Probing QCD with entangled states

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Based on quant-ph/2402.18499

Probing the CGC and QCD matter at hadron colliders
(in celebration of Edmond's 60th birthday),
March 24-27, 2025, Florence, Italy.

"Thus, the task is, not so much to see what no one has yet seen; but to think what nobody has yet thought, about that which everybody sees," E. Schrödinger.

Dirac's Last Paper

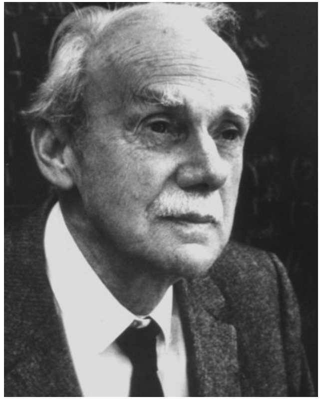
International Journal of Theoretical Physics, Vol. 23, No. 8, 1984

The Future of Atomic Physics

P. A. M. Dirac

Florida State University, Tallahassee, Florida

Received July 1, 1983



We have a theory in which infinite factors appear when we try to solve the equations. These infinite factors are swept into a renormalization procedure. The result is a theory which is not based on strict mathematics, but is rather a set of working rules.

Many people are happy with this situation because it has a limited amount of success. But this is not good enough. *Physics must be based on strict mathematics*. One can conclude that the fundamental ideas of the existing theory are wrong. A new mathematical basis is needed.

The rules of analysis

Physicist often omit limit operations. The element infinity is treated as a regular number.

1) ***Limit operations are generally non-commutative***

$$\lim_{x\to\infty}\lim_{y\to\infty}\frac{x}{x+y}=0\neq 1=\lim_{y\to\infty}\lim_{x\to\infty}\frac{x}{x+y}$$

2) ***Limit of a sum is inequivalent to the sum of limits***

$$\lim_{x\to\infty}f(x)+\lim_{x\to\infty}g(x)\ \neq\ \lim_{x\to\infty}\left(f(x)+g(x)\right)$$

3) ***Limit of a division is inequivalent to the divided limits***

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}\neq\frac{\lim_{x\to\infty}f(x)}{\lim_{x\to\infty}g(x)}$$

4) $\infty\notin\mathbb{R}$ ***: infinity does not belong to R***

$$\left(\lim_{\epsilon\to 0}\frac{1}{i\epsilon}\right)^*\neq\lim_{\epsilon\to 0}\left(\frac{1}{i\epsilon}\right)^*=-\lim_{\epsilon\to 0}\frac{1}{i\epsilon}$$

"Make no apologies for making excursions into other fields, because the separation of fields is merely a human convenience, and an unnatural. Nature is not interested in our separations, and many of the interesting phenomena bridge the gaps between fields, R. P. Feynman.

Mathematical well-known secrets

1) ***Interchange of integration and limit is generally not allowed.***

For example, taking $f_n(x) = nxe^{-nx^2}$:

$$\int_0^1 dx \lim_{n\to\infty} f_n(x) = 0 \qquad \lim_{n\to\infty}\int_0^1 dx f_n(x) = \lim_{n\to\infty}\left(-\frac{1}{2}e^{-nx^2}\Big|_{x=0}^{x=1}\right) = \frac{1}{2}$$

In fact, according to Fatou's lemma:

$$\int dx \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} \int dx\, f_n(x)$$

2) ***Interchange of limit and differentiation is generally not allowed.***

$$\frac{d}{dx}\sum_{n=0}^{\infty} f_n(x) \neq \sum_{n=0}^{\infty}\frac{d}{dx}f_n(x) \qquad \frac{d}{dx}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\frac{d}{dx}f_n(x)$$

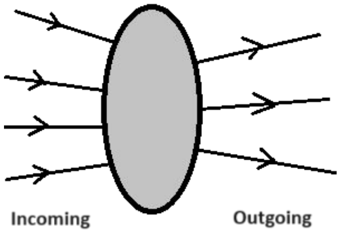
3) ***Interchange of integration and summation generally not allowed.***

$$\int_0^1 dx \sum_{n=0}^{\infty} f_n(x) \neq \sum_{n=0}^{\infty}\int_0^1 dx\, f_n(x)$$

 # Does all these rules affect the calculation of physical observables?

Scattering a la LSZ

Scattering amplitudes are defined as the overlap between the initial and final states



Our current computation approach is essentially based on the LSZ reduction formula:

$$\langle p_1,\ldots,p_n|\, q_1,\ldots,q_m\rangle$$
$$= \int \prod_{i=1}^{n}\left[dx_i \frac{ie^{-ip_i\cdot x_i}(\partial_i^2+m_i^2)}{(2\pi)^{3/2}\sqrt{Z_i}}\right]\prod_{j=1}^{m}\left[dx_j \frac{ie^{iq_j\cdot y_j}(\partial_j^2+m_j^2)}{(2\pi)^{3/2}\sqrt{Z_j}}\right]\langle\Omega|\,\mathrm{T}\phi(x_1)\ldots\phi(x_n)\,\varphi(y_1)\ldots\varphi(y_m)\,|\Omega\rangle$$

The underlying assumption: the incoming and outgoing states are fully on shell states.

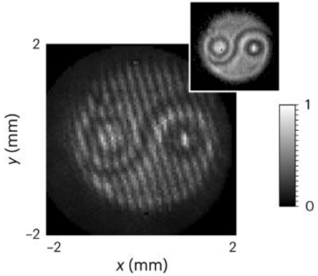
$$|p_1,\ldots,p_n\rangle = |p_1\rangle\otimes|p_2\rangle\otimes\ldots\otimes|p_n\rangle = a^\dagger(p_1)\ldots a^\dagger(p_n)\,|0\rangle$$
$$|q_1,\ldots,q_m\rangle = |q_1\rangle\otimes|q_2\rangle\otimes\ldots\otimes|q_m\rangle = a^\dagger(q_1)\ldots a^\dagger(q_m)\,|0\rangle$$
$$\sqrt{Z_i} = \||p_i\rangle\| \equiv \sqrt{\langle p_i|p_i\rangle} \qquad \sqrt{Z_j} = \||q_j(k)\rangle\|$$

