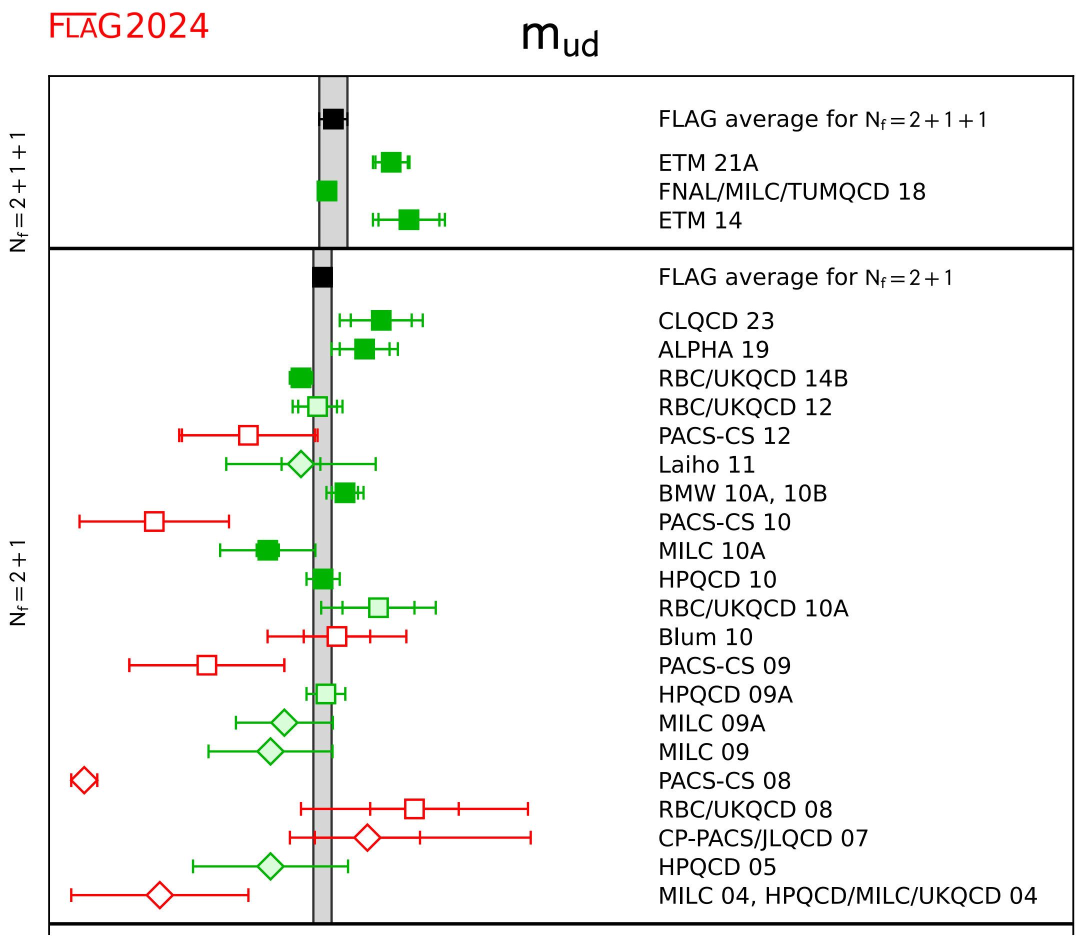
—> use a conservative # + err, also for the scale

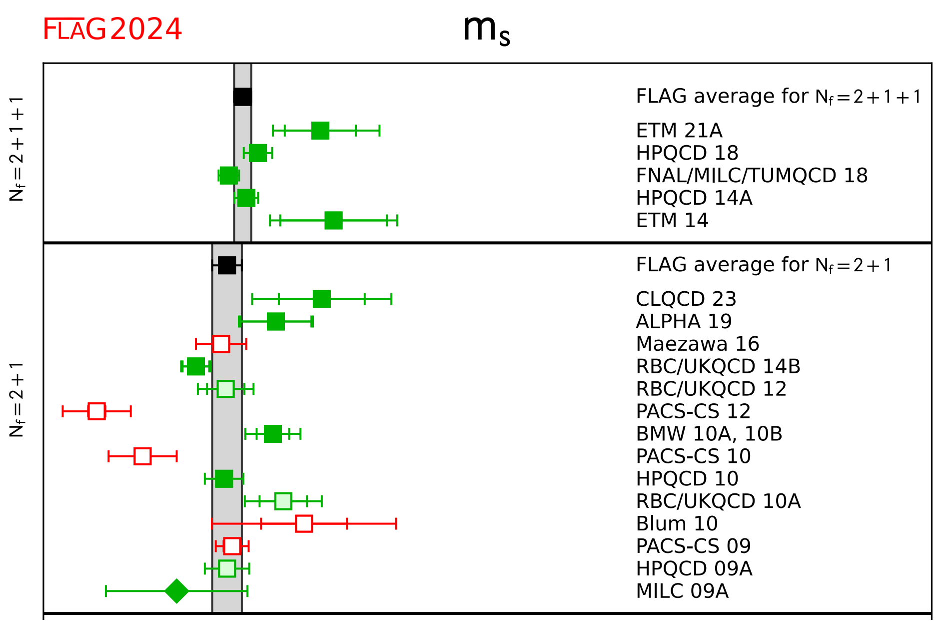


Impact / relevance 2

Light quark masses

$$m_{ud} = \frac{1}{\mathcal{S}} \times [M_\pi^2]_{\text{exp}} \times [\frac{m_{ud}\,\mathcal{S}}{M_\pi^2}]_{\text{lat}}\,, \qquad \frac{\delta m_{ud}}{m_{ud}} \approx \frac{\delta\mathcal{S}}{\mathcal{S}}$$



