Conformal BK equation at QCD Wilson-Fisher point

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$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

- LLA for DIS in pQCD $\Rightarrow$ BFKL
  - (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$): describes proliferation of gluons.

- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK eqn
  - background field method: describes recombination process.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \, A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\not{z}^4} \frac{(x-y)^2 \;\not{z}^4}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal properties of light-like Wilson lines

Wilson lines are invariant under the (Möbius) SL(2,C) group

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$
$$\bar{\hat{S}}_- \equiv \frac{i}{2}(K^1 - iK^2), \quad \bar{\hat{S}}_0 \equiv \frac{i}{2}(D - iM^{12}), \quad \bar{\hat{S}}_+ \equiv \frac{i}{2}(P^1 + iP^2)$$

form SL(2,C) algebra

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm\,, \qquad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_0\,,$$
$$[\bar{\hat{S}}_0, \bar{\hat{S}}_\pm] = \pm\bar{\hat{S}}_\pm\,, \qquad [\bar{\hat{S}}_+, \bar{\hat{S}}_-] = 2\bar{\hat{S}}_0\,.$$

Momentum operator $\hat{P}$, angular momentum operator $\hat{M}$, dilatation operator $\hat{D}$, and special conformal generator $\hat{K}$.

Conformal properties of light-like Wilson lines

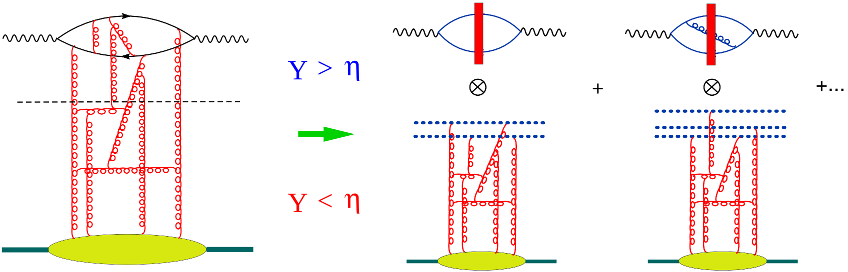
$$i[\hat{P}^\mu, \hat{A}^\alpha] = \partial^\mu \hat{A}^\alpha, \quad i[\hat{D}, \hat{A}^\alpha] = (x_\mu \partial^\mu + 1)\hat{A}^\alpha,$$
$$i[\hat{M}^{\mu\nu}, \hat{A}^\alpha] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\hat{A}^\alpha - (g^{\nu\alpha}\hat{A}^\mu - g^{\mu\alpha}\hat{A}^\nu)$$
$$i[K^\mu, A^\alpha] \; = \; (2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu + 2x^\mu)A^\alpha - 2x_\nu(g^{\nu\alpha}A^\mu - g^{\mu\alpha}A^\nu)$$

Complex notation

$$z \equiv z^1 + iz^2, \quad \bar{z} \equiv z^1 - iz^2, \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial z^1} - i\frac{\partial}{\partial z^2}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial z^2}\Big)$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2 \partial_z \hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z \hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z \hat{U}(z,\bar{z})$$
$$[\hat{\bar{S}}_-, \hat{U}(z,\bar{z})] = \bar{z}^2 \partial_{\bar{z}} \hat{U}(z,\bar{z}), \quad [\hat{\bar{S}}_0, \hat{U}(z,\bar{z})] = \bar{z}\partial_{\bar{z}} \hat{U}(z,\bar{z}), \quad [\hat{\bar{S}}_+, \hat{U}(z,\bar{z})] = -\partial_{\bar{z}} \hat{U}(z,\bar{z})$$

$U(z,\bar{z})$ lie in the standard representation of conformal group $SL(2,C)$



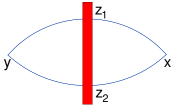
The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

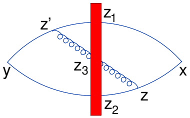
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

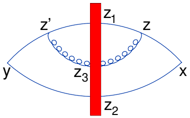
$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$



NLO Impact Factor diagrams: $I^{\rm NLO}$





$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, \left[I_1^{\mu\nu}(z_1,z_2,z_3) + I_2^{\mu\nu}(z_1,z_2,z_3)\right] \times[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

where $I_2^{\mu\nu}(z_1,z_2,z_3)$ is finite and conformal, while

$$I_1^{\mu\nu}(z_1,z_2,z_3) = \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3}$$

is rapidity divergent. $\alpha \propto k^+$

How to get the NLO Impact factor

$$\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$+ \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}] + \dots$$

$\Rightarrow$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} - \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO}$$

$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\, I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

$$\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO} = \frac{\alpha_s}{2\pi^2}\int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]\int_0^{e^\eta}\frac{d\alpha}{\alpha}$$

How to get the NLO Impact factor

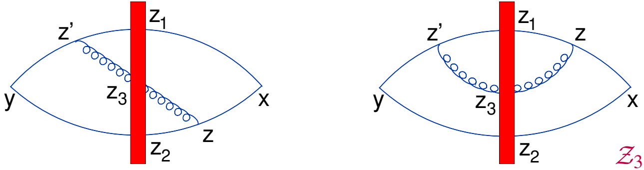
$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\ I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]$$
$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\ \Big\{I_2^{\mu\nu}(z_1,z_2,z_3) + \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\int_0^{+\infty}\frac{d\alpha}{\alpha}\,e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta}\frac{d\alpha}{\alpha}\Big]\Big\}$$
$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

$$\Big[\int_0^{+\infty}\frac{d\alpha}{\alpha}\,e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta}\frac{d\alpha}{\alpha}\Big] \to -\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C$$

where $\sigma = e^\eta$ and C is the Euler constant

$$\mathcal{Z}_3 \equiv \frac{(x-z_3)^2_\perp}{x^+} - \frac{(y-z_3)^2_\perp}{y^+}$$

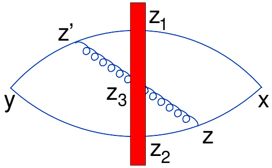
$\mathcal{Z}_3$ is not conformal invariant in the transverse 2-d coordinate space, but QCD at tree level has to be conformal invariant.