Mathematically, these states must belong to a *separable Hilbert space (=complete normed space).*

Higher dimensional entanglement

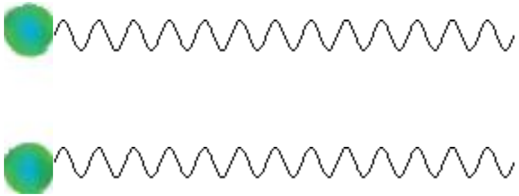
A more exotic type of states are partially off-shell and cannot be represented as a simple factorized form. Their fundamental definition involves convolution:

$$|\Phi(k)\rangle = \int dp\, f(k,p)\, |\gamma(p)\rangle\, |\gamma(k-p)\rangle$$



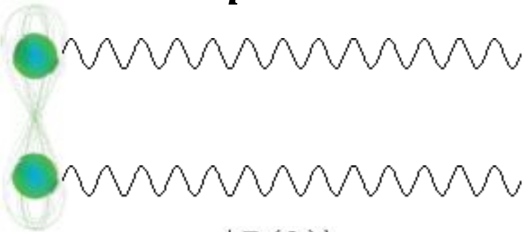
Advances in high-dimensional quantum entanglement, M Erhard, M Krenn, A Zeilinger

Two separable photons



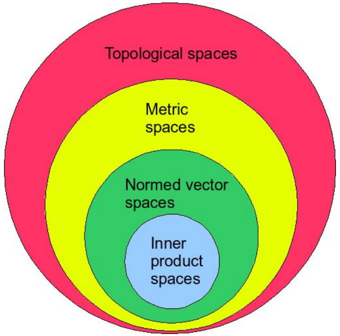
$$|\Psi(k,p)\rangle = |\gamma(k)\rangle\, |\gamma(p)\rangle$$

Biphoton



$$|\Phi(k)\rangle$$

Left: a separable state of two photons – can be normalized by introducing factor of ***Z*** for each photon. *Right:* an entangled (convoluted) biphoton state – no consistent ***Z*** can be introduced as soon as it evolves.



Dealing with these states is outside of the scope of LSZ theorem.

The Schrödinger equation

The Schrödinger equation (postulated in 1925) is the operatorial differential equation that governs the wave dynamics of a quantum mechanical system,

$$\frac{d}{dt}|\Psi(t)\rangle = -i\hat{H}\,|\Psi(t)\rangle$$

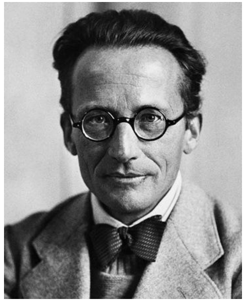
The formal solution, as long as the Hamiltonian involves no time dependence, is given in the form of evolution operator:

$$|\Psi(t)\rangle = \hat{U}(t,\,t_0)\,|\Psi(t_0)\rangle\,, \qquad \hat{U}(t,\,t_0) = e^{-i\hat{H}(t-t_0)}$$

For time dependent Hamiltonians it is customary to use the solution

$$\hat{U}(t,\,t_0) = \hat{T}\exp\left[-i\int_{t_0}^{t} dt'\,\hat{H}(t')\right]$$

In the expanded form (Dyson series):



$$\hat{U}(t,\,t_0) = \hat{\mathbf{1}} - i\int_{t_0}^{t} dt'\,\hat{H}(t') - \int_{t_0}^{t} dt'\,\hat{H}(t')\int_{t_0}^{t'} dt''\,\hat{H}(t'') + \ldots$$

How do we solve it?

The basic idea is to regard the operatorial differential equation as finite dimensional matrix version with bounded elements.

Integration of both sides following iterative procedure is based on *Picard–Lindelöf theorem*,

$$\hat{U}(t, t_0) = \hat{\mathbf{1}} - i\int_{t_0}^{t} dt'\, \hat{H}(t')\,\hat{U}(t', t_0)$$

$$= \hat{\mathbf{1}} - i\int_{t_0}^{t} dt'\, \hat{H}(t')\left[\hat{\mathbf{1}} - i\int_{t_0}^{t'} dt''\, \hat{H}(t'')\,\hat{U}(t'', t_0)\right] = \ldots = \sum_{n=0}^{\infty} u_n(t, t_0)$$

__Requirements for applicability:__

1) Finite (closed interval) time evolution.
2) Evolution according to a bounded function / finite dimensional Hamiltonian.

 # Can the iterative method be justified for asymptotic times / unbounded Hamiltonians?

The "verification"

By applying the derivative "term by term" it follows that

$$\frac{d\hat{U}(t,t_0)}{dt} = \frac{d}{dt}\sum_{n=0}^{\infty}\hat{u}_n(t) = \sum_{n=0}^{\infty}\frac{d\hat{u}_n(t)}{dt}$$

$$= \frac{d}{dt}(\hat{\mathbf{1}}) - \frac{d}{dt}\left(i\int_{t_0}^{t} dt'\,\hat{H}(t')\right) - \frac{d}{dt}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'')\right) + \ldots = -i\hat{H}(t)\,\hat{U}(t,t_0)$$

Where we applied two types of exchanges:

$$\frac{d}{dt}\sum_{n=0}^{\infty}\hat{u}_n(t) \rightarrow \sum_{n=0}^{\infty}\frac{d}{dt}\hat{u}_n(t) \qquad\qquad \frac{d}{dx}\left(\int_{x_0}^{x} dx'\,\hat{\mathcal{O}}(x')\right) = \hat{\mathcal{O}}(x)$$

<u>Requirements:</u>

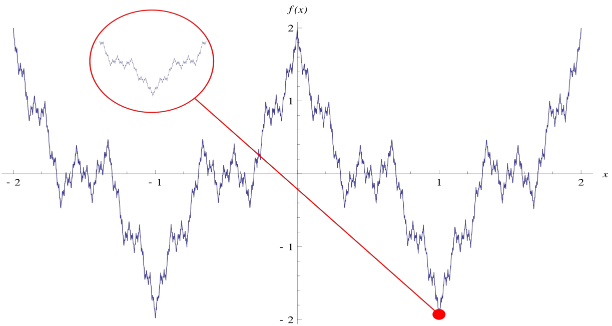
1) Uniform convergence of the infinite series.
2) Each term must be differentiable by itself.