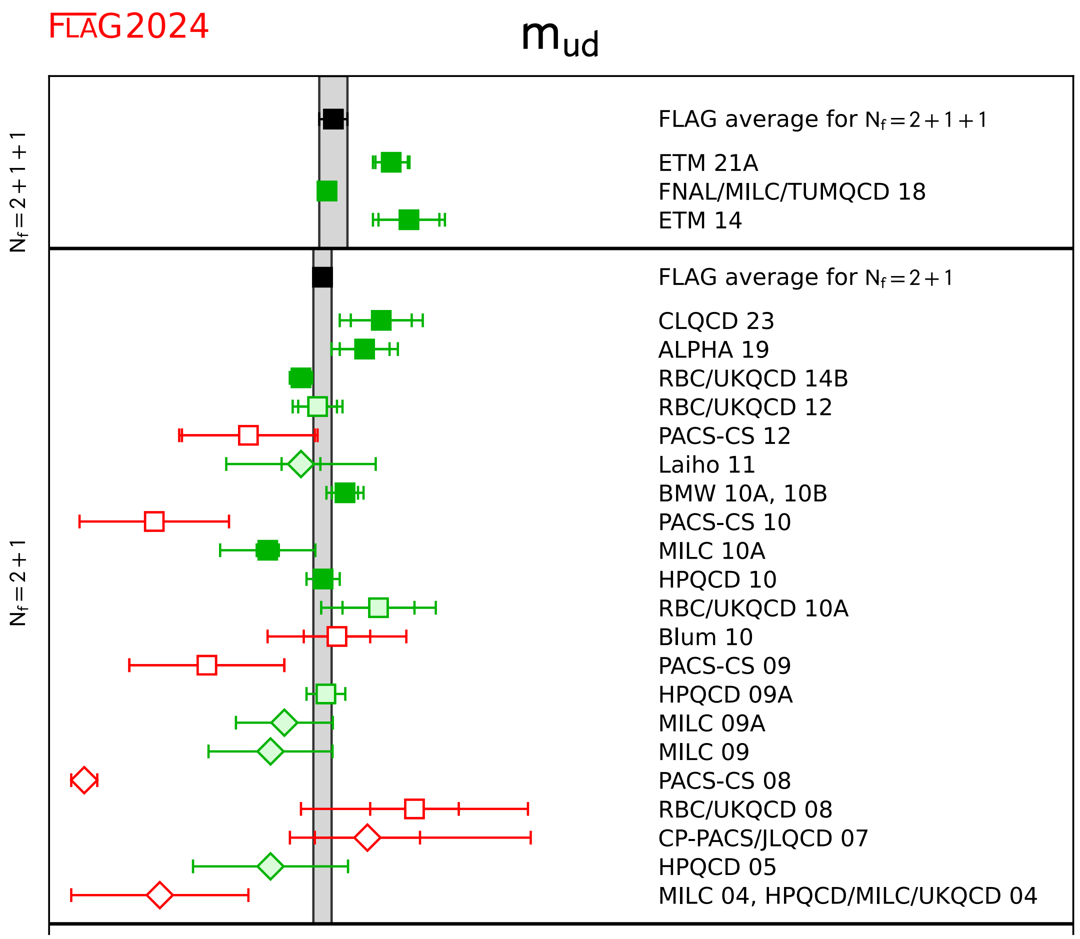


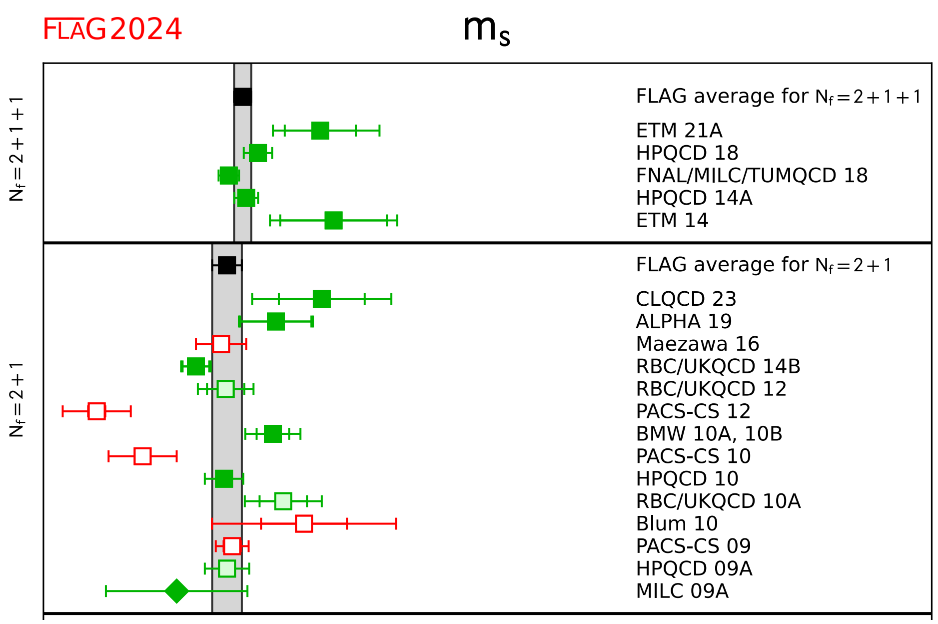
Spread partially due to scales?
not investigated by FLAG

Impact / relevance 2

Light quark masses

$$m_{ud} = \frac{1}{\mathcal{S}} \times [M_\pi^2]_{\rm exp} \times [\frac{m_{ud}\,\mathcal{S}}{M_\pi^2}]_{\rm lat}\,, \qquad \frac{\delta m_{ud}}{m_{ud}} \approx \frac{\delta \mathcal{S}}{\mathcal{S}}$$





Spread partially due to scales?
not investigated by FLAG

relevant at the level of these errors (2+1+1)