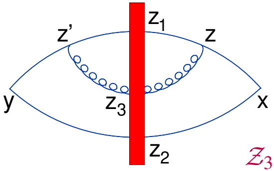


$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I_{\mu\nu}^{\rm LO}\times\frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.





$$\mathcal{Z}_3 \equiv \frac{(x-z_3)^2_\perp}{x^+} - \frac{(y-z_3)^2_\perp}{y^+}$$

$$I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}_{\mu\nu} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf} \;=\; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\,\frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\Big]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \;+\; O(\alpha_s^2)$$

the impact factor becomes conformal at the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = -\ I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z^2_{12}}{z^2_{13}z^2_{23}} \ln\frac{z^2_{12}e^{2\eta}as^2}{z^2_{13}z^2_{23}}\mathcal{Z}^2_3 + {\rm conf.}$$

the energy dependence matrix element of Wilson-line operators

$\Rightarrow\quad a = -\frac{2x^+y^+}{(x-y)^2}$

The new NLO impact factor is conformally invariant.

NLO IF in coordinate space for BK   I. Balitsky and G.A.C. (2010)

NLO IF in momentum space for BFKL   I. Balitsky and G.A.C. (2012)

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = -\, I^{\rm LO}_{\mu\nu}\,\frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

NLO IF in coordinate space for BK   I. Balitsky and G.A.C. (2010)

NLO IF in momentum space for BFKL   I. Balitsky and G.A.C. (2012)

proton-Nucleus collision at NLO   G.A.C., BW Xiao, F. Yuan (2011 - 2012)

to cure negativity of cross-section perhaps one could use similar constraint as the one used in NLO impact-factor

Evolution parameter: from η to a

In conformal $\mathcal{N}=4$ SYM theory one can construct the composite conformal dipole operator order by order in perturbation theory.

$$a \to e^{-2\eta} a \;\Rightarrow\; [\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\}]_a^{\mathrm{conf}} = 0$$

$$2a\frac{d}{da}[\mathrm{Tr}\{U_x U_y^\dagger\}]_a^{\mathrm{conf}} \sim \alpha_s K_{LO}\Big(1 + (\frac{11}{3}N_c - \frac{2}{3}n_f)\mathrm{run.coup.}\Big) + \alpha_s^2 K_{NLO}^{\mathrm{conf}}$$

Evolution equation for color dipoles in $\mathcal{N} = 4$

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\ \frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

BK at NLO

$b = \frac{11}{3}N_c - \frac{2}{3}n_f \qquad\qquad X = x - z \qquad Y = y - z$

$$\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\, \Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c \mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1 + \frac{\alpha_s}{4\pi}\Big(b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} + (\frac{67}{9} - \frac{\pi^2}{3})N_c - \frac{10}{9}n_f\Big)\Big]$$
$$-\frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\}$$
$$-(z' \to z)]\frac{1}{(z-z')^4}\Big[-2 + \frac{X'^2Y^2 + Y'^2X^2 - 4(x-y)^2(z-z')^2}{2(X'^2Y^2 - Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$
$$+[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \to z)]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\} + n_f\text{−terms}\Big)$$

NLO kernel = Running coupling terms + Non-conformal term + Conformal term

Evolution equation for color dipoles in $\mathcal{N} = 4$

$$
\begin{aligned}
&2a\frac{d}{da}[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}_a \\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}\Big]\Big\} \\
&\times\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}] \\
&-\ \frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2} \\
&\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Conformal

NLO evolution of composite "conformal" dipoles in QCD

$$2a\frac{d}{da}[\mathrm{tr}\{\hat U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\ \Big([\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1 + \frac{\alpha_s}{4\pi}\big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2} + N_c(\frac{67}{9}-\frac{\pi^2}{3}) - \frac{10}{9}n_f\big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\ \Big\{\Big[-2 + \frac{{z_{14}}^2z_{23}^2 + {z_{24}}^2z_{13}^2 - 4z_{12}^2z_{34}^2}{2({z_{14}}^2z_{23}^2 - {z_{24}}^2z_{13}^2)}\ln\frac{{z_{14}}^2z_{23}^2}{{z_{24}}^2z_{13}^2}\Big]$$

$$\times\ [\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\hat U_{z_4}\hat U^\dagger_{z_2}\hat U_{z_3}\hat U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\ \frac{z_{12}^2z_{34}^2}{z_{13}^2{z_{24}}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2} + \Big(1 + \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2 - z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2{z_{24}}^2}{{z_{14}}^2z_{23}^2}\Big]$$

$$\times\ [\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_4}\hat U_{z_3}\hat U^\dagger_{z_2}\hat U_{z_4}\hat U^\dagger_{z_3}\} - (z_4\to z_3)] + n_f-\mathrm{terms}\Big)$$

$b = \frac{11}{3}N_c - \frac{2}{3}n_f$ I. Balitsky and G.A.C

$\mathrm{K}_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $\mathrm{K}_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

BFKL / BK equation in d-dimension

$X \equiv x_\perp - z_\perp \qquad Y \equiv y_\perp - z_\perp \qquad\qquad d_\perp = 2 - 2\epsilon$

$$\frac{d}{d\eta}\mathcal{U}(x,y) = \frac{\alpha_s N_c \Gamma^2(1-\varepsilon)}{2\pi^{2-2\varepsilon}\mu^{2\varepsilon}}\int d^{2-2\varepsilon}z \left[\frac{X_i}{(X_\perp^2)^{1-\varepsilon}} - \frac{Y_i}{(Y_\perp^2)^{1-\varepsilon}}\right]\left[\frac{X_i}{(X_\perp^2)^{1-\varepsilon}} - \frac{Y_i}{(Y_\perp^2)^{1-\varepsilon}}\right]$$
$$\times\Big[\mathcal{U}(x,z) + \mathcal{U}(z,y) - \mathcal{U}(x,y) - \mathcal{U}(x,z)\mathcal{U}(z,y)\Big]$$

Not conformal invariant $\Rightarrow$ BFKL cannot be solved by power-like eigenfunction

Idea: Go to Wilson-Fisher point and restore conformal invariance

$\varepsilon_* \simeq -\frac{\alpha_s}{4\pi}b_0 - \frac{\alpha_s^2}{16\pi^2}b_1 + \cdots \;\Rightarrow\; \beta(\alpha_s) = 0$