In QFT calculations the series is asymptotically convergent and these conditions may be invalid:

$$\int_{t_0}^{t} dt'\,\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'')|\psi\rangle = \infty\,|\psi\rangle$$

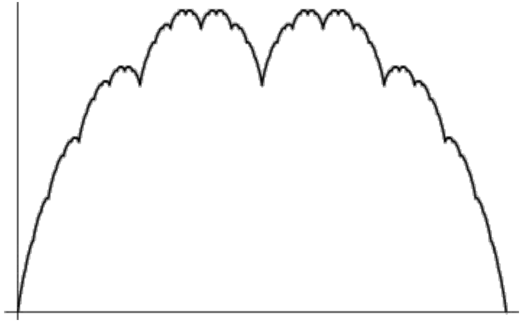
Continuous but non-differentiable behaviour

Weierstrass function

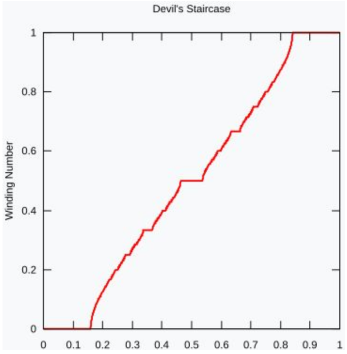


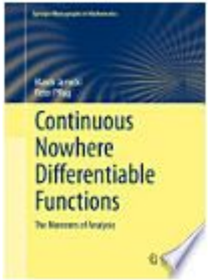
$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

Takagi function



Devil's staircase



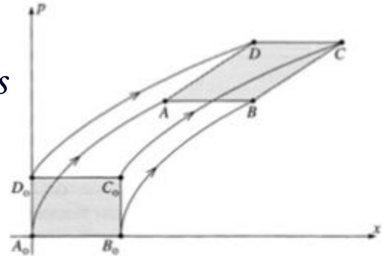


The '*monsters of analysis*' cannot be Taylor approximated and fail the current assumptions of the standard formalism of QM / QFT.

Non-smooth behavior is the true nature of the quantum phenomena.

The principle of unitarity

In classical mechanics, the preservation of the evolving phase space is described by Liouville's theorem:

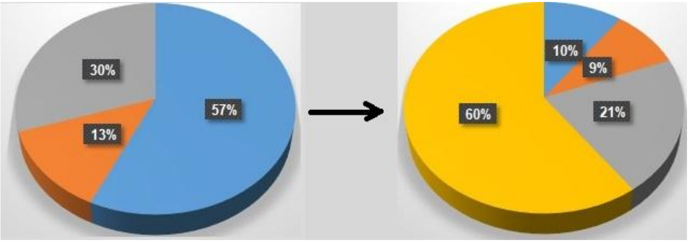


"As the systems contained in a region of phase space evolve according to clas mechanics, the volume occupied remains constant."

The quantum analogue of this principle is the Born principle –

"The evolution of the wave function is such that its norm is preserved at any given time"

$$\sum_i p_i(t_1) = \sum_i p_i(t_2) = 100\%$$



No unitarity → illogical interpretation.

The ‘unitarity conception’

The Dyson series after replacing the iterative integration by the productive one:

$$\hat{U}(t,\, t_0) \,=\, \hat{\mathbf{1}} \,-\, i \int_{t_0}^{t} dt'\, \hat{H}(t') \,-\, \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t')\, \hat{H}(t'') \,+\, \ldots$$

Applying the adjoint operator “term by term”:

$$\hat{U}^{\dagger}(t,\, t_0) \,=\, \hat{\mathbf{1}} \,+\, i \left(\int_{t_0}^{t} dt'\, \hat{H}(t') \right)^{\dagger} \,-\, \left(\int_{t_0}^{t} dt'\, \hat{H}(t') \int_{t_0}^{t'} dt'' \hat{H}(t'') \right)^{\dagger} \,+\, \ldots$$

Rewriting under the assumption of (naive) applying the conjugation:

$$\left(\int_{t_0}^{t} dt'\, \hat{H}(t') \right)^{\dagger} = \int_{t_0}^{t} dt'\, \hat{H}(t') \qquad\qquad \left(\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t')\, \hat{H}(t'') \right)^{\dagger} = \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t'')\, \hat{H}(t')$$

So, as it seems,

$$\hat{U}^{\dagger}(t, t_0) \,=\, \hat{\mathbf{1}} \,+\, i \int_{t_0}^{t} dt'\, \hat{H}(t') \,-\, \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t'')\, \hat{H}(t') \,+\, \ldots$$

Finally, by taking products we arrive at:

$$\hat{U}^{\dagger}(t, t_0)\, \hat{U}(t, t_0) \,=\, \hat{\mathbf{1}} \,+\, \left(\int_{t_0}^{t} dt'\, \hat{H}(t') \right)^{2} \,-\, \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t')\, \hat{H}(t'') \,-\, \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t'')\, \hat{H}(t') \,+\, \ldots$$

Applying the *Fubini's theorem* for interchanging the order of integrations:

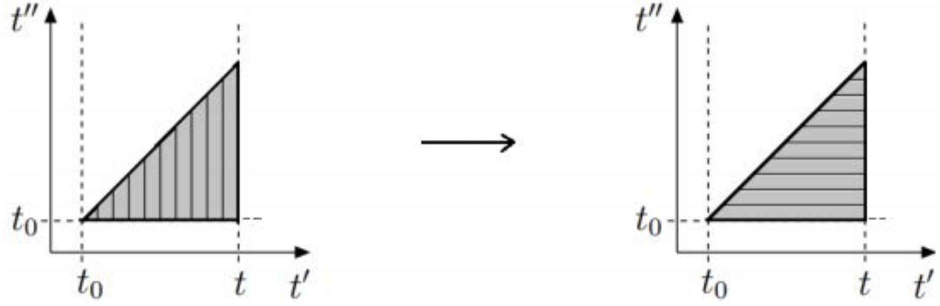
$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t'')\, \hat{H}(t') = \int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, \hat{H}(t')\, \hat{H}(t'')$$