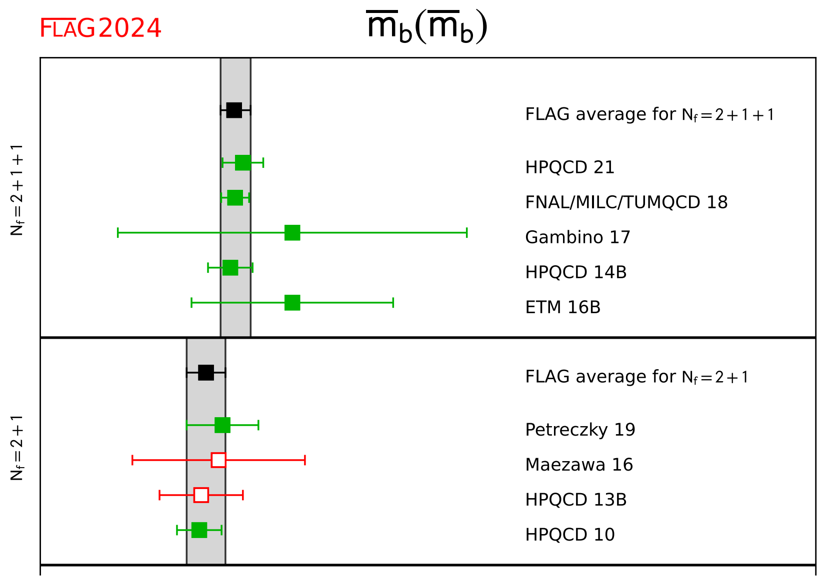
ETM 21A	[7]	A	★	★	★	★	–	$3.636(66)(^{+60}_{-57})$	$98.7(2.4)(^{+4.0}_{-3.2})$
HPQCD 18[†]	[17]	A	★	★	★	★	–		94.49(96)
FNAL/MILC/TUMQCD 18	[9]	A	★	★	★	★	–	3.404(14)(21)	92.52(40)(56)

Impact / relevance 2

Heavy quark masses

b-quark, approximately, HQET-inspired:

$$m_b = m_B^{\mathrm{exp}} + \mathcal{S} \times \left[\frac{m_b - m_B}{\mathcal{S}}\right]_{\mathrm{lat}}, \quad \frac{\delta m_b}{m_b} \approx \frac{1}{10}\frac{\delta \mathcal{S}}{\mathcal{S}}$$



1/10 suppression —> not very relevant

anyway, different problems for b-quarks

but some very small uncertainties are cited:

FNAL/MILC/TUM 18	[9]	2+1+1	A	★	○	★	–	✓	4.201(12)(1)(8)(1)

Impact / relevance 3.1

muon g-2: a_μ^{hvp}

S. Kuberski, Santa Fe workshop 2023 :

SCALE DEPENDENCIES

- Scale enters via muon mass in $\tilde{K}(t)$. Determine the scale dependence via

$$\frac{\partial (a_\mu^{\text{hvp}})^i}{\partial_\Lambda} = \left(\frac{\alpha}{\pi}\right)^2 \sum_0^\infty dt \left[\left(\frac{\partial}{\partial_\Lambda}\widetilde{K}(t)\right) W^i(t;t_0;t_1) + \widetilde{K}(t)\left(\frac{\partial}{\partial_\Lambda} W^i(t;t_0;t_1)\right)\right] G(t)$$

- Using a parametrization of the R-ratio, the Mainz group estimated $\frac{\Delta a_\mu^{\text{hvp},\Lambda}}{a_\mu^{\text{hvp}}} \approx 1.8 \frac{\Delta\Lambda}{\Lambda}$ [Della Morte et al., 1705.01775] $\rightarrow$ What about the windows?

- My **rough estimates** for $\frac{\Delta (a_\mu^{\text{hvp},\Lambda})^i \Lambda}{(a_\mu^{\text{hvp}})^i \Delta\Lambda}$ at m_π^{phys}:

$\delta a_\mu^{\text{hvp}}$	$\delta(a_\mu^{\text{hvp}})^{\text{SD}}$	$\delta(a_\mu^{\text{hvp}})^{\text{ID}}$	$\delta(a_\mu^{\text{hvp}})^{\text{LD}}$
1.8	0.0	0.5	2.7

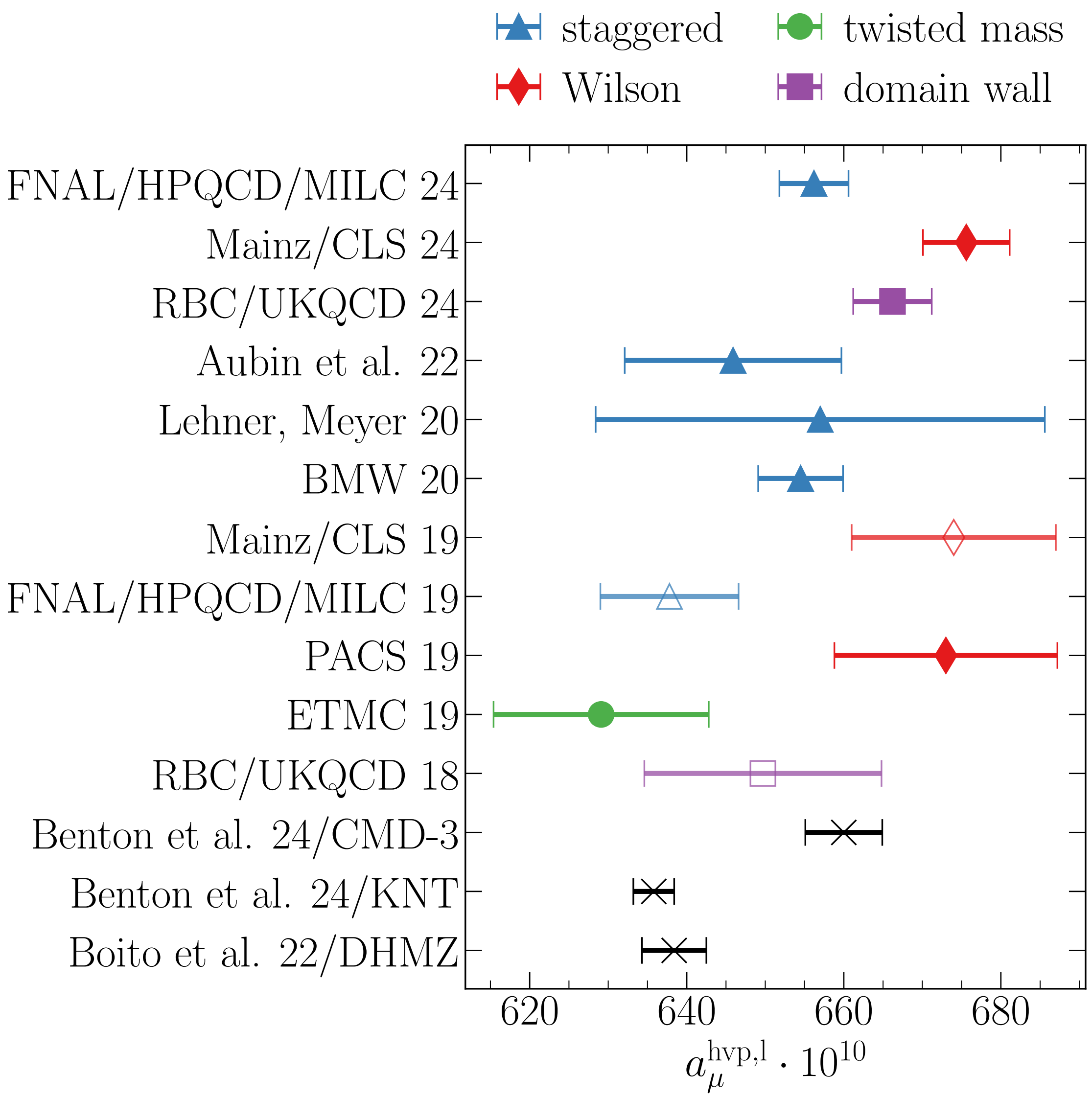
- Need a highly precise scale setting for precision in a_μ^{hvp}.