BK equation at Wilson-Fisher point

I. Balitsky and G.A.C (2024)

Expand in ε and α_s $\Rightarrow$ Consider NLO corrections (for conf. comp. operator)

$$\frac{d}{d\eta}\mathcal{U}(x,y) = \frac{\alpha_s N_c \Gamma(1-\varepsilon)}{2\pi^{2-\varepsilon}}\int d^{2-2\varepsilon}z\left(\frac{(x-y)_\perp^2}{X^2Y^2}\right)^{1-\varepsilon}\Bigg\{1+ \\ +\varepsilon\frac{\alpha_s}{4\pi}\left(\ln\frac{(x-y)_\perp^2\mu^2}{4}+2\gamma_E-\frac{X^2-Y^2}{(x-y)_\perp^2}\ln\frac{X^2}{Y^2}\right)\Bigg\}$$

$$2a\frac{d}{da}[\mathcal{U}(x,y)]_a^{\mathrm{conf}} = \frac{\alpha_s N_c}{2\pi^2}\int d^{2-2\varepsilon}z\left(\frac{(x-y)_\perp^2}{X^2Y^2}\right)^{1-\varepsilon}\Bigg\{1+ \\ +b_0\frac{\alpha_s}{4\pi}\left(\ln\frac{(x-y)_\perp^2\mu^2}{4}+2\gamma_E-\frac{X^2-Y^2}{(x-y)_\perp^2}\ln\frac{X^2}{Y^2}\right)+\frac{\alpha_s N_c}{4\pi}\left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10n_f}{9N_c}\right)\Bigg\} \\ \times\Big[\mathcal{U}(x,z)+\mathcal{U}(z,y)-\mathcal{U}(x,y)-\mathcal{U}(x,z)\mathcal{U}(z,y)\Big]+\frac{\alpha_s^2}{16\pi^4}K_{conf}+\mathcal{O}(\alpha_s^3)$$

Notice: expansion in ε does not reproduce the term $\ln\frac{X^2}{(x-y)_\perp^2}\ln\frac{Y^2}{(x-y)_\perp^2}$

BK equation at Wilson-Fisher point

$\varepsilon_* \simeq -\frac{\alpha_s}{4\pi} b_0$ $\quad\quad$ $b_0 = \frac{11}{3} N_c - \frac{2}{2} n_f$

$$\frac{d}{d\eta}\mathcal{U}(x,y) = \frac{\alpha_s N_c \Gamma(1-\varepsilon_*)}{2\pi^{2-\varepsilon_*}} \int d^{2-2\varepsilon} z \left(\frac{(x-y)^2_\perp}{X^2 Y^2}\right)^{1-\varepsilon} \Big[1 +$$

$$+ \frac{\alpha_s N_c}{4\pi}\left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10 n_f}{9 N_c}\right)\Big]\Big[\mathcal{U}(x,z) + \mathcal{U}(z,y) - \mathcal{U}(x,y) - \mathcal{U}(x,z)\mathcal{U}(z,y)\Big]$$

$$\times + \frac{\alpha_s^2}{16\pi^4} K_{conf} + \mathcal{O}(\alpha_s^3, \alpha_s^2 \varepsilon_*, \alpha^2 \varepsilon_*^2)$$

We obtain the NLO running coupling from the LO calculation (without the non conformal terms)!

Pomeron intercept at Wilson-Fisher point

Forward linearized equation: BFKL

$$\frac{d}{d\eta}\mathcal{U}(z) = \frac{\alpha_s N_c \Gamma(1-\varepsilon_*)}{2\pi^{2-\varepsilon_*}}\int d^{2-2\varepsilon_*}z'\ \Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10n_f}{9N_c}\Big)\Big]\Big(\frac{z^2}{(z-z')^2 z'^2}\Big)^{1-\varepsilon_*}$$
$$\times\ \big[2\mathcal{U}(z')-\mathcal{U}(z)\big]+\frac{\alpha_s^2 N_c^2}{4\pi^3}\int dz'\,\mathcal{K}_{\rm conf}(z,z')\,\mathcal{U}(z')\ +\ O(\alpha_s^3,\alpha_s^2\varepsilon_*,\alpha_s\varepsilon_*^2)$$

Conformal invariance restored: Solve by power-like eigenfunctions

$$\mathcal{U}(|z|)\ =\ \frac{\Gamma\big(\frac{d_\perp}{2}\big)}{\pi^{\frac{d_\perp}{2}}}\int d^{d_\perp}z'\int\frac{d\nu}{2\pi}(z'^2)^{-\frac{d_\perp}{2}-\frac{d_\perp}{4}+i\nu}(z^2)^{\frac{d_\perp}{2}-\frac{d_\perp}{4}-i\nu}\mathcal{U}(|z'|)$$

BK equation at Wilson-Fisher point

using the evolution equation in d-dimension

$$\mathcal{U}(|z|) \;=\; \frac{\Gamma\left(\frac{d_\perp}{2}\right)}{\pi^{\frac{d_\perp}{2}}}\int d^{d_\perp}z'\int\frac{d\nu}{2\pi}(z'^2)^{-\frac{d_\perp}{2}-\frac{d_\perp}{4}+i\nu}(z^2)^{\frac{d_\perp}{2}-\frac{d_\perp}{4}-i\nu}\;e^{\eta\bar{\aleph}\left(\frac{d_\perp}{4}-i\nu\right)}\mathcal{U}(|z'|)$$

$$\bar{\aleph}(\xi) \;=\; \frac{\alpha_s N_c}{\pi}\Big[\bar{\chi}(\xi)+\frac{\alpha_s N_c}{4\pi}\bar{\delta}(\xi)+O(\alpha_s^2,\alpha_s\varepsilon_*,\varepsilon_*^2)\Big]$$
$$\bar{\chi}(\xi) \;=\; \psi(1-\varepsilon_*)-\gamma_E-\psi(\xi)-\psi\big(1-\varepsilon_*-\xi\big)$$

Pomeron intercept (at $\nu = 0$)

$$\frac{\alpha_s N_c}{\pi}\chi\Big(\frac{1}{2}-\frac{\varepsilon_*}{2}\Big) = \frac{\alpha_s N_c}{\pi}\Big[4\ln 2-\frac{\pi^2}{3}\varepsilon_*+O(\varepsilon_*^2)\Big]$$