After adding integrals:

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \hat{H}(t')\, \hat{H}(t'') + \int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, \hat{H}(t')\, \hat{H}(t'') = \left(\int_{t_0}^{t} dt'\, \hat{H}(t')\right)^2$$

So, at least allegedly, *U* is a unitary operator,

$$\hat{U}^{\dagger}(t, t_0)\, \hat{U}(t, t_0) = \hat{\mathbf{1}}$$



The underlying assumptions

1) ***Exact self-adjoint:*** $$\mathcal{O}^\dagger(t) = \mathcal{O}(t)$$

2) ***Linearity:*** $$\int_{t_0}^{t_1} dt\, \hat{\mathcal{O}}_1(t) + \int_{t_0}^{t_1} dt\, \hat{\mathcal{O}}_2(t) = \int_{t_0}^{t_1} dt\, \left(\hat{\mathcal{O}}_1(t) + \hat{\mathcal{O}}_2(t)\right)$$

3) ***Additivity:*** $$\int_{t_0}^{t_1} dt\, \hat{\mathcal{O}}(t) + \int_{t_1}^{t_2} dt\, \hat{\mathcal{O}}(t) = \int_{t_0}^{t_2} dt\, \hat{\mathcal{O}}(t)$$

A sufficient condition to ensures the validity of the unitarity argument:

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \left\| \hat{H}(t'')\, \hat{H}(t')\, |\psi\rangle \right\|_{\mathcal{Y}}^2 < \infty$$

In QFT this is condition is often not satisfied.

Failure of one of the above conditions means that U is not an exact unitary operator.

Failing the Fubini's theorem

Integration over infinity may well yield something finite (without regularization). Exchanging the ordering of integrations is allowed only if the condition of absolute convergence applies,

$$\int dx \left(\int dy\, f(x,y) \right) = \int dy \left(\int dx\, f(x,y) \right) \quad \textbf{if} \quad \int dx \int dy\, |f(x,y)| < \infty$$

Violation of the condition will result in inequivalent values:

$$\int_0^2 \left(\int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\, dy \right) dx = -\frac{1}{20} \qquad \int_0^1 \left(\int_0^2 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\, dx \right) dy = \frac{1}{5}$$

$$\int_1^\infty \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\, dx\, dy = \frac{\pi}{4} \qquad \int_1^\infty \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\, dy\, dx = -\frac{\pi}{4}$$

Checkout: "*Fubini Counterexample (full version)*" @ Dr Payam (youtube).

Dr Peyam
@drpeyam
150K subscribers

The unitary solution

The solution for the Schrödinger equation is given by a unitarized version of ***U***:

$$\hat{\mathcal{P}}(t,\, t_0) \,\equiv\, \hat{\mathcal{N}}(t,\, t_0)\, \hat{U}(t,\, t_0)$$

With the self-adjoint normalization operator:

$$\hat{\mathcal{N}}(t,\, t_0) \,\equiv\, \sqrt{\hat{U}^{\dagger -1}(t,\, t_0)\, \hat{U}^{-1}(t,\, t_0)}$$

See "*On the exact solution for the Schrodinger equation,*" hep-ph/2402.18499.

Note that unitarity is **manifest** ("no need for an argument"):

$$\hat{\mathcal{P}}^{\dagger}(t,\, t_0)\, \hat{\mathcal{P}}(t,\, t_0) \,=\, \hat{U}^{\dagger}(t,\, t_0)\, \hat{U}^{\dagger -1}(t,\, t_0)\, \hat{U}^{-1}(t,\, t_0)\, \hat{U}(t,\, t_0) \,=\, \hat{\mathbf{1}}$$

<u>Main claim</u>: believing that $\hat{\mathcal{N}}(t, t_0) \,=\, \hat{\mathbf{1}}$ is a **wrong oversimplification.**

The inverse of U

The inverse of ***U*** to second perturbative order reads

$$\hat{U}^{-1}(t,t_0) = \hat{\mathbf{1}} + i\int_{t_0}^{t} dt'\,\hat{H}(t') - \left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^2 + \int_{t_0}^{t} dt'\,\hat{H}(t')\int_{t_0}^{t'} dt''\,\hat{H}(t'') + \ldots$$

Then

$$\begin{aligned}
&\hat{U}^{-1}(t,t_0)\,\hat{U}(t,t_0)\\
&= \hat{\mathbf{1}} - i\textstyle\int_{t_0}^{t} dt'\,\hat{H}(t') + i\int_{t_0}^{t} dt'\,\hat{H}(t') - \left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^2 + \left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^2\\
&- \textstyle\int_{t_0}^{t} dt'\,\hat{H}(t')\int_{t_0}^{t'} dt''\,\hat{H}(t'') + \int_{t_0}^{t} dt'\,\hat{H}(t')\int_{t_0}^{t'} dt''\,\hat{H}(t'') = \hat{\mathbf{1}}.
\end{aligned}$$

Identical cancellation of terms without any assumption required!

$$\hat{U}^{-1} \neq \hat{U}^{\dagger}$$

The new perturbative expansion

The expansion for $\boldsymbol{N}$ reads

$$\hat{\mathcal{N}}(t,t_0) = \hat{\mathbf{1}} + \frac{i}{2}\int_{t_0}^{t} dt'\,\hat{H}(t') - \frac{i}{2}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^{\dagger} - \frac{3}{8}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^{2} - \frac{3}{8}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^{\dagger 2}$$
$$+ \frac{1}{4}\left|\int_{t_0}^{t} dt'\,\hat{H}(t')\right|^{2} + \frac{1}{2}\int_{t_0}^{t} dt'\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'') + \frac{1}{2}\left(\int_{t_0}^{t} dt'\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'')\right)^{\dagger} + \ldots$$

Then,

$$\hat{\mathcal{P}}(t,t_0) = \hat{\mathbf{1}} - \frac{i}{2}\int_{t_0}^{t} dt'\,\hat{H}(t') - \frac{i}{2}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^{\dagger} + \frac{1}{8}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^{2} - \frac{3}{8}\left(\int_{t_0}^{t} dt'\,\hat{H}(t')\right)^{\dagger 2}$$
$$- \frac{1}{4}\left|\int_{t_0}^{t} dt'\,\hat{H}(t')\right|^{2} - \frac{1}{2}\int_{t_0}^{t} dt'\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'') + \frac{1}{2}\left(\int_{t_0}^{t} dt'\hat{H}(t')\int_{t_0}^{t'} dt''\hat{H}(t'')\right)^{\dagger} + \ldots$$

Reduce to Dyson series under the assumption of finite dimensional Hilbert space.