Impact / relevance 3.2

a_μ^{hvp} continued

light-quark connected contribution

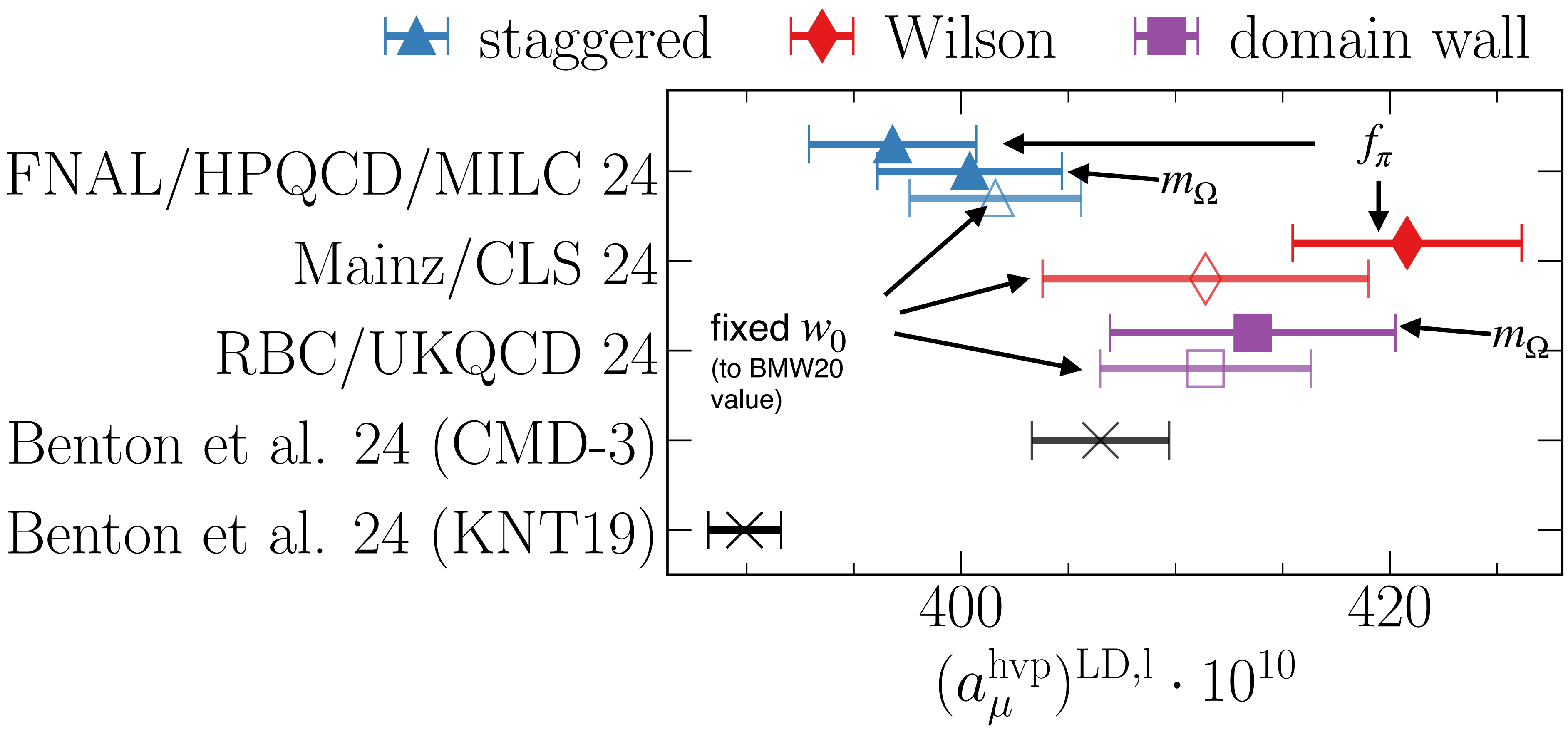
compilation by
S. Kuberski



Impact / relevance 3.3

a_μ^{hvp} continued: light-quark connected contribution, long distance window

compilation by
S. Kuberski



correlation to t_0, w_0

be aware of continuum + finite volume extrapolations

no LD in BMW 2024;
I'm sure they want to help clarify the situation and still publish their updated LD contribution

Baryon matrix elements

Quantites of interest (nucleon): Weak charges, g_A, g_S, g_T. Form factors, electromagnetic, G_E, G_M, axial, G_A, $G_{\tilde{P}}$, G_P, ... Moments of unpolarised, helicity and transversity PDFs, $g^q_{A,S,T}$, $\langle x \rangle_{q,\Delta q,\delta q}$, ... Quasi- and pseudo-PDFs.