BFKL at the saddle point approximation

DIS at high-energy $\quad -q^2 = Q^2 \gg P^2 \qquad s = (P+q)^2 \gg Q^2$

$$\sigma^{\gamma^* p}(x_B, Q^2) = \int d\nu\, F(\nu)\, x_B^{-\aleph(\nu)-1} \left(\frac{Q^2}{P^2}\right)^{\frac{1}{2}+i\nu}$$

$\aleph(\gamma)$ is the BFKL pomeron intercept. $\qquad \gamma = \frac{1}{2} = i\nu$

Saddle point approximation:

$$\sigma^{\gamma^* p}(x_B, Q^2) \sim \left(\frac{1}{x_B}\right)^{\bar{\alpha}_s 4 \ln 2}$$

From local operators to Light-ray operators

N-th moment of the structure function is

$$M_N = \int_0^1 dx_B\, x_B^{N-1} \sigma^{\gamma^* p}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma\, \frac{F(\gamma)}{N-1-\aleph(\gamma)} \left(\frac{Q^2}{P^2}\right)^{\gamma}$$

$$\aleph(\gamma) = \bar{\alpha}_s \left(2\psi(1) - \psi(\gamma) - \sum_{m=1}^{N} \frac{1}{m-\gamma} - \psi(N+1-\gamma)\right)$$

The BFKL is given as a sum over all the residues

- Leading residue: $\aleph(\gamma) \to \frac{\bar{\alpha}_s}{\gamma-1}$ $\qquad \bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$
- Next-to-Leading residue: $\aleph(\gamma) \to \frac{\bar{\alpha}_s}{\gamma-2}$

Closing the contour on the poles we get the anomalous dimensions of the leading and higher twist operators at the *nonphysical point* $N=1$.

Analytic continuation in the complex plane

$$\int_0^1 dx_B\, x_B^{n-1}\sigma^{\gamma^* p}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma\, \frac{F(\gamma)}{\omega - \aleph(\gamma)} \left(\frac{Q^2}{P^2}\right)^{\gamma}$$

Analytic continuation: $N-1 \to \omega$ complex continuous variable

$\Rightarrow$ Residues $\omega = \aleph(\gamma) \simeq \frac{\bar{\alpha}_s}{\gamma-1}$;

$$\mathrm{Leading-Twist} \qquad \gamma(\alpha_s, \omega) = \frac{\bar{\alpha}_s}{\omega} + \mathcal{O}(\alpha_s^2), \quad \sigma(\omega, Q^2) \sim \left(\frac{Q^2}{P^2}\right)^{\frac{\bar{\alpha}_s}{\omega}}$$

$\alpha_s \ln\frac{1}{x_B} \sim 1 \;\; \to \frac{\alpha_s}{\omega} \sim 1 \qquad x_B \to 0 \;\Leftrightarrow\; \omega \to 0 \qquad \Rightarrow$ resummation: BFKL eq.

Thus, we get the analytic continuation of anomalous dimension at the *nonphysical point* $N \to 1$ of twist-2 gluon operator $F^a_{\xi+}\nabla^{N-2}F^{\xi a}{}_{+}$

twist-2 gluon operator $F^a_{\xi+}\nabla^{N-2}F^{\xi a}{}_{+}$

Anomalous dimension (gluon only)

$$\gamma_N = \frac{\alpha_s N_c}{\pi}\left(-\frac{1}{N(N-1)} - \frac{1}{(N+1)(N+2)} + \psi(N+1) + \gamma_E - \frac{11}{12}\right) + \mathcal{O}(\alpha_s^2)$$

BFKL gives $\gamma(N,\alpha_s)$ at $N \to 1$

$$\gamma_N = \left[A_N^{\text{LO BFKL}} + (N-1)B_N^{NLO\ BFKL} + \dots\right]\left(\frac{\alpha_s N_c}{\pi(N-1)}\right)^n$$

- LO BFKL: Jaroszewicz (1982)
- NLO BFKL: Fadin and Lipatov; Camici, Ciafaloni (1998)
- NNLO BFKL in $\mathcal{N}=4$ SYM: Gromov *et al*(2016)

at N=1 $F^a_{-i}\nabla_-^{-1}F_-^{ai}$ is a non-local operator

We want an operator language

Light-ray operator as analytic continuation of local operator

Gluon light-ray (LR) operator of twist 2

$$F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_{-}(x_+ + x_\perp)$$

Analytic continuation of local operator

$$\begin{aligned}
F^a_{-i}\nabla_-^{j-2}F^{ai}_- &= \frac{\Gamma(j-1)}{2\pi i}\int_{\text{Hankel}} du\, u^{1-j} F^a_{-i}\, e^{-u\nabla_-} F^{ai}_-(0) \\
&= \frac{\Gamma(j-1)}{2\pi i}\int_{\text{Hankel}} dx_+\, x_+^{1-j} F^a_{-i}(x_+)[x_+, 0]F^{a-}_-(0) \\
&= \frac{1}{\Gamma(2-j)}\int_0^{+\infty} dx_+\, u_+^{1-j} F^a_{-i}(x_+)[x_+, 0]F^{ai}_-(0)
\end{aligned}$$

Evolution equation for light-ray operator

Forward light-ray operator

$$F(L_+, x_\perp) = \int dx_+ F^a_{-i}(L_+ + x_\perp)[L_+ + x_+, x_+]^{ab}_x F^{bi}_-(x_+ + x_\perp)$$

Evolution equation

$$\mu^2 \frac{d}{d\mu^2} F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab}_x F^b_-{}^i(x_+ + x_\perp)$$
$$= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+ K(x'_+, x_+; z'_+, z_+; \alpha_s) F^a_{-i}(z'_+ + x_\perp)[z'_+, z_+]^{ab}_x F^b_-{}^i(z_+ + x_\perp)$$

$$[x'_+, x_+]_x = [x'_+ + x_\perp, x_+ + x_\perp] = \mathrm{Pexp}\{ig \int_{x_+}^{x'_+} dw_+ A_-(w_+, x_\perp)\}$$