When does *N* become non-trivial?

1) Generalized functions

Introducing delta function to Dyson series leads to indeterminate expansion.

$$\int_0^x dx'\,\delta(x'-a)\,\Theta(x'-a) \longrightarrow \texttt{Indeterminate}$$

The unitarity argument fails due to "no additivity" and "no Fubini":

$$\int_{t_0}^{t'=T} dt'\,\delta(t'-T) + \int_{t'=T}^{t} dt'\,\delta(t'-T) \nrightarrow \int_{t_0}^{t} dt'\,\delta(t'-T) \qquad \int_{t_0}^{t} dt'\,\delta(t'-T) \int_{t_0}^{t'} dt''\delta(t''-T) \nrightarrow \int_{t_0}^{t} dt'\,\delta(t'-T) \int_{t'}^{t} dt''\delta(t''-T)$$

"Sur l'impossibilité de la multiplication des distributions," L. Schwartz

2) Infinitely dimensional operators

Dealing with the operators $\hat{x},\ \hat{p}$

3) Asymptotic time evolution

Studying the case $t-t_0=\infty$ as in the definition of the S-matrix

$$\lim_{t_1-t_0\to\infty}\hat{U}(t_1,t_0)\lim_{t_1-t_0\to\infty}\hat{U}^\dagger(t_1,t_0) \neq \lim_{t_1-t_0\to\infty}\hat{U}(t_1,t_0)\,\hat{U}^\dagger(t_1,t_0)$$

Why we care about infinite dimensions?

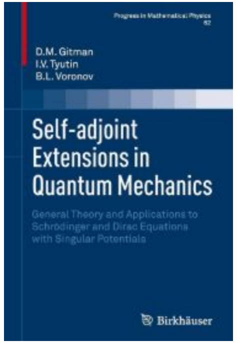
Once truncating the dimension one can no longer recover the translation and scale invariance that are required for full generality. Moreover, the uncertainty relation,

$$[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}] = i\hbar\hat{\mathbf{1}}$$

cannot be fulfilled. On finite spaces one can trace both side:

$$tr\left([\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}]\right) = i\hbar\, tr\left(\hat{\mathbf{1}}\right)$$

Since on finite dimensions the trace is cyclic operation, for any ***A*** and ***B***:



$$tr\,[\boldsymbol{A}, \boldsymbol{B}] = tr\,(\boldsymbol{A}\boldsymbol{B}) - tr\,(\boldsymbol{B}\boldsymbol{A}) = 0$$

That leads to inconsistent:

$$0 = i\hbar\,\dim\left(\mathcal{H}\right)$$

Therefore, at least one of the operators must belong to an infinitely dimensional space.

Solving Schrodinger via *P*

By realizing that singular Hamiltonian can be treated as a complex deformation

$$\hat{H}(t) = \hat{\mathcal{H}}(t) - i\hat{\mathcal{J}}(t) \qquad \lim_{\epsilon\to 0}\frac{1}{x - i\epsilon} = P.v.\left(\frac{1}{x}\right) + i\pi\delta(x)$$

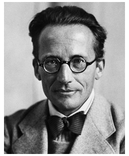
The resulting equations

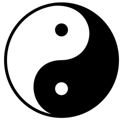
$$\frac{d\hat{U}(t,t_0)}{dt} = -i\left(\hat{\mathcal{H}}(t) - i\hat{\mathcal{J}}(t)\right)\hat{U}(t,t_0)$$

$$\frac{d\hat{\mathcal{N}}(t,t_0)}{dt} = -i\left[\hat{\mathcal{H}}(t), \hat{\mathcal{N}}(t,t_0)\right] + \hat{\mathcal{N}}(t,t_0)\,\hat{\mathcal{J}}(t)$$

Then,

$$\begin{aligned}\frac{d\hat{\mathcal{P}}(t,t_0)}{dt} &= \hat{\mathcal{N}}(t,t_0)\left(-i\left(\hat{\mathcal{H}}(t) - i\hat{\mathcal{J}}(t)\right)\hat{U}(t,t_0)\right)\\ &+ \left(-i\left[\hat{\mathcal{H}}(t), \hat{\mathcal{N}}(t,t_0)\right] + \hat{\mathcal{N}}(t,t_0)\,\hat{\mathcal{J}}(t)\right)\hat{U}(t,t_0) = -i\hat{\mathcal{H}}(t)\,\hat{\mathcal{P}}(t,t_0)\end{aligned}$$





Examples

The non-Hermitian skin effect (NHSE)

Let us describe a lattice of finite size ***l*** with unequal amplitudes for jumping to each side,

$$\hat{H} = \sum_{i=1}^{l} E a_i^\dagger a_i + \sum_{i=1}^{l-1} \left[(t_i - \gamma_i) a_i^\dagger a_{i+1} + (t_i + \gamma_i) a_{i+1}^\dagger a_i \right]$$

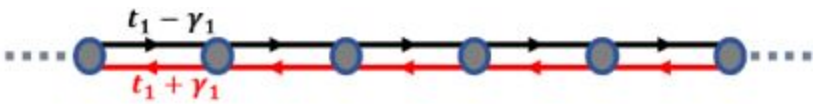
This Hamiltonian is non-Hermitian, $\hat{H}^\dagger \neq \hat{H}$. Due to no explicit time dependence,

$$\hat{U}(t, t_0) = \exp(-i(\hat{\mathcal{H}} - i\hat{\mathcal{J}})\Delta t)$$

In which unitarity is decaying. According to the definition $\hat{\mathcal{N}}(t, t_0) = \exp(\hat{\mathcal{J}}\Delta t)$, Then:

$$\hat{\mathcal{P}}(t, t_0) = \exp(\hat{\mathcal{J}}\Delta t) \exp(-i(\hat{\mathcal{H}} - i\hat{\mathcal{J}})\Delta t) = \exp(-i\hat{\mathcal{H}}\Delta t)$$

Note that intuitively we know that in the long run the particle should reach the end of the lattice, which is respected only in the new approach.