Dimensionless quantities, that, generally, have a fairly mild dependence on the light quark mass.

Uncertainties at a few percent or higher. The signal to noise problem and excited state contamination are the main difficulties, ...

Exceptions: [CalLat][1805.12130] g_A 1%, [1912.08321] 0.74%, aim for 0.2%. Don't need scale to $< 1\%$.

Sigma term $\sigma_{\pi N} = \frac{1}{2}(m_u + m_d)\langle N|u\bar{u} + d\bar{d}|N\rangle \propto m_\pi^2$ [MeV]. High precision not required.

Radii $\langle r^2 \rangle = -\frac{6}{G}\left.\frac{dG}{dQ^2}\right|_{Q^2=0}$ [fm^2], $\langle r_E^2 \rangle^{p-n}$ diverges as $m_\pi^2 \to 0$.

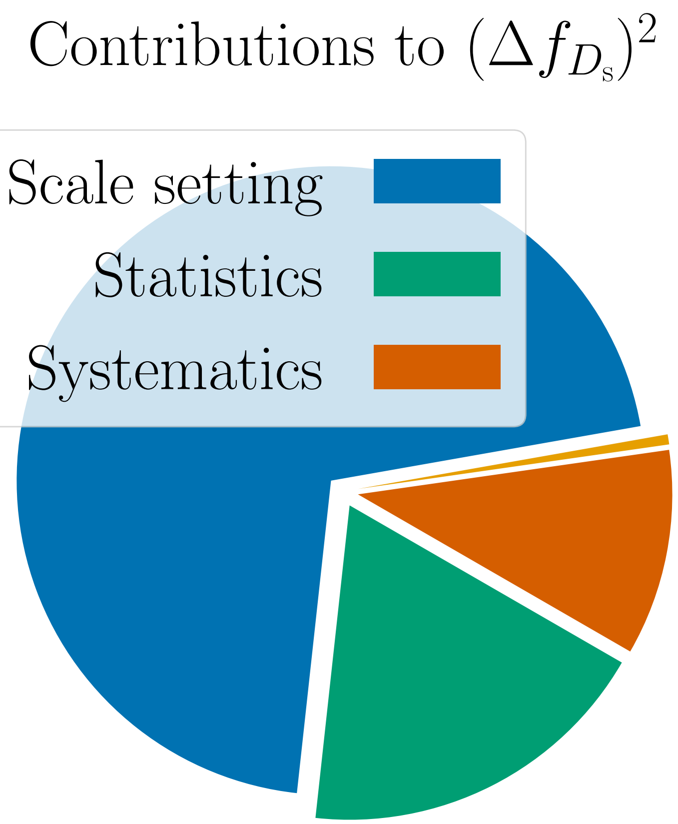
Proton $\langle r_E^2 \rangle^{1/2,p}$: tension between muonic hydrogen and some ep scattering results $\sim 5\%$. If want $\langle r_E^2 \rangle^{1/2,p}$ with $\sim 1\%$, need scale with better precision. Currently, [Mainz][2309.06590] $\langle r_E^2 \rangle^{1/2,p}$ with 2%.

Impact / relevance 5

Finite temperature physics

Decay constants

Regensburg / Münster computation of f_{D_s} [2405.04506]



should also be relevant here: (2+1+1)

Collaboration	Ref.	N_f	pu.	co.	ch.	fin.	re.	he.	f_D	f_{D_s}	f_{D_s}/f_D
ETM 21B	[453]	2+1+1	C	★	★	★	★	✓	210.1(2.4)	248.9(2.0)	1.1838(115)
FNAL/MILC 17 ▽▽	[20]	2+1+1	A	★	★	★	★	✓	212.1(0.6)	249.9(0.5)	1.1782(16)

Impact / relevance 6

Decay constants

note f_B, f_{B_s} presently not as precise
(heavy quark problem)

Impact / relevance: summary

relevance depends on the quantity

both on its precision and on its sensitivity to scales

very relevant for g-2 (in fact a bit worrying)

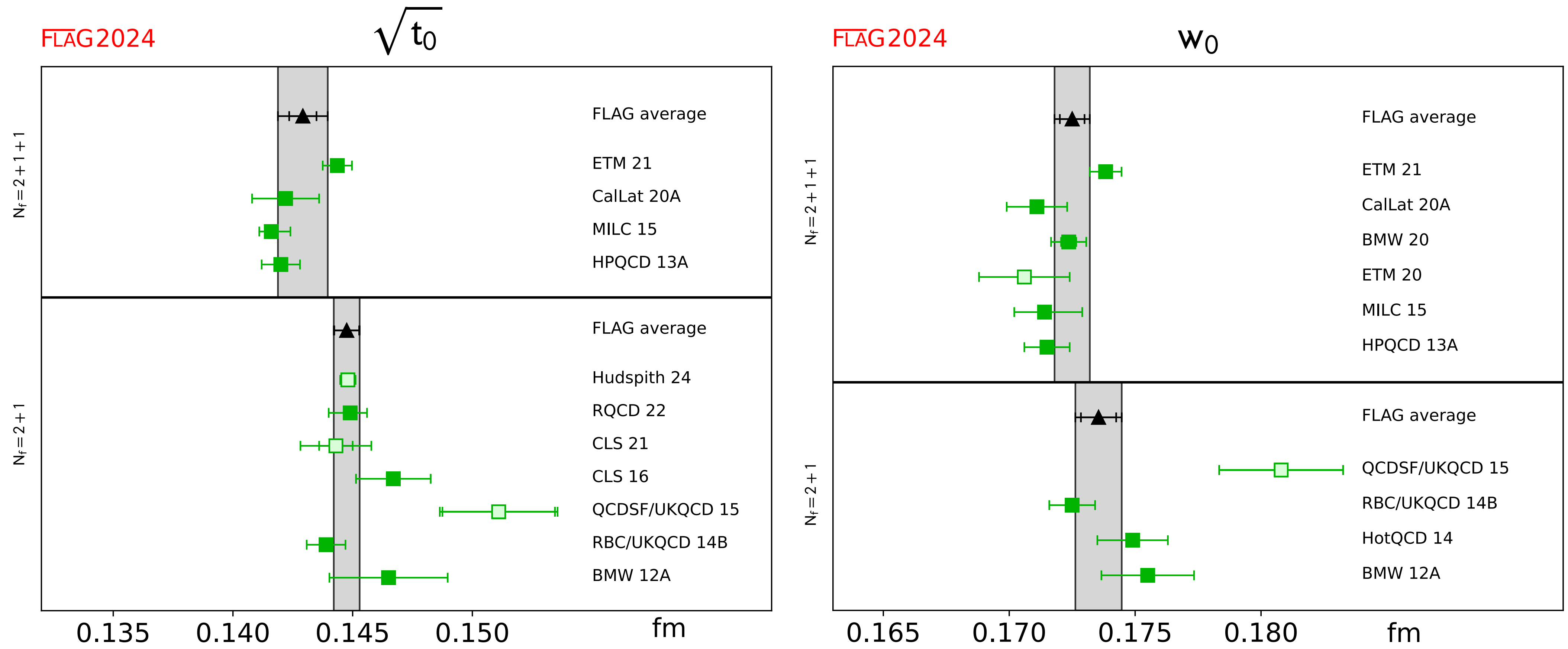
— reduced by new BMW: very long distance from phenomenology