Evolution in forward local operators

forward local operators

$$F(L_+,x_\perp)=\sum_{n=2}^{\infty}\frac{L_+^{n-2}}{(n-2)!}\mathcal{O}_n^g(x_\perp)\,,\qquad \mathcal{O}_n^g\equiv\int dx_+F^a_{-i}\nabla_-^{n-2}F_-^{ai}(x_+,x_\perp)$$

$$\mu^2\frac{d}{d\mu^2}F(L_+,x_\perp)=\int_0^1du\,K_{gg}(u,\alpha_s)F(uL_+,x_\perp)$$

$$\Rightarrow\ \mu^2\frac{d}{\mu^2}\mathcal{O}_N^g=-\gamma_N(\alpha_s)\mathcal{O}_N^g\,,\qquad \gamma_N(\alpha_s)=-\int_0^1du\,u^{N-2}K_{gg}(u,\alpha_s)$$

DGLAP kernel $(\bar{u}=1-u)$

$$u^{-1}K_{gg}(u)=\frac{2\alpha_sN_c}{\pi}\Big(u\bar{u}+\Big[\frac{1}{u\bar{u}}\Big]_+-2+\frac{11}{12}\delta(\bar{u})\Big)+\mathcal{O}(\alpha_s^2)$$

$j = \frac{1}{2} + i\nu$

$$\mathcal{F}_j(x_\perp) = \int_0^{+\infty} dL_+ \, L_+^{1-j} F(L_+, x_\perp)$$

Evolution equation

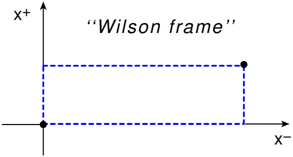
$$\mu^2 \frac{d}{d\mu^2} \mathcal{F}_j(z_\perp) = \int_0^1 du \, K_{gg}(u, \alpha_s) u^{j-2} \mathcal{F}_j(z_\perp)$$

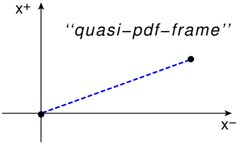
$\Rightarrow$ $\gamma_j(\alpha_s)$ is an analytic continuation of $\gamma_N(\alpha_s)$ from the "DGLAP side

How to get the anomalous dimension γ_j at $j \to 1$ from an operator language "from BFKL side"

Correlation function of

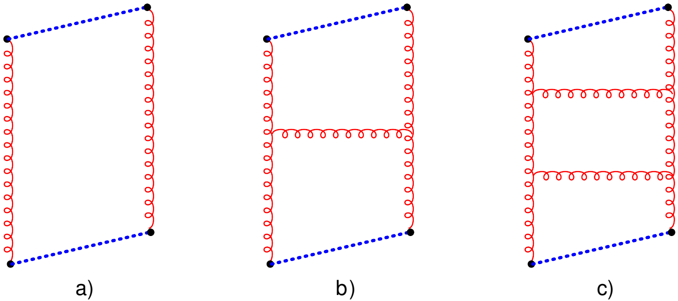
- Wilson frame built from light-ray operator
  - I. Balitsky, V. Kazakov, E. Sobko (2013)
- quasi-pdf frame built from light-ray operator
  - G.A.C. (gluon (2019) and quark (2021))





However, there are many integrals are complicated at $d_\perp \neq 2$

Example: Correlation Function of quasi-pdf LR operators



Easier way: Regge+DGLAP limit of Correlation Function of 4 scalar operators

Forward correlation function of 4 scalar operators (*e.g.*$\psi\bar{\psi}$) in $d = 4 - \varepsilon_*$ in QCD

$$A(L_+, L_-; x_{1\perp}, x_{2\perp}, x_{3\perp}, x_{4\perp}) = (x_{12}^2 x_{34}^2)^{2-2\varepsilon} (\mu^4 x_{12}^2 x_{34}^2)^{\gamma_{\mathcal{O}}}$$
$$\times \int dx_{2+} dx_{3+} \langle \mathcal{O}(L_+ + x_{2+}, x_{1\perp}) \mathcal{O}(x_{2+}, x_{2\perp}) \mathcal{O}(L_- + x_{4-}, x_{3\perp}) \mathcal{O}(x_{4-}, x_{4\perp}) \rangle$$

- Regge limit: $L_+ L_- \to \infty$
- DGLAP limit: $x_{12}^2 \to 0$

Plan:

Comparing Regge+light-cone vs light-cone+Regge limits to relate the BFKL Pomeron intercept to the anomalous dimensions of gluon LR operators in the limit

$$j - 1 = \omega \to 0, \qquad \frac{\alpha_s}{\omega} \sim 1$$

Correlation function in CFT

In CFT the 4-points CF depends only on the 2 conformal ratios: $A(R, r)$

$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho\rho' x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \to \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$
$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

Pomeron in CFT: "BFKL" representation of 4-point CF

Cornalba (2007); Cornalba, Costa, Penedones (2008); Costa, Gonsalves, Penedones (2012)

$$A(R,r) \overset{s\sim\rho\rho'\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\aleph(\alpha_s,\nu))F(\alpha_s,\nu)\Omega(r,\nu)R^{\aleph(\alpha_s,\nu)/2}$$

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ is a signature factor

$$\Omega(r,\nu) = \frac{2\nu\sinh 2\pi\nu\Gamma\left(2-\frac{d}{2}+2i\nu\right)\Gamma\left(\frac{d}{2}-1-2i\nu\right)\Gamma(d-2)}{2^{d-1}\pi^{\frac{d+1}{2}}\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}C^{\frac{d}{2}-1}_{-\frac{d}{2}+1+2i\nu}\left(\frac{1}{2\sqrt{r}}\right)$$

is a solution of the Laplace equation for H_{d-1} hyperboloid
$(\partial^2_{H_{d-1}}+\nu^2+1)\Omega(r,\nu)=0$

Dynamics encoded in:
Pomeron intercept: $\aleph(g^2,\nu)$
Pomeron residue: $F(g^2,\nu)$