The free particle

The Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

The domain is

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \qquad\qquad |E| < \infty$$

The Hilbert space for 1-dim unbounded momentum space is the extended real set,

$$\mathcal{H} = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

Which is smaller than the entire Hilbert space by the cover of the space

$$\mathcal{D}(\hat{H}) = \mathbb{R} \subset \mathcal{H}$$

$$\mathcal{C} \equiv \mathcal{H} \setminus \mathcal{D}(\hat{H}) = \{-\infty, \infty\}$$

On the residual space "C" Stone theorem is inapplicable and ***N*** is non-trivial

$$\sqrt{\left[\exp\left(-i\Delta t\,\hat{H}\right)\right]^{\dagger-1}\left[\exp\left(-i\Delta t\,\hat{H}\right)\right]^{-1}}\,|\psi(p \in \mathcal{C})\rangle \neq \hat{\mathbf{1}}\,|\psi(p \in \mathcal{C})\rangle$$

Hermitian vs self-adjoint

The definitions below coincide only on finite dimensional spaces.

<u>The self-adjoint operator:</u>

$$\langle\varphi|\,\hat{\mathcal{O}}^\dagger\,|\psi\rangle = \langle\varphi|\,\hat{\mathcal{O}}\,|\psi\rangle \qquad\qquad \mathcal{D}(\hat{\mathcal{O}}^\dagger) = \mathcal{D}(\hat{\mathcal{O}})$$

quant-ph/9907069
by F. Gieres

for any $|\varphi\rangle\,,|\psi\rangle \in \mathcal{H}$, then $\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}$.

<u>The Hermitian operator:</u>

$$\langle\varphi|\,\hat{\mathcal{O}}^\dagger\,|\psi\rangle = \langle\varphi|\,\hat{\mathcal{O}}\,|\psi\rangle$$

But $\mathcal{D}(\hat{\mathcal{O}}) \subseteq \mathcal{D}(\hat{\mathcal{O}}^\dagger)$.

Any self-adjoint operator is Hermitian, but an Hermitian operator is not necessarily self-adjoint.

The typical textbook example is the momentum operator:

$$\langle\varphi|\,\hat{\boldsymbol{p}}^\dagger\,|\psi\rangle = \langle\varphi|\,\hat{\boldsymbol{p}}\,|\psi\rangle$$

Since $\mathcal{D}(\hat{\boldsymbol{p}}) \subset \mathcal{D}(\hat{\boldsymbol{p}}^\dagger)$, at the operator level $\hat{\boldsymbol{p}}^\dagger \neq \hat{\boldsymbol{p}}$.

The ‘ad-hoc’ normalization method

By using the Dyson expansion

$$\hat{U}(0,-\infty)\left|e^-\right\rangle = \left(\hat{1} - \int d\Pi_{e_2^-\gamma, e_1^-} \left|e_2^-\gamma\right\rangle \frac{\left\langle e_2^-\gamma\right| \hat{H}_{int} \left|e_1^-\right\rangle}{E_{e_2^-\gamma} - E_{e_1^-} - i\epsilon} \left\langle e_1^-\right| - \int d\Pi_{e_2^-, e_3^-\gamma, e_1^-} \left|e_2^-\right\rangle \frac{\left\langle e_2^-\right| \hat{H}_{int} \left|e_3^-\gamma\right\rangle \left\langle e_3^-\gamma\right| \hat{H}_{int} \left|e_1^-\right\rangle}{(E_{e_2^-} - E_{e_1^-} - i\epsilon_2)(E_{e_3^-\gamma} - E_{e_1^-} - i\epsilon_1)} \left\langle e_1^-\right|\right)\left|e^-\right\rangle$$

The last term creates a problem,

$$\left\langle e^-(k)\right| \hat{U}(0,-\infty) \left|e^-(k)\right\rangle = \infty$$

Unitary time evolution implies isometry (norm preserving). Restoring the norm involves ad-hoc truncation of this term and reintroduction via the normalization condition,

$$\left\| \frac{1}{\sqrt{Z}} \left|e^-(k)\right\rangle_{t=0} \right\| = \left\| \left|e^-(k)\right\rangle_{t=-\infty} \right\|$$

Then

$$\frac{1}{\sqrt{Z}}\hat{U}(0,-\infty)\left|e^-(k)\right\rangle = \hat{\mathcal{P}}(0,-\infty)\left|e^-(k)\right\rangle$$

How to normalize an entangled state?

As long as the state is pure or mixed, the action of ***N*** is equivalent to the rescaling

$$|\psi(t)\rangle \longrightarrow \frac{1}{\mathcal{Z}_\psi}|\psi(t)\rangle$$

Now let us take an initially normalized entangled state:

$$|\Phi(k)\rangle = \int dp\, f(p)\, |\phi(p)\rangle\, |\varphi(k-p)\rangle$$

The normalization of the evolved state is currently assumed to be fixed by:

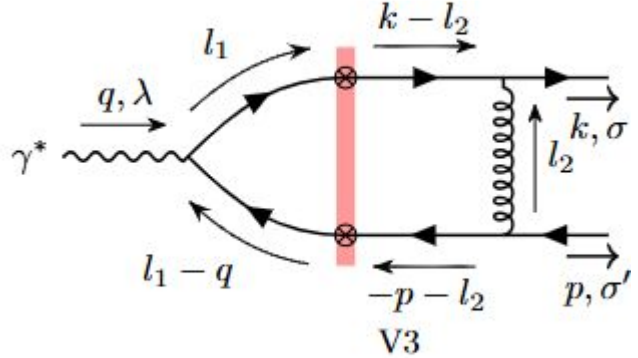
$$\hat{U}(\infty,0)\,|\Phi(k)\rangle \longrightarrow \frac{1}{\mathcal{Z}_\phi\,\mathcal{Z}_\varphi}\hat{U}(\infty,0)\,|\Phi(k)\rangle$$

The prescription above can reproduce only the self-energy contributions of each participating component but not the exchange contributions.

$$\hat{U}(\infty,0)\,|\Phi(k)\rangle \longrightarrow \sqrt{\hat{U}^{\dagger-1}(\infty,0)\hat{U}^{-1}(\infty,0)}\,\hat{U}(\infty,0)\,|\Phi(k)\rangle$$

Dijet production in DIS at NLO

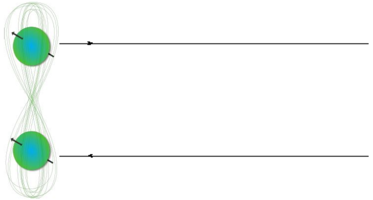
Out of ~20 diagrams involved in NLO calculation one involves the asymptotic time evolution of entangled quark anti-quark pair.