25% of error squared in new $\alpha_s(m_Z)$

dominant in some decay constant computations

FLAG 2024: 1

GF scales



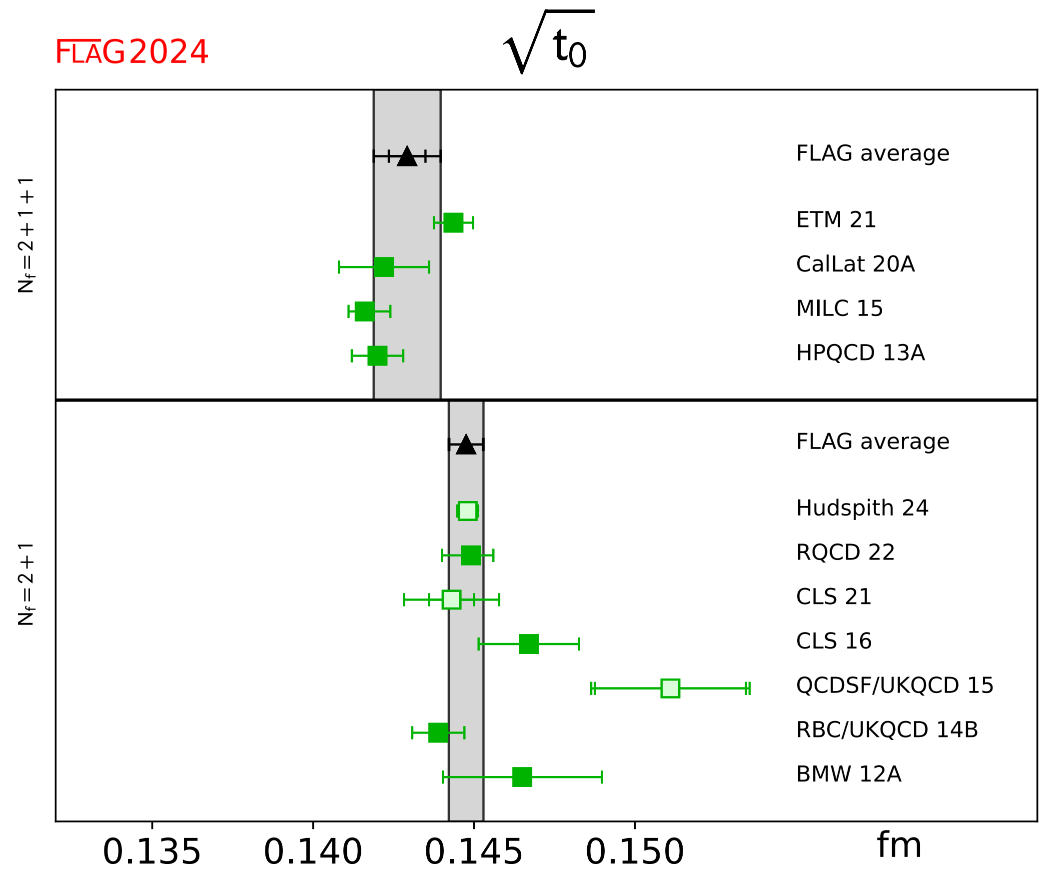
agreement could be better —> stretching of errors of averages