Light-cone limit to “BFKL” rep. of 4-point CF

$$R \xrightarrow{Regge+LC} \frac{x_{1_+}x_{2_+}x_{3_-}x_{4_-}}{x^2_{12_\perp}x^2_{34_\perp}}$$

$$r \xrightarrow{Regge+LC} \frac{x^2_{12_+}(x_{3_-}x^2_{14\perp} - x_{4_-}x^2_{13_\perp})^2}{x_{1_+}x_{2_+}x_{3_-}x_{4_-}x^2_{12_\perp}x^2_{34_\perp}}$$

$$A(L_+, L_-; x_{1\perp}, x_{2\perp}, x_{3\perp}, x_{4\perp}) = (x^2_{12}x^2_{34})^{2-2\varepsilon}(\mu^4 x^2_{12}x^2_{34})^{\gamma_{\mathcal{O}}}$$
$$\times \int dx_{2+}dx_{3+}\langle \mathcal{O}(L_+ + x_{2+}, x_{1\perp})\mathcal{O}(x_{2+}, x_{2\perp})\mathcal{O}(L_- + x_{4-}, x_{3\perp})\mathcal{O}(x_{4-}, x_{4\perp})\rangle$$
$$= \frac{i\alpha_s^2}{8}\pi^2 \int_0^1 d\nu \int_{\frac{d}{4}-i\infty}^{\frac{d}{4}+i\infty} d\xi\, f_+(\aleph(\xi))\tilde{F}(\xi)$$
$$\times \frac{(\bar{\nu}\nu)^{1-\varepsilon_*-\xi+\frac{\aleph(\xi)}{2}}}{[x^2_{13\perp}\nu + x^2_{14\perp}\bar{\nu}]^{\aleph(\xi)}} \left(\frac{x^2_{12\perp}x^2_{34\perp}}{[x^2_{13\perp}\nu + x^2_{14\perp}\bar{\nu}]^2}\right)^{1-\varepsilon_*-\xi-\frac{\aleph(\xi)}{2}} (L_+L_-)^{1+\aleph(\xi)}$$

$\xi \equiv \frac{d}{4} - i\nu = \frac{1}{2} - i\nu - \frac{\varepsilon_*}{2}$

at $d_\perp = 2$ and in $\mathcal{N} = 4$ I.Balitsky and G.A.C. (2009) but now we are in QCD

$$
\begin{aligned}
A(L_+, L_-; x_{1_\perp}, x_{2_\perp}, x_{3_\perp}, x_{4_\perp}) &= [x_{12}^2 x_{34}^2]^{4-2\varepsilon_* + \gamma_{\mathcal{O}}} (\mu^2)^{2\gamma_{\mathcal{O}}} \\
&\times \int dx_{2_+} dx_{3_-} \, \langle \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp}) \mathcal{O}(L_- + x_{4_-}, x_{3_\perp}) \mathcal{O}(x_{4_-}, x_{4_\perp}) \rangle \\
&= \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{dj}{2\pi} (1 + e^{i\pi j})(L_+ L_-)^j F(j, \alpha_s) (x_{12_\perp}^2 x_{34_\perp}^2)^{1-\varepsilon_* + \frac{\gamma_j}{2}} \\
&\times \int_0^1 du \, \frac{(\bar{u}u)^{j+\frac{\gamma_j}{2}-\varepsilon_*}}{(x_{03_\perp}^2 \bar{u} + x_{04_\perp}^2 u)^{1+j+\gamma_j - 2\varepsilon_*}}
\end{aligned}
$$

where

$$
F(j, \alpha_s) \;=\; c(j, \alpha_s) d(j, \alpha_s) \frac{i\pi^2 e^{i\pi(\frac{\gamma_j}{2} - \varepsilon_*)}}{4^{j-1}} \frac{\Gamma(1 + 2j - \gamma_j + 2\varepsilon_*)}{\Gamma^2(1 + j + \frac{\gamma_j}{2} - \varepsilon_*)}
$$

Anomalous dimensions at $j \to 1$ from comparison of the two limits

$$A(L_+, L_-; x_{1_\perp}, x_{2_\perp}, x_{3_\perp}, x_{4_\perp})$$

$$\stackrel{LC+Regge}{=} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{dj}{2\pi}(1+e^{i\pi j})(L_+L_-)^j F(j,\alpha_s)(x^2_{12_\perp} x^2_{34_\perp})^{1-\varepsilon_*+\frac{\gamma_j}{2}}$$

$$\times \int_0^1 du\, \frac{(\bar{u}u)^{j+\frac{\gamma_j}{2}-\varepsilon_*}}{(x^2_{03_\perp}\bar{u} + x^2_{04_\perp}u)^{1+j+\gamma_j-2\varepsilon_*}}$$

$$\stackrel{Regge+LC}{=} \frac{i\alpha_s^2}{8}\pi^2 \int_0^1 d\nu \int_{\frac{d}{4}-i\infty}^{\frac{d}{4}+i\infty} d\xi\, f_+(\aleph(\xi))\tilde{F}(\xi)$$

$$\times \frac{(\bar{\nu}\nu)^{1-\varepsilon_*-\xi+\frac{\aleph(\xi)}{2}}}{[x^2_{13\perp}\nu + x^2_{14\perp}\bar{\nu}]^{\aleph(\xi)}} \left(\frac{x^2_{12\perp}x^2_{34\perp}}{[x^2_{13\perp}\nu + x^2_{14\perp}\bar{\nu}]^2}\right)^{1-\varepsilon_*-\xi-\frac{\aleph(\xi)}{2}} (L_+L_-)^{1+\aleph(\xi)}$$

The two representations are the same if we make the following identifications

$$\omega \equiv j-1 = \bar{\aleph}(\xi,\alpha_s), \qquad \gamma_j(\alpha_s) = -2\xi - \bar{\aleph}(\xi) = -2\xi - \omega$$

Anomalous dimensions at $j \to 1$ from comparison of the two limits

$$\omega \;\equiv\; j-1 \;=\; \bar{\aleph}(\xi,\alpha_s), \qquad \gamma_j(\alpha_s) \;=\; -2\xi - \bar{\aleph}(\xi) \;=\; -2\xi - \omega$$

to put the above identification as the one in Lipatov-Fadin and Camici-Ciafaloni, use $\quad \tilde{\gamma}_w = -\frac{\gamma_j}{2}$

$$\Rightarrow \quad \omega \;=\; \bar{\aleph}\Big(\tilde{\gamma}_\omega - \frac{\omega}{2}\Big)$$