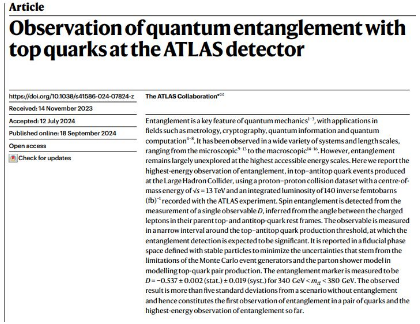
Dijet impact factor in DIS at next-to-leading order in the Color Glass Condensate, P. Caucal, F. Salazar, R. Venugopalan.

One-Loop Corrections to Dihadron Production in DIS at Small x, F. Bergabo, J. Jalilian-Marian.

The computation requires analytical continuation. JIMWLK is not apparent unless combining together with both the instantaneous and conjugate contribution.



$$|\psi(k)\rangle = \int dp\, f(k,p)\; |q(p)\rangle\, |\bar{q}(k-p)\rangle$$



Article

Observation of quantum entanglement with top quarks at the ATLAS detector

https://doi.org/10.1038/s41586-024-07824-z

Received: 14 November 2023

Accepted: 12 July 2024

Published online: 18 September 2024

Open access

Check for updates

The ATLAS Collaboration*

Entanglement is a key feature of quantum mechanics[1–3], with applications in fields such as metrology, cryptography, quantum information and quantum computation[4–8]. It has been observed in a wide variety of systems and length scales, ranging from the microscopic[9–13] to the macroscopic[14–16]. However, entanglement remains largely unexplored at the highest accessible energy scales. Here we report the highest-energy observation of entanglement, in top–antitop quark events produced at the Large Hadron Collider, using a proton–proton collision dataset with a centre-of-mass energy of $\sqrt{s} = 13$ TeV and an integrated luminosity of 140 inverse femtobarns (fb)$^{-1}$ recorded with the ATLAS experiment. Spin entanglement is detected from the measurement of a single observable D, inferred from the angle between the charged leptons in their parent top- and antitop-quark rest frames. The observable is measured in a narrow interval around the top–antitop quark production threshold, at which the entanglement detection is expected to be significant. It is reported in a fiducial phase space defined with stable particles to minimize the uncertainties that stem from the limitations of the Monte Carlo event generators and the parton shower model in modelling top-quark pair production. The entanglement marker is measured to be $D = -0.537 \pm 0.002$ (stat.) ± 0.019 (syst.) for 340 GeV $< m_{t\bar{t}} <$ 380 GeV. The observed result is more than five standard deviations from a scenario without entanglement and hence constitutes the first observation of entanglement in a pair of quarks and the highest-energy observation of entanglement so far.

$$
\begin{aligned}
\mathcal{N}_{\mathrm{V3}}^{\lambda=0,\sigma\sigma'}(\boldsymbol{r}_{xy}) = {} & \frac{\alpha_s}{\pi}\int_0^{z_q}\frac{\mathrm{d}z_g}{z_g}(-2)(z_q z_{\bar{q}})^{3/2}\delta^{\sigma,-\sigma'}QK_0(\bar{Q}_{\mathrm{V3}}r_{xy})\left(1-\frac{z_g}{z_q}\right)\left(1+\frac{z_g}{z_{\bar{q}}}\right)\\
& \times\left\{\left[(1+z_g)\left(1-\frac{z_g}{z_q}\right)\right]e^{i(\boldsymbol{P}_\perp+z_g(\boldsymbol{k}_\perp+\boldsymbol{p}_\perp))\cdot\boldsymbol{r}_{xy}}K_0(-i\Delta_{\mathrm{V3}}r_{xy})\right.\\
& -\left[1-\frac{z_g}{2z_q}+\frac{z_g}{2z_{\bar{q}}}-\frac{z_g^2}{2z_q z_{\bar{q}}}\right]e^{i\frac{z_g}{z_q}\boldsymbol{k}_\perp\cdot\boldsymbol{r}_{xy}}\mathcal{J}_{\odot}\left(\boldsymbol{r}_{xy},\left(1-\frac{z_g}{z_q}\right)\boldsymbol{P}_\perp,\Delta_{\mathrm{V3}}\right)\\
& \left.+\sigma\left[\frac{z_g}{z_q}-\frac{z_g}{z_{\bar{q}}}+\frac{z_g^2}{z_q z_{\bar{q}}}\right]e^{i\frac{z_g}{z_q}\boldsymbol{k}_\perp\cdot\boldsymbol{r}_{xy}}\mathcal{J}_{\otimes}\left(\boldsymbol{r}_{xy},\left(1-\frac{z_g}{z_q}\right)\boldsymbol{P}_\perp,\Delta_{\mathrm{V3}}\right)\right\}\\
& +(q\leftrightarrow\bar{q})\,,
\end{aligned}
$$