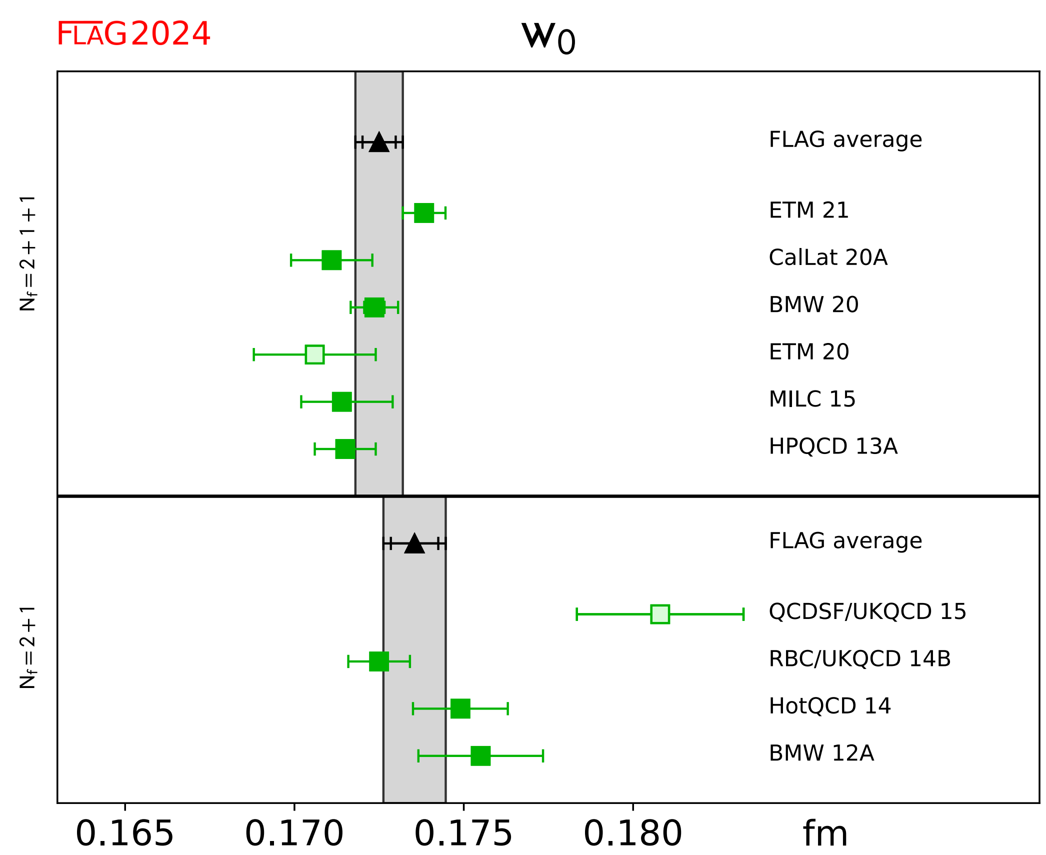
2+1+1: $\chi^2/\mathrm{dof} = 3.3$ 2+1+1: $\chi^2/\mathrm{dof} = 2.6$

2+1: $\chi^2/\mathrm{dof} = 1.1$ 2+1: $\chi^2/\mathrm{dof} = 1.5$

FLAG 2024: 1

GF scales



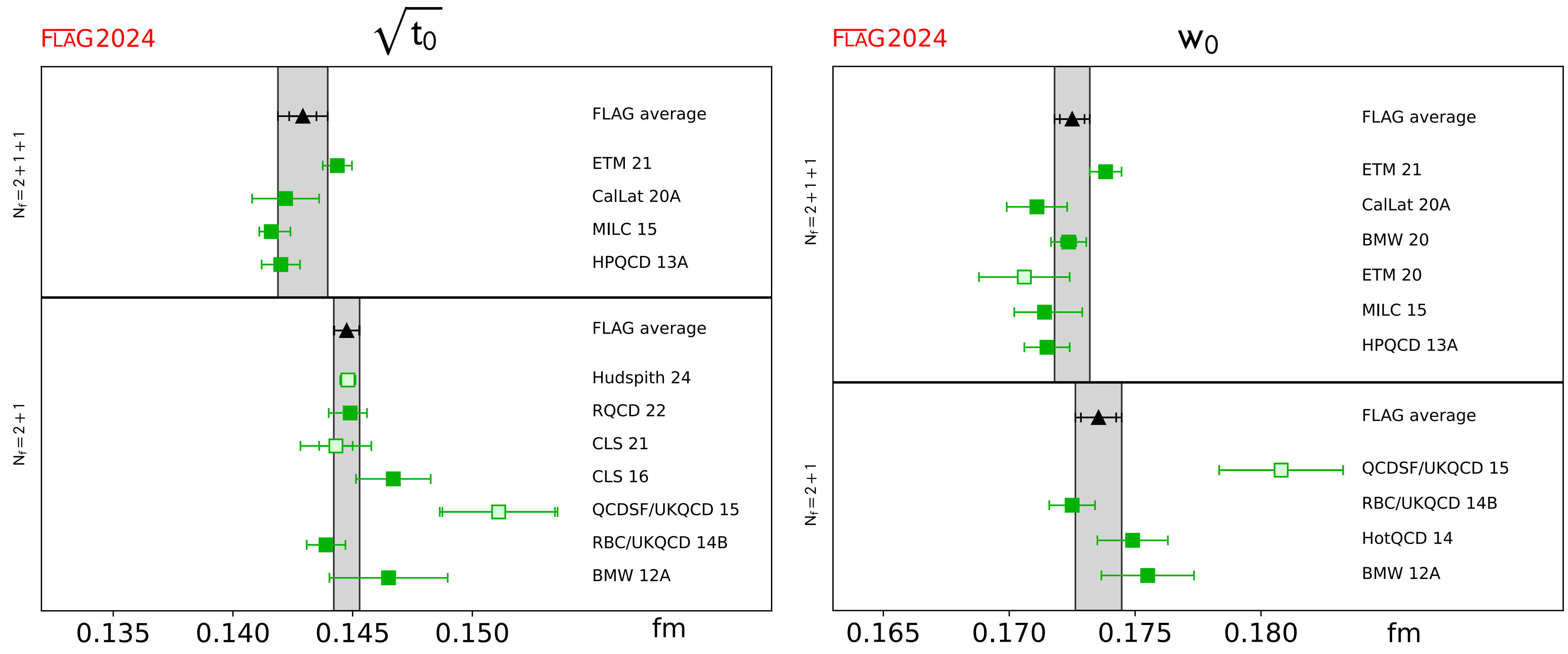


difference 2+1+1 vs. 2+1 seems too large

expecting a small sea-quark $1/m_c$ effect (see ALPHA papers)