Solve at small ω and $\tilde{\gamma} \simeq 0$. With the NLO accuracy

$$\omega \;\simeq\; \bar{\aleph}(\tilde{\gamma}_\omega) - \frac{\omega}{2}\bar{\aleph}'(\tilde{\gamma}_\omega) \;\simeq\; \bar{\aleph}(\tilde{\gamma}_\omega) - \frac{1}{2}\bar{\aleph}(\tilde{\gamma}_\omega)\bar{\aleph}'(\tilde{\gamma}_\omega)$$

which can be rewritten as

$$\omega \;=\; \frac{\alpha_s N_c}{\pi}\Big\{\bar{\chi}(\tilde{\gamma}_\omega) + \frac{\alpha_s N_c}{4\pi}\Big[\bar{\delta}(\tilde{\gamma}_\omega) - 2\bar{\chi}(\tilde{\gamma}_\omega)\bar{\chi}'(\tilde{\gamma}_\omega)\Big]\Big\}$$

Anomalous dimensions at $j \to 1$ from comparison of the two limits

In the BFKL limit

$$j - 1 = \omega \to 0, \quad \frac{\alpha_s}{\omega} \sim 1$$

the anomalous dimensions are represented as a sum of series in $\frac{\alpha_s}{\omega}$

$$\tilde{\gamma}_\omega = \sum a_n \left(\frac{\alpha_s}{\omega}\right)^n + \omega \sum b_n \left(\frac{\alpha_s}{\omega}\right)^n + \omega^2 \sum c_n \left(\frac{\alpha_s}{\omega}\right)^n + ...$$

So we expand in power of $\tilde{\gamma}$

$$\chi(\gamma) = \psi(1-\varepsilon_*) - \gamma_E - \psi(\gamma) - \psi(1-\varepsilon_* - \gamma) = \frac{1}{\gamma} + 2\varepsilon_*\zeta(3)\gamma + ... = \frac{1}{\gamma} + O(\gamma)$$

$$\omega = \frac{\alpha_s N_c}{\pi}\left\{\frac{1}{\tilde{\gamma}_\omega} + \frac{\alpha_s N_c}{4\pi}\left[-\frac{11}{3}\frac{1}{\tilde{\gamma}_\omega^2} + 2\zeta(3) - \frac{395}{27} + \frac{11}{18}\pi^2 + O(\tilde{\gamma}_\omega)\right]\right\}$$

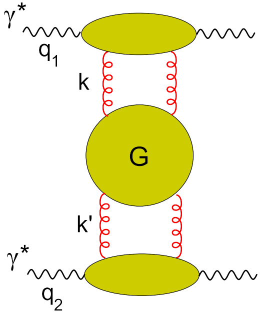
which is exactly equation (23) of Fadin-Lipatov NLO BFKL paper (1998) the corresponding anomalous dimension is

$$\tilde{\gamma}_\omega = \frac{\alpha_s N_c}{\omega} + O\Big(\frac{\alpha_s}{\omega}\Big)^4$$
$$+\omega\Big[-\frac{11}{12}\frac{\alpha_s N_c}{\pi\omega} - \Big(\frac{\alpha_s N_c}{\pi\omega}\Big)^3\frac{1}{4}\Big(\frac{395}{27} - \frac{11}{18}\pi^2 - 2\zeta(3)\Big) + O\Big(\frac{\alpha_s}{\omega}\Big)^4\Big]$$

We see that at the critical point $d = 4 - 2\varepsilon_*$ we get the same anomalous dimensions as at $d = 4$ in accordance with analysis of Braun, Manashov, Moch, Strohmaier (2016-2017)

Conclusions

- BK in d-dimension is useful if we go to the Wilson-Fisher point
  - BFKL/BK are conformal invariant and eigenfunctions are power-like
  - We recovered the NLO running coupling from LO BK (without the double log term)
- At Wilson-Fisher point $d = 4 - 2\varepsilon_*$, BFKL gives the correct anomalous dimensions of light-ray operator $F_{+i}\nabla^{j-2}$ as $j \to 1$ (in all orders in α_s)
- The formalism could allow us to calculate the running coupling part of the NNLO BFKL/BK



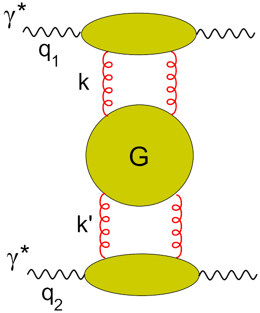
$$\frac{\partial}{\partial Y} G(k,k',Y) = \int d^2q\, K(k,q)\, G(q,k',Y)$$

$$G(k,k',Y=0) = \frac{1}{2\pi k}\delta(k-k')$$

$k \equiv |\vec{k}_\perp|$ and $k' \equiv |\vec{k}'_\perp|$

$Y = \ln \frac{s}{k\,k'}$ and $s = (q_1+q_2)^2$

Balitsky-Fadin-Kuraev-Lipatov equation



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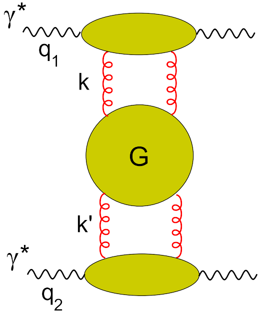

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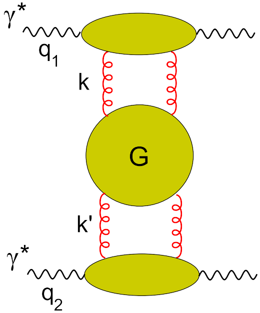

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- Resum $(\alpha_s\, Y)^n \longrightarrow$ LO BFKL eq.
- Resum $\alpha_s\,(\alpha_s\, Y)^n \longrightarrow$ NLO BFKL eq.
- The kernel is real and symmetric: $K(k,k') = K(k',k) \Rightarrow K(k,k')$ is Hermitian and the eigenvalues are real.

LO BFKL equation

$$\frac{\partial}{\partial Y}G(k,k',Y) = \int d^2q\, K^{\rm LO}(k,q)\, G(q,k',Y)$$

$$\int d^2q\, K^{\rm LO}(k,q)\ (q^2)^{-1+\gamma} = \bar{\alpha}_\mu\, \chi_0(\gamma)(k^2)^{-1+\gamma} \qquad \bar{\alpha}_\mu \equiv \frac{\alpha_\mu N_c}{\pi}$$

- $(k^2)^{-1+\gamma}$ are eigenfunctions.