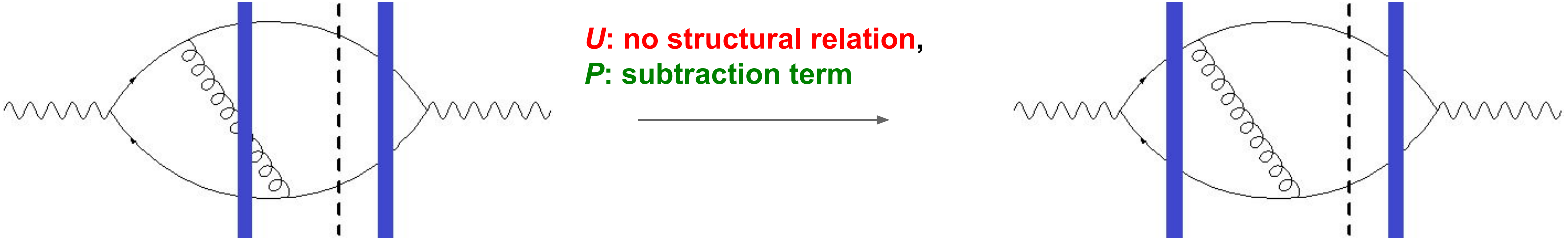
with

$$
\begin{aligned}
\bar{Q}_{\mathrm{V3}}^2 &= (z_q-z_g)(z_{\bar{q}}+z_g)Q^2\,.\\
\boldsymbol{P}_\perp &= z_{\bar{q}}\boldsymbol{k}_\perp - z_q\boldsymbol{p}_\perp\,,\\
\Delta_{\mathrm{V3}}^2 &= \left(1-\frac{z_g}{z_q}\right)\left(1+\frac{z_g}{z_{\bar{q}}}\right)\boldsymbol{P}_\perp^2\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{\odot}(\boldsymbol{r}_\perp,\boldsymbol{K}_\perp,\Delta) &= \int\frac{\mathrm{d}^2\boldsymbol{l}_\perp}{(2\pi)}\frac{2\boldsymbol{l}_\perp\cdot\boldsymbol{K}_\perp\;e^{i\boldsymbol{l}_\perp\cdot\boldsymbol{r}_\perp}}{\boldsymbol{l}_\perp^2\left[(\boldsymbol{l}_\perp-\boldsymbol{K}_\perp)^2-\Delta^2-i\epsilon\right]}\\
\mathcal{J}_{\otimes}(\boldsymbol{r}_\perp,\boldsymbol{K}_\perp,\Delta) &= \int\frac{\mathrm{d}^2\boldsymbol{l}_\perp}{(2\pi)}\frac{(-i)\boldsymbol{l}_\perp\times\boldsymbol{K}_\perp\;e^{i\boldsymbol{l}_\perp\cdot\boldsymbol{r}_\perp}}{\boldsymbol{l}_\perp^2\left[(\boldsymbol{l}_\perp-\boldsymbol{K}_\perp)^2-\Delta^2-i\epsilon\right]}
\end{aligned}
$$

NLO cross sections: *U* vs *P*

The difference between *U* and *P* emerges when computing the gluon exchange with shockwave prior to the gluon emission.



The simple normalization (as an overall prefactor) will fail to generate the term:

$$\hat{\mathcal{N}}(0,\infty) = \int dp\, dl\, f(k,p,l)\ |q(p-l)\rangle\, |\overline{q}(k-p+l)\rangle\, \langle q(p)|\, \langle \overline{q}(k-p)|$$

- ***Based on U***: leads to extremely complicated integrals,

$$\langle \Phi(k)|\, \hat{U}(0,-\infty) - \hat{\mathbf{1}}\, |\Phi(k)\rangle\Big|_{g^2} \sim \int_{-\infty}^{0} dt'\, \hat{H}(t') \int_{-\infty}^{t'} dt''\, \hat{H}(t'') \sim \frac{\langle k|\, \hat{H}\, |j\rangle \langle j|\, \hat{H}\, |i\rangle}{(E_k - E_i)(E_j - E_i)}$$

- ***Based on P***:

$$\langle \Phi(k)|\, \hat{P}(0,-\infty) - \hat{\mathbf{1}}\, |\Phi(k)\rangle\Big|_{g^2} \sim \frac{1}{2}\left|\int_{-\infty}^{0} dt'\, \hat{H}(t')\right|^2 \sim \frac{\langle k|\, \hat{H}\, |j\rangle \langle j|\, \hat{H}\, |i\rangle}{2(E_k - E_j)(E_j - E_i)}$$

A sloppy trick

As discussed in Tuomas talk:

$$\frac{\delta(M_X{}^2 - M_2^2)}{(M_2^2 - M_1^2 + i\delta)(M_2^2 - M_0^2 + i\delta)} + \frac{\delta(M_X{}^2 - M_1^2)}{(M_1^2 - M_0^2 + i\delta)(M_1^2 - M_2^2 - i\delta)} + \frac{\delta(M_X{}^2 - M_0^2)}{(M_0^2 - M_1^2 - i\delta)(M_0^2 - M_2^2 - i\delta)}$$

(Note: sign of $i\delta$ essential)

$$= \frac{1}{2\pi i}\left[\frac{1}{(M_X{}^2 - M_0^2 - i\delta)(M_X{}^2 - M_1^2 - i\delta)(M_X{}^2 - M_2^2 - i\delta)} - \text{c.c.}\right]$$

Where is the cheat?

1) Omitting the limit operation.
2) Using a common regulator for all the denominators.
3) Rewriting the sum of limits as a limit of sum.

More rigorously,

$$\frac{1}{(M_0^2 - M_1^2 - i\delta)(M_0^2 - M_2^2 - i\delta)} \longrightarrow \lim_{\delta_1 \to 0} \frac{1}{M_0^2 - M_1^2 - i\delta_1} \lim_{\delta_2 \to 0} \frac{1}{M_0^2 - M_2^2 - i\delta_2}$$

Simplification via Sokhotski theorem

$$\lim_{\epsilon \to 0} \frac{1}{f(x) - i\epsilon} = P.v.\left(\frac{1}{f(x)}\right) - i\pi\delta(f(x))$$

Summary

1) The current treatment of QFT is based on '*conception of unitarity*' - a low level of mathematical rigorousness. A manifestly unitary and non-linear formalism is proposed.

2) The new perturbative expansion shows that JIMWLK is the universal high energy limit of quantum behavior.

3) Main implications – quantum chaos, entangled states, and non-perturbative physics. Experimental tests to appear.

HAPPY BIRTHDAY EDMOND!