FLAG 2024: 1

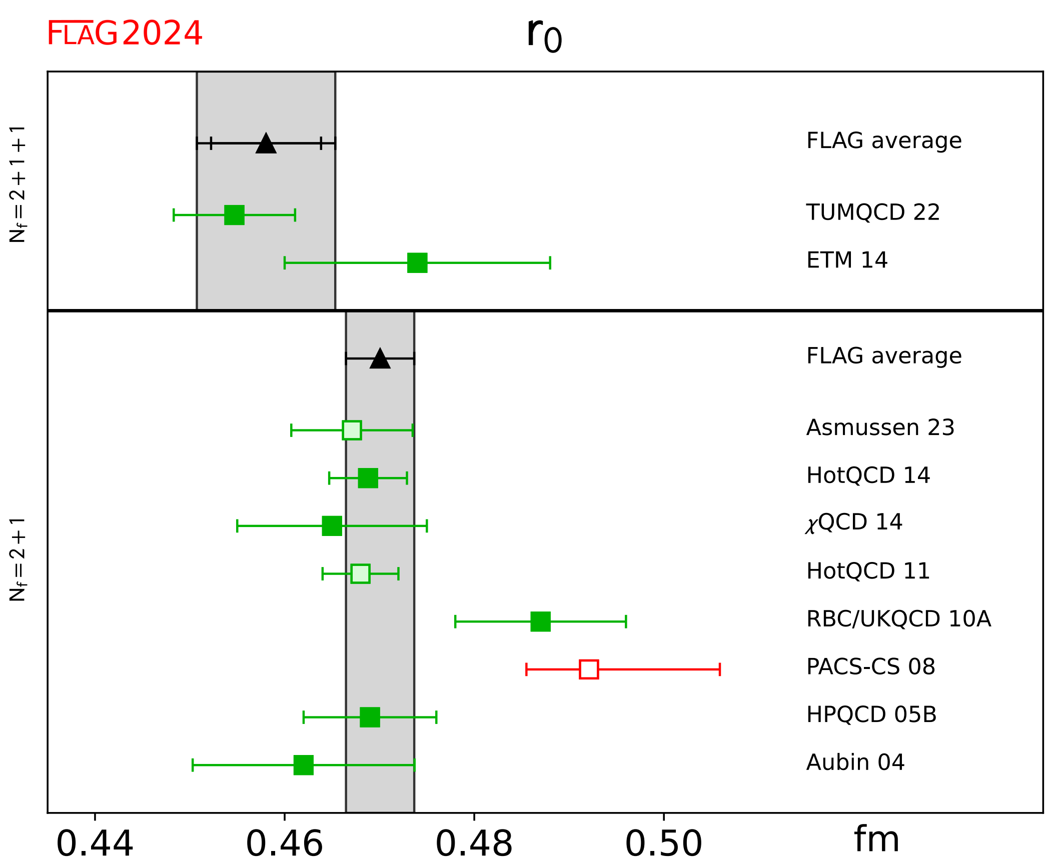
GF scales

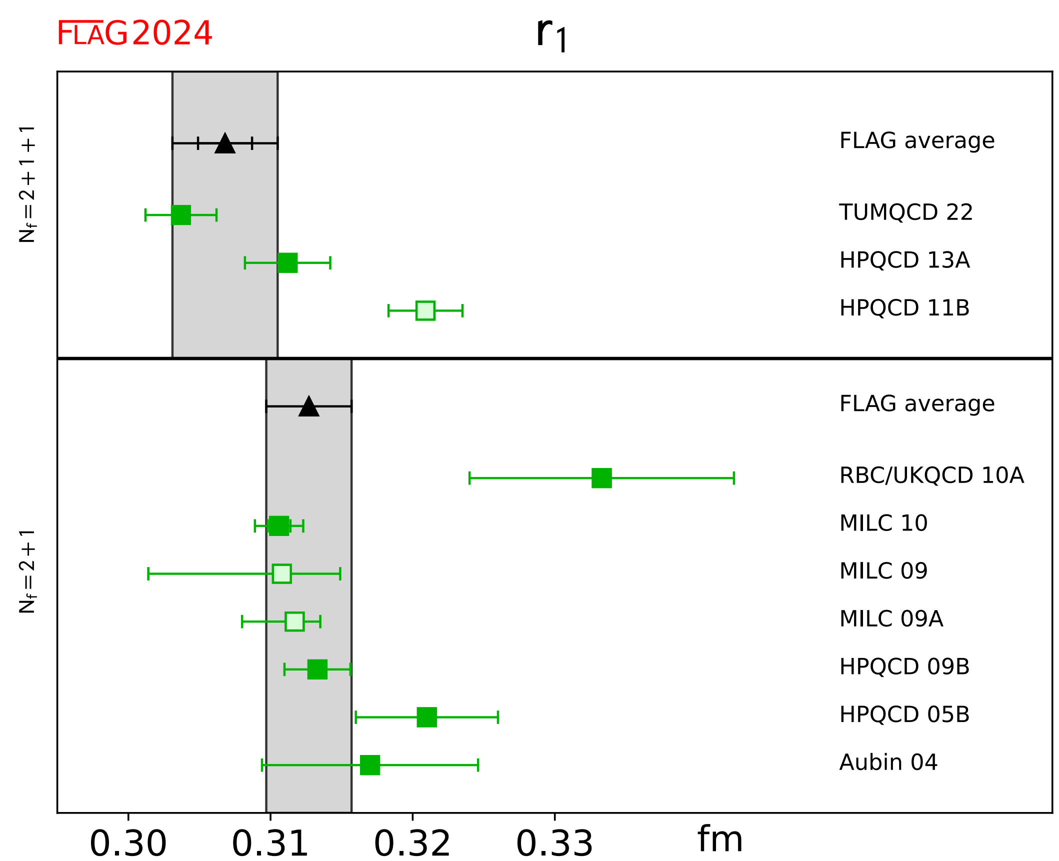


assuming a small sea-quark $1/m_c$ effect

quite some difference rooted vs. local formulations

Potential scales





agreement could be better —> stretching of errors of averages

2+1+1: $\chi^2/\mathrm{dof} = 2.0$
2+1: $\chi^2/\mathrm{dof} = 1.3$

2+1+1: $\chi^2/\mathrm{dof} = 3.7$
2+1: $\chi^2/\mathrm{dof} = 2.5$

my guess: likely due to excited state effects

This workshop

Discuss status again and the way forward

updates (a number of computations are quite old)

discuss the challenges

- excited states
- continuum extrapolations

(iso)QCD definition and relevance

…