LO BFKL equation

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- $(k^2)^{-1+\gamma}$ are eigenfunctions.
- For $\gamma = \frac{1}{2} + i\nu$ and ν real parameter $\Rightarrow$ $(k^2)^{-1+\gamma}$ form a complete set.
- $\Rightarrow$ LO eigenvalues $\chi_0(\nu) = 2\,\psi(1) - \psi(\frac{1}{2} + i\nu) - \psi(\frac{1}{2} - i\nu)$ are real and sym. $\nu \leftrightarrow -\nu$
- LO BFKL is Conformal invariant.

$$G(k, k', Y) = \int \frac{d\nu}{2\pi^2\, k\, k'} \left(\frac{k^2}{k'^2}\right)^{i\nu} e^{\bar{\alpha}_\mu \chi_0(\nu)\, Y}$$

BFKL equation in the $\mathcal{N}$=4 SYM case

- In $\mathcal{N} = 4$ SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$ are eigenfunctions at any order.

$$K(q,k) = \alpha_{\rm SYM} K^{\rm LO}(q,k) + \alpha^2_{\rm SYM} K^{\rm NLO}(q,k) + \dots$$

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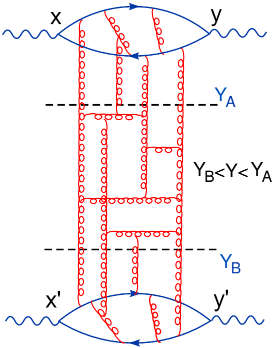
$$\int d^2q\, K(q,k)\, (q^2)^{-\frac{1}{2}+i\nu} = [\alpha_{\rm SYM}\chi_0(\nu) + \alpha^2_{\rm SYM}\chi_1(\nu)\dots]\,(k^2)^{-\frac{1}{2}+i\nu}$$

$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2 k\, k'}\, e^{[\alpha_{\rm SYM}\chi_0(\nu)+\alpha^2_{\rm SYM}\chi_1(\nu)\dots]} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

- The eigenvalues $\bar\alpha_\mu\,\chi_0(\nu) + \bar\alpha^2_\mu\,\chi_1^{\rm SYM}(\nu) + \dots$ are real and symmetric for $\nu \leftrightarrow -\nu$.

$\gamma^*\gamma^*$ scattering cross-section at LO



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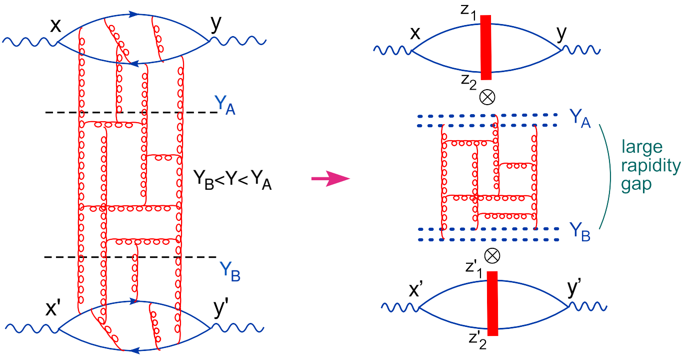
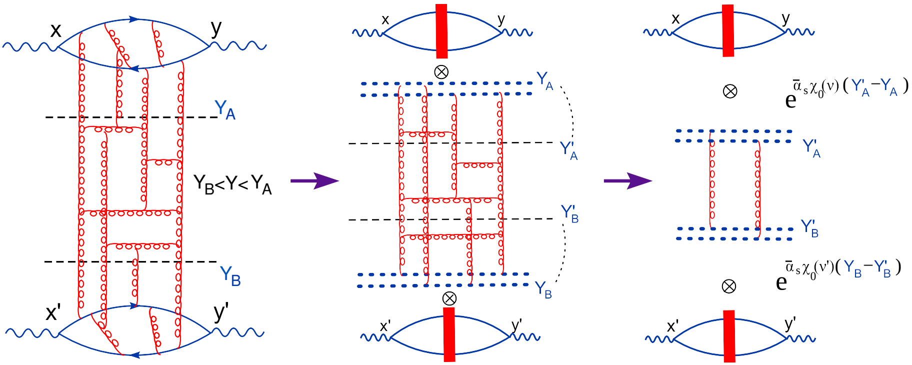


$$\langle j^\alpha(x) j^\beta(y) j^\rho(x') j^\lambda(y')\rangle \propto I_A^{\alpha\beta}(x,y;z_1,z_2)\, I_B^{\rho\lambda}(x',y';z'_1,z'_2)$$
$$\otimes\langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle$$

$\gamma^*\gamma^*$ scattering cross-section at LO



$$\langle j^\alpha(x)j^\beta(y)j^\rho(x')j^\lambda(y')\rangle \propto I_A^{\alpha\beta}(x,y;z_1,z_2)\,I_B^{\rho\lambda}(x',y';z'_1,z'_2)$$
$$\otimes\langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\rangle_A\langle \mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle_A$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1,q_2) \propto i\frac{\alpha_s^2}{Q_1Q_2}\int d\nu\, I_{\rm LO}^{\alpha\beta}(\nu)\, I_{\rm LO}^{\rho\lambda}(\nu)\,\left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar\alpha_\mu\,\chi_0(\nu)\,(Y_A-Y_B)}$$

$$Y_A = \frac{1}{2}\ln\frac{s}{Q_1^2}, \qquad Y_B = -\frac{1}{2}\ln\frac{s}{Q_2^2}, \qquad s=(q_1+q_2)^2$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1,q_2) \propto i\frac{\alpha_s^2}{Q_1Q_2}\int d\nu\, I_{\rm LO}^{\alpha\beta}(\nu)\, I_{\rm LO}^{\rho\lambda}(\nu)\,\left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar\alpha_\mu\,\chi_0(\nu)\,\ln\frac{s}{Q_1Q_2}}$$

NLO structure functions for DIS off a large nucleus

$$(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int\frac{d^2z_1 d^2z_2}{z_{12}^4}\,\Big\{I^{\mu\nu}_{\rm LO}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$$

$$+\int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)}-2C\Big)I^{\mu\nu}_{\rm LO}+\ I^{\mu\nu}_2\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\}-N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}$$

where

$$(I_2)_{\mu\nu}(z_1,z_2,z_3)\ =\frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)}+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}$$

$$+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big]+\frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big]$$

$$+(\zeta_1\leftrightarrow\zeta_2)\Big\}$$